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AP CALCULUS AB & BC

The Complete AP Calculus Course.

Clear explanations, worked examples, full practice sets with answers and explanations, challenge problems, and three full-length practice exams.

INSIDE

Limits · Derivatives · Applications of Derivatives · Integrals · Applications of Integrals · Techniques of Integration · Series, Parametrics, Vectors, and Polar · Extensions · Practice and Review · Challenge Problems · Formula Reference · Three Practice Exams

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SCHOLAR COACH AP CALCULUS COMPLETE COURSE

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Twelve parts of a complete AP Calculus course, from limits to three full practice exams.

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Each section has an explanation, worked examples, and practice problems. Answers and explanations close every chapter.

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SCHOLAR COACH AP CALCULUS COMPLETE COURSE

Welcome to AP Calculus.

Everything you need in one place: the ideas, the methods, the practice, and the exams.

Welcome

How to Use This Book

A complete AP Calculus course: clear explanations, worked examples, thousands of practice problems with full explanations, and three practice exams.

What is inside

Chapters 1 to 5 cover the Calculus AB course: limits, derivatives and their applications, integrals and their applications, and differential equations. Chapters 6 and 7 add the Calculus BC topics: integration techniques, L'Hopital's rule, improper integrals, series, parametric equations, vector-valued functions, and polar coordinates. Chapter 8 offers extensions for after the exam, Chapter 9 is mixed practice and review, Chapter 10 holds challenge problems, the Appendix is a formula reference, and three full practice exams close the book.

If you are taking	Study
Calculus AB	Chapters 1 to 5, Chapter 9 sets 9.1 to 9.8, and Practice Exams 1 and 3.
Calculus BC	Chapters 1 to 7, all of Chapter 9, and Practice Exam 2 (plus Exams 1 and 3 for AB-level review).
Honors or first-year calculus	Chapters 1 to 5 in order, then Chapters 6 to 8 as extensions.

A routine that works

1. **Read.** Study the key boxes, then work each example yourself before reading its solution.
2. **Practice.** Do every Set A problem, then Sets B and C. Write full solutions on paper.
3. **Check.** Compare with the answers and explanations at the end of the chapter. Mark mistakes and note the reason.
4. **Review.** Return to missed problems the next day, then do the multiple-choice, free-response, and chapter test sections.

Calculator conventions

Problems marked **[No calculator]** have clean exact answers and are meant to be solved by hand. Problems marked **[Calculator]** allow a graphing calculator: give decimal answers correct to three places unless a problem says otherwise. On the exam, calculator use is for graphing, solving equations, numerical derivatives and definite integrals only; you must still show the setup.

Justifications

Free-response answers earn credit only when reasons are shown. Name the theorem or test you use, check its hypotheses, and state your conclusion in a full sentence that includes units when the problem has a context.

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Chapter 1: Limits.

13 sections with explanations and worked examples, then 265 practice problems. Answers and full explanations are at the end of the chapter.

1.1 Functions and Their Graphs

Review the language of functions: domain, range, transformations, composition, symmetry, and inverses. Every limit and derivative you meet later is built on these ideas.

A **function** assigns to each input exactly one output. Calculus studies how outputs change as inputs change, so before any limit or derivative you need to be fluent with the basic vocabulary: domain, range, graphs, transformations, composition, symmetry, and inverses.

Domain and Range

The **domain** of f is the set of all inputs x for which $f(x)$ is defined (as a real number). The **range** is the set of all outputs $f(x)$ that actually occur.

Unless a problem states otherwise, the domain is every real number except those that cause (1) division by zero, (2) an even root of a negative number, or (3) the logarithm of a number that is zero or negative.

EXAMPLE 1

Find the domain of $f(x) = \dfrac{\sqrt{x-3}}{x-5}$ and write it in interval notation.

SOLUTION

The square root needs $x - 3 \geq 0$, so $x \geq 3$.

The denominator is zero when $x = 5$, so $x \neq 5$.

Domain: $[3, 5) \cup (5, \infty)$.

Transformations of $y = f(x)$

$y = f(x) + k$ shifts the graph up k units (down if $k < 0$). $y = f(x - h)$ shifts it right h units (left if $h < 0$).

$y = a\,f(x)$ stretches vertically by the factor $|a|$ and reflects across the x-axis if $a < 0$. $y = f(bx)$ shrinks the graph horizontally by the factor $|b|$ (it stretches if $|b| < 1$) and reflects across the y-axis if $b < 0$.

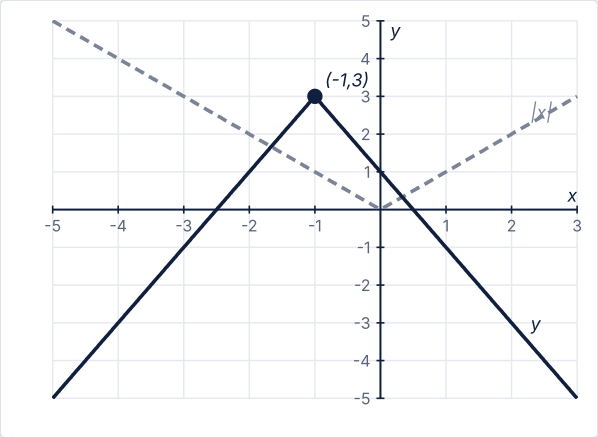
Horizontal changes act on x before f is applied, so they work in the opposite direction from what the sign first suggests.

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EXAMPLE 2

Describe how the graph of $y = -2|x+1| + 3$ is obtained from $y = |x|$, then give its vertex, domain, and range.



The graph of y = -2|x+1| + 3 (solid) and y = |x| (dashed).

SOLUTION

$|x+1|$: shift $y = |x|$ left 1 unit.

$-2|x+1|$: stretch vertically by 2 and reflect across the x-axis, so the V opens downward.

$+3$: shift up 3 units.

The vertex moves from $(0,0)$ to $(-1,3)$. Domain: all real numbers. Range: $y \le 3$, that is $(-\infty, 3]$.

Composition, Symmetry, and Inverses

Composition: $(f \circ g)(x) = f(g(x))$. Its domain is all x in the domain of g for which $g(x)$ is in the domain of f.

Even: $f(-x) = f(x)$ for every x in the domain (graph symmetric about the y-axis). **Odd:** $f(-x) = -f(x)$ (graph symmetric about the origin).

Inverse: If f is one-to-one (each output comes from only one input), then f^{-1} exists and $f^{-1}(b) = a$ exactly when $f(a) = b$. Then $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. The graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

EXAMPLE 3

Let $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-2}$. Find $f(g(x))$ and $g(f(x))$ with their domains.

SOLUTION

$f(g(x)) = (\sqrt{x-2})^2 + 1 = x - 2 + 1 = x - 1$. But g requires $x \ge 2$, so the domain is $[2, \infty)$, not all reals.

$g(f(x)) = \sqrt{(x^2+1) - 2} = \sqrt{x^2 - 1}$. This needs $x^2 \ge 1$, so the domain is $(-\infty, -1] \cup [1, \infty)$.

EXAMPLE 4

Find the inverse of $f(x) = \dfrac{2x+1}{x-3}$.

SOLUTION

Write $y = \dfrac{2x+1}{x-3}$ and swap the variables: $x = \dfrac{2y+1}{y-3}$.

Clear the denominator: $xy - 3x = 2y + 1$, so $y(x-2) = 3x + 1$.

$f^{-1}(x) = \dfrac{3x+1}{x-2}$, with $x \ne 2$.

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EXAMPLE 5

Determine whether each function is even, odd, or neither: (a) $f(x) = x^3 - 4x$, (b) $g(x) = x^4 + x$, (c) $h(x) = \dfrac{|x|}{x^2 + 1}$.

SOLUTION

(a) $f(-x) = -x^3 + 4x = -(x^3 - 4x) = -f(x)$, so f is **odd**.

(b) $g(-x) = x^4 - x$, which equals neither $g(x)$ nor $-g(x)$, so **neither**.

(c) $h(-x) = \dfrac{|-x|}{(-x)^2 + 1} = \dfrac{|x|}{x^2 + 1} = h(x)$, so h is **even**.

Common mistakes

(1) $f^{-1}(x)$ means the inverse function, not $\dfrac{1}{f(x)}$. (2) $f(g(x))$ and $g(f(x))$ are usually different. (3) When you simplify a composition, keep the domain of the original expression: $(\sqrt{x})^2$ simplifies to x only for $x \geq 0$.

Practice

SET A Skills

Work each problem. [No calculator]

1. Find the domain of $f(x) = \sqrt{5 - 2x}$.
2. Find the domain of $g(x) = \dfrac{x + 4}{x^2 - 9}$.
3. Find the domain of $h(x) = \ln(x^2 - 4)$.
4. Find the range of $f(x) = -2(x - 1)^2 + 5$.
5. Starting from $y = \sqrt{x}$, describe the transformations that produce $y = -\sqrt{x + 3} + 2$, and state the domain and range of the result.
6. Let $f(x) = x^2 - 3x$ and $g(x) = 2x + 1$. Find (a) $f(g(x))$ and (b) $g(f(x))$, fully simplified.
7. Classify each function as even, odd, or neither.
   (a) $p(x) = x^5 + 3x$
   (b) $q(x) = x^4 + x$
   (c) $r(x) = \dfrac{x^2}{|x| + 1}$
8. Let $f(x) = \begin{cases} x^2 - 1 & x < 2 \\ 3x - 5 & x \geq 2 \end{cases}$. Find $f(-3)$, $f(2)$, and $f(5)$.
9. Find the inverse function.
   (a) $f(x) = 3x - 7$
   (b) $g(x) = (x - 2)^3 + 1$

SET B Applications and Mixed Practice

Use a calculator where helpful and round decimals to three places. [Calculator]

10. The radius of a spherical balloon (in centimeters) is $r(t) = 2 + 0.5t$ after t seconds, and its volume is $V(r) = \frac{4}{3}\pi r^3$. Write V as a function of t and find the volume at $t = 4$ seconds. Give your answer to three decimal places.
11. An open-top box has a square base of side x cm and volume $500\ \text{cm}^3$. Its surface area is the base plus four sides.
    (a) Show that $S(x) = x^2 + \dfrac{2000}{x}$.
    (b) State the domain of S for this context.
    (c) Find $S(8)$ to three decimal places.
12. The temperature conversion $F(C) = \frac{9}{5}C + 32$ gives degrees Fahrenheit from degrees Celsius. Find the inverse function $C(F)$ and use it to convert 100 degrees Fahrenheit to Celsius, to three decimal places.
13. An electric bill (in dollars) for k kilowatt-hours is $C(k) = \begin{cases} 0.12k & 0 \leq k \leq 300 \\ 36 + 0.20(k - 300) & k > 300 \end{cases}$.
    (a) Find $C(250)$ and $C(450)$.
    (b) For what k is $C(k) = 80$?

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14. A ball is thrown upward from a height of 1.5 m so that $h(t) = -4.9t^2 + 20t + 1.5$ meters after t seconds, until it lands at $h = 0$.
(a) Find the time it lands, to three decimal places.
(b) State the domain and range of h for this context (three decimal places).

15. An oil slick spreads in a circle whose radius in meters is $r(t) = 3t$ after t minutes. Write the area as a function of t and find the area at $t = 2.5$ minutes. Give your answer to three decimal places.

16. Let $f(x) = x^3 + 2x$. Explain why f has an inverse, then use your calculator to find $f^{-1}(5)$ to three decimal places.

SET C Challenge and AP-Style Problems

Justify your answers. [No calculator]

17. Let $f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x+1}$. Find the domain of $f(g(x))$.

18. Let $f(x) = 2x + 3$.
(a) Find $g(x)$ such that $f(g(x)) = x^2 + 5$.
(b) Find $h(u)$ such that $h(f(x)) = 4x^2 + 12x + 7$.

19. (a) Suppose f is odd and $f(0)$ is defined. Explain why $f(0) = 0$.
(b) Show that the product of an even function and an odd function is odd.

20. Let $f(x) = x^3 + x + 1$.
(a) Find $f^{-1}(3)$ and $f^{-1}(11)$.
(b) Explain why f has an inverse.

21. Let $f(x) = \begin{cases} x + k & x < 1 \\ x^2 - 3 & x \geq 1 \end{cases}$. Find the value of k for which the graph of f has no break at $x = 1$.

1.2 Average Rate of Change and the Slope of a Line

Slope measures steepness for lines, and average rate of change extends it to curves through secant lines. The difference quotient packages that idea and leads straight to the derivative.

The **slope** of a line measures how fast y changes as x changes. When a curve is not a line, we can still measure the average change over an interval. That idea, the average rate of change, is the stepping stone to the derivative.

Slope and Equations of Lines

The slope of the line through (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Point-slope form: $y - y_1 = m(x - x_1)$. Slope-intercept form: $y = mx + b$. Parallel lines have equal slopes. Perpendicular lines (neither vertical) have slopes that multiply to -1.

EXAMPLE 1

Find the equation of the line through $(-2, 5)$ and $(4, -1)$.

SOLUTION

$m = \dfrac{-1 - 5}{4 - (-2)} = \dfrac{-6}{6} = -1.$

Point-slope with $(-2, 5)$: $y - 5 = -1(x + 2)$.

Simplify: $y = -x + 3$. Check with the other point: $-4 + 3 = -1$.

Average Rate of Change and the Secant Line

The **average rate of change** of f on $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}.$$

It is the slope of the **secant line** through $(a, f(a))$ and $(b, f(b))$.

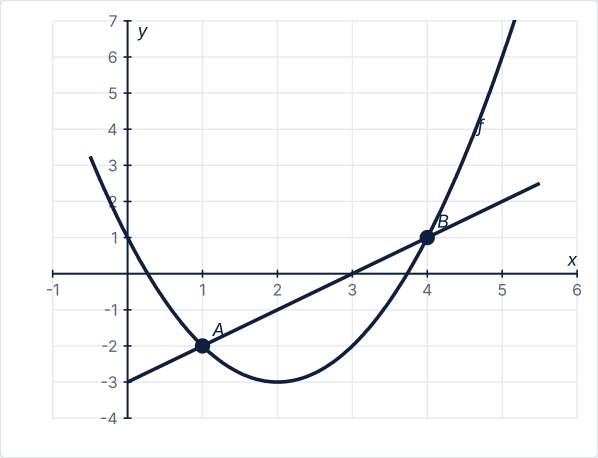
Its units are (units of f) per (units of x), for example meters per second.

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EXAMPLE 2

Let $f(x) = x^2 - 4x + 1$. Find the average rate of change on $[1, 4]$ and the equation of the secant line.



The parabola y = x^2 - 4x + 1 and the secant line through A(1, -2) and B(4, 1).

SOLUTION

$f(1) = 1 - 4 + 1 = -2$ and $f(4) = 16 - 16 + 1 = 1$.

Average rate of change $= \dfrac{1 - (-2)}{4 - 1} = \dfrac{3}{3} = 1$.

Secant line through $(1, -2)$ with slope 1: $y + 2 = x - 1$, so $y = x - 3$.

t (s)	2	6	10
s (m)	7	27	63

EXAMPLE 3

The table shows the distance s (in meters) that a cart has rolled down a track after t seconds. Find the average velocity on $[2, 6]$ and on $[6, 10]$, with units.

SOLUTION

On $[2, 6]$: $\dfrac{s(6) - s(2)}{6 - 2} = \dfrac{27 - 7}{4} = 5$ meters per second.

On $[6, 10]$: $\dfrac{s(10) - s(6)}{10 - 6} = \dfrac{63 - 27}{4} = 9$ meters per second.

The cart is speeding up, since the second average is larger.

The Difference Quotient

The **difference quotient** of f at x is

$$\frac{f(x + h) - f(x)}{h}, \qquad h \neq 0.$$

It is the average rate of change of f on $[x, x + h]$. To simplify it: substitute carefully (use parentheses), expand, cancel terms, then factor h out of the numerator and cancel it.

EXAMPLE 4

Simplify the difference quotient of $f(x) = 3x^2 - 2x$.

SOLUTION

$f(x + h) = 3(x + h)^2 - 2(x + h) = 3x^2 + 6xh + 3h^2 - 2x - 2h$.

Subtract $f(x) = 3x^2 - 2x$: $f(x + h) - f(x) = 6xh + 3h^2 - 2h$.

Divide by h: $\dfrac{h(6x + 3h - 2)}{h} = 6x + 3h - 2$.

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EXAMPLE 5

Simplify the difference quotient of $f(x) = \dfrac{1}{x}$.

SOLUTION

$\dfrac{\frac{1}{x+h} - \frac{1}{x}}{h}$. Use the common denominator $x(x+h)$ in the numerator: $\dfrac{1}{x+h} - \dfrac{1}{x} = \dfrac{x - (x+h)}{x(x+h)} = \dfrac{-h}{x(x+h)}$.

Divide by h: $\dfrac{-h}{x(x+h)} \cdot \dfrac{1}{h} = -\dfrac{1}{x(x+h)}$.

Common mistakes

(1) Writing $f(x+h) = f(x) + h$. Substitute $x+h$ for every x instead. (2) Forgetting that $(x+h)^2 = x^2 + 2xh + h^2$, not $x^2 + h^2$. (3) Reversing the order of subtraction in the numerator and denominator: use $f(b) - f(a)$ over $b - a$, not over $a - b$.

Practice

SET A Skills

Work each problem. [No calculator]

1. Find the slope of the line through $(3, -4)$ and $(-1, 4)$, and write its equation in slope-intercept form.
2. The line ℓ has equation $3x - 2y = 8$. Write the equation of the line through $(1, 5)$ that is
   (a) parallel to ℓ,
   (b) perpendicular to ℓ.
3. Find the average rate of change of $f(x) = x^3 - 2x$ on $[-1, 3]$.
4. Find the average rate of change of $g(x) = \sqrt{x+1}$ on $[3, 8]$.
5. Find the average rate of change of $h(x) = \dfrac{2}{x}$ on $[1, 4]$.
6. Find an equation of the secant line to $f(x) = x^2 + 2x$ through the points with $x = 0$ and $x = 3$.
7. The table gives the position $s(t)$ of a particle moving on a line (s in meters, t in seconds).
   (a) Find the average velocity on $[2, 9]$.
   (b) On which of the intervals $[0, 2]$, $[2, 5]$, $[5, 9]$ is the average velocity greatest?

t	0	2	5	9
$s(t)$	3	11	26	42

8. Simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ for $f(x) = x^2 + 5x$.
9. Simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ for $f(x) = 4 - 3x^2$.

SET B Applications and Mixed Practice

Round decimal answers to three places. [Calculator]

10. A cooling liquid has temperature $T(t) = 20 + 60e^{-0.1t}$ degrees Celsius after t minutes. Find the average rate of change of T on $[0, 5]$ to three decimal places, with units.
11. The table shows the population of a bacterial colony (in thousands) t hours after it was first measured.
    (a) Find the average rate of change on $[0, 5]$ and on $[5, 10]$, with units.
    (b) Find the average rate of change on $[0, 10]$. Is it the average of your two answers in (a)? Explain.

t (hours)	0	5	10
P (thousands)	1.2	1.5	1.98

12. A drone's distance from its launch point is $s(t) = 8\sqrt{t}$ meters after t seconds. Find its average velocity on $[2, 7]$ to three decimal places.

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13. Let $f(x) = x^3$. Use a calculator to find the average rate of change of f on
(a) $[2, 2.1]$, (b) $[2, 2.01]$, (c) $[2, 2.001]$ (three decimal places).
(d) What value do the slopes of these secant lines seem to approach?

14. The cost, in dollars, of producing q items is $C(q) = 500 + 12q + 0.02q^2$.
(a) Find the average rate of change of cost from $q = 100$ to $q = 150$, with units.
(b) Find the average cost per item $\dfrac{C(150)}{150}$ to three decimal places.

15. The concentration of a reactant is $A(t) = \dfrac{2}{1 + 0.5t}$ moles per liter after t seconds.
(a) Find the average rate of change on $[0, 4]$ and on $[4, 10]$ to three decimal places.
(b) What do the answers say about the speed of the reaction?

16. The concentration of a medicine in the blood is $c(t) = \dfrac{5t}{t^2 + 1}$ milligrams per liter, t in hours.
(a) Find the average rate of change on $[0.5, 2]$.
(b) Does your answer mean the concentration stayed constant on that interval? Explain.
(c) Find the average rate of change on $[0, 1]$.

SET C Challenge and AP-Style Problems

Show your reasoning. [No calculator]

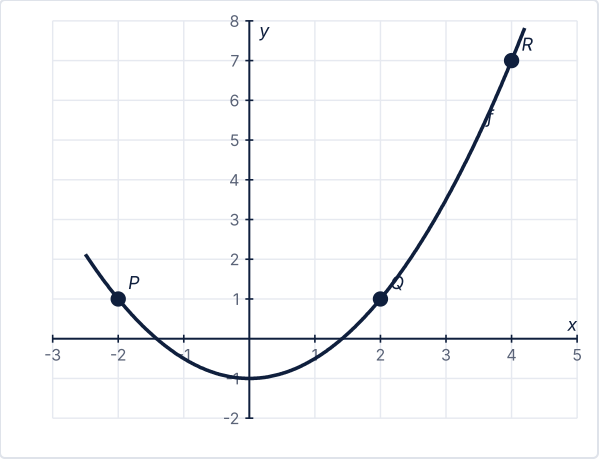
17. For which value of a is the average rate of change of $f(x) = x^2$ on $[a, a + 2]$ equal to 6?

18. A function f satisfies $f(1) = 5$, the average rate of change of f on $[1, 3]$ is 4, and the average rate of change on $[3, 6]$ is -1. Find $f(6)$.

19. Simplify the difference quotient of $f(x) = \sqrt{2x + 1}$ so that h no longer appears in a denominator. (Hint: multiply by the conjugate.)

20. (a) Show that for $f(x) = px^2 + qx + r$, the average rate of change on $[m, n]$ equals $p(m + n) + q$.
(b) Use this to find the average rate of change of $f(x) = 3x^2 - 5x + 1$ on $[2, 6]$.

21. The graph of $f(x) = \frac{x^2}{2} - 1$ is shown with the points $P(-2, 1)$, $Q(2, 1)$, and $R(4, 7)$.
(a) Find the slope of secant line PR.
(b) Find the slope of secant line QR.
(c) Find the slope of secant line PQ and explain what it means about the graph.



Graph of f with points P, Q, R.

1.3 Evaluating Limits Algebraically

Learn to find limits exactly using limit laws, direct substitution, factoring, conjugates, and complex fractions. These skills are the working tools behind every derivative definition.

A limit describes the value a function approaches as the input approaches a number. Most limits on the AP exam are found with algebra: either by substitution or by rewriting the expression so that the trouble disappears.

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Limit Laws and Direct Substitution

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ both exist (are finite numbers), and c is a constant. Then

$\lim_{x\to a}[f(x) \pm g(x)] = L \pm M$, $\lim_{x\to a}[c\,f(x)] = cL$, $\lim_{x\to a}[f(x)g(x)] = LM$, $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$, $\lim_{x\to a}[f(x)]^n = L^n$.

Direct substitution: if f is a polynomial, or a rational function whose denominator is not 0 at a, or a root, trigonometric, exponential, or logarithmic function defined at a, then $\lim_{x\to a} f(x) = f(a)$.

EXAMPLE 1

Find $\lim_{x\to 2} \dfrac{x^2+3x-1}{x+1}$.

SOLUTION

The denominator at $x = 2$ is $3 \neq 0$, so direct substitution works.

$\dfrac{4+6-1}{3} = \dfrac{9}{3} = 3.$

The Form 0/0 and What to Do

If substitution gives $\dfrac{0}{0}$, the limit may still exist. The function agrees with a simpler expression for all $x \neq a$, and a limit ignores what happens at $x = a$ itself. So simplify first, then substitute. Three standard tools:

(1) **Factor and cancel** a common factor $(x - a)$. (2) **Multiply by the conjugate** when a square root appears. (3) **Combine fractions** in a complex fraction and cancel.

EXAMPLE 2

Find $\lim_{x\to 3} \dfrac{x^2-9}{x^2-2x-3}$.

SOLUTION

Substitution gives $\dfrac{0}{0}$.

Factor: $\dfrac{(x-3)(x+3)}{(x-3)(x+1)} = \dfrac{x+3}{x+1}$ for $x \neq 3$.

Limit $= \dfrac{3+3}{3+1} = \dfrac{3}{2}$.

EXAMPLE 3

Find $\lim_{x\to 0} \dfrac{\sqrt{x+9}-3}{x}$.

SOLUTION

Substitution gives $\dfrac{0}{0}$. Multiply top and bottom by the conjugate $\sqrt{x+9}+3$.

Numerator: $(x+9)-9 = x$. So the expression is $\dfrac{x}{x\left(\sqrt{x+9}+3\right)} = \dfrac{1}{\sqrt{x+9}+3}$ for $x \neq 0$.

Limit $= \dfrac{1}{3+3} = \dfrac{1}{6}$.

EXAMPLE 4

Find $\lim_{x\to 0} \dfrac{\frac{1}{x+4} - \frac{1}{4}}{x}$.

SOLUTION

Combine the numerator: $\dfrac{1}{x+4} - \dfrac{1}{4} = \dfrac{4-(x+4)}{4(x+4)} = \dfrac{-x}{4(x+4)}$.

Divide by x: $\dfrac{-x}{4(x+4)} \cdot \dfrac{1}{x} = \dfrac{-1}{4(x+4)}$ for $x \neq 0$.

Limit $= \dfrac{-1}{4(4)} = -\dfrac{1}{16}$.

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EXAMPLE 5

Suppose $\lim_{x\to a} f(x) = 4$ and $\lim_{x\to a} g(x) = -2$. Find $\lim_{x\to a} \dfrac{[f(x)]^2 + 3g(x)}{f(x)\,g(x)}$.

SOLUTION

Both limits exist and the denominator limit is $4(-2) = -8 \neq 0$, so the laws apply.

Numerator: $4^2 + 3(-2) = 10$. Denominator: -8.

Limit $= \dfrac{10}{-8} = -\dfrac{5}{4}$.

Common mistakes

(1) Writing $\dfrac{0}{0} = 0$ or $\dfrac{0}{0} = 1$: it is neither, it means you must simplify. (2) Cancelling terms instead of factors, such as cancelling x in $\dfrac{x+3}{x}$. (3) If substitution gives a nonzero number over 0, such as $\dfrac{5}{0}$, there is no cancellation to do: the limit does not exist as a number (Section 1.6).

Practice

SET A ### Skills

Evaluate each limit. [No calculator]

1. Evaluate $\lim_{x\to -1} (2x^3 - x + 4)$.
2. Evaluate $\lim_{x\to 4} \dfrac{x^2 - 16}{x - 4}$.
3. Evaluate $\lim_{x\to -2} \dfrac{x^2 + x - 2}{x^2 + 3x + 2}$.
4. Evaluate $\lim_{x\to 5} \dfrac{\sqrt{x+4} - 3}{x - 5}$.
5. Evaluate $\lim_{x\to 1} \dfrac{x - 1}{\sqrt{x+3} - 2}$.
6. Evaluate $\lim_{x\to 0} \dfrac{\frac{1}{x+3} - \frac{1}{3}}{x}$.
7. Evaluate $\lim_{h\to 0} \dfrac{(2+h)^3 - 8}{h}$.
8. Evaluate $\lim_{x\to 2} \dfrac{x^3 - 8}{x^2 - 4}$.
9. Suppose $\lim_{x\to a} f(x) = -3$ and $\lim_{x\to a} g(x) = 5$. Find each limit.
   (a) $\lim_{x\to a} [2f(x) - g(x)]$
   (b) $\lim_{x\to a} [f(x)g(x)]$
   (c) $\lim_{x\to a} \sqrt{[f(x)]^2 + 16}$
   (d) $\lim_{x\to a} \dfrac{f(x)}{g(x)}$

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SET B Applications and Mixed Practice

Round decimal answers to three places. [Calculator]

10. Let $f(x) = \dfrac{x^3 - 27}{x - 3}$.

(a) Why can you not substitute $x = 3$?

(b) Use a calculator to find $f(2.99)$, $f(2.999)$, $f(3.001)$, and $f(3.01)$ to three decimal places.

(c) Evaluate the limit exactly by algebra.

11. Use a table of values with $x = 0.99$ and $x = 1.01$ (three decimal places) to estimate $\lim\limits_{x \to 1} \dfrac{x - 1}{\sqrt{x + 8} - 3}$, then find the exact value by multiplying by a conjugate.

12. A dropped object falls $s(t) = 4.9t^2$ meters in t seconds. Its average velocity on $[2, 2 + h]$ is $\dfrac{s(2 + h) - s(2)}{h}$.

(a) Compute this for $h = 0.1$, $h = 0.01$, $h = 0.001$ (three decimal places).

(b) Simplify the quotient algebraically and find the limit as $h \to 0$. What does it represent?

13. A sensor reports a rate $R(t) = \dfrac{t^2 - 25}{t - 5}$ milligrams per hour, but the formula gives $\frac{0}{0}$ at $t = 5$. What value of $R(5)$ would make the reported rate fit smoothly with the values near $t = 5$? Justify with a limit.

14. Let $g(x) = \dfrac{\frac{1}{x} - \frac{1}{5}}{x - 5}$. Compute $g(4.99)$ and $g(5.01)$ to three decimal places, then find $\lim\limits_{x \to 5} g(x)$ exactly.

15. The area of a circle is $A(r) = \pi r^2$.

(a) For $r = 3$, compute $\dfrac{A(3 + h) - A(3)}{h}$ for $h = 0.1$ and $h = 0.01$ (three decimal places).

(b) Find $\lim\limits_{h \to 0} \dfrac{A(3 + h) - A(3)}{h}$ exactly and to three decimal places.

16. Evaluate $\lim\limits_{x \to -3} \dfrac{x^2 + 2x - 3}{x^2 - 9}$. Give the exact value and its decimal form to three places.

SET C Challenge and AP-Style Problems

Show your reasoning. [No calculator]

17. Find the value of k for which $\lim\limits_{x \to 2} \dfrac{x^2 + kx - 10}{x - 2}$ exists, and find the limit.

18. Find constants a and b such that $\lim\limits_{x \to 1} \dfrac{\sqrt{ax + b} - 2}{x - 1} = \dfrac{3}{4}$.

19. Let $f(x) = x^3$. Evaluate $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$ as a function of x, then find its value at $x = -2$.

20. A student writes: $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3} = \dfrac{0}{0} = 1$. Explain what is wrong and give the correct value.

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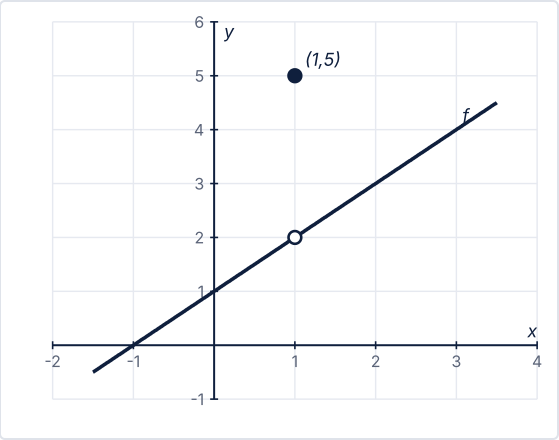
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21. Let $f(x) = \dfrac{x^2 - 1}{x - 1}$ for $x \neq 1$ and $f(1) = 5$. The graph is shown.

(a) Find $\lim_{x \to 1} f(x)$.

(b) Find $f(1)$.

(c) Explain why they are different.



Graph of f: a line with the point at x = 1 moved.

1.4 One-Sided Limits and Piecewise Functions

Study limits from the left and right, and use them to decide whether a limit exists. You will work with piecewise definitions, absolute value, and greatest-integer jumps.

Many functions are defined by different formulas on different intervals, and many have jumps. To describe their behavior at a break point we look at the left side and the right side separately.

One-Sided Limits and the Existence of a Limit

$\lim_{x \to a^-} f(x) = L$ means $f(x)$ approaches L as x approaches a through values **less than** a. $\lim_{x \to a^+} f(x) = L$ means the same through values **greater than** a.

Existence test: $\lim_{x \to a} f(x)$ exists and equals L if and only if both one-sided limits exist and

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

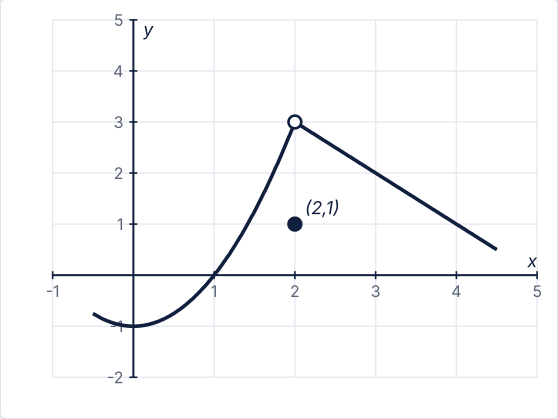
If the one-sided limits differ, or either fails to exist, the two-sided limit does not exist.

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EXAMPLE 1

Let $f(x) = \begin{cases} x^2 - 1 & x < 2 \\ 1 & x = 2 \\ 5 - x & x > 2 \end{cases}$. Find $\lim_{x \to 2^-} f(x)$, $\lim_{x \to 2^+} f(x)$, $\lim_{x \to 2} f(x)$, and $f(2)$.



The function in Example 1: both pieces head to height 3 at x = 2.

SOLUTION

For $x < 2$ use $x^2 - 1$: $\lim_{x \to 2^-} f(x) = 4 - 1 = 3$.

For $x > 2$ use $5 - x$: $\lim_{x \to 2^+} f(x) = 5 - 2 = 3$.

The one-sided limits agree, so $\lim_{x \to 2} f(x) = 3$. But $f(2) = 1$, which does not matter for the limit.

Absolute Value and Greatest Integer

$|u| = u$ if $u \geq 0$ and $|u| = -u$ if $u < 0$. So $\frac{|x|}{x} = 1$ for $x > 0$ and -1 for $x < 0$.

The greatest integer function $\lfloor x \rfloor$ is the largest integer less than or equal to x. For example $\lfloor 2.9 \rfloor = 2$, $\lfloor 3 \rfloor = 3$, $\lfloor -0.5 \rfloor = -1$. It jumps by 1 at each integer.

To find a one-sided limit of an expression with absolute values, replace each $|u|$ by u or $-u$ according to the sign of u on that side.

EXAMPLE 2

Find $\lim_{x \to 3} \frac{|x - 3|}{x - 3}$.

SOLUTION

For $x > 3$, $|x - 3| = x - 3$, so the quotient is 1. Thus the right limit is 1.

For $x < 3$, $|x - 3| = -(x - 3)$, so the quotient is -1. The left limit is -1.

The one-sided limits differ, so the limit **does not exist**.

EXAMPLE 3

Find the value of k so that $\lim_{x \to 2} f(x)$ exists, where $f(x) = \begin{cases} kx - 1 & x < 2 \\ x^2 + k & x \geq 2 \end{cases}$.

SOLUTION

Left limit: $2k - 1$. Right limit: $2^2 + k = 4 + k$.

Set them equal: $2k - 1 = 4 + k$, so $k = 5$.

Check: both one-sided limits equal 9.

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EXAMPLE 4

Find (a) $\lim_{x\to 3^-}\lfloor x\rfloor$, (b) $\lim_{x\to 3^+}\lfloor x\rfloor$, (c) $\lim_{x\to 2.5}\lfloor x\rfloor$.

SOLUTION

(a) Just left of 3 (for example 2.99) the value is 2. So the limit is 2.

(b) Just right of 3 the value is 3. So the limit is 3.

(c) Near 2.5 every x in $(2,3)$ gives 2, so the limit is 2. (The two-sided limit at $x=3$ does not exist.)

EXAMPLE 5

Find $\lim_{x\to -1}\dfrac{|x+1|}{x^2-1}$.

SOLUTION

Factor: $x^2-1=(x+1)(x-1)$.

For $x>-1$: $\dfrac{x+1}{(x+1)(x-1)}=\dfrac{1}{x-1}\to\dfrac{1}{-2}=-\dfrac{1}{2}$.

For $x<-1$: $\dfrac{-(x+1)}{(x+1)(x-1)}=\dfrac{-1}{x-1}\to\dfrac{-1}{-2}=\dfrac{1}{2}$.

The one-sided limits differ, so the limit does not exist.

Common mistakes

(1) Using the value $f(a)$ to decide the limit: the limit depends only on values near a. (2) Using the wrong piece: for $x\to a^-$ use the formula valid for $x<a$. (3) Concluding the limit exists after finding only one side. AP justification: "Since $\lim_{x\to a^-}f(x)=3$ and $\lim_{x\to a^+}f(x)=4$ are not equal, $\lim_{x\to a}f(x)$ does not exist."

Practice

SET A ### Skills

Find each limit or state that it does not exist. [No calculator]

1. Let $f(x)=\begin{cases}2x+1 & x<1\\ 5-x & x\ge 1\end{cases}$. Find $\lim_{x\to 1^-}f(x)$, $\lim_{x\to 1^+}f(x)$, $\lim_{x\to 1}f(x)$, and $f(1)$.
2. Find each limit, or state that it does not exist.
   (a) $\lim_{x\to 0^-}\dfrac{|x|}{x}$
   (b) $\lim_{x\to 0^+}\dfrac{|x|}{x}$
   (c) $\lim_{x\to 0}\dfrac{|x|}{x}$
3. Find $\lim_{x\to 4^-}\sqrt{4-x}$. Does $\lim_{x\to 4}\sqrt{4-x}$ exist? Explain.
4. Find $\lim_{x\to -2^+}\dfrac{|x+2|}{x^2+5x+6}$, $\lim_{x\to -2^-}\dfrac{|x+2|}{x^2+5x+6}$, and say whether the two-sided limit exists.
5. Find each limit.
   (a) $\lim_{x\to 3^-}\lfloor x\rfloor$
   (b) $\lim_{x\to 3^+}\lfloor x\rfloor$
   (c) $\lim_{x\to 3.5}\lfloor x\rfloor$
   (d) $\lim_{x\to -1^-}\lfloor x\rfloor$
6. Find the value of k for which $\lim_{x\to 1}f(x)$ exists, where $f(x)=\begin{cases}3x+k & x<1\\ x^2+4 & x\ge 1\end{cases}$.
7. Let $f(x)=\begin{cases}\dfrac{x^2-4}{x-2} & x<2\\ 3x-2 & x>2\end{cases}$. Find $\lim_{x\to 2}f(x)$.

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8. Find each limit.
   (a) $\lim\limits_{x\to 3^+} \dfrac{x^2-9}{|x-3|}$
   (b) $\lim\limits_{x\to 3^-} \dfrac{x^2-9}{|x-3|}$
   (c) Does $\lim\limits_{x\to 3} \dfrac{x^2-9}{|x-3|}$ exist?
9. Rewrite $f(x) = |2x-6| + x$ as a piecewise function without absolute values, and find $\lim\limits_{x\to 3} f(x)$.

SET B Applications and Mixed Practice

Round decimal answers to three places. [Calculator]

10. A parking garage charges 3 dollars for each hour or part of an hour, so the cost for staying t hours ($0 < t \le 6$) is $P(t) = 3\lceil t \rceil$ dollars, where $\lceil t \rceil$ is the smallest integer greater than or equal to t.
    (a) Find $P(2)$, $\lim\limits_{t\to 2^-} P(t)$, and $\lim\limits_{t\to 2^+} P(t)$.
    (b) Does $\lim\limits_{t\to 2} P(t)$ exist? Explain.
11. A tax is $T(x) = \begin{cases} 0.10x & 0 \le x \le 10000 \\ 1000 + 0.15(x-10000) & x > 10000 \end{cases}$ dollars on an income of x dollars.
    (a) Find $\lim\limits_{x\to 10000^-} T(x)$ and $\lim\limits_{x\to 10000^+} T(x)$.
    (b) Does $\lim\limits_{x\to 10000} T(x)$ exist? (c) Find $T(12500)$.
12. A valve controls a flow rate $R(t) = \begin{cases} 0.8t & 0 \le t < 5 \\ 10 - t & t \ge 5 \end{cases}$ liters per second.
    (a) Find $\lim\limits_{t\to 5^-} R(t)$ and $\lim\limits_{t\to 5^+} R(t)$.
    (b) Does the flow rate change smoothly at $t = 5$? Explain.
    (c) Find $R(8)$.
13. A voltage is $V(t) = \begin{cases} 0 & t < 0 \\ 12\left(1 - e^{-2t}\right) & t \ge 0 \end{cases}$ volts.
    (a) Find $\lim\limits_{t\to 0^-} V(t)$ and $\lim\limits_{t\to 0^+} V(t)$.
    (b) Find $V(0.5)$ and $V(1.5)$ to three decimal places.
14. Let $f(x) = \dfrac{|x^2-4|}{x-2}$.
    (a) Compute $f(1.99)$ and $f(2.01)$ to three decimal places.
    (b) Find $\lim\limits_{x\to 2^-} f(x)$, $\lim\limits_{x\to 2^+} f(x)$ exactly, and decide if $\lim\limits_{x\to 2} f(x)$ exists.
15. A shipping fee is $C(w) = 5 + 2\lfloor w \rfloor$ dollars for a package of w kilograms.
    (a) Find $\lim\limits_{w\to 3^-} C(w)$, $\lim\limits_{w\to 3^+} C(w)$, and $C(3)$.
    (b) Find $C(3.7)$.
16. Let $f(x) = \begin{cases} e^x & x < 0 \\ 1 + \sin x & x \ge 0 \end{cases}$ (radians).
    (a) Find $\lim\limits_{x\to 0^-} f(x)$ and $\lim\limits_{x\to 0^+} f(x)$.
    (b) Does $\lim\limits_{x\to 0} f(x)$ exist? (c) Find $f(-0.1)$ and $f(0.1)$ to three decimal places.

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SET C Challenge and AP-Style Problems

Justify your answers. [No calculator]

17. Find constants a and b so that $\lim_{x\to 1} f(x)$ and $\lim_{x\to 3} f(x)$ both exist, where $f(x) = \begin{cases} x+2 & x<1 \\ ax+b & 1\le x<3 \\ x^2-2 & x\ge 3 \end{cases}$.

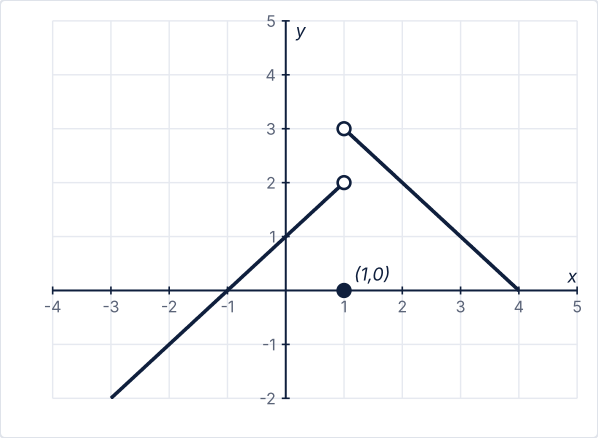
18. If $\lim_{x\to a}|f(x)|$ exists, must $\lim_{x\to a} f(x)$ exist? Give an example to support your answer.

19. The graph of f is shown.

(a) Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

(b) Does $\lim_{x\to 1} f(x)$ exist? Find $f(1)$.

(c) Find $\lim_{x\to 3} f(x)$.



Graph of f.

20. Evaluate $\lim_{x\to 0} \dfrac{|x-1|-|x+1|}{x}$.

21. Find each limit, or state that it does not exist.

(a) $\lim_{x\to 2^-} \lfloor x^2 \rfloor$

(b) $\lim_{x\to 2^+} \lfloor x^2 \rfloor$

(c) $\lim_{x\to 0} \lfloor x^2 \rfloor$

(d) $\lim_{x\to 0} \lfloor x \rfloor$

1.5 Limits from Graphs and Tables

Read limits from graphs, estimate them from tables, and learn to recognize when a limit does not exist. You will practice separating the limit from the function value.

Graphs and tables give a way to find or estimate limits without a formula. The key idea never changes: a limit asks what output value the function approaches as the input gets close to a, from both sides, and it ignores what happens exactly at a.

Reading Limits from a Graph

To find $\lim_{x\to a} f(x)$: put one finger on the graph on each side of $x=a$ and slide toward $x=a$. Read the height each finger approaches.

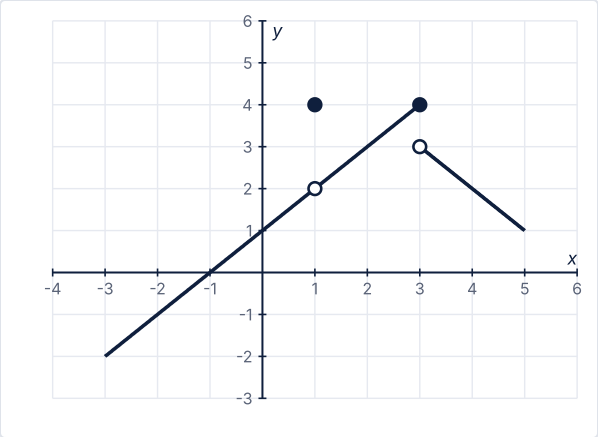
An **open circle** marks a height that is approached but not attained. A **closed circle** marks the actual value $f(a)$. The limit may differ from $f(a)$, and may exist even if $f(a)$ is undefined.

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EXAMPLE 1

Use the graph of f to find $\lim_{x\to 1} f(x)$, $f(1)$, $\lim_{x\to 3^-} f(x)$, $\lim_{x\to 3^+} f(x)$, $\lim_{x\to 3} f(x)$, and $f(3)$.



Graph of f used in Example 1.

SOLUTION

Near $x = 1$ the graph approaches height 2 from both sides (the open circle), so $\lim_{x\to 1} f(x) = 2$. The solid dot says $f(1) = 4$.

At $x = 3$ the left branch approaches 4 and the right branch starts near 3: $\lim_{x\to 3^-} f(x) = 4$ and $\lim_{x\to 3^+} f(x) = 3$.

The one-sided limits differ, so $\lim_{x\to 3} f(x)$ does not exist. The solid dot gives $f(3) = 4$.

Estimating Limits from a Table

Choose x-values that approach a from both sides, such as $a \pm 0.1$, $a \pm 0.01$, $a \pm 0.001$. If the values of $f(x)$ settle toward one number L from both sides, estimate $\lim_{x\to a} f(x) = L$.

A table gives evidence, not proof, and it can mislead. The limit **does not exist** when (1) the left and right values approach different numbers (a jump), (2) the values grow without bound (Section 1.6), or (3) the values oscillate and never settle.

EXAMPLE 2

The table shows values of $f(x) = \dfrac{x-2}{x^2-4}$. Estimate $\lim_{x\to 2} f(x)$.

SOLUTION

From the left the values $0.2564,\ 0.2506,\ 0.2501$ decrease toward 0.25.

From the right the values $0.2439,\ 0.2494,\ 0.2499$ increase toward 0.25.

Both sides agree: the limit is about $0.25 = \frac{1}{4}$. (Check: $f(x) = \frac{1}{x+2}$ for $x \neq 2$, so the limit is $\frac{1}{4}$.)

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

EXAMPLE 3

Explain why $\lim_{x\to 0} \sin\left(\dfrac{1}{x}\right)$ does not exist.

SOLUTION

As $x \to 0$, $\dfrac{1}{x}$ becomes arbitrarily large, so $\sin\left(\dfrac{1}{x}\right)$ keeps cycling through all values from -1 to 1.

At $x = \dfrac{1}{\pi/2 + 2\pi n}$ the value is 1, and at $x = \dfrac{1}{3\pi/2 + 2\pi n}$ the value is -1. Both lists of x-values approach 0 as n grows.

Since the outputs do not approach a single number, the limit does not exist.

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EXAMPLE 4

Use the graph in Example 1 to find $\lim_{x \to 1} f(f(x))$ and compare it to $f(f(1))$.

SOLUTION

As $x \to 1$, the inner value $f(x) = x + 1$ approaches 2. Near input 2 the outer function also follows the line $y = x + 1$, so f of something approaching 2 approaches 3.

So $\lim_{x \to 1} f(f(x)) = 3$.

But $f(1) = 4$ and $f(4) = 6 - 4 = 2$, so $f(f(1)) = 2$. The limit and the value differ.

Common mistakes

(1) Reading the closed dot instead of the open circle. (2) Averaging the left and right values when they disagree: the limit simply does not exist. (3) Trusting a table that uses only one side of a, or values that are not close enough to a.

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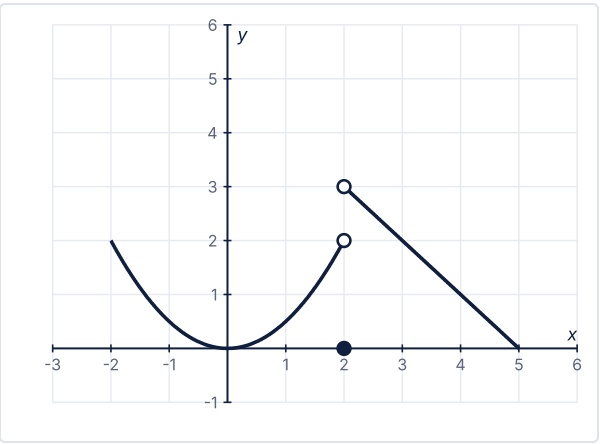
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Practice

SET A Skills

Use the graph or table given. [No calculator]

1. The graph of f is shown.
   (a) $\lim_{x\to 2^-} f(x)$
   (b) $\lim_{x\to 2^+} f(x)$
   (c) $\lim_{x\to 2} f(x)$
   (d) $f(2)$
   (e) $\lim_{x\to 4} f(x)$

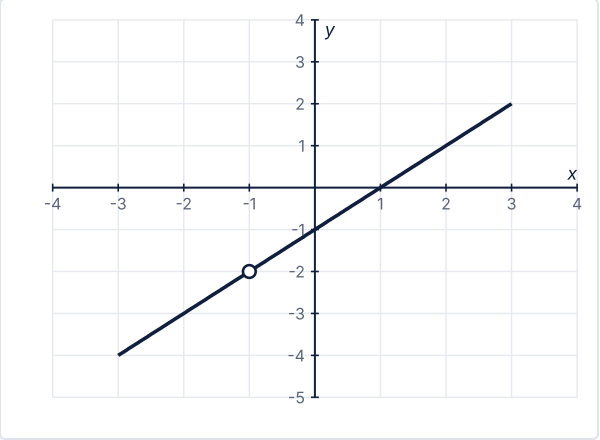


Graph of f.

2. Values of $f(x) = \dfrac{e^x - 1}{x}$ are given. Estimate $\lim_{x\to 0} f(x)$.

x	-0.1	-0.01	0.01	0.1
$f(x)$	0.9516	0.9950	1.0050	1.0517

3. The graph of g has a hole.
   (a) $\lim_{x\to -1} g(x)$
   (b) $g(-1)$
   (c) $\lim_{x\to 2} g(x)$



Graph of g.

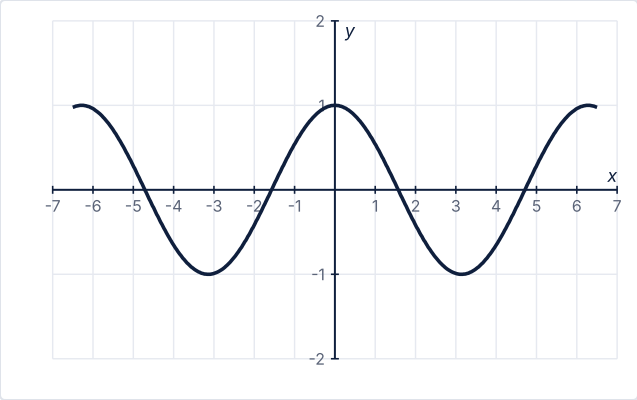
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4. Values of a function f are given near $x = 3$.
   (a) Estimate $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$.
   (b) Does $\lim_{x \to 3} f(x)$ exist?

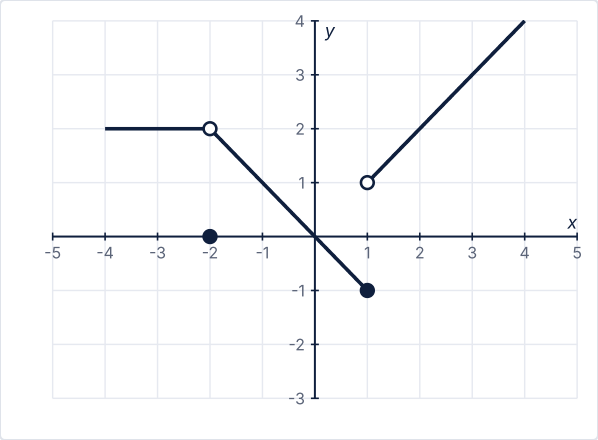
x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	1.9	1.99	1.999	4.001	4.01	4.1

5. Use the graph of $y = \cos x$ to evaluate each limit.
   (a) $\lim_{x \to 0} \cos x$
   (b) $\lim_{x \to \pi/2} \cos x$
   (c) $\lim_{x \to \pi} \cos x$



Graph of y = cos x (x in radians).

6. True or false: if $f(2)$ is undefined, then $\lim_{x \to 2} f(x)$ does not exist. Justify with an example.
7. The graph of f is shown.
   (a) $\lim_{x \to -2} f(x)$
   (b) $f(-2)$
   (c) $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$
   (d) $\lim_{x \to 3} f(x)$



Graph of f.

8. Sketch a graph of a function f with $\lim_{x \to 2} f(x) = 3$, $f(2) = 1$, $\lim_{x \to 4^-} f(x) = 0$, and $\lim_{x \to 4^+} f(x) = 2$. Describe your sketch in words.
9. Use the graph of f from the first problem of this set to find each limit, or state that it does not exist.
   (a) $\lim_{x \to 2^-} [2f(x) + 1]$
   (b) $\lim_{x \to 2^+} \sqrt{f(x) + 1}$
   (c) $\lim_{x \to 2} [f(x)]^2$

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SET B Applications and Mixed Practice

Use a calculator for numerical work and give decimals to three places. [Calculator]

10. Let $f(x) = \dfrac{2^x - 1}{x}$.

(a) Use a calculator to find $f(-0.01)$, $f(0.01)$, $f(-0.001)$, and $f(0.001)$ to three decimal places.

(b) Estimate $\lim_{x \to 0} f(x)$.

11. The distance fallen by a probe is $s(t) = t^2 + 2t$ meters after t seconds. Average velocity on $[t_1, t_2]$ is the change in s divided by the change in t.

(a) Find the average velocity on $[2.9, 3]$ and on $[3, 3.1]$.

(b) Estimate the instantaneous velocity at $t = 3$ by finding the limit of these averages as the interval shrinks to $t = 3$.

12. Let $f(x) = \dfrac{x^3 - 1}{x - 1}$.

(a) Compute $f(0.999)$ and $f(1.001)$ to three decimal places.

(b) Estimate $\lim_{x \to 1} f(x)$ and then confirm it exactly by factoring.

13. Let $f(x) = \sin\left(\dfrac{\pi}{x}\right)$ (radians).

(a) Compute $f(0.1)$, $f(0.4)$, and $f\left(\frac{2}{7}\right)$.

(b) Use these values to explain why $\lim\limits_{x \to 0} f(x)$ does not exist.

14. Let $f(x) = \dfrac{\cos x - 1}{x^2}$.

(a) Compute $f(0.1)$ and $f(0.01)$ to three decimal places.

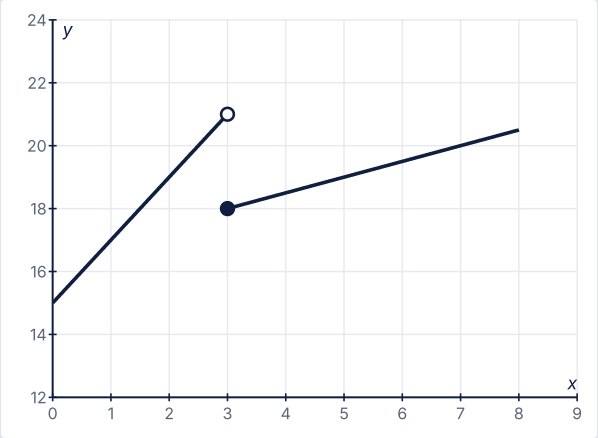
(b) Estimate $\lim_{x \to 0} f(x)$.

15. A furnace controller resets partway through the day. The graph shows room temperature T against time h.

(a) Find $\lim_{h \to 3^-} T$ and $\lim_{h \to 3^+} T$.

(b) Find $T(3)$ and say whether $\lim_{h \to 3} T$ exists.

(c) Find $\lim_{h \to 6} T$.



Room temperature T (degrees Celsius) against time h (hours).

16. Let $f(x) = (1 + x)^{1/x}$.

(a) Compute $f(0.1)$, $f(0.01)$, $f(0.001)$, and $f(-0.001)$ to three decimal places.

(b) Estimate $\lim_{x \to 0} f(x)$ to three decimal places and identify the constant.

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SET C Challenge and AP-Style Problems

Justify your answers. [No calculator]

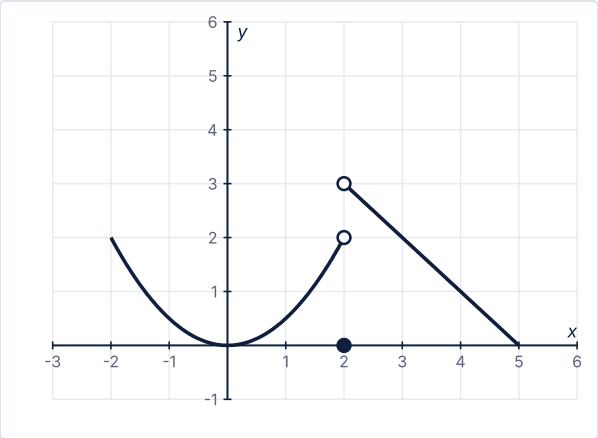
17. The graph of f is shown: $f(x) = \frac{x^2}{2}$ for $-2 \le x < 2$, $f(2) = 0$, and $f(x) = 5 - x$ for $2 < x \le 5$. Find each limit.

(a) $\lim_{x \to 0} f(f(x))$

(b) $\lim_{x \to 2^-} f(f(x))$

(c) $\lim_{x \to 2^+} f(f(x))$

(d) Does $\lim_{x \to 2} f(f(x))$ exist?



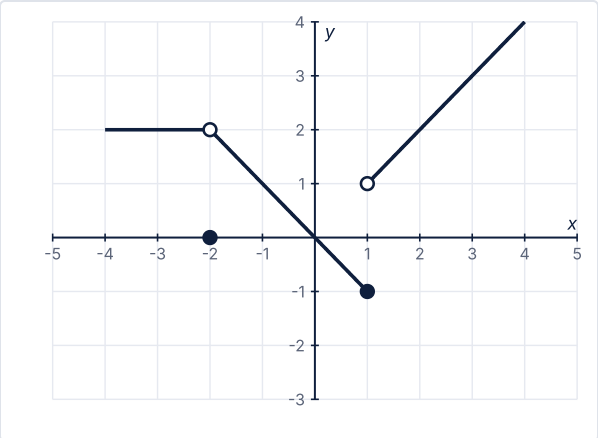
Graph of f.

18. A student computes $f(x) = \sin\left(\frac{\pi}{x}\right)$ at $x = 0.1,\ 0.01,\ 0.001$ and finds 0 every time, and concludes that $\lim_{x \to 0} f(x) = 0$. Explain the error.

19. Use the graph of f (shown).

(a) For which values c in $(-4, 4)$ does $\lim_{x \to c} f(x)$ fail to exist?

(b) For which values c in $(-4, 4)$ does the limit exist but differ from $f(c)$?



Graph of f.

20. Suppose $\lim_{x \to 2} f(x) = 5$. Explain why $f(x) > 4$ for all $x \neq 2$ sufficiently close to 2.

21. The table shows $f(x) = \dfrac{3^x - 2^x}{x}$ near 0. Estimate $\lim_{x \to 0} f(x)$ to one decimal place.

x	-0.01	0.01
$f(x)$	0.4018	0.4091

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1.6 Infinite Limits and Vertical Asymptotes

Learn what it means for a limit to be infinite, how to find vertical asymptotes of rational and trigonometric functions, and how sign analysis tells you which way a function blows up.

Sometimes the outputs of a function do not settle on a number but grow without bound. This is reported with the symbols ∞ and $-\infty$, and it signals a vertical asymptote in the graph.

Infinite Limits and Vertical Asymptotes

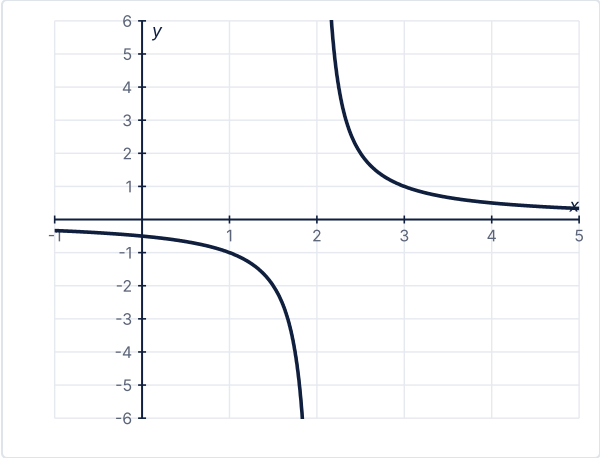
$\lim_{x\to a} f(x) = \infty$ means $f(x)$ increases without bound as $x \to a$. $\lim_{x\to a} f(x) = -\infty$ means $f(x)$ decreases without bound. The same notation is used for one-sided limits.

The line $x = a$ is a **vertical asymptote** of f if at least one of $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ is ∞ or $-\infty$.

Writing $\lim f(x) = \infty$ describes how the limit fails to exist: ∞ is not a number, and the limit does not exist as a real number.

EXAMPLE 1

Find $\lim_{x\to 2^-} \dfrac{1}{x-2}$, $\lim_{x\to 2^+} \dfrac{1}{x-2}$, and $\lim_{x\to 2} \dfrac{1}{(x-2)^2}$.



Graph of y = 1/(x - 2): the line x = 2 is a vertical asymptote.

SOLUTION

For x slightly less than 2, $x-2$ is a small **negative** number, so $\dfrac{1}{x-2}$ is a large negative number: the left limit is $-\infty$.

For x slightly greater than 2, $x-2$ is a small positive number: the right limit is ∞. So $\lim_{x\to 2} \dfrac{1}{x-2}$ does not exist.

$(x-2)^2$ is a small positive number on both sides, so $\lim_{x\to 2} \dfrac{1}{(x-2)^2} = \infty$.

Finding Vertical Asymptotes of a Rational Function

For $f(x) = \dfrac{p(x)}{q(x)}$ in lowest terms (all common factors cancelled), the vertical asymptotes are the lines $x = a$ where $q(a) = 0$.

A common factor $(x-a)$ that cancels gives a **hole** at $x = a$ (the limit is a finite number), not an asymptote.

To find the direction of the unbounded behavior, use **sign analysis**: when $x \to a^{\pm}$, find the sign of each factor near a and combine. A small positive denominator with a positive numerator gives $+\infty$; a small negative denominator with a positive numerator gives $-\infty$.

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EXAMPLE 2

Find all vertical asymptotes and holes of $f(x) = \dfrac{x^2 - x - 6}{x^2 - 4}$, and the one-sided limits at each vertical asymptote.

SOLUTION

Factor: $f(x) = \dfrac{(x-3)(x+2)}{(x-2)(x+2)} = \dfrac{x-3}{x-2}$ for $x \neq \pm 2$.

At $x = -2$ the factor cancels: a hole, with $\lim_{x \to -2} f(x) = \dfrac{-5}{-4} = \dfrac{5}{4}$.

At $x = 2$ the numerator is -1. As $x \to 2^-$, $x - 2$ is a small negative number, so $f \to \dfrac{-1}{0^-} = \infty$. As $x \to 2^+$, $f \to \dfrac{-1}{0^+} = -\infty$.

Vertical asymptote: $x = 2$ only.

EXAMPLE 3

Find $\lim_{x \to -1^-}$, $\lim_{x \to -1^+}$, $\lim_{x \to 2^-}$, $\lim_{x \to 2^+}$ of $g(x) = \dfrac{x - 5}{(x+1)(x-2)}$.

SOLUTION

Near $x = -1$ the other factors are $\dfrac{x-5}{x-2} \to \dfrac{-6}{-3} = 2 > 0$. So $g \approx \dfrac{2}{x+1}$: left limit $-\infty$, right limit ∞.

Near $x = 2$ the other factors are $\dfrac{x-5}{x+1} \to \dfrac{-3}{3} = -1 < 0$. So $g \approx \dfrac{-1}{x-2}$: left limit ∞ (negative over negative), right limit $-\infty$.

Other Functions with Vertical Asymptotes

$\tan x$ and $\sec x$ have vertical asymptotes at $x = \dfrac{\pi}{2} + n\pi$ (n any integer); $\cot x$ and $\csc x$ have them at $x = n\pi$. Also $\lim_{x \to 0^+} \ln x = -\infty$, so $x = 0$ is a vertical asymptote of $y = \ln x$.

For example, $\lim_{x \to (\pi/2)^-} \tan x = \infty$ and $\lim_{x \to (\pi/2)^+} \tan x = -\infty$, because $\tan x = \dfrac{\sin x}{\cos x}$ with $\sin x \to 1$ and $\cos x \to 0^+$ (left) or 0^- (right).

EXAMPLE 4

Find $\lim_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{x^2} \right)$.

SOLUTION

Each term is unbounded, with opposite signs near 0^+, so combine into one fraction: $\dfrac{1}{x} - \dfrac{1}{x^2} = \dfrac{x-1}{x^2}$.

As $x \to 0$ the numerator tends to -1 and $x^2 \to 0^+$ (positive from both sides).

So the limit is $\dfrac{-1}{0^+} = -\infty$.

Common mistakes

(1) Treating ∞ as a number: $\infty - \infty$ is not 0. Combine terms first. (2) Calling every zero of the denominator an asymptote: cancel common factors first, since they produce holes. (3) Forgetting the sign: $\dfrac{1}{x-2}$ goes to different infinities on the two sides, while $\dfrac{1}{(x-2)^2}$ does not. AP justification: "Since $\lim_{x \to 2^-} f(x) = \infty$, the line $x = 2$ is a vertical asymptote."

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Practice

SET A Skills

Find each limit or asymptote. [No calculator]

1. Find each limit.
   (a) $\lim\limits_{x\to 3^-} \dfrac{1}{x-3}$
   (b) $\lim\limits_{x\to 3^+} \dfrac{1}{x-3}$
   (c) $\lim\limits_{x\to 3} \dfrac{1}{x-3}$
2. Find $\lim\limits_{x\to -2} \dfrac{5}{(x+2)^2}$.
3. Find the vertical asymptotes of $f(x) = \dfrac{x+1}{x^2-x-6}$.
4. Find the vertical asymptotes and any hole of $f(x) = \dfrac{x^2-9}{x^2-5x+6}$, and give the limit at the hole.
5. Find each limit.
   (a) $\lim\limits_{x\to 4^-} \dfrac{3-x}{x-4}$
   (b) $\lim\limits_{x\to 4^+} \dfrac{3-x}{x-4}$
6. Find each limit.
   (a) $\lim\limits_{x\to 0^+} \dfrac{x-2}{x^3}$
   (b) $\lim\limits_{x\to 0^-} \dfrac{x-2}{x^3}$
7. Find each limit.
   (a) $\lim\limits_{x\to (\pi/2)^-} \tan x$
   (b) $\lim\limits_{x\to (\pi/2)^+} \tan x$
   (c) $\lim\limits_{x\to 0^+} \ln x$
8. Find $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{x^4}\right)$.
9. Values of $f(x) = \dfrac{1}{(x-1)^2}$ are given. Describe the behavior of f near $x = 1$ and write the limit statement.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	100	10000	1000000	1000000	10000	100

SET B Applications and Mixed Practice

Round decimal answers to three places where asked. [Calculator]

10. The electric force on a small charge is $F(r) = \dfrac{8}{r^2}$ newtons at a distance r meters from a fixed charge.
    (a) Find $F(0.1)$ and $F(0.01)$.
    (b) Find $\lim\limits_{r\to 0^+} F(r)$ and interpret.
11. A lens of focal length 10 cm forms an image at distance $v(u) = \dfrac{10u}{u-10}$ cm when an object is u cm away.
    (a) Compute $v(10.1)$, $v(10.01)$, and $v(9.9)$.
    (b) Find $\lim\limits_{u\to 10^+} v(u)$ and $\lim\limits_{u\to 10^-} v(u)$.
12. The cost, in thousands of dollars, of removing p percent of a pollutant is $C(p) = \dfrac{80p}{100-p}$ for $0 \le p < 100$.
    (a) Find $C(95)$, $C(99)$, and $C(99.9)$.
    (b) Find $\lim\limits_{p\to 100^-} C(p)$ and interpret.

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13. The time-dilation factor for speed v (as a fraction of the speed of light) is $\gamma(v) = \dfrac{1}{\sqrt{1 - v^2}}$ for $0 \le v < 1$.
    (a) Find $\gamma(0.9)$, $\gamma(0.99)$, and $\gamma(0.999)$ to three decimal places.
    (b) Find $\lim_{v \to 1^-} \gamma(v)$.

14. A rotating beam of light hits a straight shore at distance $d(\theta) = 3\tan\theta$ kilometers along the shore, where θ is the angle in radians from the perpendicular, $0 \le \theta < \frac{\pi}{2}$.
    (a) Find $d(1.5)$ and $d(1.57)$ to three decimal places.
    (b) Find $\lim_{\theta \to (\pi/2)^-} d(\theta)$.

15. Let $f(x) = \dfrac{2x + 1}{x^2 - 2x - 8}$.
    (a) Find the vertical asymptotes.
    (b) Find the four one-sided limits at them.
    (c) Use a calculator to find $f(4.001)$ and $f(3.999)$ to three decimal places.

16. Let $f(x) = \dfrac{x^2 - 1}{x^2 - 3x + 2}$.
    (a) Find the vertical asymptote and the hole.
    (b) Compute $f(0.999)$ and $f(1.001)$ to three decimal places and compare with the limit at the hole.

SET C Challenge and AP-Style Problems

Justify your answers. [No calculator]

17. For which values of a does $f(x) = \dfrac{x + a}{x^2 - 3x + 2}$ have exactly one vertical asymptote?

18. Write a rational function with a vertical asymptote at $x = -1$ with $\lim_{x \to -1^-} f(x) = \infty$ and $\lim_{x \to -1^+} f(x) = -\infty$, a vertical asymptote at $x = 3$ with $\lim_{x \to 3} f(x) = -\infty$, and a hole at $x = 2$.

19. Find $\lim_{x \to -3} \dfrac{x}{|x + 3|}$.

20. Explain why $f(x) = \dfrac{x^2 - 4}{x - 2}$ has no vertical asymptote but $g(x) = \dfrac{x^2 + 4}{x - 2}$ does.

21. Find each limit.
    (a) $\lim_{x \to 0^+} \dfrac{1 + \cos x}{\sin x}$
    (b) $\lim_{x \to 0^-} \dfrac{1 + \cos x}{\sin x}$
    (c) $\lim_{x \to \pi^+} \csc x$
    (d) $\lim_{x \to \pi^-} \cot x$

1.7 Limits at Infinity and Horizontal Asymptotes

You will learn to find the end behavior of a function by evaluating limits as x grows without bound, and to use those limits to identify horizontal and slant asymptotes.

A **limit at infinity** describes what happens to $f(x)$ as x grows without bound in the positive or negative direction. These limits capture the **end behavior** of a function, and they lead directly to horizontal and slant asymptotes.

Limits at Infinity and Horizontal Asymptotes

$\lim_{x \to \infty} f(x) = L$ means $f(x)$ gets arbitrarily close to L as x increases without bound. $\lim_{x \to -\infty} f(x) = L$ is defined the same way for x decreasing without bound.
If either limit equals a finite number L, the line $y = L$ is a **horizontal asymptote** of the graph. A function can have 0, 1 or 2 horizontal asymptotes.

Basic fact: for any real $p > 0$, $\lim_{x \to \infty} \dfrac{1}{x^p} = 0$, and $\lim_{x \to -\infty} \dfrac{1}{x^p} = 0$ whenever x^p is defined.

To evaluate a limit at infinity of a quotient, divide the numerator and denominator by the highest power of x that appears in the denominator, then use $\frac{1}{x^p} \to 0$.

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Rational Functions: Comparing Degrees

Let $f(x) = \dfrac{p(x)}{q(x)}$ where p has degree m and q has degree k, with leading coefficients a and b.

If $m < k$, then $\lim_{x\to\pm\infty} f(x) = 0$ (horizontal asymptote $y = 0$).

If $m = k$, then $\lim_{x\to\pm\infty} f(x) = \dfrac{a}{b}$ (horizontal asymptote $y = \frac{a}{b}$).

If $m > k$, then $|f(x)| \to \infty$ and there is no horizontal asymptote. When $m = k + 1$ the graph has a **slant asymptote**, found by long division.

EXAMPLE 1

Evaluate $\lim\limits_{x\to\infty} \dfrac{3x^2 - 5x + 2}{4x^2 + x - 7}$.

SOLUTION

The degrees are equal, so divide every term by x^2: $\dfrac{3 - \frac{5}{x} + \frac{2}{x^2}}{4 + \frac{1}{x} - \frac{7}{x^2}}$.

Each term with x in the denominator tends to 0, so the limit is $\dfrac{3}{4}$.

Common mistake

For $x < 0$, $\sqrt{x^2} = |x| = -x$, not x. When a limit as $x \to -\infty$ involves a square root, rewrite $\sqrt{x^2}$ as $-x$ before dividing. Forgetting this changes the sign of the answer.

EXAMPLE 2

Find $\lim\limits_{x\to\infty} \dfrac{\sqrt{4x^2+1}}{x+3}$ and $\lim\limits_{x\to-\infty} \dfrac{\sqrt{4x^2+1}}{x+3}$.

SOLUTION

For $x > 0$: $\sqrt{4x^2+1} = x\sqrt{4 + 1/x^2}$. Dividing by x gives $\dfrac{\sqrt{4+1/x^2}}{1+3/x} \to \dfrac{2}{1} = 2$.

For $x < 0$: $\sqrt{4x^2+1} = |x|\sqrt{4+1/x^2} = -x\sqrt{4+1/x^2}$. Dividing by x gives $\dfrac{-\sqrt{4+1/x^2}}{1+3/x} \to -2$.

So the graph has two horizontal asymptotes, $y = 2$ and $y = -2$.

EXAMPLE 3

Evaluate $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x} - x\right)$.

SOLUTION

Direct substitution gives the form $\infty - \infty$, so multiply by the conjugate over itself.

$\dfrac{(x^2+x) - x^2}{\sqrt{x^2+x}+x} = \dfrac{x}{\sqrt{x^2+x}+x}$.

Divide top and bottom by x (with $x > 0$): $\dfrac{1}{\sqrt{1+1/x}+1} \to \dfrac{1}{2}$.

Growth Rates as x Approaches Infinity

For every $p > 0$ and every base $a > 1$: $\lim_{x\to\infty} \dfrac{\ln x}{x^p} = 0$ and $\lim_{x\to\infty} \dfrac{x^p}{a^x} = 0$.

In words, $\ln x$ grows more slowly than any positive power of x, and any exponential a^x with $a > 1$ grows faster than any power of x. When two terms compete, the faster-growing term dominates.

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EXAMPLE 4

Evaluate $\lim_{x\to\infty} \dfrac{e^x + x^5}{3e^x - x^2}$.

SOLUTION

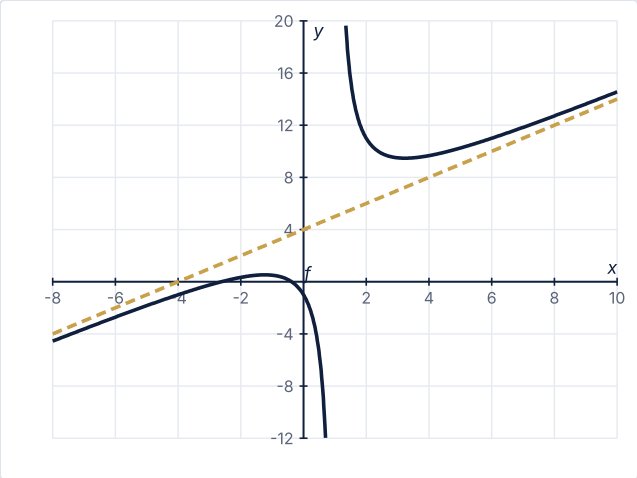
Both x^5 and x^2 are negligible next to e^x. Divide top and bottom by e^x: $\dfrac{1 + x^5 e^{-x}}{3 - x^2 e^{-x}}$.

Since $\frac{x^p}{e^x} \to 0$ for every power p, the limit is $\dfrac{1}{3}$.

When the numerator of a rational function has degree exactly one more than the denominator, the graph approaches an oblique line, called a slant asymptote. Long division reveals it: $f(x) = (\text{linear}) + \dfrac{\text{remainder}}{\text{denominator}}$, and the remainder term tends to 0.

EXAMPLE 5

Find the slant asymptote of $f(x) = \dfrac{x^2 + 3x + 1}{x - 1}$ and describe the vertical asymptote.



The graph of f (navy) and its slant asymptote y = x + 4 (gold).

SOLUTION

Long division: $x^2 + 3x + 1 = (x - 1)(x + 4) + 5$, so $f(x) = x + 4 + \dfrac{5}{x - 1}$.

As $x \to \pm\infty$, $\frac{5}{x-1} \to 0$, so $f(x)$ approaches the line $y = x + 4$. This is the slant asymptote.

At $x = 1$ the denominator is 0 and the numerator is 5, so $x = 1$ is a vertical asymptote.

Practice

SET A Skills

Evaluate each limit or find each asymptote. [No calculator]

1. $\lim_{x\to\infty} \dfrac{5x^2 - 3x + 1}{2x^2 + 7}$
2. $\lim_{x\to\infty} \dfrac{4x + 1}{x^3 - 2}$
3. $\lim_{x\to-\infty} \dfrac{3x^3 - x}{x^2 + 4}$
4. $\lim_{x\to\infty} \dfrac{\sqrt{9x^2 + x}}{2x - 1}$
5. $\lim_{x\to-\infty} \dfrac{\sqrt{16x^2 + 1}}{2x + 5}$
6. $\lim_{x\to\infty} \dfrac{2e^x + 1}{e^x - 3}$
7. $\lim_{x\to\infty} \dfrac{\sin x}{x}$
8. $\lim_{x\to\infty} \left(\sqrt{x^2 + 6x} - x\right)$
9. Find every horizontal asymptote of $f(x) = \dfrac{3x - 1}{\sqrt{x^2 + 4}}$.

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SET B Applications and Mixed Practice

Give decimal answers to three decimal places. [Calculator]

10. The concentration of a medicine in the blood is $C(t) = \dfrac{8t}{t^2 + 4}$ milligrams per liter, t hours after an injection.

(a) Find $\lim_{t \to \infty} C(t)$ and interpret it.

(b) Find the larger value of t for which $C(t) = 0.1$. Give your answer to three decimal places.

11. A culture has $P(t) = \dfrac{5000}{1 + 49e^{-0.4t}}$ bacteria after t hours.

(a) Find $\lim_{t \to \infty} P(t)$ and interpret it.

(b) At what time is the population exactly half of that limit? Give your answer to three decimal places.

12. [Calculator] The table gives values of $g(x) = x \sin\left(\frac{1}{x}\right)$ (radian mode).

(a) Use the table to estimate $\lim_{x \to \infty} g(x)$.

(b) Justify your estimate exactly by substituting $u = \frac{1}{x}$.

x	10	100	1000	10000
$g(x)$	0.998	1.000	1.000	1.000

13. A falling object has speed $v(t) = 40\left(1 - e^{-0.25t}\right)$ meters per second after t seconds.

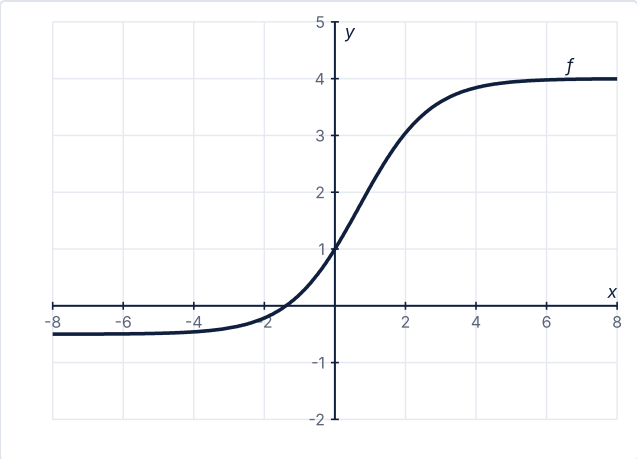
(a) Find the terminal speed $\lim_{t \to \infty} v(t)$.

(b) At what time does the speed reach 90% of the terminal speed? Give your answer to three decimal places.

14. [Calculator] The graph shows $f(x) = \dfrac{4e^x - 1}{e^x + 2}$.

(a) From the graph, estimate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

(b) Confirm both values algebraically.



Graph of f.

15. The average cost per unit of producing q units of a chemical is $A(q) = \dfrac{500 + 12q}{q}$ dollars per unit.

(a) Find $\lim_{q \to \infty} A(q)$.

(b) What does the result say about the cost of producing very large batches?

16. A cup of liquid cools according to $T(t) = 22 + 58e^{-0.15t}$ degrees Celsius, t minutes after it is poured.

(a) Find the horizontal asymptote of T and interpret it.

(b) After what time is the temperature within 1 degree of the horizontal asymptote? Give your answer to three decimal places.

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SET C Challenge and AP-Style Problems

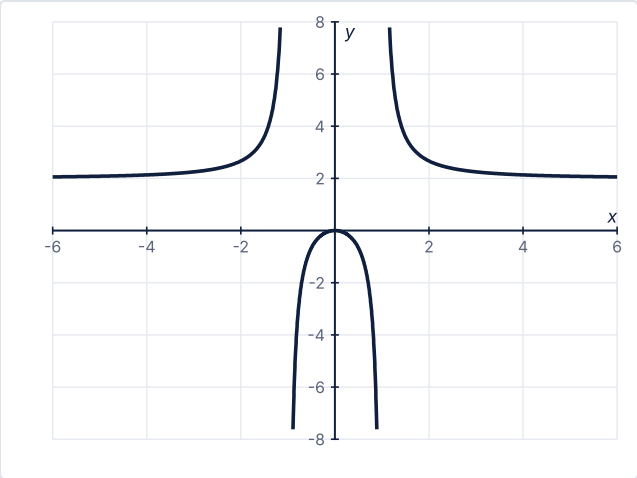
Show your reasoning. [No calculator]

17. Find the constant a for which $\lim_{x\to\infty}\left(\sqrt{4x^2+ax}-2x\right)=3$. [No calculator]

18. Find constants a and b such that $\lim_{x\to\infty}\left[\dfrac{x^2+1}{x+1}-(ax+b)\right]=0$.

19. Evaluate $\lim_{x\to\infty}\dfrac{3x+\sin x}{x-2\cos x}$ and justify each step.

20. The graph of $f(x)=\dfrac{2x^2}{x^2-1}$ is shown.
    (a) Find all horizontal asymptotes.
    (b) Find all vertical asymptotes.
    (c) Find $\lim_{x\to1^+}f(x)$.
    (d) Does the graph ever cross its horizontal asymptote? Justify.



Graph of f.

21. Evaluate $\lim_{x\to\infty}\dfrac{x^3+2^x}{3^x-x^5}$. Name the growth comparison you use.

1.8 Exponential and Logarithmic Functions

You will review the laws of exponents and logarithms, learn the number e and the natural logarithm, solve exponential and logarithmic equations, and use exponential models and their limits.

Exponential and logarithmic functions model growth and decay in biology, chemistry and finance, and they appear in many limits and derivatives later in the course. This section reviews the algebra you need and the key limits involving e^x and $\ln x$.

Laws of Exponents and Logarithms

For $a>0$: $a^m a^n=a^{m+n}$, $\dfrac{a^m}{a^n}=a^{m-n}$, $(a^m)^n=a^{mn}$, $a^{-n}=\dfrac{1}{a^n}$.

For $b>0$, $b\neq1$, $M>0$, $N>0$: $\log_b(MN)=\log_b M+\log_b N$, $\log_b\dfrac{M}{N}=\log_b M-\log_b N$, $\log_b(M^r)=r\log_b M$.

Inverse relationship: $\log_b(b^x)=x$ for all real x, and $b^{\log_b x}=x$ for $x>0$. Change of base: $\log_b x=\dfrac{\ln x}{\ln b}$.

The Number e and the Natural Logarithm

The number $e\approx2.71828$ is defined by $e=\lim_{t\to0}(1+t)^{1/t}$. The natural logarithm is $\ln x=\log_e x$, the inverse of e^x.

$y=e^x$ has domain all reals, range $y>0$ and horizontal asymptote $y=0$ on the left. $y=\ln x$ has domain $x>0$, range all reals and vertical asymptote $x=0$.

Limits: $\lim_{x\to\infty}e^x=\infty$, $\lim_{x\to-\infty}e^x=0$, $\lim_{x\to\infty}\ln x=\infty$, $\lim_{x\to0^+}\ln x=-\infty$. Both e^x and $\ln x$ are continuous on their domains.

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Common mistakes

$\ln(a+b)$ is not $\ln a + \ln b$, and $\ln(MN) = \ln M + \ln N$ only when $M, N > 0$. When you solve a logarithmic equation, always check each answer in the original equation: combining logs can introduce extraneous solutions, because $\ln x$ is only defined for $x > 0$.

EXAMPLE 1

Solve $e^{2x} - 5e^x + 6 = 0$.

SOLUTION

The equation is quadratic in $u = e^x$: $u^2 - 5u + 6 = 0$.

Factor: $(u-2)(u-3) = 0$, so $e^x = 2$ or $e^x = 3$.

Take logarithms: $x = \ln 2$ or $x = \ln 3$. Both are valid since e^x takes every positive value.

EXAMPLE 2

Solve $\ln x + \ln(x-2) = \ln 3$.

SOLUTION

The domain requires $x > 0$ and $x - 2 > 0$, so $x > 2$.

Combine: $\ln[x(x-2)] = \ln 3$. Since $\ln$ is one-to-one, $x^2 - 2x = 3$.

So $x^2 - 2x - 3 = (x-3)(x+1) = 0$. Only $x = 3$ satisfies $x > 2$, so reject $x = -1$. The solution is $x = 3$.

EXAMPLE 3

Evaluate $\lim_{x \to \infty} \ln\left(\dfrac{3x+1}{x+5}\right)$.

SOLUTION

The quotient inside has equal degrees, so $\frac{3x+1}{x+5} \to 3$.

Since $\ln$ is continuous at 3, we may move the limit inside: $\ln 3$.

EXAMPLE 4

Describe the behavior of $f(x) = \ln x - 3$ as $x \to 0^+$ and as $x \to \infty$.

SOLUTION

As $x \to 0^+$, $\ln x \to -\infty$, so $f(x) \to -\infty$ and $x = 0$ is a vertical asymptote.

As $x \to \infty$, $\ln x \to \infty$ slowly, so $f(x) \to \infty$. There is no horizontal asymptote.

Exponential Growth and Decay Models

A quantity that changes at a rate proportional to its size satisfies $Q(t) = Q_0 e^{kt}$, where Q_0 is the initial amount.

If $k > 0$ the quantity grows (doubling time $\frac{\ln 2}{k}$). If $k < 0$ it decays (half-life $\frac{\ln 2}{|k|}$). Equivalent form: $Q(t) = Q_0 b^t$ with $b = e^k$.

To find k, use two data points and take a logarithm.

EXAMPLE 5

A sample of a radioactive substance has mass 80 mg. After 6 hours the mass is 50 mg. Write a model $A(t) = 80e^{kt}$, then find the mass after 15 hours and the half-life (three decimal places where needed). [Calculator]

SOLUTION

Use $A(6) = 50$: $e^{6k} = \frac{5}{8}$, so $k = \frac{1}{6}\ln\frac{5}{8} \approx -0.078$.

At $t = 15$: $A(15) = 80\left(\frac{5}{8}\right)^{15/6} = 80\left(\frac{5}{8}\right)^{2.5} \approx 24.705$ mg.

Half-life: $e^{kt} = \frac{1}{2}$ gives $t = \frac{\ln 2}{-k} = \frac{6\ln 2}{\ln 1.6} \approx 8.849$ hours.

Exponential growth is eventually faster than any power function, and logarithmic growth is slower than any positive power. You met these comparisons for limits at infinity in the previous section, and they are used constantly with e^x and $\ln x$.

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Practice

SET A Skills

Solve, simplify or evaluate. [No calculator]

1. Evaluate $\ln\left(e^{5}\right) + e^{\ln 7}$.
2. Write $3\ln 2 - \ln 4 + \ln 5$ as a single logarithm and simplify.
3. Solve $3^{2x-1} = 27$.
4. Solve $e^{3x} = 5$. Give an exact answer.
5. Solve $\ln(2x - 1) = 3$. Give an exact answer.
6. Solve $\log_2 x + \log_2(x - 6) = 4$.
7. Solve $e^{2x} - 7e^{x} + 10 = 0$.
8. Evaluate $\lim_{x \to -\infty} \dfrac{3e^{x} + 2}{e^{x} + 4}$.
9. Evaluate $\lim_{x \to \infty} \ln\left(\dfrac{6x^{2} + 1}{2x^{2} - x}\right)$.

SET B Applications and Mixed Practice

Give decimal answers to three decimal places. [Calculator]

10. A bacteria colony has $N(t) = 200e^{0.35t}$ cells after t hours. When does the colony reach 5000 cells? Answer to three decimal places.
11. A radioactive sample with a half-life of 12 hours has $A(t) = 60\left(\frac{1}{2}\right)^{t/12}$ milligrams after t hours.
    (a) How much remains after 30 hours?
    (b) When will 5 milligrams remain? Answer to three decimal places.
12. An account holds 2500 dollars and grows continuously, $A(t) = 2500e^{0.04t}$ dollars after t years. How long does it take for the balance to double? Answer to three decimal places.
13. The table shows an exponential function $f(x) = a \cdot b^{x}$.
    (a) Find a and b.
    (b) Find $f(6)$.
    (c) Solve $f(x) = 1000$ to three decimal places.

x	0	1	2	3
$f(x)$	3	6	12	24

14. Atmospheric pressure at height h kilometers is $P(h) = 101.3e^{-0.12h}$ kilopascals.
    (a) Find $\lim_{h \to \infty} P(h)$.
    (b) At what height is the pressure 50 kilopascals? Answer to three decimal places.
15. The concentration of a drug is $C(t) = 12e^{-0.3t}$ milligrams per liter after t hours. After how many hours does the concentration first fall below 1 milligram per liter? Answer to three decimal places.
16. The pH of a solution is $-\log_{10}[H^{+}]$, where $[H^{+}]$ is the hydrogen ion concentration in moles per liter.
    (a) Find the pH when $[H^{+}] = 3.2 \times 10^{-5}$.
    (b) Find the pH when $[H^{+}] = 2.5 \times 10^{-4}$.
    (c) Which solution is more acidic?

SET C Challenge and AP-Style Problems

Show your reasoning. Problems marked [No calculator] must have exact answers.

17. [No calculator] A function $f(x) = Ae^{kx}$ satisfies $f(0) = 4$ and $f(2) = 36$. Find A, k and $f(5)$.
18. Solve $4^{x} - 5 \cdot 2^{x} + 4 = 0$.
19. Solve $\ln\left(x^{2} - 4\right) = \ln(3x)$ and explain why any solution you reject is not valid.
20. Let $f(x) = e^{2x-1} + 3$. Find $f^{-1}(x)$ and state its domain.
21. Evaluate $\lim_{x \to \infty} \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$ and $\lim_{x \to -\infty} \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$.

1.9 Special Trigonometric Limits and the Squeeze Theorem

You will learn the Squeeze Theorem and the special limits for sin x over x and (1 - cos x) over x, and use them to evaluate trigonometric limits that algebra alone cannot handle.

Trigonometric functions are not algebraic, so limits such as $\frac{\sin x}{x}$ as $x \to 0$ cannot be found by factoring. They are handled by two tools: the Squeeze Theorem, and a pair of special limits. All angles in this section are measured in radians.

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The Squeeze Theorem

Suppose $g(x) \le f(x) \le h(x)$ for all x in an open interval containing c (except possibly at $x = c$), and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. Then $\lim_{x\to c} f(x) = L$.

The same holds as $x \to \infty$ or $x \to -\infty$ if the inequality holds for all large $|x|$. The hypothesis requires the same limit L for both bounds.

Special Trigonometric Limits (radians)

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0 \qquad \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

Consequences: $\lim_{x\to 0} \dfrac{\tan x}{x} = 1$ and, for constants $a, b \neq 0$, $\lim_{x\to 0} \dfrac{\sin ax}{bx} = \dfrac{a}{b}$.

These work for any expression in place of x that tends to 0: $\lim_{u\to 0} \frac{\sin u}{u} = 1$.

Common mistakes

The limit $\lim \frac{\sin x}{x} = 1$ is only for $x \to 0$ and radians. As $x \to \infty$, $\frac{\sin x}{x} \to 0$. Also, $\frac{\sin 5x}{x}$ tends to 5, not 1: first match the argument, $\frac{\sin 5x}{x} = 5 \cdot \frac{\sin 5x}{5x}$.

EXAMPLE 1

Evaluate $\lim_{x\to 0} \dfrac{\sin 5x}{3x}$.

SOLUTION

Make the argument of sine match the denominator: $\dfrac{\sin 5x}{3x} = \dfrac{5}{3} \cdot \dfrac{\sin 5x}{5x}$.

As $x \to 0$, $5x \to 0$, so $\frac{\sin 5x}{5x} \to 1$ and the limit is $\dfrac{5}{3}$.

EXAMPLE 2

Evaluate (a) $\lim_{x\to 0} \dfrac{\tan 3x}{\sin 7x}$ and (b) $\lim_{x\to 0} \dfrac{1-\cos 4x}{x^2}$.

SOLUTION

(a) $\dfrac{\tan 3x}{\sin 7x} = \dfrac{\sin 3x}{\cos 3x \, \sin 7x} = \dfrac{3}{7} \cdot \dfrac{\sin 3x}{3x} \cdot \dfrac{7x}{\sin 7x} \cdot \dfrac{1}{\cos 3x} \to \dfrac{3}{7} \cdot 1 \cdot 1 \cdot 1 = \dfrac{3}{7}$.

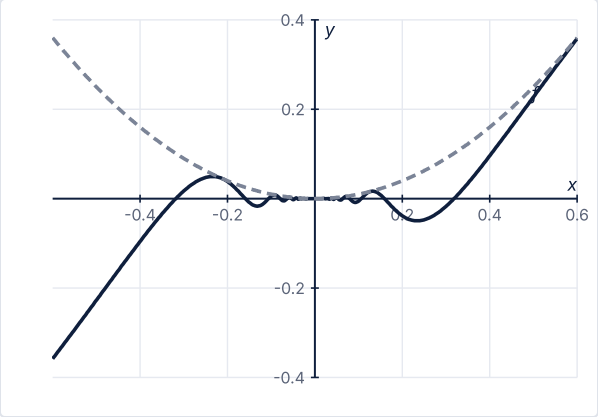
(b) $\dfrac{1-\cos 4x}{x^2} = 16 \cdot \dfrac{1-\cos 4x}{(4x)^2} \to 16 \cdot \dfrac{1}{2} = 8$.

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EXAMPLE 3

Show that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.



The graph of f is squeezed between y = x^2 and y = -x^2.

SOLUTION

We cannot use $\lim \sin \frac{1}{x}$, which does not exist. Use the Squeeze Theorem instead.

For $x \neq 0$, $-1 \leq \sin \frac{1}{x} \leq 1$. Multiplying by $x^2 \geq 0$ keeps the direction: $-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$.

Both $-x^2$ and x^2 tend to 0 as $x \to 0$, so the middle expression tends to 0.

Proof Idea: (1 - cos x)/x

Multiply by the conjugate: $\frac{1-\cos x}{x} = \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} = \frac{\sin^2 x}{x(1+\cos x)} = \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$.

As $x \to 0$ this tends to $1 \cdot \frac{0}{2} = 0$. Dividing once more by x gives $\frac{1-\cos x}{x^2} = \frac{1}{1+\cos x}\left(\frac{\sin x}{x}\right)^2 \to \frac{1}{2}$.

EXAMPLE 4

Suppose $3-(x-2)^2 \leq f(x) \leq 3+(x-2)^2$ for all real x. Find $\lim_{x\to 2} f(x)$, and decide whether f must be continuous at 2.

SOLUTION

As $x \to 2$, both bounds tend to 3. By the Squeeze Theorem, $\lim_{x\to 2} f(x) = 3$.

At $x = 2$ the inequality reads $3 \leq f(2) \leq 3$, so $f(2) = 3$. Then $\lim_{x\to 2} f(x) = f(2)$, so f is continuous at 2.

The theorem says nothing about continuity elsewhere: f could behave wildly away from $x = 2$.

EXAMPLE 5

Evaluate $\lim_{x\to\infty} \frac{\sin x + 2x}{x + \cos x}$.

SOLUTION

Divide top and bottom by x: $\frac{\frac{\sin x}{x} + 2}{1 + \frac{\cos x}{x}}$.

Since $\left|\frac{\sin x}{x}\right| \leq \frac{1}{x}$ and $\left|\frac{\cos x}{x}\right| \leq \frac{1}{x}$, both tend to 0 by the Squeeze Theorem.

The limit is $\frac{0+2}{1+0} = 2$.

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Practice

SET A Skills

Evaluate each limit. [No calculator]

1. $\lim_{x\to 0} \dfrac{\sin 7x}{2x}$
2. $\lim_{x\to 0} \dfrac{\tan 6x}{x}$
3. $\lim_{x\to 0} \dfrac{1-\cos 3x}{x}$
4. $\lim_{x\to 0} \dfrac{1-\cos 2x}{x^2}$
5. $\lim_{x\to 0} \dfrac{x}{\sin 3x}$
6. $\lim_{x\to 0} \dfrac{\sin^2 5x}{x^2}$
7. $\lim_{x\to 0} \dfrac{\sin x}{x^2+x}$
8. $\lim_{x\to 0} x\cos\left(\dfrac{5}{x}\right)$. Name the theorem you use.
9. $\lim_{x\to \infty} \dfrac{\cos x}{x^2+1}$

SET B Applications and Mixed Practice

Use radian mode and give decimal answers to three decimal places. [Calculator]

10. The table gives values of $f(x)=\dfrac{\sin x}{x}$ with x in radians.

    (a) Estimate $\lim_{x\to 0^+} f(x)$.

    (b) A calculator set to degree mode gives $\dfrac{\sin(0.1)}{0.1}\approx 0.017$. Explain why this does not contradict the limit 1.

x	0.5	0.1	0.01
$f(x)$	0.959	0.998	1.000

11. The table shows $g(x)=\dfrac{1-\cos x}{x^2}$ (radians).

    (a) Estimate $\lim_{x\to 0} g(x)$.

    (b) Confirm with the identity $\dfrac{1-\cos x}{x^2}=\dfrac{1}{1+\cos x}\left(\dfrac{\sin x}{x}\right)^2$.

x	0.5	0.1	0.01
$g(x)$	0.490	0.500	0.500

12. A circle has radius 10 cm. A central angle θ (in radians) cuts off an arc of length 10θ and a chord of length $20\sin\frac{\theta}{2}$.

    (a) Show $\lim_{\theta\to 0} \dfrac{\text{arc}}{\text{chord}} = 1$.

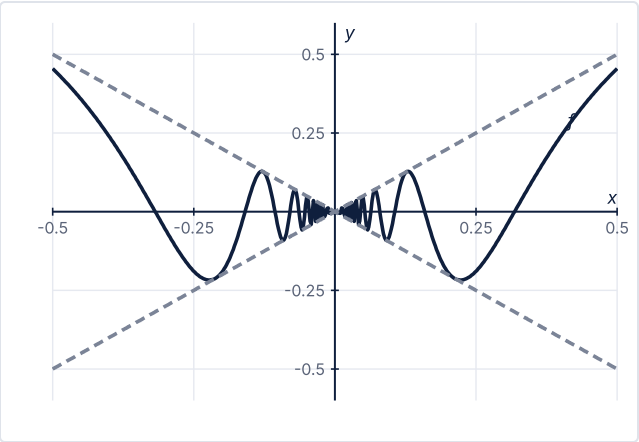
    (b) Evaluate the ratio for $\theta = 0.2$ to three decimal places.

13. Physicists replace $\sin\theta$ by θ for small angles. For $\theta = 0.3$ radians find the percent error $\dfrac{\theta-\sin\theta}{\sin\theta}\times 100$ to three decimal places, and explain why the approximation is good for small θ.

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14. The graph shows $f(x) = x\sin\left(\frac{1}{x}\right)$ for $x \neq 0$ together with the dashed graphs of $y = |x|$ and $y = -|x|$.
(a) Explain why $-|x| \le f(x) \le |x|$.
(b) Find $\lim_{x\to 0} f(x)$.



Graph of f squeezed between y = |x| and y = -|x|.

15. A surveyor measures the angle of elevation $\theta = 0.04$ radian to the top of a pole 50 meters away, so the height is $50\tan\theta$.
(a) Compute $50\tan 0.04$ to three decimal places.
(b) Compare with 50θ and explain using $\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1$.

16. An isosceles triangle has two sides of length 1 and apex angle θ (radians), so its area is $A = \frac{1}{2}\sin\theta$.
(a) Find $\lim_{\theta\to 0}\dfrac{A}{\theta}$.
(b) Evaluate $\frac{A}{\theta}$ for $\theta = 0.1$ to three decimal places.

SET C Challenge and AP-Style Problems

Justify your answers. [No calculator]

17. [No calculator] Evaluate $\lim_{x\to 0}\dfrac{\sin 3x\,\tan 2x}{x^2\cos 5x}$.

18. Evaluate $\lim_{x\to 0}\dfrac{1-\cos x}{x\sin x}$.

19. Evaluate $\lim_{x\to 0}\dfrac{x^2\sin(1/x)}{\sin x}$ and justify.

20. Suppose $2x - 1 \le f(x) \le x^2$ for all x in the interval $(0, 4)$.
(a) Find $\lim_{x\to 1} f(x)$ and name the theorem.
(b) Is f continuous at $x = 1$? Justify.

21. Find all values of k such that $\lim_{x\to 0}\dfrac{1-\cos kx}{x^2} = 8$.

1.10 Continuity and Types of Discontinuity

You will learn the three-part definition of continuity, classify removable, jump and infinite discontinuities, and choose constants that make a piecewise function continuous.

Informally, a function is continuous if you can draw its graph without lifting your pencil. Precisely, continuity is a statement about limits, and the AP exam expects you to justify it with the definition.

Continuity at a Point

A function f is **continuous at** $x = c$ if all three conditions hold: (1) $f(c)$ is defined; (2) $\lim_{x\to c} f(x)$ exists; (3) $\lim_{x\to c} f(x) = f(c)$.

f is continuous on an open interval if it is continuous at every point of it. On a closed interval $[a, b]$ it must also be continuous from the right at a ($\lim_{x\to a^+} f(x) = f(a)$) and from the left at b.

If any condition fails, f is **discontinuous** at c.

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Types of Discontinuity

Removable: $\lim_{x\to c} f(x)$ exists (finite) but is not equal to $f(c)$, or $f(c)$ is undefined. A hole in the graph. Redefine $f(c)$ to the limit to remove it.

Jump: both one-sided limits exist (finite) but are different, so $\lim_{x\to c} f(x)$ does not exist.

Infinite: at least one one-sided limit is ∞ or $-\infty$ (a vertical asymptote).

A function may also fail to have a limit because it oscillates, such as $\sin\frac{1}{x}$ at 0.

Continuous Functions You Can Rely On

Polynomials, e^x, $\sin x$ and $\cos x$ are continuous for all real x. Rational functions, $\tan x$, $\ln x$ and $\sqrt{x}$ are continuous at every point of their domains.

If f and g are continuous at c, so are $f \pm g$, fg, and $\frac{f}{g}$ when $g(c) \neq 0$. If g is continuous at c and f is continuous at $g(c)$, then $f(g(x))$ is continuous at c, and $\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right)$.

EXAMPLE 1

Let $f(x) = \dfrac{x^2-4}{x-2}$ for $x \neq 2$ and $f(2) = 3$. Is f continuous at 2? If not, classify and repair the discontinuity.

SOLUTION

Condition (1): $f(2) = 3$ is defined.

Condition (2): for $x \neq 2$, $f(x) = x + 2$, so $\lim_{x\to 2} f(x) = 4$ exists.

Condition (3): $4 \neq 3$, so f is not continuous at 2. The limit exists, so the discontinuity is **removable**. Redefining $f(2) = 4$ makes f continuous.

EXAMPLE 2

Find and classify the discontinuities of $f(x) = \dfrac{x^2-x-6}{x^2-9}$.

SOLUTION

Factor: $f(x) = \dfrac{(x-3)(x+2)}{(x-3)(x+3)}$. The denominator is 0 at $x = 3$ and $x = -3$.

At $x = 3$: cancel $(x-3)$, so $f(x) = \frac{x+2}{x+3} \to \frac{5}{6}$. The limit is finite: **removable**.

At $x = -3$: the factor $(x+3)$ does not cancel. The numerator $x^2 - x - 6$ equals $9 + 3 - 6 = 6 \neq 0$ at $x = -3$ while the denominator tends to 0, so $|f| \to \infty$: **infinite** discontinuity.

EXAMPLE 3

Find the value of k that makes $f(x) = \begin{cases} kx+1, & x < 2 \\ x^2 - k, & x \geq 2 \end{cases}$ continuous at $x = 2$.

SOLUTION

Left limit: $\lim_{x\to 2^-}(kx+1) = 2k+1$. Right limit and value: $\lim_{x\to 2^+}(x^2-k) = f(2) = 4-k$.

Continuity requires the one-sided limits to be equal: $2k+1 = 4-k$, so $k = 1$.

Check: with $k = 1$ both sides give 3.

Common mistake

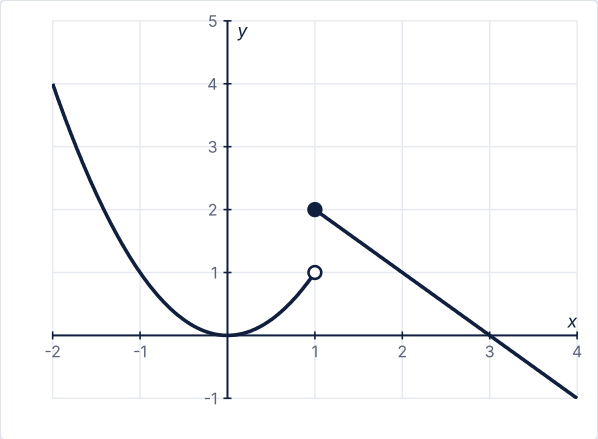
A hole in the graph and a vertical asymptote are both discontinuities, but they are different types. Cancelling a common factor does not make the original function continuous at that point: $\frac{x^2-4}{x-2}$ is still undefined at $x = 2$ until you define a value there.

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EXAMPLE 4

Let $g(x) = \begin{cases} x^2, & x < 1 \\ 3 - x, & x \geq 1 \end{cases}$. Is g continuous at $x = 1$? Justify with limits.



Graph of g: a jump at x = 1.

SOLUTION

Left: $\lim_{x \to 1^-} x^2 = 1$. Right: $\lim_{x \to 1^+} (3 - x) = 2$.

The one-sided limits are different, so $\lim_{x \to 1} g(x)$ does not exist. Therefore g is not continuous at 1: a **jump discontinuity** of size 1.

Note $g(1) = 2$ equals the right-hand limit, so g is continuous from the right at 1 but not from the left.

EXAMPLE 5

State the intervals on which $f(x) = \dfrac{\ln(x-1)}{x^2 - 9}$ is continuous, and justify.

SOLUTION

$\ln(x-1)$ requires $x > 1$. The denominator is 0 at $x = \pm 3$, and only $x = 3$ lies in $x > 1$.

f is a quotient of continuous functions, continuous wherever the denominator is nonzero and $x > 1$.

So f is continuous on $(1, 3)$ and $(3, \infty)$.

Practice

SET A Skills

Decide continuity, classify discontinuities or find the constants. [No calculator]

1. Let $f(x) = \dfrac{x^2 - 25}{x - 5}$ for $x \neq 5$ and $f(5) = 7$. Is f continuous at $x = 5$? If not, classify the discontinuity.
2. Find all discontinuities of $f(x) = \dfrac{x + 1}{x^2 - 5x + 6}$ and classify each.
3. Show that $f(x) = \dfrac{x^2 - 1}{x + 1}$ has a removable discontinuity at $x = -1$ and state how to remove it.
4. Find k so that $f(x) = \begin{cases} 3x + k, & x < 2 \\ x^2 - 1, & x \geq 2 \end{cases}$ is continuous at $x = 2$.
5. Find a so that $g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ a, & x = 2 \end{cases}$ is continuous at $x = 2$.
6. Let $h(x) = \dfrac{|x - 3|}{x - 3}$ for $x \neq 3$. Find $\lim_{x \to 3^-} h(x)$, $\lim_{x \to 3^+} h(x)$ and classify the discontinuity at $x = 3$.
7. State the largest interval(s) on which $f(x) = \sqrt{x - 2}$ is continuous.
8. State the intervals on which $g(x) = \dfrac{\ln x}{x - 1}$ is continuous.
9. Use continuity to evaluate $\lim_{x \to 1} e^{(x^2 - 1)/(x - 1)}$.

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SET B Applications and Mixed Practice

Give decimal answers to three decimal places. [Calculator]

10. A shipping company charges 4 dollars for a package up to and including 1 kilogram, 6.50 dollars for weights over 1 and up to 2 kilograms, and 9 dollars for weights over 2 and up to 3 kilograms. Let $C(w)$ be the cost.

(a) Find $\lim_{w\to 1^-} C(w)$ and $\lim_{w\to 1^+} C(w)$.

(b) Is C continuous at $w=1$? Classify any discontinuity.

(c) Is C continuous from the left at $w=1$?

11. A tax is $T(x)=0.10x$ dollars on an income of x dollars for $0\le x\le 10000$, and $T(x)=1000+0.15(x-10000)$ for $x>10000$.

(a) Show that T is continuous at $x=10000$.

(b) Find $T(25000)$.

12. A particle has velocity $v(t)=\begin{cases} t^2, & 0\le t<3 \\ ct-3, & t\ge 3 \end{cases}$. Find c so that v is continuous at $t=3$.

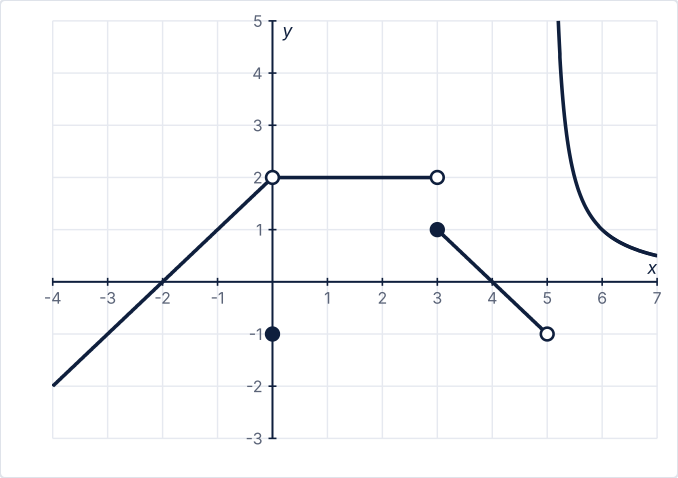
13. The table shows values of $f(x)=\dfrac{e^x-1}{x}$ for $x\neq 0$.

(a) Estimate $\lim_{x\to 0} f(x)$.

(b) What value should $f(0)$ be given to make f continuous at 0?

x	-0.1	-0.01	0.01	0.1
$f(x)$	0.952	0.995	1.005	1.052

14. The graph of f on $[-4,7]$ is shown. At $x=0$ the open circle is at $(0,2)$ and the closed dot is at $(0,-1)$; at $x=3$ the open circle is at $(3,2)$ and the closed dot is at $(3,1)$; the open circle at $x=5$ is $(5,-1)$. Classify the discontinuities at $x=0$, $x=3$ and $x=5$ and give the relevant limits.



Graph of f.

15. A supplier charges 10 dollars per unit for orders of up to and including 50 units, and 8 dollars per unit for the entire order if more than 50 units are bought. Let $C(q)$ be the total cost.

(a) Find $C(50)$ and $\lim_{q\to 50^+} C(q)$.

(b) Classify the discontinuity at $q=50$.

(c) What is the smallest whole number of units above 50 whose total cost exceeds $C(50)$?

16. Find k so that $g(x)=\begin{cases} \dfrac{1-\cos 2x}{x^2}, & x\neq 0 \\ k, & x=0 \end{cases}$ is continuous at 0, then check $g(0.01)$ to three decimal places.

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SET C Challenge and AP-Style Problems

Justify with the definition of continuity. Problems marked [No calculator] must have exact answers.

17. [No calculator] Find a and b so that $f(x) = \begin{cases} 2x + a, & x < 1 \\ b, & x = 1 \\ x^2 + 3, & x > 1 \end{cases}$ is continuous at $x = 1$.

18. [No calculator] Find the value of a for which $\lim_{x \to 3} \dfrac{x^2 + ax - 6}{x - 3}$ exists, and then find that limit and the value $f(3)$ that makes $f(x) = \frac{x^2+ax-6}{x-3}$ continuous at 3.

19. [No calculator] Let $f(x) = \dfrac{\sin(3x - 6)}{x - 2}$ for $x \neq 2$ and $f(2) = 3$. Use the definition of continuity to decide whether f is continuous at $x = 2$.

20. [No calculator] Find and classify every discontinuity of $f(x) = \dfrac{x^2 - 1}{x^3 - x}$.

21. If $|f|$ is continuous at $x = c$, must f be continuous at c? Give a proof or a counterexample.

1.11 The Intermediate Value Theorem

You will learn the statement and hypotheses of the Intermediate Value Theorem and use it to prove that equations have solutions, with the justification style the AP exam expects.

The graph of a continuous function has no gaps, so to get from one height to another it must pass through every height in between. The Intermediate Value Theorem turns this idea into a tool for proving that an equation has a solution, even when you cannot solve it exactly.

The Intermediate Value Theorem (IVT)

If f is **continuous on the closed interval** $[a, b]$ and N is any number strictly between $f(a)$ and $f(b)$, then there is at least one number c in the open interval (a, b) with $f(c) = N$.

Both hypotheses matter: f must be continuous at every point of $[a, b]$, and N must lie between the values at the endpoints. The conclusion is only that a number c exists, not what it is or how many there are.

Showing a Root Exists

To show $f(x) = 0$ has a solution in (a, b): check that f is continuous on $[a, b]$, and show that $f(a)$ and $f(b)$ have opposite signs. Then $N = 0$ lies between them.

To show $f(x) = g(x)$ has a solution, apply the IVT to $h(x) = f(x) - g(x)$.

AP-style justification

Write a sentence of this form: Since f is continuous on $[a, b]$ and $f(a) < N < f(b)$, the Intermediate Value Theorem guarantees a number c in (a, b) with $f(c) = N$. You must state that f is continuous and show the two endpoint values; omitting the continuity statement loses credit.

EXAMPLE 1

Show that $x^3 + 2x - 5 = 0$ has a solution in $(1, 2)$.

SOLUTION

Let $f(x) = x^3 + 2x - 5$. It is a polynomial, so it is continuous on $[1, 2]$.

$f(1) = 1 + 2 - 5 = -2 < 0$ and $f(2) = 8 + 4 - 5 = 7 > 0$.

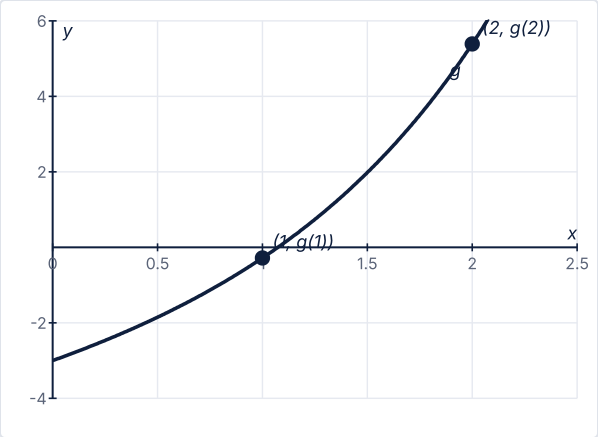
Since 0 is between -2 and 7, the IVT guarantees a c in $(1, 2)$ with $f(c) = 0$.

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EXAMPLE 2

Show that $e^x = 4 - x$ has a solution in $(1, 2)$.



g(x) = e^x + x - 4 changes sign between x = 1 and x = 2.

SOLUTION

Move everything to one side: let $g(x) = e^x + x - 4$. It is continuous for all x.

$g(1) = e - 3 \approx -0.282 < 0$ and $g(2) = e^2 - 2 \approx 5.389 > 0$.

By the IVT there is c in $(1, 2)$ with $g(c) = 0$, i.e. $e^c = 4 - c$.

EXAMPLE 3

A continuous function f on $[0, 10]$ has $f(0) = -3$, $f(4) = 2$, $f(7) = 9$, $f(10) = 6$. Which of the following equations does the IVT guarantee a solution for on $(0, 10)$: $f(x) = 0$, $f(x) = 5$, $f(x) = 12$?

SOLUTION

$f(x) = 0$: $f(0) = -3 < 0 < 2 = f(4)$, so yes, a solution in $(0, 4)$.

$f(x) = 5$: $f(4) = 2 < 5 < 9 = f(7)$, so yes, a solution in $(4, 7)$.

$f(x) = 12$: 12 is above every listed value, so the IVT guarantees nothing about $f(x) = 12$.

EXAMPLE 4

Let $f(x) = \dfrac{1}{x}$ on $[-1, 1]$. Then $f(-1) = -1 < 0 < 1 = f(1)$. Does $f(x) = 0$ have a solution in $(-1, 1)$?

SOLUTION

No. The equation $\frac{1}{x} = 0$ has no solution at all.

This does not contradict the IVT: f is not defined at $x = 0$, so it is not continuous on $[-1, 1]$, and the hypothesis fails.

Lesson: the continuity hypothesis cannot be dropped.

EXAMPLE 5

Show that $f(x) = x^2 + \cos x$ takes the value 2 on $[0, 2]$. [Radian mode]

SOLUTION

f is continuous on $[0, 2]$ (sum of continuous functions).

$f(0) = 0 + 1 = 1$ and $f(2) = 4 + \cos 2 \approx 3.584$.

Since $1 < 2 < 3.584$, the IVT gives c in $(0, 2)$ with $f(c) = 2$.

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Common mistakes

The IVT proves existence only; it does not say the solution is unique. Also it cannot be used to show that a value outside the interval from $f(a)$ to $f(b)$ is attained. And a sign change of f across an interval where f is not continuous guarantees nothing.

Practice

SET A Skills

Use the Intermediate Value Theorem and justify. [No calculator]

1. Show that $x^3 - 4x + 2 = 0$ has a solution in the interval $(0, 1)$.
2. Show that $x^4 - 3x - 5 = 0$ has a solution in $[1, 2]$.
3. Show that the equation $\cos x = x$ has a solution in $(0, 1)$. [Radian mode]
4. Show that $\ln x = 2 - x$ has a solution in $(1, 2)$.
5. Show that $\sqrt{x} = e^{-x}$ has a solution in $(0, 1)$.
6. The function f is continuous on $[1, 9]$ and the table gives selected values. For each value c, decide whether the Intermediate Value Theorem guarantees a solution of $f(x) = c$ in the interval $(1, 9)$.
   (a) $c = 0$
   (b) $c = 3$
   (c) $c = 7$

x	1	3	6	9
$f(x)$	-2	1	-4	5

7. Does the IVT guarantee a solution of $f(x) = 0$ on $[0, 3]$ for $f(x) = \dfrac{1}{x - 2}$, even though $f(0) = -\frac{1}{2} < 0 < 1 = f(3)$? Explain.
8. A function f is continuous on $[1, 5]$ with $f(1) = 2$ and $f(5) = 10$. Must f take the value 6 on $(1, 5)$? Must it take the value 12?
9. Show that $f(x) = x^3 - 6x + 2$ has three real zeros by locating each in an interval of the form $(n, n + 1)$ for integers n.

SET B Applications and Mixed Practice

Justify existence with the IVT, and give decimal answers to three decimal places. [Calculator]

10. The temperature $T(t)$ in degrees Celsius is continuous for $0 \le t \le 24$ hours. Selected values are shown. Show that $T(t) = 15$ at least twice on $(0, 24)$.

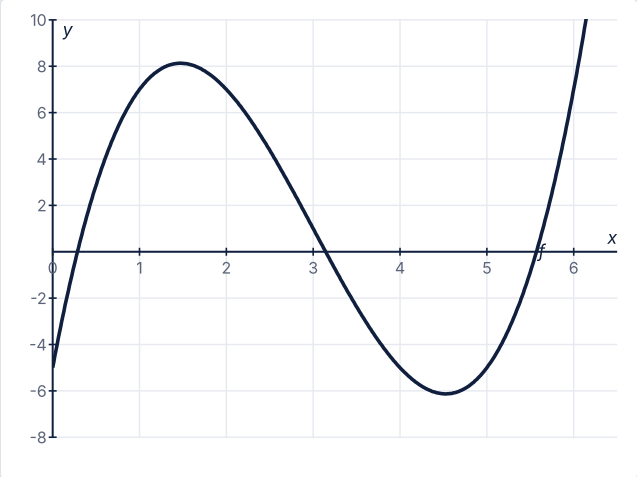
t (hours)	0	6	12	18	24
$T(t)$	8	6	19	14	9

11. Let $f(x) = x^3 + x - 3$. (a) Show that f has a zero in $(1, 2)$. (b) Use a calculator to approximate the zero to three decimal places.
12. Show that $\cos x = x^2$ has a solution in $(0, 1)$ (radians), and approximate it to three decimal places.
13. A sphere of radius r centimeters has surface area $S(r) = 4\pi r^2$. Show that some radius in $(1, 3)$ gives a surface area of exactly 100 square centimeters, and find it to three decimal places.

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14. The graph shows $f(x) = x^3 - 9x^2 + 20x - 5$ on $[0, 6.5]$.
(a) From the graph, how many zeros does f appear to have?
(b) Evaluate f at the integers 0 through 6 and use the IVT to confirm one zero in each of three intervals of length 1.



Graph of f.

15. Define $f(x) = \begin{cases} x - 3, & 0 \le x < 2 \\ x + 2, & 2 \le x \le 4 \end{cases}$. Note $f(0) = -3 < 0 < 6 = f(4)$, yet $f(x) = 0$ has no solution on $[0, 4]$. Explain why this does not contradict the IVT.

16. A mass on a spring has position $s(t) = 3\cos 2t + t$ centimeters at time t seconds. Show that $s(t) = 0$ at least twice on $[0, 2]$ (radian mode) and approximate both times to three decimal places.

SET C Challenge and AP-Style Problems

Write complete justifications. [No calculator]

17. The function f is continuous on $[0, 1]$ and $0 \le f(x) \le 1$ for every x in $[0, 1]$. Prove that $f(c) = c$ for some c in $[0, 1]$.

18. The function f is continuous on $[0, 2\pi]$ and $f(0) = f(2\pi)$. Prove that there is a c in $[0, \pi]$ with $f(c) = f(c + \pi)$.

19. For which values of k do $f(0)$ and $f(1)$ have opposite signs for $f(x) = x^3 - 3x + k$? What does the IVT then guarantee?

20. Prove that $x^5 + x = 1$ has exactly one real solution.

21. A continuous function f on $[-1, 5]$ satisfies $f(-1) = -3$, $f(0) = 2$, $f(2) = -1$, $f(5) = 6$. What is the least number of solutions of (a) $f(x) = 0$ and (b) $f(x) = 1$ that the IVT guarantees on $(-1, 5)$?

1.12 Multiple-Choice Practice on Limits

Thirty multiple-choice questions covering limits, asymptotes, continuity, the Intermediate Value Theorem and exponential and logarithmic functions.

Work Part A without a calculator and Part B with one. Choose the single best answer. Explanations follow in the answer section.

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SET A Part A [No calculator]

Part A: 20 questions. [No calculator]

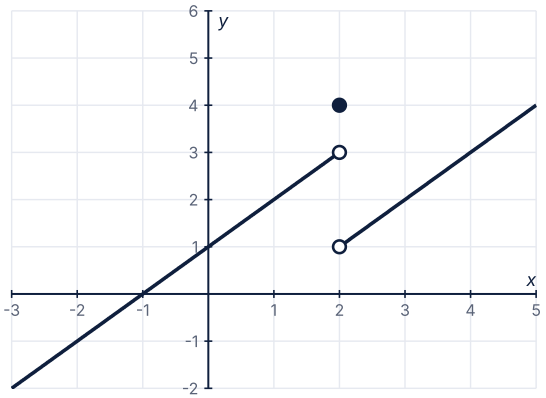
1. [No calculator] $\lim_{x\to 3} \dfrac{x^2-9}{x^2-2x-3} =$

   (A) $\frac{3}{2}$ (B) 0 (C) 2
   (D) 3 (E) does not exist

2. [No calculator] $\lim_{x\to 0} \dfrac{\sqrt{x+9}-3}{x} =$

   (A) 0 (B) $\frac{1}{3}$ (C) $\frac{1}{6}$
   (D) 3 (E) does not exist

3. [No calculator] The graph of f is shown, with $f(2)=4$ marked by the filled dot. Which statement is true?



Graph of f.

   (A) $\lim_{x\to 2} f(x) = 3$ (B) $\lim_{x\to 2} f(x) = 1$ (C) $\lim_{x\to 2} f(x) = 4$
   (D) f is continuous at $x=2$ (E) $\lim_{x\to 2} f(x)$ does not exist

4. [No calculator] The table gives values of $f(x)=\dfrac{x^2-16}{x-4}$. What is $\lim_{x\to 4} f(x)$?

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	7.9	7.99	7.999	8.001	8.01	8.1

   (A) 0 (B) 8 (C) 4
   (D) 16 (E) does not exist

5. [No calculator] $\lim_{x\to\infty} \dfrac{4x^3-x}{2x^3+5x^2} =$

   (A) 0 (B) $\frac{1}{2}$ (C) 4 (D) 2 (E) ∞

6. [No calculator] $\lim_{x\to-\infty} \dfrac{\sqrt{9x^2+1}}{x-2} =$

   (A) 3 (B) -3 (C) 0 (D) $-\infty$ (E) $-\frac{1}{3}$

7. [No calculator] How many vertical asymptotes does the graph of $f(x)=\dfrac{x-1}{x^2-3x+2}$ have, and where?

   (A) One, at $x=2$ (B) One, at $x=1$ (C) Two, at $x=1$ and $x=2$
   (D) None (E) One, at $x=-2$

8. [No calculator] $\lim_{x\to 0} \dfrac{\tan 5x}{2x} =$

   (A) 0 (B) $\frac{2}{5}$ (C) 1 (D) $\frac{5}{2}$ (E) 5

9. [No calculator] $\lim_{x\to 0} \dfrac{1-\cos 4x}{x\sin x} =$

   (A) 0 (B) 2 (C) 8 (D) 4 (E) 16

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10. [No calculator] Suppose $5 - |x-3| \le f(x) \le 5 + (x-3)^2$ for all x near 3. Then $\lim_{x\to 3} f(x) =$

(A) 0 **(B)** 3 **(C)** 8
(D) cannot be determined **(E)** 5

11. [No calculator] $\lim_{x\to 2^-} \dfrac{x+1}{x-2} =$

(A) ∞ **(B)** 0 **(C)** 3 **(D)** $-\infty$ **(E)** -3

12. [No calculator] For which value of k is $f(x) = \begin{cases} kx^2 + 1, & x \le 2 \\ 3x - k, & x > 2 \end{cases}$ continuous at $x = 2$?

(A) 0 **(B)** 1 **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{4}$ **(E)** 2

13. [No calculator] The function $f(x) = \dfrac{x^2-4}{x+2}$ is not defined at $x = -2$. What type of discontinuity does it have there?

(A) jump **(B)** infinite
(C) oscillating **(D)** none; f is continuous there
(E) removable

14. [No calculator] The function f is continuous on $[1,5]$ with $f(1) = -2$ and $f(5) = 3$. Which statement must be true?

(A) $f(x) = 0$ has at least one solution in $(1, 5)$ **(B)** $f(x) = 5$ has a solution in $(1, 5)$
(C) f is increasing on $[1, 5]$ **(D)** $f(3) = 0$
(E) $f(x) = 0$ has exactly one solution in $(1, 5)$

15. [No calculator] The solution set of $\ln x + \ln(x-3) = \ln 4$ is

(A) $\{-1\}$ **(B)** $\{-1, 4\}$ **(C)** $\{4\}$ **(D)** $\{1\}$ **(E)** $\{7\}$

16. [No calculator] $\lim_{x\to\infty} \dfrac{3e^x - 2}{e^x + 5} =$

(A) $-\frac{2}{5}$ **(B)** 0 **(C)** 3 **(D)** $\frac{3}{5}$ **(E)** ∞

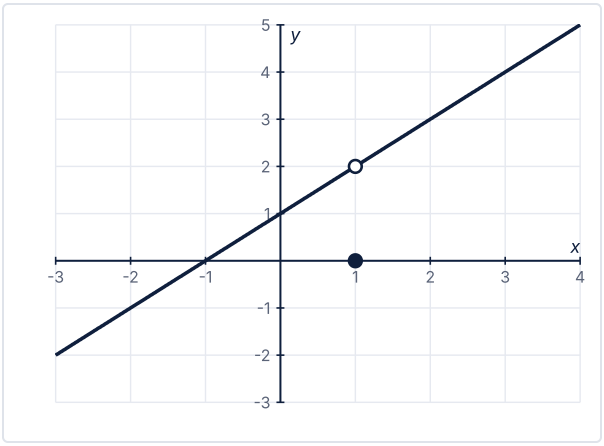
17. [No calculator] The slant asymptote of $f(x) = \dfrac{x^2 + 2x - 3}{x - 2}$ is

(A) $y = x + 4$ **(B)** $y = x + 2$ **(C)** $y = x - 4$ **(D)** $y = x$ **(E)** $y = x + 5$

18. [No calculator] $\lim_{x\to 0} x^2 \cos\dfrac{1}{x} =$

(A) 1 **(B)** 0 **(C)** -1
(D) ∞ **(E)** does not exist

19. [No calculator] The graph of f is shown. The point $(1, 2)$ is an open circle and $f(1) = 0$ is the filled dot. Which statement is true?



Graph of f.

(A) $\lim_{x\to 1} f(x) = 0$ **(B)** $\lim_{x\to 1} f(x)$ does not exist **(C)** f is continuous at $x = 1$
(D) $\lim_{x\to 1} f(x) = 1$ **(E)** $\lim_{x\to 1} f(x) = 2$ but $f(1) = 0$

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20. [No calculator] $\lim_{h\to 0}\dfrac{(2+h)^3-8}{h}=$

(A) 0 **(B)** 6 **(C)** 8
(D) 12 **(E)** does not exist

SET B Part B [Calculator]

Part B: 10 questions. [Calculator] Give numerical answers to three decimal places where applicable.

21. [Calculator] $\lim_{x\to\infty}\left(1+\dfrac{3}{x}\right)^x=$ (to three decimal places)

(A) 3.000 **(B)** 2.718 **(C)** 20.086 **(D)** 8.155 **(E)** 1.000

22. [Calculator] The solution of $2e^{0.3t}=15$ is $t\approx$

(A) 6.716 **(B)** 0.904 **(C)** 9.027 **(D)** 2.015 **(E)** 7.500

23. [Calculator] The table shows $g(x)=\dfrac{e^{2x}-1}{x}$. Which value does $\lim_{x\to 0}g(x)$ appear to equal?

x	0.1	0.01	0.001
$g(x)$	2.214	2.020	2.002

(A) 0 **(B)** 1 **(C)** 2.214 **(D)** e **(E)** 2

24. [Calculator] The equation $x^3+x-5=0$ has exactly one real solution (since x^3+x is increasing). Its value, to three decimal places, is

(A) 1.250 **(B)** 1.516 **(C)** 1.618 **(D)** 1.732 **(E)** 1.415

25. [Calculator] A population is $P(t)=\dfrac{5000}{1+49e^{-0.4t}}$. At what time t does $P(t)=4000$?

(A) 9.730 **(B)** 6.598 **(C)** 10.000 **(D)** 13.195 **(E)** 24.612

26. [Calculator] The table shows $h(x)=\dfrac{\ln x}{x-1}$. To make h continuous at $x=1$, the value of $h(1)$ should be

x	0.9	0.99	1.01	1.1
$h(x)$	1.054	1.005	0.995	0.953

(A) 1 **(B)** 0 **(C)** e **(D)** $\ln 2$ **(E)** -1

27. [Calculator] For $f(x)=\dfrac{6x+1}{2x+3}$, what is the smallest integer N such that $|f(x)-3|<0.01$ for all $x>N$?

(A) 398 **(B)** 400 **(C)** 797 **(D)** 399 **(E)** 800

28. [Calculator] The function $v(t)=\begin{cases}2e^{t}, & t<1\\ ct, & t\geq 1\end{cases}$ is continuous at $t=1$ when $c\approx$

(A) 2.000 **(B)** 5.437 **(C)** 2.718 **(D)** 1.359 **(E)** 7.389

29. [Calculator] The equation $e^x=3x$ has two solutions. The larger one is, to three decimal places,

(A) 0.619 **(B)** 1.099 **(C)** 1.512 **(D)** 1.386 **(E)** 2.000

30. [Calculator] A radioactive sample of 100 mg has half-life 8 days. The mass left after 20 days is

(A) 25.000 mg **(B)** 12.500 mg **(C)** 40.000 mg **(D)** 5.000 mg **(E)** 17.678 mg

1.13 Free-Response Practice on Limits

Four multi-part free-response questions that combine tables, graphs, limits, continuity and the Intermediate Value Theorem, written the way the exam asks them.

Each question is worth 9 points. Write complete sentences for every justification: name the theorem, check its hypotheses, then state the conclusion. Questions 2 and 3 are [No calculator]; Question 4 is [Calculator], and Question 1 mixes both.

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QUESTION 1 Temperature data (9 points)

Show your work. Justify every conclusion with a theorem or a limit.

1. The temperature $T(t)$, in degrees Celsius, of a liquid in a reaction vessel is a continuous function of time t, in minutes, for $0 \le t \le 10$. Selected values are given in the table.

(a) Find the average rate of change of T on the interval $2 \le t \le 7$. Include units. [No calculator]

(b) Is there a time t in $2 < t < 5$ at which $T(t) = 55$? Is there a time in $5 < t < 7$ at which $T(t) = 55$? Justify your answers. Explain why $T(t) = 55$ for at least two different times in $0 < t < 10$.

(c) For $t \ge 10$ the temperature is modeled by $T(t) = 120 - \dfrac{500}{t}$. Find $\lim_{t \to \infty} T(t)$ and give a physical meaning of the answer.

(d) Show that T is continuous at $t = 10$ when the table value and the model are used together. Then find the time $t > 10$ at which the model gives $T(t) = 100$.

t (min)	0	2	5	7	10
$T(t)$ (degrees C)	20	35	60	48	70

QUESTION 2 Reading a graph (9 points)

Show your work. No calculator is allowed on this question. [No calculator]

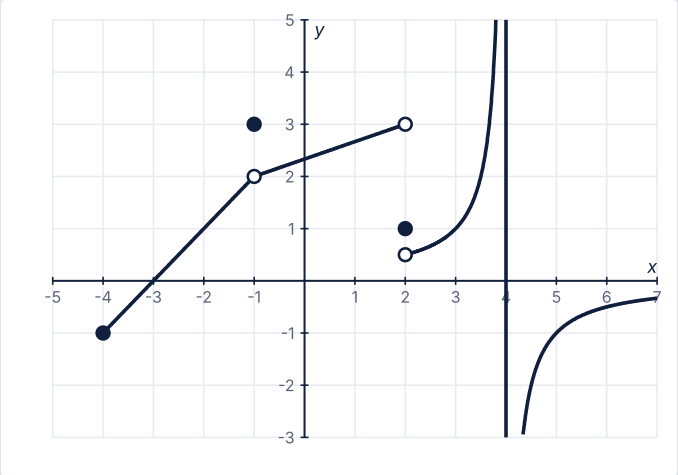
2. The graph of a function f is shown. The domain of f is $x \ge -4$, $x \ne 4$. On the piece with $x > 2$, $f(x) = \dfrac{1}{4 - x}$ (this formula continues for all $x > 4$).

(a) Find $\lim_{x \to -1} f(x)$ and $f(-1)$. Is f continuous at $x = -1$? If not, name the type of discontinuity.

(b) Find $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$. Does $\lim_{x \to 2} f(x)$ exist? Name the type of discontinuity at $x = 2$.

(c) Find $\lim_{x \to 4^-} f(x)$, $\lim_{x \to 4^+} f(x)$ and $\lim_{x \to \infty} f(x)$. Write the equation of each asymptote of the graph of f.

(d) Evaluate $\lim_{x \to -1} \left(f(x)^2 + 3f(x)\right)$. Let $g(x) = f(x)$ for $x \ne -1$ and $g(-1) = k$. For what value of k is g continuous at $x = -1$?



Graph of f. The vertical line x = 4 is an asymptote.

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QUESTION 3 **Damped signal model (9 points)**

Show your work. No calculator is allowed on this question. [No calculator]

3. A model for the strength $f(x)$ of a damped signal at a distance x meters from a source, for $x \geq 0$, is

$$f(x) = \begin{cases} \dfrac{\sqrt{x+7}-3}{x-2}, & 0 \leq x < 2 \\ c, & x = 2 \\ \dfrac{m\sin(x-2)}{x^2-4}, & x > 2 \end{cases}$$

where m and c are constants.

(a) Find $\lim_{x\to 2^-} f(x)$. Show the algebra that removes the indeterminate form.

(b) Find $\lim_{x\to 2^+} f(x)$ in terms of m. For what value of m does $\lim_{x\to 2} f(x)$ exist?

(c) With that value of m, for what value of c is f continuous at $x = 2$? Verify each condition of continuity.

(d) Use the Squeeze Theorem to find $\lim_{x\to\infty} f(x)$ for the value of m found in part (b).

QUESTION 4 **Bacteria culture (9 points)**

Show your work. A calculator is allowed on this question; give decimals to three places unless stated. [Calculator]

4. The number of bacteria $N(t)$ in a culture, in thousands, at time t hours is modeled by $N(t) = \dfrac{900}{1+8e^{-0.4t}}$ for $t \geq 0$. [Calculator]

(a) Find $N(0)$ and the average rate of change of N on $0 \leq t \leq 5$. Include units.

(b) Find $\lim_{t\to\infty} N(t)$, and explain what it means in this context.

(c) At what time t does the culture reach 450 thousand bacteria? Give your answer to three decimal places.

(d) Let $Q(t) = \dfrac{N(t)-N(5)}{t-5}$. The table gives values of $Q(t)$ for t near 5. Use the table to estimate $\lim_{t\to 5} Q(t)$ to the nearest tenth, and give the units. Explain why $Q(5)$ itself cannot be found directly from the formula for Q.

t	4.9	4.99	5.01	5.1
$Q(t)$	89.775	89.851	89.865	89.918

Answers Chapter 1: Answers and Explanations

Answers for every practice problem, numbered to match. Read the explanation even when your answer is correct.

1.1 Functions and Their Graphs

SET A · SKILLS

1. Need $5 - 2x \geq 0$, so $x \leq \frac{5}{2}$. Domain: $(-\infty, \frac{5}{2}]$.

2. Denominator $x^2 - 9 = (x-3)(x+3)$ is zero at $x = \pm 3$. Domain: all real x except 3 and -3.

3. Need $x^2 - 4 > 0$, so $|x| > 2$. Domain: $(-\infty, -2) \cup (2, \infty)$.

4. $(x-1)^2 \geq 0$, so $-2(x-1)^2 \leq 0$ and $f(x) \leq 5$, with 5 attained at $x = 1$. Range: $(-\infty, 5]$.

5. Shift left 3, reflect across the x-axis, shift up 2. The domain is $x + 3 \geq 0$, i.e. $[-3, \infty)$. Since $\sqrt{x+3} \geq 0$, $-\sqrt{x+3} + 2 \leq 2$: range $(-\infty, 2]$.

6. (a) $f(2x+1) = (2x+1)^2 - 3(2x+1) = 4x^2 + 4x + 1 - 6x - 3 = 4x^2 - 2x - 2$.

(b) $g(x^2 - 3x) = 2(x^2 - 3x) + 1 = 2x^2 - 6x + 1$.

7. (a) Odd: $p(-x) = -x^5 - 3x = -p(x)$.

(b) Neither: $q(-x) = x^4 - x$ is not $q(x)$ or $-q(x)$.

(c) Even: $r(-x) = \dfrac{x^2}{|-x|+1} = r(x)$.

8. $f(-3) = (-3)^2 - 1 = 8$ (since $-3 < 2$). $f(2) = 3(2) - 5 = 1$ (since $2 \geq 2$). $f(5) = 15 - 5 = 10$.

9. (a) Swap: $x = 3y - 7$, so $f^{-1}(x) = \dfrac{x+7}{3}$.

(b) Swap: $x = (y-2)^3 + 1$, so $y - 2 = \sqrt[3]{x-1}$ and $g^{-1}(x) = \sqrt[3]{x-1} + 2$.

SET B · APPLICATIONS AND MIXED PRACTICE

10. $V(t) = \frac{4}{3}\pi(2 + 0.5t)^3$. At $t = 4$: $r = 4$, so $V = \frac{4}{3}\pi(64) = \frac{256}{3}\pi \approx 268.083\ \text{cm}^3$.

11. (a) Height $= \dfrac{500}{x^2}$. Surface area $= x^2 + 4x \cdot \dfrac{500}{x^2} = x^2 + \dfrac{2000}{x}$.

(b) $x > 0$, that is $(0, \infty)$.

(c) $S(8) = 64 + 250 = 314.000\ \text{cm}^2$.

12. Solve $C = \frac{5}{9}(F - 32)$, so $C(F) = \frac{5}{9}(F - 32)$. Then $C(100) = \frac{5}{9}(68) \approx 37.778$ degrees Celsius.

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13. (a) $C(250) = 0.12(250) = 30.000$ dollars. $C(450) = 36 + 0.20(150) = 66.000$ dollars.
(b) $80 > 36$, so use the second piece: $36 + 0.20(k - 300) = 80$, so $k - 300 = 220$ and $k = 520$ kilowatt-hours.

14. (a) Solve $-4.9t^2 + 20t + 1.5 = 0$: $t = \dfrac{20 + \sqrt{400 + 29.4}}{9.8} \approx 4.155$ s (the negative root is rejected).
(b) Domain $[0, 4.155]$. The maximum is at $t = \frac{20}{9.8}$, where $h \approx 21.908$, so the range is $[0, 21.908]$.

15. $A(r) = \pi r^2$, so $A(t) = \pi(3t)^2 = 9\pi t^2$. At $t = 2.5$: $9\pi(6.25) = 56.25\pi \approx 176.715$ m^2.

16. f is increasing for all x (the sum of two increasing functions), so it is one-to-one and has an inverse. We need $x^3 + 2x = 5$; graphing $y = x^3 + 2x - 5$ and finding its zero gives $f^{-1}(5) \approx 1.328$.

SET C · CHALLENGE AND AP-STYLE PROBLEMS

17. Need $x + 1 \geq 0$ for g, and $g(x) \neq 2$ for f: $\sqrt{x+1} = 2$ at $x = 3$. Domain: $[-1, 3) \cup (3, \infty)$.

18. (a) $2g + 3 = x^2 + 5$, so $g(x) = \dfrac{x^2 + 2}{2}$.
(b) Note $4x^2 + 12x + 7 = (2x + 3)^2 - 2$. So $h(u) = u^2 - 2$.

19. (a) Odd means $f(-x) = -f(x)$. At $x = 0$: $f(0) = -f(0)$, so $2f(0) = 0$ and $f(0) = 0$.
(b) If f is even and g is odd, $(fg)(-x) = f(-x)g(-x) = f(x)(-g(x)) = -(fg)(x)$, so fg is odd.

20. (a) $f(1) = 3$, so $f^{-1}(3) = 1$. $f(2) = 8 + 2 + 1 = 11$, so $f^{-1}(11) = 2$.
(b) f is a sum of increasing functions (x^3, x, and a constant), so it is increasing everywhere and passes the horizontal line test: it is one-to-one.

21. For no break, the two pieces must meet at $x = 1$: $1 + k = 1^2 - 3 = -2$, so $k = -3$.

1.2 Average Rate of Change and the Slope of a Line

SET A · SKILLS

1. $m = \dfrac{4 - (-4)}{-1 - 3} = \dfrac{8}{-4} = -2$. Then $y + 4 = -2(x - 3)$, so $y = -2x + 2$.

2. The slope of ℓ: $y = \frac{3}{2}x - 4$, so $m = \frac{3}{2}$.
(a) Parallel: $y - 5 = \frac{3}{2}(x - 1)$, so $y = \frac{3}{2}x + \frac{7}{2}$.
(b) Perpendicular slope $-\frac{2}{3}$: $y - 5 = -\frac{2}{3}(x - 1)$, so $y = -\frac{2}{3}x + \frac{17}{3}$.

3. $f(-1) = -1 + 2 = 1$ and $f(3) = 27 - 6 = 21$. Average rate $= \dfrac{21 - 1}{3 - (-1)} = \dfrac{20}{4} = 5$.

4. $g(3) = 2$ and $g(8) = 3$. Average rate $= \dfrac{3 - 2}{8 - 3} = \dfrac{1}{5}$.

5. $h(1) = 2$ and $h(4) = \frac{1}{2}$. Average rate $= \dfrac{\frac{1}{2} - 2}{3} = \dfrac{-3/2}{3} = -\dfrac{1}{2}$.

6. $f(0) = 0$ and $f(3) = 9 + 6 = 15$, so the slope is $\dfrac{15}{3} = 5$. The secant line is $y = 5x$.

7. (a) $\dfrac{s(9) - s(2)}{9 - 2} = \dfrac{42 - 11}{7} = \dfrac{31}{7}$ meters per second.
(b) The averages are $\frac{11-3}{2} = 4$, $\frac{26-11}{3} = 5$, and $\frac{42-26}{4} = 4$. The greatest is on $[2, 5]$.

8. $f(x + h) = (x + h)^2 + 5(x + h) = x^2 + 2xh + h^2 + 5x + 5h$. Subtract $f(x)$: $2xh + h^2 + 5h$. Divide by h: $2x + h + 5$.

9. $f(x + h) = 4 - 3(x^2 + 2xh + h^2)$. Subtract $f(x)$: $-6xh - 3h^2$. Divide by h: $-6x - 3h$.

SET B · APPLICATIONS AND MIXED PRACTICE

10. $T(0) = 80$ and $T(5) = 20 + 60e^{-0.5} \approx 56.392$. Average rate $= \dfrac{56.392 - 80}{5} \approx -4.722$ degrees Celsius per minute.

11. (a) $[0, 5]$: $\dfrac{1.5 - 1.2}{5} = 0.06$ thousand per hour. $[5, 10]$: $\dfrac{1.98 - 1.5}{5} = 0.096$ thousand per hour.
(b) $\dfrac{1.98 - 1.2}{10} = 0.078$ thousand per hour. This equals the average of 0.06 and 0.096 because the two intervals have equal length; it would not in general.

12. $\dfrac{8\sqrt{7} - 8\sqrt{2}}{5} \approx \dfrac{21.166 - 11.314}{5} \approx 1.970$ meters per second.

13. (a) $\dfrac{9.261 - 8}{0.1} = 12.610$.
(b) $\dfrac{8.120601 - 8}{0.01} \approx 12.060$.
(c) $\dfrac{8.012006001 - 8}{0.001} \approx 12.006$.
(d) The slopes approach 12.

14. (a) $C(100) = 1900$ and $C(150) = 2750$. Average rate $= \dfrac{850}{50} = 17$ dollars per item.
(b) $\dfrac{2750}{150} \approx 18.333$ dollars per item.

15. (a) $A(0) = 2$, $A(4) = \frac{2}{3}$, $A(10) = \frac{1}{3}$. On $[0, 4]$: $\dfrac{2/3 - 2}{4} \approx -0.333$. On $[4, 10]$: $\dfrac{1/3 - 2/3}{6} \approx -0.056$ moles per liter per second.
(b) The concentration falls, but more slowly later: the reaction slows down.

16. (a) $c(0.5) = \dfrac{2.5}{1.25} = 2$ and $c(2) = \dfrac{10}{5} = 2$, so the average rate is 0 milligrams per liter per hour.
(b) No. The concentration rose above 2 and then came back down. An average rate of 0 only compares the endpoints.
(c) $c(1) = 2.5$ and $c(0) = 0$, so the average rate is 2.500 milligrams per liter per hour.

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SET C · CHALLENGE AND AP-STYLE PROBLEMS

17. On $[a, a+2]$: $\dfrac{(a+2)^2 - a^2}{2} = \dfrac{4a+4}{2} = 2a+2$. Setting $2a+2=6$ gives $a=2$.

18. $\dfrac{f(3)-5}{2} = 4$, so $f(3) = 13$. Then $\dfrac{f(6)-13}{3} = -1$, so $f(6) = 10$.

19. Numerator: $\sqrt{2x+2h+1} - \sqrt{2x+1}$. Multiply by its conjugate over itself: $\dfrac{(2x+2h+1)-(2x+1)}{h\left(\sqrt{2x+2h+1}+\sqrt{2x+1}\right)} = \dfrac{2}{\sqrt{2x+2h+1}+\sqrt{2x+1}}$.

20. (a) $f(n) - f(m) = p(n^2-m^2) + q(n-m) = (n-m)\,[p(n+m)+q]$. Dividing by $n-m$ gives $p(m+n)+q$.
(b) $p=3$, $q=-5$: $3(2+6)-5 = 19$.

21. (a) $\dfrac{7-1}{4-(-2)} = 1$.
(b) $\dfrac{7-1}{4-2} = 3$.
(c) $\dfrac{1-1}{2-(-2)} = 0$: the secant line PQ is horizontal, because P and Q have the same height.

1.3 Evaluating Limits Algebraically

SET A · SKILLS

1. Polynomial, so substitute: $2(-1) - (-1) + 4 = -2 + 1 + 4 = 3$.

2. Factor: $\dfrac{(x-4)(x+4)}{x-4} = x+4$ for $x \neq 4$. Limit $= 8$.

3. Substitution gives $\frac{0}{0}$. Factor: $\dfrac{(x+2)(x-1)}{(x+2)(x+1)} = \dfrac{x-1}{x+1}$. Limit $= \dfrac{-3}{-1} = 3$.

4. Multiply by $\dfrac{\sqrt{x+4}+3}{\sqrt{x+4}+3}$: numerator becomes $(x+4)-9 = x-5$, which cancels. Limit $= \dfrac{1}{\sqrt{9}+3} = \dfrac{1}{6}$.

5. Multiply by $\dfrac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$: the denominator becomes $(x+3)-4 = x-1$, which cancels with the numerator, leaving $\sqrt{x+3}+2$. Limit $= 2+2 = 4$.

6. Numerator: $\dfrac{3-(x+3)}{3(x+3)} = \dfrac{-x}{3(x+3)}$. Dividing by x: $\dfrac{-1}{3(x+3)}$. Limit $= -\dfrac{1}{9}$.

7. $(2+h)^3 - 8 = 12h + 6h^2 + h^3$. Divide by h: $12 + 6h + h^2 \to 12$.

8. $\dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \dfrac{x^2+2x+4}{x+2}$. Limit $= \dfrac{12}{4} = 3$.

9. (a) $2(-3) - 5 = -11$.
(b) $(-3)(5) = -15$.
(c) $\sqrt{9+16} = 5$.
(d) $\dfrac{-3}{5}$ (the limit of the denominator is $5 \neq 0$).

SET B · APPLICATIONS AND MIXED PRACTICE

10. (a) Substituting gives $\frac{0}{0}$, which is undefined.
(b) $f(2.99) \approx 26.910$, $f(2.999) \approx 26.991$, $f(3.001) \approx 27.009$, $f(3.01) \approx 27.090$.
(c) $\dfrac{(x-3)(x^2+3x+9)}{x-3} = x^2+3x+9 \to 27$.

11. At $x = 0.99$ the quotient is about 5.998 and at $x = 1.01$ about 6.002, so the limit seems to be 6. Exactly: multiply by $\dfrac{\sqrt{x+8}+3}{\sqrt{x+8}+3}$ to get $\dfrac{(x-1)(\sqrt{x+8}+3)}{x-1} = \sqrt{x+8}+3 \to 6$.

12. (a) $h = 0.1$: 20.090. $h = 0.01$: 19.649. $h = 0.001$: 19.605 (meters per second).
(b) $\dfrac{4.9\left[(2+h)^2 - 4\right]}{h} = \dfrac{4.9(4h+h^2)}{h} = 4.9(4+h) \to 19.6$. This is the instantaneous velocity at $t=2$ seconds: 19.6 m/s.

13. For $t \neq 5$, $R(t) = \dfrac{(t-5)(t+5)}{t-5} = t+5$, and $\lim_{t\to 5} R(t) = 10$. So $R(5) = 10$ milligrams per hour fits smoothly.

14. $g(4.99) \approx -0.040$ and $g(5.01) \approx -0.040$. Exactly: the numerator is $\dfrac{5-x}{5x}$, so $g(x) = \dfrac{-1}{5x}$ for $x \neq 5$, and the limit is $-\dfrac{1}{25} = -0.040$.

15. (a) $h = 0.1$: $\pi(6.1) \approx 19.164$. $h = 0.01$: $\pi(6.01) \approx 18.881$.
(b) $\dfrac{\pi(3+h)^2 - 9\pi}{h} = \pi(6+h) \to 6\pi \approx 18.850$.

16. Factor: $\dfrac{(x+3)(x-1)}{(x+3)(x-3)} = \dfrac{x-1}{x-3}$. Limit $= \dfrac{-4}{-6} = \dfrac{2}{3} \approx 0.667$.

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17. For the limit to exist the numerator must be 0 at $x = 2$ (otherwise a nonzero number over 0): $4 + 2k - 10 = 0$, so $k = 3$. Then $\dfrac{x^2 + 3x - 10}{x - 2} = \dfrac{(x+5)(x-2)}{x-2} = x + 5 \to 7$.

18. The numerator must vanish at $x = 1$: $\sqrt{a+b} = 2$, so $a + b = 4$. Rationalize: $\dfrac{(ax+b) - 4}{(x-1)(\sqrt{ax+b}+2)}$; with $b = 4 - a$ the top is $a(x-1)$, giving $\dfrac{a}{\sqrt{ax+b}+2} \to \dfrac{a}{4}$. So $\dfrac{a}{4} = \dfrac{3}{4}$, $a = 3$, $b = 1$.

19. $\dfrac{(x+h)^3 - x^3}{h} = \dfrac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + 3xh + h^2 \to 3x^2$. At $x = -2$: $3(4) = 12$.

20. $\dfrac{0}{0}$ is an indeterminate form, not a number, so it cannot be set equal to 1. Factor: $\dfrac{(x-3)(x+3)}{x-3} = x + 3$ for $x \neq 3$, so the limit is 6.

21. (a) For $x \neq 1$, $f(x) = x + 1$, so $\lim_{x \to 1} f(x) = 2$.
    (b) $f(1) = 5$.
    (c) A limit depends only on values of f near 1, not at 1. The function value at 1 was defined separately and does not follow the line.

1.4 One-Sided Limits and Piecewise Functions

SET A · SKILLS

1. Left: $2(1) + 1 = 3$. Right: $5 - 1 = 4$. The one-sided limits differ, so $\lim_{x \to 1} f(x)$ does not exist. $f(1) = 5 - 1 = 4$.

2. (a) For $x < 0$, $|x| = -x$, so the quotient is -1: limit -1.
   (b) For $x > 0$ the quotient is 1: limit 1.
   (c) Does not exist, since $-1 \neq 1$.

3. As $x \to 4^-$, $4 - x \to 0^+$, so $\sqrt{4-x} \to 0$. The two-sided limit does not exist because $\sqrt{4-x}$ is undefined for $x > 4$, so there is no right-hand side to approach from.

4. Since $x^2 + 5x + 6 = (x+2)(x+3)$: for $x > -2$ the quotient is $\dfrac{1}{x+3} \to 1$. For $x < -2$, $|x+2| = -(x+2)$, giving $\dfrac{-1}{x+3} \to -1$. Limits 1 and -1 differ, so the two-sided limit does not exist.

5. (a) 2
   (b) 3
   (c) 3 (near 3.5, $\lfloor x \rfloor = 3$)
   (d) -2 (just left of -1, such as -1.01, the floor is -2)

6. Left: $3(1) + k$. Right: $1 + 4 = 5$. Setting $3 + k = 5$ gives $k = 2$.

7. For $x < 2$: $\dfrac{(x-2)(x+2)}{x-2} = x + 2 \to 4$. For $x > 2$: $3x - 2 \to 4$. Both sides give 4, so the limit is 4.

8. (a) For $x > 3$: $\dfrac{(x-3)(x+3)}{x-3} = x + 3 \to 6$.
   (b) For $x < 3$: $\dfrac{(x-3)(x+3)}{-(x-3)} = -(x+3) \to -6$.
   (c) No, since $6 \neq -6$.

9. If $x \geq 3$, $2x - 6 \geq 0$ so $f(x) = 2x - 6 + x = 3x - 6$. If $x < 3$, $f(x) = -(2x-6) + x = 6 - x$. As $x \to 3^+$: $3(3) - 6 = 3$. As $x \to 3^-$: $6 - 3 = 3$. The limit is 3.

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10. (a) $P(2) = 3(2) = 6$. Just left of 2 (e.g. 1.9), $\lceil t \rceil = 2$, so the left limit is 6. Just right of 2, $\lceil t \rceil = 3$, so the right limit is 9.
    (b) No, since the one-sided limits 6 and 9 differ.

11. (a) Left: $0.10(10000) = 1000$. Right: $1000 + 0.15(0) = 1000$.
    (b) Yes, the limit is 1000 dollars.
    (c) $T(12500) = 1000 + 0.15(2500) = 1375$ dollars.

12. (a) Left: $0.8(5) = 4$. Right: $10 - 5 = 5$.
    (b) No: the rates 4 and 5 differ, so the flow jumps by 1 liter per second at $t = 5$.
    (c) $R(8) = 2$ liters per second.

13. (a) Left limit 0. Right: $12(1 - e^0) = 0$. Both are 0.
    (b) $V(0.5) = 12(1 - e^{-1}) \approx 7.585$ volts. $V(1.5) = 12(1 - e^{-3}) \approx 11.403$ volts.

14. (a) $f(1.99) \approx -3.990$ and $f(2.01) \approx 4.010$.
    (b) For $x > 2$ near 2, $x^2 - 4 > 0$ so $f = x + 2 \to 4$. For $x < 2$ near 2 (with $x > -2$), $x^2 - 4 < 0$, so $|x^2 - 4| = -(x^2 - 4)$ and $f = -(x+2) \to -4$. The limits differ, so the two-sided limit does not exist.

15. (a) Left: $5 + 2(2) = 9$. Right: $5 + 2(3) = 11$. $C(3) = 5 + 2(3) = 11$.
    (b) $C(3.7) = 5 + 2(3) = 11$ dollars.

16. (a) Left: $e^0 = 1$. Right: $1 + \sin 0 = 1$.
    (b) Yes, the limit is 1.
    (c) $f(-0.1) = e^{-0.1} \approx 0.905$ and $f(0.1) = 1 + \sin 0.1 \approx 1.100$.

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17. At $x = 1$: left limit $1 + 2 = 3$, right limit $a + b$, so $a + b = 3$. At $x = 3$: left limit $3a + b$, right limit $9 - 2 = 7$, so $3a + b = 7$. Subtract: $2a = 4$, $a = 2$, $b = 1$.
18. No. Let $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$. Then $|f(x)| = 1$ for all x, so $\lim_{x \to 0} |f(x)| = 1$, but the one-sided limits of f at 0 are -1 and 1, so $\lim_{x \to 0} f(x)$ does not exist.
19. (a) Left: following $y = x + 1$ toward $x = 1$ gives 2. Right: following $y = 4 - x$ gives 3.
    (b) No, since $2 \neq 3$. $f(1) = 0$ (the solid point).
    (c) Near $x = 3$ the graph is $y = 4 - x$, so the limit is 1.
20. Near 0 we have $|x - 1| = 1 - x$ and $|x + 1| = x + 1$. The quotient is $\dfrac{(1 - x) - (x + 1)}{x} = \dfrac{-2x}{x} = -2$, so the limit is -2.
21. (a) As $x \to 2^-$, $x^2 \to 4$ from below, so $\lfloor x^2 \rfloor = 3$. Limit 3.
    (b) As $x \to 2^+$, x^2 is slightly above 4, so the floor is 4. Limit 4.
    (c) Near 0, $0 \leq x^2 < 1$, so the floor is 0. Limit 0.
    (d) Left of 0 the floor is -1, right of 0 it is 0. Does not exist.

1.5 Limits from Graphs and Tables

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1. (a) The left branch approaches height 2 (open circle), so the limit is 2.
   (b) The right branch starts at height 3: limit 3.
   (c) Does not exist, since $2 \neq 3$.
   (d) The solid dot is at 0, so $f(2) = 0$.
   (e) Near $x = 4$ the graph is the line $y = 5 - x$, so the limit is 1.
2. The values approach 1 from both sides (about 0.995 and 1.005 at ± 0.01), so the limit is about 1.
3. (a) Both sides approach height -2 at the open circle: limit -2.
   (b) $g(-1)$ is undefined (open circle with no solid dot).
   (c) Near $x = 2$ the graph is $y = x - 1$, so the limit is 1.
4. (a) From the left the values approach 2; from the right they approach 4.
   (b) No. The one-sided limits 2 and 4 differ.
5. (a) 1
   (b) 0
   (c) -1
6. False. For example $f(x) = \dfrac{x^2 - 4}{x - 2}$ is undefined at 2, but $f(x) = x + 2$ for $x \neq 2$, so $\lim_{x \to 2} f(x) = 4$. A limit does not depend on the value at the point.
7. (a) Both the flat piece (from the left) and the line $y = -x$ (from the right) approach height 2 at $x = -2$, so the limit is 2.
   (b) The solid dot is at 0, so $f(-2) = 0$.
   (c) Left: -1. Right: 1.
   (d) Near $x = 3$ the graph is $y = x$, so the limit is 3.
8. One valid sketch: put an open circle at $(2, 3)$ with the graph passing through it from both sides, and a solid dot at $(2, 1)$. Near $x = 4$ let the left piece approach height 0 and the right piece start near height 2 (open circles at both $(4, 0)$ and $(4, 2)$, or a solid dot at one of them).
9. (a) The left limit of f is 2, so the limit is $2(2) + 1 = 5$.
   (b) The right limit of f is 3, so the limit is $\sqrt{3 + 1} = 2$.
   (c) Does not exist: the one-sided limits of $[f(x)]^2$ are 4 and 9, which differ.

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10. (a) $f(-0.01) \approx 0.691$, $f(0.01) \approx 0.696$, $f(-0.001) \approx 0.693$, $f(0.001) \approx 0.693$.
    (b) The limit is about 0.693 (it is $\ln 2$).
11. (a) $s(2.9) = 14.21$, $s(3) = 15$, $s(3.1) = 15.81$. $[2.9, 3]$: $\dfrac{0.79}{0.1} = 7.9$. $[3, 3.1]$: $\dfrac{0.81}{0.1} = 8.1$ (meters per second).
    (b) The averages straddle 8 and close in on it: the velocity at $t = 3$ is 8 m/s. (Algebra: the average on $[3, 3 + h]$ is $8 + h$.)
12. (a) $f(0.999) \approx 2.997$ and $f(1.001) \approx 3.003$.
    (b) The limit appears to be 3. Exactly: $\dfrac{(x - 1)(x^2 + x + 1)}{x - 1} = x^2 + x + 1 \to 3$.
13. (a) $f(0.1) = \sin(10\pi) = 0$, $f(0.4) = \sin(2.5\pi) = 1$, $f\left(\frac{2}{7}\right) = \sin(3.5\pi) = -1$.
    (b) There are inputs arbitrarily close to 0 where $f = 1$ ($x = \frac{2}{4n+1}$), where $f = -1$ ($x = \frac{2}{4n+3}$), and where $f = 0$. The outputs do not approach one number, so the limit does not exist.
14. (a) $f(0.1) \approx -0.500$ and $f(0.01) \approx -0.500$.
    (b) The limit is about $-0.500 = -\frac{1}{2}$.

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15. (a) Left: 21 (open circle). Right: 18.
(b) $T(3) = 18$. The limit does not exist since $21 \neq 18$.
(c) Near $h = 6$: $T = 18 + 0.5(3) = 19.5$. Limit 19.5 degrees Celsius.
16. (a) $f(0.1) \approx 2.594$, $f(0.01) \approx 2.705$, $f(0.001) \approx 2.717$, $f(-0.001) \approx 2.720$.
(b) The values from both sides squeeze toward about 2.718, the constant e.

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17. (a) As $x \to 0$, $f(x) = \frac{x^2}{2} \to 0$ from above, and near 0 the graph is continuous, so $f(f(x)) \to f(0) = 0$.
(b) As $x \to 2^-$, $f(x) \to 2$ from below, so the outer function sees inputs approaching 2 from the left: limit 2.
(c) As $x \to 2^+$, $f(x) = 5 - x \to 3$ from below, and $f(3) = 2$ with f following $5 - x$ near 3: limit 2.
(d) Yes, both one-sided limits are 2, so the limit is 2 (even though f itself has no limit at 2).
18. The values $x = 0.1,\ 0.01,\ 0.001$ are of the form $\frac{1}{10^n}$, for which $\frac{\pi}{x} = 10^n\pi$ is a multiple of π, so the sine is 0. Other inputs near 0, such as $x = \frac{2}{4n+1}$, give 1, and $x = \frac{2}{4n+3}$ give -1. The outputs do not settle, so the limit does not exist. A table only gives evidence, and a poorly chosen set of inputs can mislead.
19. (a) Only $c = 1$ (left limit -1, right limit 1).
(b) Only $c = -2$: the limit is 2 but $f(-2) = 0$. At $c = 1$ the limit does not exist, and at every other c the limit equals $f(c)$.
20. Since the outputs get arbitrarily close to 5, they eventually are within 1 of 5, that is between 4 and 6. So for x close enough to 2 ($x \neq 2$), $4 < f(x) < 6$.
21. The values are about 0.40 on both sides, so the limit is approximately 0.4 (it equals $\ln \frac{3}{2} \approx 0.405$).

1.6 Infinite Limits and Vertical Asymptotes

SET A · SKILLS

1. (a) $-\infty$ (denominator small and negative)
(b) ∞
(c) Does not exist, since the one-sided limits differ.
2. $(x+2)^2$ is a small positive number on both sides and the numerator is 5, so the limit is ∞.
3. Factor the denominator: $(x-3)(x+2)$. Nothing cancels with $x+1$, so the vertical asymptotes are $x = 3$ and $x = -2$.
4. $f(x) = \dfrac{(x-3)(x+3)}{(x-3)(x-2)} = \dfrac{x+3}{x-2}$. Hole at $x = 3$ with limit $\dfrac{6}{1} = 6$. Vertical asymptote: $x = 2$ only.
5. (a) Numerator $\to -1$, denominator $\to 0^-$: $\dfrac{-1}{0^-} = \infty$.
(b) Numerator $\to -1$, denominator $\to 0^+$: $-\infty$.
6. (a) Numerator $\to -2$, $x^3 \to 0^+$: $-\infty$.
(b) Numerator $\to -2$, $x^3 \to 0^-$: ∞.
7. (a) ∞ ($\sin x \to 1$, $\cos x \to 0^+$)
(b) $-\infty$ ($\cos x \to 0^-$)
(c) $-\infty$
8. Combine: $\dfrac{x^2-1}{x^4}$. The numerator tends to -1 and $x^4 \to 0^+$, so the limit is $-\infty$.
9. As x approaches 1 from either side, $f(x)$ increases without bound. So $\lim\limits_{x\to 1} f(x) = \infty$ and $x = 1$ is a vertical asymptote.

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10. (a) $F(0.1) = 800$ and $F(0.01) = 80000$ newtons.
(b) The limit is ∞: the force grows without bound as the charges are brought together.
11. (a) $v(10.1) = \dfrac{101}{0.1} = 1010$. $v(10.01) = 10010$. $v(9.9) = \dfrac{99}{-0.1} = -990$ (centimeters).
(b) Right: ∞ (numerator tends to 100, denominator to 0^+). Left: $-\infty$. The image moves infinitely far away as the object nears the focal point.
12. (a) $C(95) = \dfrac{7600}{5} = 1520$, $C(99) = 7920$, $C(99.9) = 79920$ thousand dollars.
(b) The limit is ∞ (numerator $\to 8000$, denominator $\to 0^+$): removing all of the pollutant would cost an unbounded amount.
13. (a) $\gamma(0.9) \approx 2.294$, $\gamma(0.99) \approx 7.089$, $\gamma(0.999) \approx 22.366$.
(b) $1 - v^2 \to 0^+$, so $\gamma \to \infty$.
14. (a) $d(1.5) \approx 42.304$ km and $d(1.57) \approx 3767.297$ km.
(b) $\tan\theta \to \infty$, so the limit is ∞.
15. (a) $x^2 - 2x - 8 = (x-4)(x+2)$; the numerator $2x+1$ is not 0 at 4 or -2, so both $x = 4$ and $x = -2$ are vertical asymptotes.
(b) At $x = 4$: $\dfrac{2x+1}{x+2} \to \dfrac{9}{6} > 0$, so left $-\infty$, right ∞. At $x = -2$: $\dfrac{2x+1}{x-4} \to \dfrac{-3}{-6} > 0$, so left $-\infty$, right ∞.
(c) $f(4.001) \approx 1500.083$ and $f(3.999) \approx -1499.917$.
16. (a) $f(x) = \dfrac{(x-1)(x+1)}{(x-1)(x-2)} = \dfrac{x+1}{x-2}$. Vertical asymptote $x = 2$; hole at $x = 1$.
(b) $f(0.999) \approx -1.997$ and $f(1.001) \approx -2.003$, approaching the limit $\dfrac{2}{-1} = -2$ at the hole.

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17. The denominator is $(x-1)(x-2)$. There are two asymptotes unless the numerator cancels one factor: $x+a=0$ at $x=1$ gives $a=-1$ (then $f=\frac{1}{x-2}$), and at $x=2$ gives $a=-2$ (then $f=\frac{1}{x-1}$). So $a=-1$ or $a=-2$.
18. One answer: $f(x)=\dfrac{2-x}{(x-2)(x+1)(x-3)^2}$. After cancelling, $f(x)=\dfrac{-1}{(x+1)(x-3)^2}$ for $x\neq 2$. Near $x=-1$: $(x-3)^2\to 16>0$, so $f\approx\dfrac{-1/16}{x+1}$: left limit ∞, right limit $-\infty$. Near $x=3$: $(x+1)\to 4$ and $(x-3)^2\to 0^+$, so $f\to-\infty$ from both sides. At $x=2$ the cancelled factor leaves a hole (at height $-\frac{1}{3}$).
19. Near -3 the numerator tends to -3 and $|x+3|\to 0^+$ from both sides. So the quotient is $\dfrac{-3}{0^+}$ and the limit is $-\infty$.
20. $f(x)=\dfrac{(x-2)(x+2)}{x-2}=x+2$ for $x\neq 2$, so f has a finite limit 4 at 2 (a hole). For g, the numerator is 8 at $x=2$ while the denominator tends to 0, so g is unbounded near 2: $x=2$ is a vertical asymptote.
21. (a) Numerator $\to 2$, $\sin x\to 0^+$: ∞.
    (b) Numerator $\to 2$, $\sin x\to 0^-$: $-\infty$.
    (c) Just past π, $\sin x<0$ and small: $\csc x\to-\infty$.
    (d) Just before π: $\cos x\to -1$ and $\sin x\to 0^+$, so $\cot x\to-\infty$.

1.7 Limits at Infinity and Horizontal Asymptotes

SET A · SKILLS

1. $\frac{5}{2}$. Equal degrees: divide by x^2, the limit is the ratio of leading coefficients.
2. 0. The denominator has higher degree, so dividing by x^3 gives $\frac{4/x^2+1/x^3}{1-2/x^3}\to 0$.
3. $-\infty$. Dividing by x^2 gives $\frac{3x-1/x}{1+4/x^2}$, and $3x\to-\infty$.
4. $\frac{3}{2}$. For $x>0$, $\sqrt{9x^2+x}=x\sqrt{9+1/x}$. Divide top and bottom by x: $\frac{\sqrt{9+1/x}}{2-1/x}\to\frac{3}{2}$.
5. -2. For $x<0$, $\sqrt{16x^2+1}=|x|\sqrt{16+1/x^2}=-x\sqrt{16+1/x^2}$. Dividing by x gives $\frac{-\sqrt{16+1/x^2}}{2+5/x}\to\frac{-4}{2}=-2$.
6. 2. Divide by e^x: $\frac{2+e^{-x}}{1-3e^{-x}}\to\frac{2}{1}=2$.
7. 0. Since $-1\le\sin x\le 1$, we have $-\frac{1}{x}\le\frac{\sin x}{x}\le\frac{1}{x}$ for $x>0$, and both bounds tend to 0.
8. 3. Multiply by the conjugate: $\frac{6x}{\sqrt{x^2+6x}+x}=\frac{6}{\sqrt{1+6/x}+1}\to\frac{6}{2}=3$.
9. $y=3$ and $y=-3$. As $x\to\infty$, $f(x)\to\frac{3}{1}=3$. As $x\to-\infty$, $\sqrt{x^2+4}\approx -x$, so $f(x)\to\frac{3}{-1}=-3$.

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10. (a) 0: divide by t^2, giving $\frac{8/t}{1+4/t^2}\to 0$. The drug is eventually cleared from the blood.
    (b) Solve $8t=0.1(t^2+4)$, i.e. $t^2-80t+4=0$, so $t=40\pm\sqrt{1596}$. The larger root is $t\approx 79.950$ hours.
11. (a) 5000: as $t\to\infty$, $e^{-0.4t}\to 0$, so $P\to\frac{5000}{1}$. This is the maximum population the culture can support.
    (b) Solve $1+49e^{-0.4t}=2$, so $e^{-0.4t}=\frac{1}{49}$ and $t=\frac{\ln 49}{0.4}\approx 9.730$ hours.
12. (a) The values approach 1.
    (b) With $u=\frac{1}{x}$, $u\to 0^+$ as $x\to\infty$, and $g=\frac{\sin u}{u}$. Since $\lim_{u\to 0}\frac{\sin u}{u}=1$, the limit is 1.
13. (a) 40 m/s, since $e^{-0.25t}\to 0$.
    (b) Need $40(1-e^{-0.25t})=36$, so $e^{-0.25t}=0.1$ and $t=\frac{\ln 10}{0.25}\approx 9.210$ seconds.
14. (a) The graph levels off near $y=4$ on the right and near $y=-0.5$ on the left.
    (b) As $x\to\infty$, divide by e^x: $\frac{4-e^{-x}}{1+2e^{-x}}\to 4$. As $x\to-\infty$, $e^x\to 0$, so $f(x)\to\frac{0-1}{0+2}=-\frac{1}{2}$.
15. (a) $A(q)=\frac{500}{q}+12\to 12$.
    (b) For very large batches, the fixed cost of 500 dollars is spread over so many units that the average cost approaches 12 dollars per unit, the cost of the variable part alone.
16. (a) $y=22$, the room temperature the liquid approaches.
    (b) Need $58e^{-0.15t}=1$, so $t=\frac{\ln 58}{0.15}\approx 27.070$ minutes.

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17. $a=12$. Multiplying by the conjugate gives $\frac{ax}{\sqrt{4x^2+ax}+2x}=\frac{a}{\sqrt{4+a/x}+2}\to\frac{a}{4}$. Set $\frac{a}{4}=3$.
18. $a=1$, $b=-1$. Long division: $\frac{x^2+1}{x+1}=x-1+\frac{2}{x+1}$. The remainder tends to 0, so $y=x-1$ is the slant asymptote.
19. 3. Divide by x: $\frac{3+\frac{\sin x}{x}}{1-\frac{2\cos x}{x}}$. Since $\left|\frac{\sin x}{x}\right|\le\frac{1}{x}$ and $\left|\frac{2\cos x}{x}\right|\le\frac{2}{x}$, both fractions tend to 0 by the Squeeze Theorem, so the limit is $\frac{3}{1}=3$.
20. (a) $y=2$, since $\frac{2x^2}{x^2-1}\to 2$ as $x\to\pm\infty$.
    (b) $x=1$ and $x=-1$, where the denominator is 0 and the numerator is 2.
    (c) $+\infty$: near $x=1^+$ the numerator is close to $2>0$ and x^2-1 is small and positive.
    (d) No. Setting $\frac{2x^2}{x^2-1}=2$ gives $2x^2=2x^2-2$, which is impossible.
21. 0. Divide by 3^x: $\frac{x^3/3^x+(2/3)^x}{1-x^5/3^x}$. Because 3^x grows faster than any power of x, $\frac{x^3}{3^x}\to 0$ and $\frac{x^5}{3^x}\to 0$, and $(2/3)^x\to 0$. The limit is $\frac{0}{1}=0$.

1.8 Exponential and Logarithmic Functions

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1. 12. $\ln(e^5) = 5$ and $e^{\ln 7} = 7$ because $\ln$ and e^x are inverse functions.
2. $\ln 10$. $3\ln 2 = \ln 8$, so the sum is $\ln \frac{8\cdot 5}{4} = \ln 10$.
3. $x = 2$. Write $27 = 3^3$, so $2x - 1 = 3$.
4. $x = \frac{\ln 5}{3}$. Take $\ln$ of both sides: $3x = \ln 5$.
5. $x = \frac{e^3+1}{2}$. Exponentiate: $2x - 1 = e^3$. The argument $e^3 > 0$, so the solution is valid.
6. $x = 8$. Combine: $\log_2[x(x-6)] = 4$, so $x^2 - 6x = 16$, i.e. $(x-8)(x+2) = 0$. The domain requires $x > 6$, so reject $x = -2$.
7. $x = \ln 2$ or $x = \ln 5$. Let $u = e^x$: $u^2 - 7u + 10 = (u-2)(u-5) = 0$, so $e^x = 2$ or $e^x = 5$.
8. $\frac{1}{2}$. As $x \to -\infty$, $e^x \to 0$, so the quotient tends to $\frac{0+2}{0+4}$.
9. $\ln 3$. The quotient inside tends to $\frac{6}{2} = 3$, and $\ln$ is continuous at 3, so the limit is $\ln 3$.

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10. Solve $e^{0.35t} = 25$: $t = \frac{\ln 25}{0.35} \approx 9.197$ hours.
11. (a) $A(30) = 60 \cdot 2^{-2.5} \approx 10.607$ mg.
    (b) Solve $\left(\frac{1}{2}\right)^{t/12} = \frac{1}{12}$: $t = 12\log_2 12 = \frac{12\ln 12}{\ln 2} \approx 43.020$ hours.
12. Solve $e^{0.04t} = 2$: $t = \frac{\ln 2}{0.04} \approx 17.329$ years. The doubling time does not depend on the starting amount.
13. (a) $a = f(0) = 3$ and $b = \frac{6}{3} = 2$, so $f(x) = 3 \cdot 2^x$.
    (b) $f(6) = 3 \cdot 64 = 192$.
    (c) $2^x = \frac{1000}{3}$, so $x = \frac{\ln(1000/3)}{\ln 2} \approx 8.381$.
14. (a) 0, since $e^{-0.12h} \to 0$.
    (b) $e^{-0.12h} = \frac{50}{101.3}$, so $h = \frac{\ln(101.3/50)}{0.12} \approx 5.884$ km.
15. Solve $12e^{-0.3t} = 1$: $t = \frac{\ln 12}{0.3} \approx 8.283$ hours. After this time $C < 1$ because C is decreasing.
16. (a) $-\log_{10}(3.2 \times 10^{-5}) \approx 4.495$.
    (b) $-\log_{10}(2.5 \times 10^{-4}) \approx 3.602$.
    (c) The second one: a higher concentration of H^+ gives a lower pH.

SET C · CHALLENGE AND AP-STYLE PROBLEMS

17. $A = 4$ from $f(0)$. Then $4e^{2k} = 36$ gives $e^{2k} = 9$, so $k = \ln 3$. Hence $f(x) = 4 \cdot 3^x$ and $f(5) = 4 \cdot 243 = 972$.
18. $x = 0$ or $x = 2$. Let $u = 2^x$ so $4^x = u^2$: $u^2 - 5u + 4 = (u-1)(u-4) = 0$, so $2^x = 1$ or $2^x = 4$.
19. $x = 4$. Equating arguments: $x^2 - 3x - 4 = 0$, so $x = 4$ or $x = -1$. For $x = -1$, $\ln(3x) = \ln(-3)$ is undefined, so it is rejected. For $x = 4$ both arguments equal $12 > 0$.
20. $f^{-1}(x) = \frac{1+\ln(x-3)}{2}$ with domain $x > 3$. Solve $y - 3 = e^{2x-1}$, so $2x - 1 = \ln(y-3)$. The domain of f^{-1} is the range of f, which is $y > 3$.
21. 1 and -1. For $x \to \infty$ divide by e^x: $\frac{1+e^{-2x}}{1-e^{-2x}} \to 1$. For $x \to -\infty$ divide by e^{-x}: $\frac{e^{2x}+1}{e^{2x}-1} \to \frac{1}{-1} = -1$.

1.9 Special Trigonometric Limits and the Squeeze Theorem

SET A · SKILLS

1. $\frac{7}{2}$. Write $\frac{\sin 7x}{2x} = \frac{7}{2} \cdot \frac{\sin 7x}{7x} \to \frac{7}{2} \cdot 1$.
2. 6. $\frac{\tan 6x}{x} = \frac{\sin 6x}{x} \cdot \frac{1}{\cos 6x} \to 6 \cdot 1$.
3. 0. Write $\frac{1-\cos 3x}{x} = 3 \cdot \frac{1-\cos 3x}{3x} \to 3 \cdot 0$.
4. 2. $\frac{1-\cos 2x}{x^2} = 4 \cdot \frac{1-\cos 2x}{(2x)^2} \to 4 \cdot \frac{1}{2}$. Alternatively use $1 - \cos 2x = 2\sin^2 x$.
5. $\frac{1}{3}$. The reciprocal of $\frac{\sin 3x}{x} \to 3$.
6. 25. $\frac{\sin^2 5x}{x^2} = \left(\frac{\sin 5x}{x}\right)^2 \to 5^2$.
7. 1. Factor $x^2 + x = x(x+1)$: $\frac{\sin x}{x} \cdot \frac{1}{x+1} \to 1 \cdot 1$.
8. 0. Since $-1 \le \cos\frac{5}{x} \le 1$, we get $-|x| \le x\cos\frac{5}{x} \le |x|$. Both bounds tend to 0, so the limit is 0 by the Squeeze Theorem.
9. 0. Since $\left|\frac{\cos x}{x^2+1}\right| \le \frac{1}{x^2+1} \to 0$, the Squeeze Theorem gives 0.

SET B · APPLICATIONS AND MIXED PRACTICE

10. (a) The values approach 1, so the limit is 1.
    (b) The limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$ holds only when x is measured in radians. In degree mode, $\sin(0.1^\circ) \approx \sin(0.001745)$, so the ratio is about $\frac{\pi}{180} \approx 0.017$.
11. (a) The values approach 0.5.
    (b) Multiply by $\frac{1+\cos x}{1+\cos x}$: $\frac{1-\cos^2 x}{x^2(1+\cos x)} = \frac{\sin^2 x}{x^2(1+\cos x)}$. As $x \to 0$ this tends to $\frac{1^2}{1+1} = \frac{1}{2}$.
12. (a) $\frac{10\theta}{20\sin(\theta/2)} = \frac{\theta/2}{\sin(\theta/2)} \to 1$ because $\frac{\sin u}{u} \to 1$ with $u = \frac{\theta}{2}$.
    (b) Arc $= 2.000$, chord $= 20\sin 0.1 \approx 1.997$, ratio ≈ 1.002.
13. $\sin 0.3 \approx 0.29552$, so the error is $\frac{0.3-0.29552}{0.29552} \times 100 \approx 1.516$ percent. Since $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, the ratio of $\sin\theta$ to θ is close to 1 when θ is small.
14. (a) Since $\left|\sin\frac{1}{x}\right| \le 1$, we have $\left|x\sin\frac{1}{x}\right| \le |x|$, which means $-|x| \le x\sin\frac{1}{x} \le |x|$.
    (b) Both bounds tend to 0, so $\lim_{x\to 0} f(x) = 0$ by the Squeeze Theorem.
15. (a) $50\tan 0.04 \approx 2.001$ meters.
    (b) $50(0.04) = 2$. They agree closely because $\tan\theta \approx \theta$ for small θ, as $\frac{\tan\theta}{\theta} = \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta} \to 1$.
16. (a) $\frac{A}{\theta^2} = \frac{1}{2} \cdot \frac{\sin\theta}{\theta} \to \frac{1}{2}$.
    (b) $\frac{0.5\sin 0.1}{0.1} \approx 0.499$.

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17. 6. Rewrite as $\frac{\sin 3x}{x} \cdot \frac{\tan 2x}{x} \cdot \frac{1}{\cos 5x} \to 3 \cdot 2 \cdot 1 = 6$.
18. $\frac{1}{2}$. Write it as $\frac{1-\cos x}{x^2} \cdot \frac{x}{\sin x} \to \frac{1}{2} \cdot 1$.
19. 0. Write it as $\frac{x}{\sin x} \cdot x \sin \frac{1}{x}$. The first factor tends to 1. The second tends to 0 by the Squeeze Theorem since $|x \sin \frac{1}{x}| \le |x|$.
20. (a) $\lim_{x\to 1}(2x-1) = 1$ and $\lim_{x\to 1} x^2 = 1$, so $\lim_{x\to 1} f(x) = 1$ by the Squeeze Theorem (the inequality holds on an open interval containing 1).
    (b) Yes. At $x = 1$ the inequality gives $1 \le f(1) \le 1$, so $f(1) = 1 = \lim_{x\to 1} f(x)$.
21. $k = 4$ or $k = -4$. $\frac{1-\cos kx}{x^2} = k^2 \cdot \frac{1-\cos kx}{(kx)^2} \to \frac{k^2}{2}$ (for $k \neq 0$). Set $\frac{k^2}{2} = 8$.

1.10 Continuity and Types of Discontinuity

SET A · SKILLS

1. No. $\lim_{x\to 5} \frac{(x-5)(x+5)}{x-5} = 10$, but $f(5) = 7 \neq 10$. The limit exists, so the discontinuity is removable (redefine $f(5) = 10$).
2. $x^2 - 5x + 6 = (x-2)(x-3)$, so $x = 2$ and $x = 3$. At each, the numerator is nonzero (3 and 4), so $|f| \to \infty$: both are infinite discontinuities (vertical asymptotes).
3. For $x \neq -1$, $f(x) = x - 1$, so $\lim_{x\to -1} f(x) = -2$. $f(-1)$ is undefined. Defining $f(-1) = -2$ makes f continuous.
4. $k = -3$. Left limit: $6 + k$. Right limit and value: $4 - 1 = 3$. Set $6 + k = 3$.
5. $a = 4$. For $x \neq 2$, $g(x) = x + 2$, so $\lim_{x\to 2} g(x) = 4$, and continuity requires $g(2) = 4$.
6. For $x < 3$, $|x-3| = -(x-3)$ so $h = -1$. For $x > 3$, $h = 1$. The one-sided limits are -1 and 1, so there is a jump discontinuity.
7. $[2, \infty)$. f is defined only for $x \ge 2$ and is continuous at every point of its domain (continuous from the right at $x = 2$).
8. The domain requires $x > 0$ and $x \neq 1$, and g is continuous wherever it is defined: $(0, 1)$ and $(1, \infty)$.
9. e^2. For $x \neq 1$, $\frac{x^2-1}{x-1} = x + 1 \to 2$. Since e^u is continuous, the limit is e^2.

SET B · APPLICATIONS AND MIXED PRACTICE

10. (a) 4 and 6.50.
    (b) The one-sided limits differ, so $\lim_{w\to 1} C(w)$ does not exist and C has a jump discontinuity at $w = 1$.
    (c) Yes: $C(1) = 4 = \lim_{w\to 1^-} C(w)$.
11. (a) $T(10000) = 1000$. Left limit: $0.10(10000) = 1000$. Right limit: $1000 + 0 = 1000$. All three agree, so T is continuous.
    (b) $T(25000) = 1000 + 0.15(15000) = 3250$ dollars.
12. $c = 4$. Left limit is 9; the right side gives $3c - 3$. Set $3c - 3 = 9$.
13. (a) The values from both sides approach 1, so the limit is about 1.
    (b) Define $f(0) = 1$.
14. $x = 0$: $\lim_{x\to 0} f(x) = 2$ from both sides but $f(0) = -1$, so removable.
    $x = 3$: left limit 2, right limit 1, so a jump discontinuity.
    $x = 5$: the left limit is -1 but as $x \to 5^+$, $f(x) \to \infty$, so an infinite discontinuity (vertical asymptote on the right).
15. (a) $C(50) = 500$, and $\lim_{q\to 50^+} 8q = 400$.
    (b) The one-sided limits (500 from the left, 400 from the right) differ: a jump discontinuity.
    (c) Need $8q > 500$, so $q > 62.5$ and $q = 63$.
16. $k = 2$, since $\frac{1-\cos 2x}{x^2} = 4 \cdot \frac{1-\cos 2x}{(2x)^2} \to 4 \cdot \frac{1}{2} = 2$. Check: $g(0.01) = \frac{1-\cos 0.02}{0.0001} \approx 2.000$.

SET C · CHALLENGE AND AP-STYLE PROBLEMS

17. $\lim_{x\to 1^-} f = 2 + a$, $\lim_{x\to 1^+} f = 4$, $f(1) = b$. Continuity needs $2 + a = 4 = b$, so $a = 2$ and $b = 4$.
18. The denominator tends to 0, so the numerator must vanish at 3: $9 + 3a - 6 = 0$, $a = -1$. Then $\frac{x^2-x-6}{x-3} = \frac{(x-3)(x+2)}{x-3} = x + 2 \to 5$. Define $f(3) = 5$.
19. $f(2) = 3$ is defined. With $u = 3(x-2) \to 0$: $f(x) = 3 \cdot \frac{\sin u}{u} \to 3$. So $\lim_{x\to 2} f(x) = 3 = f(2)$, and f is continuous at 2.
20. $x^3 - x = x(x-1)(x+1)$, so the discontinuities are $x = 0, 1, -1$. For $x \neq 0, \pm 1$, $f(x) = \frac{1}{x}$. At $x = 1$ and $x = -1$ the limits are 1 and -1: removable. At $x = 0$, $|f| \to \infty$: infinite.
21. No. Let $f(x) = 1$ for $x \ge 0$ and $f(x) = -1$ for $x < 0$. Then $|f(x)| = 1$ for all x is continuous, but f has a jump discontinuity at 0 (one-sided limits -1 and 1).

1.11 The Intermediate Value Theorem

SET A · SKILLS

1. Let $f(x) = x^3 - 4x + 2$, continuous on $[0, 1]$ (polynomial). $f(0) = 2 > 0$ and $f(1) = -1 < 0$. Since 0 lies between $f(1)$ and $f(0)$, the Intermediate Value Theorem gives c in $(0, 1)$ with $f(c) = 0$.
2. $f(x) = x^4 - 3x - 5$ is continuous. $f(1) = -7 < 0$ and $f(2) = 5 > 0$, so by the IVT there is c in $(1, 2)$ with $f(c) = 0$.
3. Let $f(x) = \cos x - x$, continuous. $f(0) = 1 > 0$. $f(1) = \cos 1 - 1 < 0$ because $\cos 1 < 1$. By the IVT there is c in $(0, 1)$ with $f(c) = 0$, i.e. $\cos c = c$.
4. Let $f(x) = \ln x + x - 2$, continuous for $x > 0$. $f(1) = -1 < 0$ and $f(2) = \ln 2 > 0$. By the IVT, $f(c) = 0$ for some c in $(1, 2)$.
5. Let $f(x) = \sqrt{x} - e^{-x}$, continuous on $[0, 1]$. $f(0) = -1 < 0$ and $f(1) = 1 - e^{-1} > 0$. By the IVT there is c in $(0, 1)$ with $f(c) = 0$.

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6. (a) Yes: $f(1) = -2 < 0 < 1 = f(3)$, so there is a solution in $(1, 3)$.
   (b) Yes: $f(6) = -4 < 3 < 5 = f(9)$, so there is a solution in $(6, 9)$.
   (c) No guarantee: 7 is larger than every tabulated value, so it does not lie between two known function values. The IVT cannot conclude anything.
7. No. f is not continuous on $[0, 3]$ because it is undefined at $x = 2$. The hypothesis fails, and indeed $\frac{1}{x-2}$ is never 0.
8. Value 6: yes, since $2 < 6 < 10$ and f is continuous (IVT). Value 12: not necessarily, since 12 is not between $f(1)$ and $f(5)$; f could stay below 12 (e.g. a straight line).
9. $f(-3) = -7$, $f(-2) = 6$: zero in $(-3, -2)$. $f(0) = 2$, $f(1) = -3$: zero in $(0, 1)$. $f(2) = -2$, $f(3) = 11$: zero in $(2, 3)$. The three intervals are disjoint and f is continuous, so there are three distinct zeros.

SET B · APPLICATIONS AND MIXED PRACTICE

10. $T(6) = 6 < 15 < 19 = T(12)$, so by the IVT $T(c) = 15$ for some c in $(6, 12)$. Also $T(18) = 14 < 15 < 19 = T(12)$, so $T(d) = 15$ for some d in $(12, 18)$. These intervals do not overlap, so $c \neq d$.
11. (a) f is continuous, $f(1) = -1 < 0$ and $f(2) = 7 > 0$, so a zero exists in $(1, 2)$ by the IVT.
    (b) $x \approx 1.213$.
12. Let $f(x) = \cos x - x^2$. $f(0) = 1 > 0$ and $f(1) = \cos 1 - 1 \approx -0.460 < 0$, so by the IVT there is a solution in $(0, 1)$. A calculator gives $x \approx 0.824$.
13. S is continuous, $S(1) = 4\pi \approx 12.566 < 100 < 113.097 \approx 4\pi(9) = S(3)$, so by the IVT such a radius exists. Solving $4\pi r^2 = 100$: $r = \sqrt{25/\pi} \approx 2.821$ cm.
14. (a) Three zeros.
    (b) $f(0) = -5$, $f(1) = 7$, $f(2) = 7$, $f(3) = 1$, $f(4) = -5$, $f(5) = -5$, $f(6) = 7$. Sign changes occur on $(0, 1)$, $(3, 4)$ and $(5, 6)$, so by the IVT (f is a polynomial, hence continuous) there is a zero in each.
15. The IVT requires f to be continuous on $[0, 4]$. Here $\lim_{x \to 2^-} f(x) = -1$ but $f(2) = 4$, so f has a jump at $x = 2$ and skips every value from -1 to 4, including 0.
16. $s(0) = 3$, $s(1) \approx -0.248$, $s(2) \approx 0.039$. By the IVT there is a zero in $(0, 1)$ and another in $(1, 2)$. Calculator: $t \approx 0.946$ and $t \approx 1.993$ seconds.

SET C · CHALLENGE AND AP-STYLE PROBLEMS

17. Let $g(x) = f(x) - x$, continuous. $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$. If either equals 0 we are done. Otherwise $g(0) > 0 > g(1)$, and the IVT gives c in $(0, 1)$ with $g(c) = 0$.
18. Let $g(x) = f(x + \pi) - f(x)$ on $[0, \pi]$, continuous. $g(0) = f(\pi) - f(0)$ and $g(\pi) = f(2\pi) - f(\pi) = f(0) - f(\pi) = -g(0)$. So $g(0)$ and $g(\pi)$ are opposite in sign or both zero; by the IVT there is c with $g(c) = 0$.
19. $f(0) = k$ and $f(1) = k - 2$. They have opposite signs exactly when $k(k - 2) < 0$, i.e. $0 < k < 2$. For those k, f (a polynomial) has a zero in $(0, 1)$.
20. Let $f(x) = x^5 + x - 1$. $f(0) = -1 < 0$ and $f(1) = 1 > 0$, so a solution exists in $(0, 1)$ by the IVT. For uniqueness: if $a < b$ then $a^5 < b^5$ and $a < b$, so $f(a) < f(b)$. A strictly increasing function takes the value 0 at most once.
21. (a) Sign changes on $(-1, 0)$, $(0, 2)$ and $(2, 5)$: at least 3 solutions.
    (b) $-3 < 1 < 2$ on $[-1, 0]$, $2 > 1 > -1$ on $[0, 2]$, $-1 < 1 < 6$ on $[2, 5]$: at least 3 solutions. In each case the three intervals do not overlap in interior points.

1.12 Multiple-Choice Practice on Limits

SET A · PART A [NO CALCULATOR]

1. A. Factor: $\frac{(x-3)(x+3)}{(x-3)(x+1)} = \frac{x+3}{x+1} \to \frac{6}{4} = \frac{3}{2}$.
2. C. Multiply by the conjugate: $\frac{x}{x(\sqrt{x+9}+3)} = \frac{1}{\sqrt{x+9}+3} \to \frac{1}{6}$.
3. E. From the left $f \to 3$, from the right $f \to 1$. The one-sided limits differ, so the limit does not exist.
4. B. The values from both sides approach 8. Algebraically $f(x) = x + 4$ for $x \neq 4$.
5. D. Equal degrees: the limit is the ratio of leading coefficients, $\frac{4}{2} = 2$.
6. B. For $x < 0$, $\sqrt{9x^2 + 1} \approx |x| \cdot 3 = -3x$. So the quotient tends to $\frac{-3x}{x} = -3$.
7. A. $f(x) = \frac{x-1}{(x-1)(x-2)}$. At $x = 1$ the factor cancels (a hole, limit -1); at $x = 2$ the denominator tends to 0 with numerator 1, so $x = 2$ is an asymptote.
8. D. $\frac{\tan 5x}{2x} = \frac{5}{2} \cdot \frac{\sin 5x}{5x} \cdot \frac{1}{\cos 5x} \to \frac{5}{2}$.
9. C. $\frac{1-\cos 4x}{x^2} \cdot \frac{x}{\sin x} \to 16 \cdot \frac{1}{2} \cdot 1 = 8$.
10. E. Both bounds tend to 5 as $x \to 3$, so $f \to 5$ by the Squeeze Theorem.
11. D. The numerator tends to $3 > 0$ and $x - 2$ is a small negative number, so the quotient is large and negative.
12. B. Need $4k + 1 = 6 - k$, so $5k = 5$ and $k = 1$.
13. E. For $x \neq -2$, $f(x) = x - 2$, so the limit is -4 (finite) while $f(-2)$ is undefined: a removable discontinuity.
14. A. 0 lies between $f(1)$ and $f(5)$, so the IVT guarantees a zero. The other statements need not hold.
15. C. $x(x - 3) = 4$ gives $x = 4$ or $x = -1$, but $\ln x$ needs $x > 0$ and $\ln(x - 3)$ needs $x > 3$. Only 4 works.
16. C. Divide by e^x: $\frac{3-2e^{-x}}{1+5e^{-x}} \to 3$.
17. A. Long division: $x^2 + 2x - 3 = (x - 2)(x + 4) + 5$, so $f(x) = x + 4 + \frac{5}{x-2}$.
18. B. $-x^2 \leq x^2 \cos\frac{1}{x} \leq x^2$ and both bounds tend to 0 (Squeeze Theorem).
19. E. The graph approaches height 2 from both sides, so the limit is 2, but the function value is 0. It is a removable discontinuity.
20. D. $(2 + h)^3 - 8 = 12h + 6h^2 + h^3$, so the quotient is $12 + 6h + h^2 \to 12$.

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SET B · PART B [CALCULATOR]

21. C. With $t = \frac{3}{x}$, $(1+t)^{3/t} = \left[(1+t)^{1/t}\right]^3 \to e^3 \approx 20.086$.

22. A. $e^{0.3t} = 7.5$, so $t = \frac{\ln 7.5}{0.3} \approx 6.716$.

23. E. The values $2.214,\ 2.020,\ 2.002$ decrease toward 2.

24. B. $f(1) = -3 < 0 < 5 = f(2)$ locates it in $(1, 2)$. A calculator gives $x \approx 1.516$.

25. D. $1 + 49e^{-0.4t} = 1.25$, so $e^{-0.4t} = \frac{0.25}{49}$ and $t = \frac{\ln 196}{0.4} \approx 13.195$.

26. A. The values on both sides approach 1, so define $h(1) = 1$.

27. D. $\lim_{x\to\infty} f(x) = 3$ and $|f(x) - 3| = \frac{8}{2x+3} < 0.01$ when $2x + 3 > 800$, i.e. $x > 398.5$. So $N = 399$ works, and $N = 398$ fails at $x = 398.2$.

28. B. Left limit $2e$, right value c. So $c = 2e \approx 5.437$.

29. C. Let $f(x) = e^x - 3x$: $f(1) < 0 < f(2)$ (indeed $f(1) = -0.28$, $f(2) = 1.39$). A calculator gives $x \approx 1.512$ (the smaller root is 0.619).

30. E. $100\left(\frac{1}{2}\right)^{20/8} = 100 \cdot 2^{-2.5} \approx 17.678$ mg.

1.13 Free-Response Practice on Limits

QUESTION 1 · TEMPERATURE DATA (9 POINTS)

1. (a) $\frac{T(7) - T(2)}{7 - 2} = \frac{48 - 35}{5} = \frac{13}{5} = 2.6$ degrees Celsius per minute.

(b) Yes for both. T is continuous on $[2, 5]$ with $T(2) = 35 < 55 < 60 = T(5)$, so by the Intermediate Value Theorem there is a t in $(2, 5)$ with $T(t) = 55$. Likewise T is continuous on $[5, 7]$ with $T(5) = 60 > 55 > 48 = T(7)$, so the IVT gives a t in $(5, 7)$ with $T(t) = 55$. These two intervals do not overlap, so the two times are different.

(c) $\lim_{t\to\infty}\left(120 - \frac{500}{t}\right) = 120 - 0 = 120$. In the long run the temperature of the liquid approaches 120 degrees Celsius.

(d) $T(10) = 70$. Since T is continuous on $[0, 10]$, $\lim_{t\to 10^-} T(t) = 70$. Also $\lim_{t\to 10^+}\left(120 - \frac{500}{t}\right) = 120 - 50 = 70$. The one-sided limits both equal $T(10) = 70$, so T is continuous at $t = 10$. Solve $120 - \frac{500}{t} = 100$: $\frac{500}{t} = 20$, so $t = 25$ minutes.

Scoring: 9 points. (a) 2 points: difference quotient 1, value with units 1. (b) 3 points: continuity and IVT hypotheses 1, both intervals checked 1, conclusion about two different times 1. (c) 2 points: limit 1, meaning 1. (d) 2 points: continuity check with one-sided limits 1, value $t = 25$ 1.

QUESTION 2 · READING A GRAPH (9 POINTS)

2. (a) From the left, $f(x) = x + 3 \to 2$. From the right, $f(x) = 2 + \frac{x+1}{3} \to 2$. So $\lim_{x\to -1} f(x) = 2$, but $f(-1) = 3$. Since the limit differs from $f(-1)$, f is not continuous at -1: a removable discontinuity.

(b) $\lim_{x\to 2^-} f(x) = 2 + \frac{3}{3} = 3$ and $\lim_{x\to 2^+} f(x) = \frac{1}{4-2} = \frac{1}{2}$. The one-sided limits differ, so $\lim_{x\to 2} f(x)$ does not exist. Jump discontinuity.

(c) As $x \to 4^-$, $4 - x \to 0^+$, so $f(x) \to +\infty$. As $x \to 4^+$, $4 - x \to 0^-$, so $f(x) \to -\infty$. As $x \to \infty$, $\frac{1}{4-x} \to 0$. Vertical asymptote $x = 4$, horizontal asymptote $y = 0$ (to the right).

(d) By the limit laws, $\lim\left(f^2 + 3f\right) = 2^2 + 3(2) = 10$. g is continuous at -1 exactly when $k = \lim_{x\to -1} f(x) = 2$.

Scoring: 9 points. (a) 3 points: limit 1, $f(-1)$ 1, conclusion with type 1. (b) 2 points: two one-sided limits 1, does not exist with type 1. (c) 2 points: one-sided infinite limits 1, limit at infinity and asymptote equations 1. (d) 2 points: value 10 with limit laws 1, $k = 2$ 1.

QUESTION 3 · DAMPED SIGNAL MODEL (9 POINTS)

3. (a) Multiply by the conjugate: $\frac{\sqrt{x+7} - 3}{x - 2} \cdot \frac{\sqrt{x+7} + 3}{\sqrt{x+7} + 3} = \frac{x - 2}{(x-2)(\sqrt{x+7} + 3)} = \frac{1}{\sqrt{x+7} + 3}$ for $x \neq 2$. The limit is $\frac{1}{3+3} = \frac{1}{6}$.

(b) $\frac{m\sin(x-2)}{(x-2)(x+2)} = m \cdot \frac{\sin(x-2)}{x-2} \cdot \frac{1}{x+2} \to m \cdot 1 \cdot \frac{1}{4} = \frac{m}{4}$. The two-sided limit exists when $\frac{m}{4} = \frac{1}{6}$, so $m = \frac{2}{3}$.

(c) $\lim_{x\to 2} f(x) = \frac{1}{6}$ exists. For continuity we need $f(2) = c$ to equal the limit, so $c = \frac{1}{6}$. Then $f(2)$ is defined, the limit exists, and the two are equal.

(d) For $x > 2$, $|\sin(x-2)| \le 1$, so $-\frac{2/3}{x^2 - 4} \le f(x) \le \frac{2/3}{x^2 - 4}$. Both bounds tend to 0 as $x \to \infty$, so by the Squeeze Theorem $\lim_{x\to\infty} f(x) = 0$.

Scoring: 9 points. (a) 3 points: conjugate 1, simplification 1, limit 1. (b) 2 points: use of the special limit 1, $m = \frac{2}{3}$ 1. (c) 2 points: $c = \frac{1}{6}$ 1, three conditions stated 1. (d) 2 points: bounds 1, conclusion 1.

QUESTION 4 · BACTERIA CULTURE (9 POINTS)

4. (a) $N(0) = \frac{900}{1+8} = 100$ thousand. $N(5) \approx 432.135$, so the average rate is $\frac{432.135 - 100}{5} \approx 66.427$ thousand bacteria per hour.

(b) As $t \to \infty$, $e^{-0.4t} \to 0$, so $N(t) \to \frac{900}{1+0} = 900$. The culture population levels off at 900 thousand bacteria; $y = 900$ is a horizontal asymptote.

(c) $\frac{900}{1+8e^{-0.4t}} = 450 \Rightarrow 1 + 8e^{-0.4t} = 2 \Rightarrow e^{-0.4t} = \frac{1}{8} \Rightarrow t = \frac{\ln 8}{0.4} \approx 5.199$ hours.

(d) The values are about 89.775, 89.851, 89.865, 89.918 on both sides of 5 and settle near 89.86, so the limit is about 89.9 thousand bacteria per hour. $Q(5)$ would be $\frac{0}{0}$, since the numerator and the denominator are both 0 at $t = 5$.

Scoring: 9 points. (a) 2 points: $N(0)$ 1, average rate with units 1. (b) 2 points: limit 1, meaning 1. (c) 2 points: equation setup 1, answer 1. (d) 3 points: estimate 1, units 1, explanation of $\frac{0}{0}$ 1.

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Chapter 2: Derivatives.

16 sections with explanations and worked examples, then 328 practice problems. Answers and full explanations are at the end of the chapter.

2.1 Exponent Rules and the Power Rule

You will learn to rewrite roots and fractions as powers and differentiate them in one step with the power rule. This is the base of almost every derivative you will compute.

Every derivative rule in this course is built on one idea: the derivative $f'(x)$ gives the slope of the graph of f at x. The power rule is the first and most-used shortcut, and with it you can differentiate any polynomial, any root and any simple fraction in one line. The only real work is to **rewrite** the function so that each term is a constant times a power of x.

Exponent Rules (algebra you need first)

For $x > 0$ and real numbers m and n:

$$x^m \cdot x^n = x^{m+n}, \qquad \frac{x^m}{x^n} = x^{m-n}, \qquad (x^m)^n = x^{mn}, \qquad x^{-n} = \frac{1}{x^n}, \qquad x^{1/n} = \sqrt[n]{x}, \qquad x^{m/n} = \left(\sqrt[n]{x}\right)^m.$$

Examples: $\frac{1}{x^3} = x^{-3}$, $\sqrt{x} = x^{1/2}$, $\frac{1}{\sqrt[3]{x}} = x^{-1/3}$, $\sqrt[3]{x^2} = x^{2/3}$. A sum in the numerator can be split term by term: $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$, but a sum in the denominator cannot.

The derivative of a function is itself a function, written $f'(x)$ or $\frac{dy}{dx}$. The next box gives the rules that let you find it without limits. You will prove the power rule for a few cases in the next section, where the definition of the derivative is introduced.

The Power Rule

If n is any real number, then

$$\frac{d}{dx}\left[x^n\right] = n\,x^{n-1}.$$

Bring the exponent down in front, then subtract 1 from the exponent. The rule holds wherever x^n and x^{n-1} are defined. For example, $\frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3}$ is defined for $x \neq 0$ even though $x^{1/3}$ is defined at $x = 0$.

Constant, Constant Multiple, Sum and Difference Rules

If c is a constant and f and g are differentiable, then

$$\frac{d}{dx}[c] = 0, \qquad \frac{d}{dx}[c\,f(x)] = c\,f'(x), \qquad \frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x).$$

A constant term disappears, a constant factor stays, and you differentiate a sum one term at a time. Note that π, e and $\sqrt{2}$ are constants, so $\frac{d}{dx}[\pi^2] = 0$ and $\frac{d}{dx}[\pi x] = \pi$.

1. **Rewrite.** Turn every root and every power in a denominator into a power of x. Split fractions with a sum in the numerator.
2. **Differentiate term by term.** Apply the power rule to each term, keeping constant multiples.
3. **Simplify.** Combine terms, and if the question started with roots or fractions, you may write the answer that way again.

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EXAMPLE 1

Differentiate $f(x) = 4x^5 - 7x^3 + 2x - 11$.

SOLUTION

Apply the rules term by term: $\frac{d}{dx}[4x^5] = 20x^4$, $\frac{d}{dx}[7x^3] = 21x^2$, $\frac{d}{dx}[2x] = 2$ and $\frac{d}{dx}[11] = 0$.

$f'(x) = 20x^4 - 21x^2 + 2$.

EXAMPLE 2

Find $\frac{dy}{dx}$ for $y = \frac{3}{x^4} + 6\sqrt{x} - \frac{2}{\sqrt[3]{x}}$.

SOLUTION

Rewrite: $y = 3x^{-4} + 6x^{1/2} - 2x^{-1/3}$.

Differentiate: $\frac{dy}{dx} = 3(-4)x^{-5} + 6 \cdot \frac{1}{2}x^{-1/2} - 2\left(-\frac{1}{3}\right)x^{-4/3}$.

Simplify: $\frac{dy}{dx} = -12x^{-5} + 3x^{-1/2} + \frac{2}{3}x^{-4/3} = -\frac{12}{x^5} + \frac{3}{\sqrt{x}} + \frac{2}{3x^{4/3}}$.

Common mistake

Do not differentiate a quotient by differentiating the top and the bottom separately. The power rule applies to a **single power** of x, so first rewrite: $\frac{1}{x^3}$ becomes x^{-3} and its derivative is $-3x^{-4}$, not $\frac{0}{3x^2}$. Also remember that the exponent decreases by 1: $\frac{d}{dx}[x^{-3}] = -3x^{-4}$, not $-3x^{-2}$.

EXAMPLE 3

Differentiate $g(x) = \frac{4x^3 - 6x + 1}{2x}$.

SOLUTION

Split the fraction (the denominator is a single term): $g(x) = \frac{4x^3}{2x} - \frac{6x}{2x} + \frac{1}{2x} = 2x^2 - 3 + \frac{1}{2}x^{-1}$.

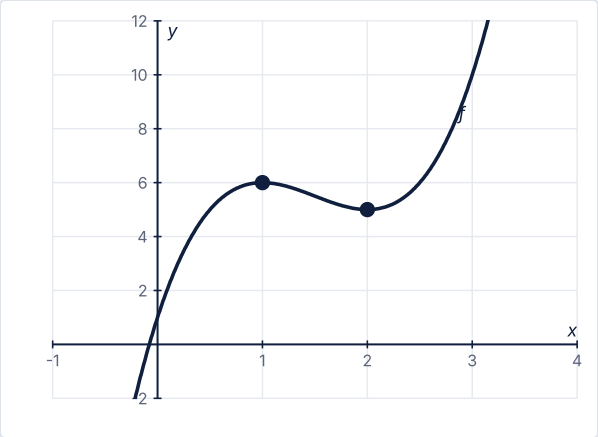
$g'(x) = 4x + 0 - \frac{1}{2}x^{-2} = 4x - \frac{1}{2x^2}$.

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EXAMPLE 4

Let $f(x) = 2x^3 - 9x^2 + 12x + 1$. Find the points on the graph of f where the tangent line is horizontal.



Graph of f with horizontal tangents at x = 1 and x = 2.

SOLUTION

A horizontal tangent has slope 0, so solve $f'(x) = 0$.

$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$, so $x = 1$ or $x = 2$.

$f(1) = 2 - 9 + 12 + 1 = 6$ and $f(2) = 16 - 36 + 24 + 1 = 5$. The points are $(1, 6)$ and $(2, 5)$.

EXAMPLE 5

The slope of the graph of $y = a\sqrt{x}$ at $x = 9$ equals 2. Find the constant a.

SOLUTION

$y = ax^{1/2}$, so $y' = \frac{a}{2}x^{-1/2} = \frac{a}{2\sqrt{x}}$.

At $x = 9$: $\frac{a}{2 \cdot 3} = 2$, so $a = 12$.

Practice

SET A Skills

Differentiate each function. Simplify where it helps. [No calculator]

1. $f(x) = 7x^4 - 3x^2 + 5x - 9$
2. $y = \frac{4}{x^3}$
3. $g(x) = 5\sqrt{x}$
4. $y = \sqrt[3]{x^2}$
5. $h(x) = \frac{x^3 - 4x}{x^2}$
6. $y = \frac{3}{2\sqrt{x}}$
7. $f(x) = (2x + 1)^2$ (expand first)
8. $y = x^2(3x - 5)$ (multiply first)
9. $f(t) = \pi t^3 + \pi^2$

SET B Applications and mixed problems

Give exact answers, plus decimals to three places where a decimal is asked for. [Calculator]

10. A particle moves along a line with position $s(t) = t^3 - 6t^2 + 9t + 2$ meters, t in seconds. Find the velocity $v(t) = s'(t)$, the velocity at $t = 2$, and the times when the particle is momentarily at rest.
11. The area of a circle of radius r cm is $A(r) = \pi r^2$. Find $A'(r)$, evaluate $A'(5)$ to three decimal places, and state the units and meaning of the result.
12. The volume of a sphere is $V(r) = \frac{4}{3}\pi r^3$. Find $V'(r)$, compare it with the surface area $4\pi r^2$, and evaluate $V'(3)$ to three decimal places.
13. Find all points on the graph of $y = x^3 - 3x$ where the tangent line has slope 9.

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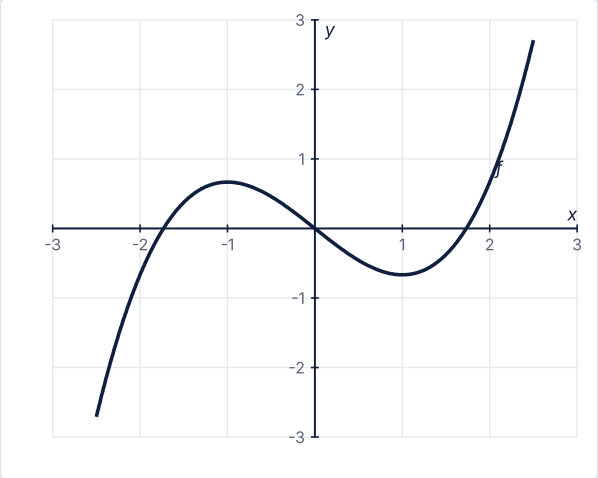
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14. The cost, in dollars, of producing q units of a chemical is $C(q) = 500 + 4q + 0.01q^2$. Find the marginal cost $C'(q)$ and evaluate $C'(120)$. Interpret the result with units.
15. A projectile's height is $h(t) = 80t - 5t^2$ meters at time t seconds. Find $h'(3)$ and the time when $h'(t) = 0$. What does that time represent?
16. The function $f(x) = ax^2 + bx$ satisfies $f(1) = 4$ and $f'(1) = 7$. Find a and b.

SET C Challenge and AP-style

Show the reasoning that supports each answer. [No calculator]

17. The cubic $p(x) = x^3 + ax^2 + bx$ has horizontal tangent lines at $x = 1$ and at $x = 3$. Find a and b.
18. Let $f(x) = x + \dfrac{4}{x}$. (a) Find all x where the slope of the graph is 0.
    (b) For which x is the slope positive? Justify.
19. A student writes $\dfrac{d}{dx}\left[\dfrac{1}{3x^2}\right] = \dfrac{1}{6x}$. Identify the error and give the correct derivative.
20. Let $f(x) = x^{1/3}$. Find $f'(x)$, state its domain, and explain what happens to the graph of f at $x = 0$ even though $f(0)$ is defined.
21. The graph of $f(x) = \frac{1}{3}x^3 - x$ is shown. (a) Estimate from the graph where the tangent lines are horizontal.
    (b) Find $f'(x)$ and verify your estimates.
    (c) Find the slope of the graph at $x = 2$.



Graph of f.

2.2 The Derivative by Definition

You will define the derivative as the limit of a difference quotient and use it to find slopes, rates and derivative formulas. This definition underlies every later rule.

In Chapter 1 you computed the slope of a secant line through two points as a difference quotient. The derivative is what that slope turns into when the two points slide together. This section defines the derivative with a limit, and every derivative rule you learn later is a shortcut for this one definition.

Definition of the Derivative (h form)

The **derivative** of f at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. If it exists, f is **differentiable** at x.

The quotient is the slope of the secant line through $(x, f(x))$ and $(x+h, f(x+h))$. The limit is the slope of the tangent line at x and is also the instantaneous rate of change of f at x.

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Definition at a Point (x to a form)

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

This is the same limit with $h = x - a$. Use it when you want the slope at one specific number a; use the h form when you want a formula for $f'(x)$. Either way the quotient is $\frac{0}{0}$ when you substitute directly, so you must simplify before taking the limit.

1. **Write the quotient.** Compute $f(x+h)$ and form $\dfrac{f(x+h) - f(x)}{h}$.
2. **Simplify until h divides out.** Expand, combine fractions or multiply by a conjugate so that a factor of h cancels from the numerator.
3. **Take the limit as $h \to 0$.** Now substitute $h = 0$. The result is $f'(x)$.

EXAMPLE 1

Use the limit definition to find $f'(x)$ for $f(x) = x^2 - 3x$.

SOLUTION

$f(x+h) = (x+h)^2 - 3(x+h) = x^2 + 2xh + h^2 - 3x - 3h.$

$\dfrac{f(x+h) - f(x)}{h} = \dfrac{2xh + h^2 - 3h}{h} = 2x + h - 3$ for $h \neq 0$.

$f'(x) = \lim_{h \to 0}(2x + h - 3) = 2x - 3.$

EXAMPLE 2

Use the definition to find $f'(x)$ for $f(x) = \dfrac{1}{x}$, $x \neq 0$.

SOLUTION

$\dfrac{f(x+h) - f(x)}{h} = \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h} = \dfrac{\frac{x-(x+h)}{x(x+h)}}{h} = \dfrac{-h}{h\,x(x+h)} = \dfrac{-1}{x(x+h)}.$

$f'(x) = \lim_{h \to 0} \dfrac{-1}{x(x+h)} = -\dfrac{1}{x^2}$. This agrees with the power rule: $\frac{d}{dx}[x^{-1}] = -x^{-2}$.

EXAMPLE 3

Use the definition to find $f'(x)$ for $f(x) = \sqrt{x}$, $x > 0$.

SOLUTION

$\dfrac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \dfrac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \dfrac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \dfrac{1}{\sqrt{x+h} + \sqrt{x}}.$

$f'(x) = \lim_{h \to 0} \dfrac{1}{\sqrt{x+h} + \sqrt{x}} = \dfrac{1}{2\sqrt{x}}.$

EXAMPLE 4

Find $f'(2)$ for $f(x) = x^3$ using the $x \to a$ form.

SOLUTION

$f'(2) = \lim_{x \to 2} \dfrac{x^3 - 8}{x - 2} = \lim_{x \to 2} \dfrac{(x-2)(x^2 + 2x + 4)}{x - 2}.$

$= \lim_{x \to 2}(x^2 + 2x + 4) = 12$. The tangent line at $x = 2$ has slope 12.

Notation and Meaning

For $y = f(x)$ the derivative is written $f'(x)$, y', $\dfrac{dy}{dx}$ or $\dfrac{d}{dx}[f(x)]$. The value at a point is $f'(a)$ or $\left.\dfrac{dy}{dx}\right|_{x=a}$.

Meaning: $f'(a)$ is the slope of the tangent line at $x = a$ and the instantaneous rate of change of f at a. Its units are (units of f) per (units of x). If $f'(a) > 0$ the function is increasing at a; if $f'(a) < 0$ it is decreasing.

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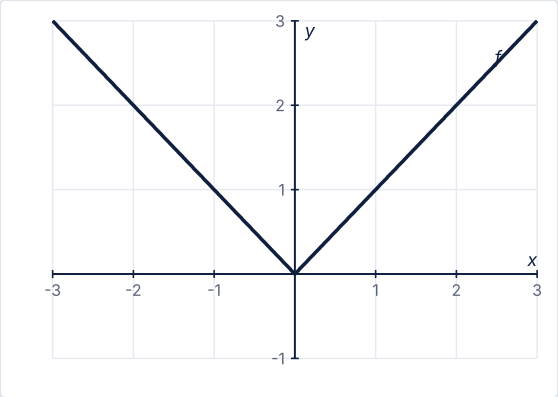
Differentiability and Continuity

If f is differentiable at $x = a$, then f is continuous at $x = a$.

The converse is false. A function can be continuous at a point and still fail to be differentiable there. This happens at a **corner** (different one-sided slopes), a **cusp** or **vertical tangent** (the slope becomes infinite) and, of course, at any discontinuity.

EXAMPLE 5

Show that $f(x) = |x|$ is not differentiable at $x = 0$.



Graph of f(x) = |x|: a corner at the origin.

SOLUTION

$f'(0) = \lim_{h\to 0} \frac{|h| - 0}{h} = \lim_{h\to 0} \frac{|h|}{h}$.

From the right ($h > 0$), $\frac{|h|}{h} = 1$. From the left ($h < 0$), $\frac{|h|}{h} = -1$.

The one-sided limits are different, so the limit does not exist. The graph has a corner at the origin: f is continuous at 0 but not differentiable there.

Common mistake

Do not substitute $h = 0$ before simplifying: you will get $\frac{0}{0}$. And remember that **differentiable implies continuous, but not the other way round**. Continuity at a point never proves differentiability there.

Practice

SET A **Skills**

Use the limit definition of the derivative. [No calculator]

1. Use the h definition to find $f'(x)$ for $f(x) = 5x - 2$.
2. Use the h definition to find $f'(x)$ for $f(x) = x^2 + 4x$.
3. Use the h definition to find $f'(x)$ for $f(x) = 3x^2 - x + 1$.
4. Use the h definition to find $f'(x)$ for $f(x) = x^3$.
5. Use the h definition to find $f'(x)$ for $f(x) = \dfrac{1}{x+2}$.
6. Use the h definition to find $f'(x)$ for $f(x) = \sqrt{x+3}$.
7. Use $\lim\limits_{x\to 3} \dfrac{f(x) - f(3)}{x - 3}$ to find $f'(3)$ for $f(x) = x^2 - 2x$.
8. Use the $x \to a$ form to find $f'(1)$ for $f(x) = \dfrac{2}{x}$.
9. The limit $\lim\limits_{h\to 0} \dfrac{(2+h)^5 - 32}{h}$ is a derivative. Identify f and a, then evaluate it using the power rule.

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SET B Applications and mixed problems

Give exact answers; give decimals to three places where asked. [Calculator]

10. A particle has position $s(t) = t^2 + 3t$ meters at time t seconds. Use the definition to find the velocity $s'(t)$, then find the velocity at $t = 2$ with units.

11. The temperature $T(t)$ of a sample is recorded at selected times. Use the data to estimate $T'(5)$, using the closest points on either side of $t = 5$, and explain the meaning of your answer with units.

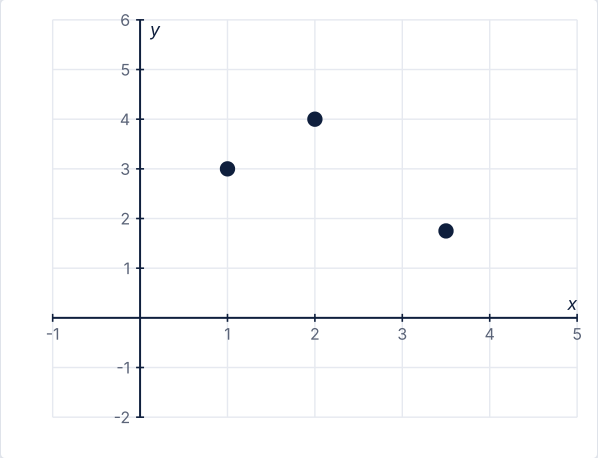
t (min)	2	4	5	6	8
$T(t)$ (degrees C)	48.0	61.2	66.1	70.8	79.5

12. Let $f(x) = 2^x$. Compute the difference quotient $\dfrac{f(1+h) - f(1)}{h}$ for $h = 0.001$ and for $h = -0.001$, each to three decimal places. Use your results to estimate $f'(1)$ to two decimal places.

13. The area of a circle is $A(r) = \pi r^2$ square centimeters when the radius is r centimeters. Use the definition to find $A'(r)$, then give $A'(4)$ to three decimal places and interpret it.

14. The cost of producing q units is $C(q) = 0.5q^2 + 10q$ dollars. Find $C(51) - C(50)$. Then use the definition (or the $x \to a$ form) to find the exact marginal cost $C'(50)$, and compare the two numbers.

15. The graph of $f(x) = -x^2 + 4x$ is shown. (a) Tell whether $f'(1)$, $f'(2)$ and $f'(3.5)$ are positive, negative or zero. (b) Use the definition to find $f'(x)$ and give the three values.



Graph of f.

16. Let $f(x) = |x - 2|$. Use one-sided limits of the difference quotient at $x = 2$ to show that $f'(2)$ does not exist.

SET C Challenge and AP-style

Justify each answer with the definition or with one-sided limits. [No calculator]

17. Let $f(x) = x|x|$. Use the definition to find $f'(0)$. Is f differentiable at 0?

18. Let $f(x) = \begin{cases} x^2, & x \le 1 \\ ax + b, & x > 1. \end{cases}$ Find a and b so that f is differentiable at $x = 1$. Justify each condition you use.

19. Evaluate $\lim\limits_{h \to 0} \dfrac{(8+h)^{1/3} - 2}{h}$ by recognizing it as a derivative.

20. Suppose f is differentiable, $f(3) = 7$ and $f'(3) = 4$. Find $\lim\limits_{x \to 3} \dfrac{f(x) - 7}{x^2 - 9}$.

21. The table gives values of a differentiable function g. (a) Estimate $g'(2)$ using the closest points on either side. (b) Compute the slopes of the four secant segments between consecutive table points. What do they suggest about how g' changes on $[1, 3]$?

x	1	1.5	2	2.5	3
$g(x)$	4.0	4.9	5.6	6.1	6.4

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2.3 Tangent and Normal Lines

You will write tangent and normal lines, find horizontal and vertical tangents, handle tangent lines through outside points, and use a tangent line to estimate function values.

The derivative gives the slope of the tangent line, so once you can compute $f'(a)$ you can write the equation of that line, the line perpendicular to it, and, as a first step toward approximation, a linear function that imitates f near a. These ideas appear on the exam in many forms, from a plain equation to a table-based estimate.

Tangent and Normal Lines

Suppose f is differentiable at $x = a$. The **tangent line** at $(a, f(a))$ has slope $f'(a)$:

$$y - f(a) = f'(a)\,(x - a).$$

The **normal line** is perpendicular to the tangent line at the same point. If $f'(a) \neq 0$ its slope is $-\dfrac{1}{f'(a)}$:

$$y - f(a) = -\frac{1}{f'(a)}\,(x - a).$$

Special cases: if $f'(a) = 0$ the tangent is the horizontal line $y = f(a)$ and the normal is the vertical line $x = a$. If the tangent is vertical, $x = a$, the normal is the horizontal line $y = f(a)$.

EXAMPLE 1

Find the equations of the tangent line and the normal line to $f(x) = x^3 - 2x + 1$ at $x = 2$.

SOLUTION

$f(2) = 8 - 4 + 1 = 5$, so the point is $(2, 5)$. $f'(x) = 3x^2 - 2$, so $f'(2) = 10$.

Tangent: $y - 5 = 10(x - 2)$, that is $y = 10x - 15$.

Normal slope $-\frac{1}{10}$: $y - 5 = -\frac{1}{10}(x - 2)$, that is $y = -\frac{1}{10}x + \frac{26}{5}$.

Horizontal and Vertical Tangents

The graph of a differentiable function has a **horizontal tangent** at the values x where $f'(x) = 0$.

A **vertical tangent** occurs at $x = a$ when f is continuous at a and $|f'(x)| \to \infty$ as $x \to a$ (for example f' is undefined at a because of a zero in the denominator). The line is $x = a$.

EXAMPLE 2

Find the horizontal tangent lines to $f(x) = x^3 - 3x^2 - 9x + 4$.

SOLUTION

$f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1) = 0$ gives $x = 3$ and $x = -1$.

$f(3) = 27 - 27 - 27 + 4 = -23$ and $f(-1) = -1 - 3 + 9 + 4 = 9$.

The horizontal tangent lines are $y = -23$ and $y = 9$.

EXAMPLE 3

Show that $f(x) = (x - 1)^{1/3}$ has a vertical tangent line at $x = 1$, and write its equation.

SOLUTION

f is continuous everywhere. $f'(x) = \frac{1}{3}(x - 1)^{-2/3} = \dfrac{1}{3\,(x - 1)^{2/3}}$.

As $x \to 1$, the denominator tends to 0^+ from both sides, so $f'(x) \to +\infty$. The tangent line at $(1, 0)$ is vertical: $x = 1$.

Common mistake

When you write the normal line, take the **negative reciprocal** of the slope, and take it of the number $f'(a)$, not of the function. A slope of 10 becomes $-\frac{1}{10}$, and a slope of $-\frac{2}{3}$ becomes $\frac{3}{2}$. Also check the point: the tangent line must pass through $(a, f(a))$, so find $f(a)$ before you use the slope.

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Tangent Line Through an Outside Point

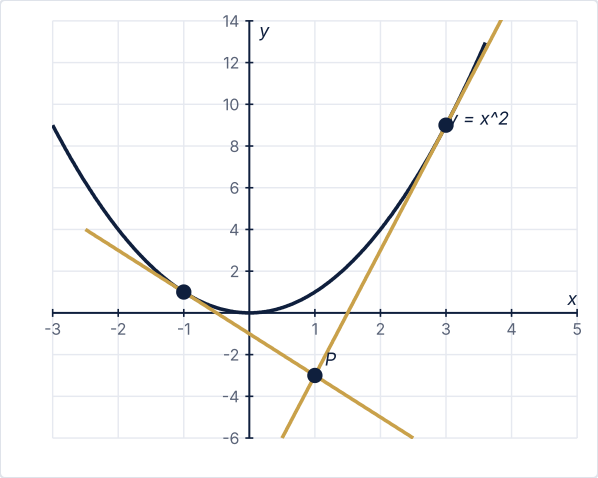
If the point $P = (p, q)$ is **not** on the graph, let the point of tangency be $(a, f(a))$ with unknown a. The tangent line there must pass through P:

$$q = f(a) + f'(a)\,(p - a).$$

Solve this equation for a. Each solution gives one tangent line. There can be 0, 1 or several.

EXAMPLE 4

Find the equations of the lines through $(1, -3)$ that are tangent to $y = x^2$.



Two tangent lines to y = x^2 pass through P(1, -3).

SOLUTION

The tangent at (a, a^2) has slope $2a$: $y - a^2 = 2a(x - a)$. Substitute $(1, -3)$: $-3 - a^2 = 2a - 2a^2$.

Then $a^2 - 2a - 3 = 0$, so $(a - 3)(a + 1) = 0$ and $a = 3$ or $a = -1$.

For $a = 3$: $y = 6x - 9$. For $a = -1$: $y = -2x - 1$. Check: both lines give $y = -3$ at $x = 1$.

Local Linear Approximation (preview)

If f is differentiable at a, the **linearization** of f at a is the tangent line function

$$L(x) = f(a) + f'(a)\,(x - a),$$

and $f(x) \approx L(x)$ for x close to a.

The closer x is to a, the better the estimate. Use a point a where both $f(a)$ and $f'(a)$ are easy to compute.

EXAMPLE 5

Use a tangent line to estimate $\sqrt{26}$. Is the estimate too large or too small?

SOLUTION

Let $f(x) = \sqrt{x}$ and $a = 25$. $f(25) = 5$ and $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(25) = \frac{1}{10}$.

$L(x) = 5 + \frac{1}{10}(x - 25)$, so $\sqrt{26} \approx L(26) = 5.1$.

The graph of $\sqrt{x}$ bends below its tangent lines, so the estimate 5.1 is slightly too large. (The true value is $5.099\ldots$)

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Practice

SET A Skills

Write each answer in point-slope or slope-intercept form. [No calculator]

1. Write the equation of the tangent line to $f(x) = x^2 + 3x$ at $x = 1$.
2. Write the equation of the tangent line to $y = \sqrt{x}$ at $x = 9$.
3. Write the equation of the tangent line to $y = \dfrac{4}{x}$ at $x = 2$.
4. Write the equation of the normal line to $y = x^3 - x$ at $x = 2$.
5. Write the equation of the normal line to $y = 2x^2 - 4x + 1$ at the point $(3, 7)$.
6. Find the points on the graph of $y = x^3 - 12x$ where the tangent line is horizontal.
7. Find the point on the graph of $y = x^2 - 6x + 5$ where the tangent line is parallel to $y = 4x + 1$.
8. Find the point where the graph of $y = x^{1/3}$ has a vertical tangent line, and write the tangent line's equation.
9. Write the equation of the tangent line to $y = x^3 + 2x$ at $x = -1$.

SET B Applications and mixed problems

Give decimals to three places where asked. [Calculator]

10. A particle's position is $s(t) = 2t^{3/2}$ meters, t in seconds. Write the tangent line to the graph of s at $t = 4$ and use it to estimate $s(4.2)$. Compare with the true value to three decimal places.
11. Use a tangent line to $f(x) = \sqrt{x}$ at a convenient point to approximate $\sqrt{50}$. Give your estimate to three decimal places.
12. Use the linearization of $f(x) = x^3$ at $a = 2$ to estimate $(1.97)^3$. Give the estimate and the exact value of 1.97^3 to three decimal places.
13. Find the equations of the two lines through $(2, 2)$ that are tangent to the parabola $y = x^2 - 4x + 7$.
14. The normal line to $y = x^2$ at $(1, 1)$ meets the parabola at a second point. Find that point.
15. The price, in dollars, that buyers will pay per unit when q units are offered is $p(q) = \dfrac{120}{\sqrt{q}}$. Find the tangent line to the graph at $q = 16$, and use it to estimate $p(18)$. Give the estimate and the true value to three decimal places.
16. Find the point on $y = \sqrt{x}$ where the tangent line is perpendicular to the line $y = -2x + 3$.

SET C Challenge and AP-style

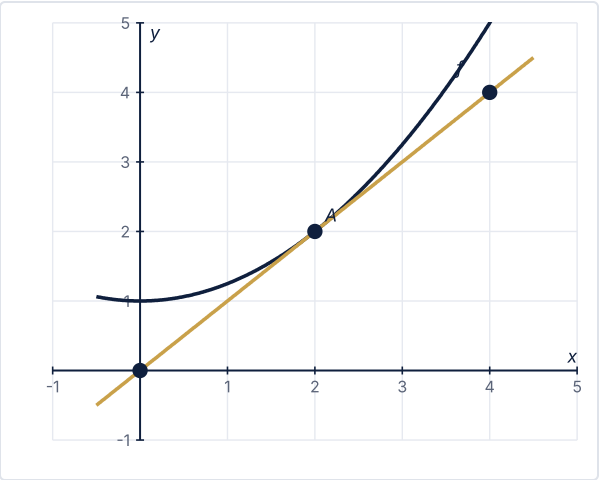
Justify each answer. [No calculator]

17. Find the equation of the tangent line to $y = x^3$ that passes through the point $(0, -2)$.
18. For what value of k does the tangent line to $y = kx^2$ at $x = 1$ pass through $(0, -1)$?
19. How many tangent lines to the graph of $y = x^2 + 1$ pass through the origin? Find each equation.

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20. The graph of f is shown, together with the line tangent to the graph at $A(2,2)$. (a) Find $f(2)$ and $f'(2)$ from the picture.
(b) Write the equation of the tangent line and use it to estimate $f(2.2)$.
(c) Is the estimate too high or too low? Explain using the graph.



A curve and its tangent line at A(2, 2). The line passes through (0, 0) and (4, 4).

21. Find the equations of all tangent lines to $f(x) = x^3 + x$ that are parallel to the line $y = 4x - 7$.

2.4 Trigonometry Refresher

You will review radians, exact unit-circle values, key identities, solving trig equations on an interval, the graphs of the trig functions and the inverse trig functions, all of which you need for derivatives.

Calculus uses angles in **radians**, and nearly every derivative and integral involving sine and cosine assumes it. Before you differentiate trigonometric functions, you need quick, exact values, a short list of identities, the shapes of the graphs and the meaning of the inverse functions. This section is a focused review of those tools.

Radians and the Unit Circle

An angle of θ radians at the center of a circle of radius r cuts off an arc of length $s = r\theta$. Conversion: $180^\circ = \pi$ radians. For a point (x, y) on the unit circle at angle θ:

$$\cos\theta = x, \qquad \sin\theta = y, \qquad \tan\theta = \frac{y}{x},$$

and $\sec\theta = \frac{1}{\cos\theta}$, $\csc\theta = \frac{1}{\sin\theta}$, $\cot\theta = \frac{1}{\tan\theta}$.
For other angles use a reference angle and the quadrant signs: in Quadrant I all functions are positive, in II only sine (and cosecant), in III only tangent (and cotangent), in IV only cosine (and secant).

θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Degrees	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Identities to Know

$$\sin^2\theta + \cos^2\theta = 1, \qquad 1 + \tan^2\theta = \sec^2\theta, \qquad 1 + \cot^2\theta = \csc^2\theta.$$
$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1.$$
$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B, \qquad \cos(A \pm B) = \cos A\cos B \mp \sin A\sin B.$$

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EXAMPLE 1

Find the exact value of (a) $\sin\frac{5\pi}{6}$, (b) $\sec\frac{2\pi}{3}$, (c) $\tan\frac{7\pi}{4}$.

SOLUTION

(a) The reference angle is $\frac{\pi}{6}$ and sine is positive in Quadrant II: $\sin\frac{5\pi}{6}=\frac{1}{2}$.

(b) $\cos\frac{2\pi}{3}=-\frac{1}{2}$, so $\sec\frac{2\pi}{3}=\frac{1}{-1/2}=-2$.

(c) The reference angle is $\frac{\pi}{4}$, in Quadrant IV where tangent is negative: $\tan\frac{7\pi}{4}=-1$.

EXAMPLE 2

If $\sin\theta=\frac{3}{5}$ and θ is in Quadrant II, find $\cos\theta$, $\tan\theta$ and $\sin 2\theta$.

SOLUTION

$\cos^2\theta=1-\frac{9}{25}=\frac{16}{25}$. In Quadrant II cosine is negative, so $\cos\theta=-\frac{4}{5}$.

$\tan\theta=\frac{3/5}{-4/5}=-\frac{3}{4}$.

$\sin 2\theta=2\sin\theta\cos\theta=2\cdot\frac{3}{5}\cdot\left(-\frac{4}{5}\right)=-\frac{24}{25}$.

1. **Reduce to one function.** Use an identity so that the equation involves a single trigonometric function of one angle.
2. **Solve for that function.** Factor or use the quadratic formula; treat $\sin x$ (or $\cos x$) as the unknown.
3. **Find all angles in the interval.** Use the unit circle: each value of sine or cosine from -1 to 1 gives two angles in $[0,2\pi)$, except at the extremes. Check the endpoints of the interval.

EXAMPLE 3

Solve $2\sin^2 x-\sin x-1=0$ for $0\le x<2\pi$.

SOLUTION

Factor: $(2\sin x+1)(\sin x-1)=0$.

$\sin x=1$ gives $x=\frac{\pi}{2}$. $\sin x=-\frac{1}{2}$ gives $x=\frac{7\pi}{6}$ and $x=\frac{11\pi}{6}$ (Quadrants III and IV).

Solutions: $x=\frac{\pi}{2},\ \frac{7\pi}{6},\ \frac{11\pi}{6}$.

Common mistake

Do not divide both sides of an equation by $\sin x$ or $\cos x$: you lose the solutions where that factor is 0. For $\sin 2x=\sin x$, write $2\sin x\cos x-\sin x=0$ and factor out $\sin x$. Also remember that $\sin^2 x$ means $(\sin x)^2$, and that $\sin(2x)\neq 2\sin x$.

Graphs of Sinusoids

For $y=A\sin\big(B(x-C)\big)+D$ or $y=A\cos\big(B(x-C)\big)+D$ (with $B>0$): the **amplitude** is $|A|$, the **period** is $\frac{2\pi}{B}$, the **phase shift** is C and the **midline** is $y=D$. The maximum is $D+|A|$ and the minimum is $D-|A|$.

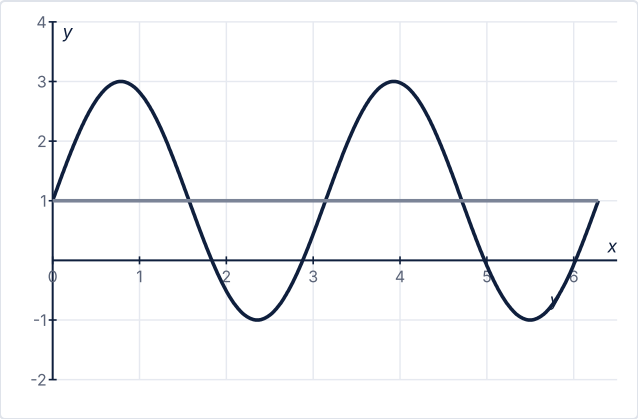
$\tan x$ has period π and vertical asymptotes at $x=\frac{\pi}{2}+k\pi$.

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EXAMPLE 4

Describe the graph of $y = 2\sin(2x) + 1$: amplitude, period, midline, maximum and minimum. Then state the x-values of the maximum on $[0, 2\pi]$.



y = 2 sin(2x) + 1 and its midline y = 1.

SOLUTION

$A = 2$, $B = 2$, $D = 1$. Amplitude 2, period $\frac{2\pi}{2} = \pi$, midline $y = 1$, maximum 3, minimum -1.

The maximum occurs when $2x = \frac{\pi}{2} + 2k\pi$, so $x = \frac{\pi}{4} + k\pi$: on $[0, 2\pi]$ at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Inverse Trigonometric Functions

$y = \arcsin x$ means $\sin y = x$ with $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. $y = \arccos x$ means $\cos y = x$ with $0 \le y \le \pi$. $y = \arctan x$ means $\tan y = x$ with $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

The domain of $\arcsin$ and $\arccos$ is $[-1, 1]$; the domain of $\arctan$ is all real numbers. The output is always an angle in the stated range, so $\arcsin(\sin x) = x$ only when x is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

EXAMPLE 5

Evaluate (a) $\arcsin\left(-\frac{1}{2}\right)$, (b) $\arccos\left(-\frac{\sqrt{2}}{2}\right)$, (c) $\sin\left(\arccos\frac{3}{5}\right)$.

SOLUTION

(a) We need an angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with sine $-\frac{1}{2}$: $-\frac{\pi}{6}$.

(b) We need an angle in $[0, \pi]$ with cosine $-\frac{\sqrt{2}}{2}$: $\frac{3\pi}{4}$.

(c) Let $\theta = \arccos\frac{3}{5}$ (Quadrant I). Then $\cos\theta = \frac{3}{5}$ and $\sin\theta = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$, so $\sin\left(\arccos\frac{3}{5}\right) = \frac{4}{5}$.

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Practice

SET A Skills

Give exact answers in radians. [No calculator]

1. Find the exact value of each.
   (a) $\sin\dfrac{7\pi}{6}$
   (b) $\cos\dfrac{5\pi}{3}$
   (c) $\tan\dfrac{3\pi}{4}$
   (d) $\csc\left(-\dfrac{\pi}{2}\right)$
2. (a) Convert 150° to radians.
   (b) Convert $\dfrac{7\pi}{4}$ radians to degrees.
   (c) Find the length of an arc of a circle of radius 6 cut off by a central angle of $\dfrac{\pi}{3}$.
3. If $\cos\theta = -\dfrac{5}{13}$ and θ is in Quadrant II, find $\sin\theta$, $\tan\theta$ and $\sin 2\theta$.
4. Simplify $\dfrac{\sin^2 x}{1-\cos x}$ for x not a multiple of 2π.
5. Simplify $(1+\tan^2 x)\cos^2 x$.
6. Solve $2\cos x = \sqrt{3}$ for $0 \le x < 2\pi$.
7. Solve $\sin x = \cos x$ for $0 \le x < 2\pi$.
8. Solve $2\sin^2 x + \sin x - 1 = 0$ for $0 \le x < 2\pi$.
9. Solve $\sin 2x = \sin x$ for $0 \le x < 2\pi$.

SET B Applications and mixed problems

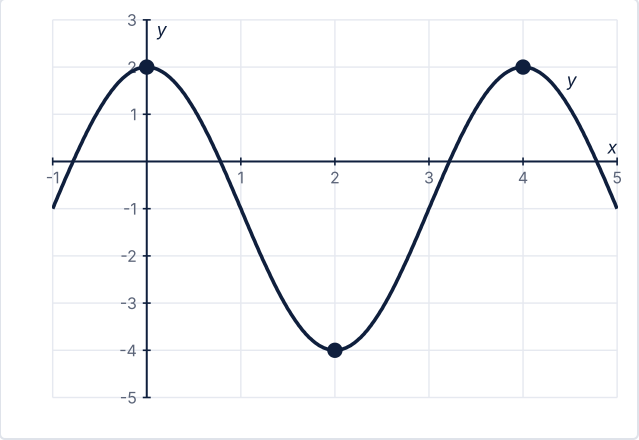
Give decimals to three places where asked, and angles in radians. [Calculator]

10. A point on a wheel has height $h(t) = 12 - 10\cos\left(\dfrac{\pi t}{15}\right)$ meters above the ground at time t seconds. Find the period, the maximum and minimum heights, and the first two times in one turn when the height is 17 meters.
11. The average daily temperature in degrees Celsius on day t of the year is modeled by $T(t) = 15 + 12\sin\left(\dfrac{2\pi(t-100)}{365}\right)$, $0 \le t \le 365$. Find the days (to three decimal places) when $T(t) = 21$.
12. Solve $3\cos x = 1$ for $0 \le x < 2\pi$. Give answers to three decimal places.
13. Solve $\tan x = 2.5$ for $0 \le x < 2\pi$. Give answers to three decimal places.
14. Solve $3\sin^2 x - 2\sin x - 1 = 0$ for $0 \le x < 2\pi$. Give answers to three decimal places.
15. Evaluate to three decimal places: (a) $\arcsin 0.3$ (b) $\arccos(-0.8)$ (c) $\arctan 5$.

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16. The graph of $y = a\cos(bx) + d$ is shown, with $b > 0$. Find a, b and d, and state the period and the range.



One and a quarter periods of a cosine curve.

SET C Challenge and AP-style

Show the reasoning for each answer. Give exact answers. [No calculator]

17. Verify the identity $\dfrac{\sin x}{1+\cos x} + \dfrac{1+\cos x}{\sin x} = \dfrac{2}{\sin x}$.

18. Solve $2\cos^2 x + 3\sin x = 3$ for $0 \le x < 2\pi$.

19. Solve $\cos 2x = \cos x$ for $0 \le x < 2\pi$.

20. Solve $\sin 2x = -\dfrac{\sqrt{3}}{2}$ for $0 \le x \le \pi$.

21. A sinusoidal function f has a maximum value of 7 at $x = \dfrac{\pi}{3}$, and the next minimum value, -1, occurs at $x = \dfrac{5\pi}{6}$. Write a formula for f of the form $A\cos\big(B(x-C)\big) + D$.

2.5 Differentiability and Continuity

Learn how to tell when a derivative exists, why differentiable functions must be continuous, and how to find constants that make piecewise functions smooth.

The derivative of f at a is the limit $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, when that limit exists. A function can fail to have a derivative at a point even when its graph looks perfectly reasonable there. In this section you learn exactly how that happens, and how continuity and differentiability are related.

Differentiability Implies Continuity

If f is differentiable at $x = a$, then f is continuous at $x = a$.

Reason: $f(a+h) - f(a) = \dfrac{f(a+h)-f(a)}{h} \cdot h \to f'(a)\cdot 0 = 0$, so $\lim_{h\to 0} f(a+h) = f(a)$.

The converse is false. Continuity is necessary for differentiability, but it is not enough. Equivalently: if f is **not** continuous at a, then f is **not** differentiable at a.

Four Ways Differentiability Fails at x = a

Corner: the one-sided derivatives exist but are different (for example $f(x) = |x|$ at 0).

Cusp: the slopes of the tangent lines approach $+\infty$ from one side and $-\infty$ from the other (for example $f(x) = x^{2/3}$ at 0).

Vertical tangent: the slopes approach $+\infty$ (or $-\infty$) from both sides, so the graph is continuous and steep (for example $f(x) = x^{1/3}$ at 0).

Discontinuity: a jump, hole, or infinite break. If f is not continuous at a, then $f'(a)$ does not exist.

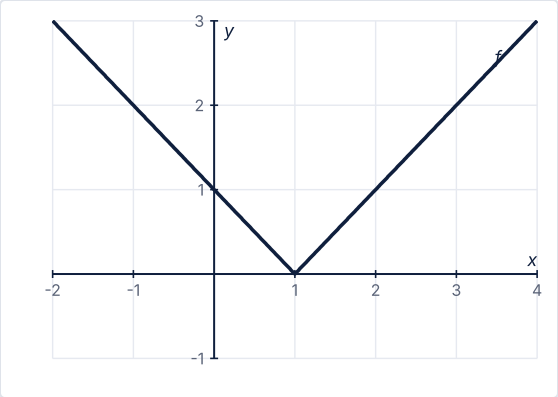
To test a piecewise function at the point where the formula changes, always do two checks in order. First, check continuity: do the left and right limits equal $f(a)$? Second, compare the one-sided derivatives: compute $\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$ and $\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$. The derivative exists exactly when they agree. When both pieces are given by differentiable formulas, the one-sided derivatives are the derivatives of the two formulas evaluated at a.

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EXAMPLE 1

Is $f(x) = |x - 1|$ differentiable at $x = 1$? Justify.



A corner at x = 1.

SOLUTION

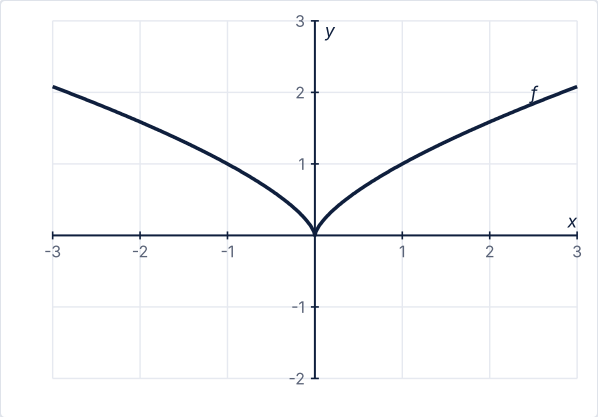
$f(1) = 0$. For $x \geq 1$, $f(x) = x - 1$, so the right-hand derivative is 1.

For $x < 1$, $f(x) = -(x - 1)$, so the left-hand derivative is -1.

The one-sided derivatives are different, so $f'(1)$ does not exist. The graph has a corner at $(1, 0)$, although f is continuous there.

EXAMPLE 2

Describe the behavior of $f(x) = x^{2/3}$ and $g(x) = x^{1/3}$ at $x = 0$.



Graph of f(x) = x^(2/3): a cusp at the origin.

SOLUTION

Both are continuous at 0 with value 0. Differentiate for $x \neq 0$: $f'(x) = \frac{2}{3x^{1/3}}$ and $g'(x) = \frac{1}{3x^{2/3}}$.

For f: as $x \to 0^+$, $f'(x) \to +\infty$; as $x \to 0^-$, $f'(x) \to -\infty$. The slopes blow up with opposite signs: a **cusp**.

For g: $g'(x) \to +\infty$ from both sides: a **vertical tangent** (the line $x = 0$).

Neither $f'(0)$ nor $g'(0)$ exists.

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EXAMPLE 3

Find constants a and b so that $f(x) = \begin{cases} ax^2 + b, & x \le 1 \\ 4x - 1, & x > 1 \end{cases}$ is differentiable at $x = 1$.

SOLUTION

Continuity at 1: the left side gives $a + b$, the right side gives $4(1) - 1 = 3$. So $a + b = 3$.

Matching slopes at 1: the left derivative is $2ax = 2a$, the right derivative is 4. So $2a = 4$ and $a = 2$.

Then $b = 3 - a = 1$. Answer: $a = 2,\ b = 1$.

EXAMPLE 4

Use the definition to decide whether $f(x) = x|x|$ is differentiable at $x = 0$.

SOLUTION

$f(0) = 0$, so $\dfrac{f(0+h) - f(0)}{h} = \dfrac{h|h|}{h} = |h|$ for $h \neq 0$.

$\lim_{h \to 0} |h| = 0$, so the limit exists. Therefore f is differentiable at 0 and $f'(0) = 0$.

Note that the graph has no corner: it passes smoothly through the origin with a horizontal tangent.

Common mistake

Matching derivative formulas is not enough. For $f(x) = \begin{cases} x^2, & x < 1 \\ 2x + 3, & x \ge 1 \end{cases}$ both pieces have derivative 2 at $x = 1$, yet f is not differentiable there, because f is not even continuous: the left limit is 1 but $f(1) = 5$. Always check continuity first. Also remember that a sharp corner does **not** mean the function is discontinuous.

Practice

SET A ### Skills

Decide differentiability and find constants as asked. [No calculator]

1. Is $f(x) = |x + 3|$ differentiable at $x = -3$? Explain.
2. Let $f(x) = \begin{cases} x^2, & x \le 2 \\ 4x - 4, & x > 2 \end{cases}$.
   (a) Is f continuous at $x = 2$?
   (b) Is f differentiable at $x = 2$? If so, find $f'(2)$.
3. Let $g(x) = \begin{cases} 2x - 1, & x < 1 \\ x^2 + 1, & x \ge 1 \end{cases}$. Is g differentiable at $x = 1$? Explain.
4. For $h(x) = \sqrt[3]{x}$, find $h'(x)$ for $x \neq 0$ and describe what happens at $x = 0$.
5. Find the value of k that makes $f(x) = \begin{cases} kx + 2, & x < 3 \\ x^2 - 1, & x \ge 3 \end{cases}$ continuous at $x = 3$.
6. True or false: if f is continuous at $x = a$, then f is differentiable at $x = a$. If false, give a counterexample.
7. Describe the graph of $f(x) = x^{2/5}$ at $x = 0$ using the behavior of $f'(x)$ as $x \to 0^-$ and $x \to 0^+$.
8. At which values of x is $f(x) = |x^2 - 4|$ not differentiable? Explain.
9. State the largest set of x-values on which $f(x) = \sqrt{x - 4}$ is differentiable, and explain why $x = 4$ is excluded.

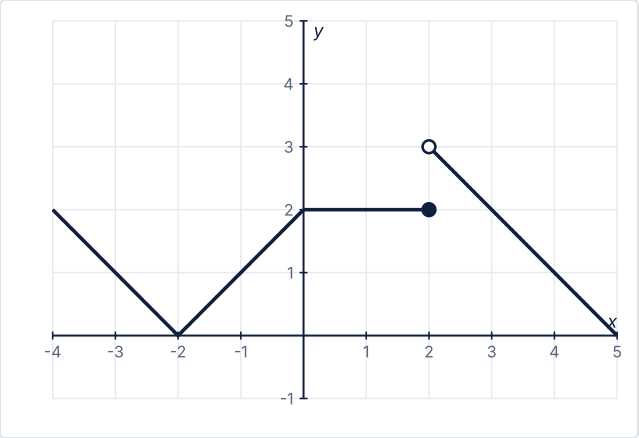
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SET B Applications and Mixed

Answer each question. [No calculator]

10. The graph of f is shown. For each of $x = -2,\ 0,\ 2,\ 4$, decide whether f is differentiable there and give a reason.



Graph of f.

11. Find a and b so that $f(x) = \begin{cases} ax^2 + bx, & x \le 1 \\ 2x^3 - x, & x > 1 \end{cases}$ is differentiable at $x = 1$.

12. Show that $f(x) = |x|^3$ is differentiable at $x = 0$ by evaluating $\lim_{h \to 0} \dfrac{f(h) - f(0)}{h}$.

13. A particle moves along a line with position $s(t) = |t^2 - 4t + 3|$ for $0 \le t \le 5$. At which times is the velocity undefined? Find the velocity at $t = 2$.

14. Is $g(x) = x|x - 3|$ differentiable at $x = 3$? Find the one-sided derivatives.

15. The function f satisfies $f(4) = 7$ and $\lim_{h \to 0} \dfrac{f(4 + h) - 7}{h} = -3$. Find $f'(4)$ and $\lim_{x \to 4} f(x)$. Justify.

16. Find constants m and c so that $f(x) = \begin{cases} 3x - x^2, & x < 2 \\ mx + c, & x \ge 2 \end{cases}$ is differentiable at $x = 2$. Then write the equation of the tangent line at $x = 2$.

SET C Challenge

Answer each question. Justify your reasoning. [No calculator]

17. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$. Use the definition to show that $f'(0)$ exists and find it.

18. Find constants a and b so that $f(x) = \begin{cases} x^2, & x \le 1 \\ ax + b, & 1 < x < 3 \\ -x^2 + 8x - 10, & x \ge 3 \end{cases}$ is differentiable for all x.

19. Suppose $f'(3) = -4$. Find $\lim_{h \to 0} \dfrac{f(3 + 2h) - f(3)}{h}$.

20. For which values of $p > 0$ is $f(x) = |x|^p$ differentiable at $x = 0$? Explain.

21. Which of the following functions is differentiable at $x = 0$?

(A) $|x|$ **(B)** $x^{2/3}$ **(C)** $x^{1/3}$ **(D)** $x|x|$ **(E)** $|x| + x$

2.6 The Product and Quotient Rules

Learn the product and quotient rules, combine them with the power rule, and use them with tables and graphs to find rates of change of products and ratios.

Most functions you meet are built from simpler ones by multiplying or dividing. The derivative of a product is **not** the product of the derivatives, and the same is true for quotients. This section gives the correct rules and shows how to combine them with the power rule.

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The Product Rule

If f and g are differentiable at x, then

$$\frac{d}{dx}\big[f(x)\,g(x)\big] = f'(x)\,g(x) + f(x)\,g'(x).$$

In words: derivative of the first times the second, plus the first times the derivative of the second. For three factors, differentiate one factor at a time: $(fgh)' = f'gh + fg'h + fgh'$.

The Quotient Rule

If f and g are differentiable at x and $g(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{\big[g(x)\big]^2}.$$

In words: low d-high minus high d-low, over low squared. The order in the numerator matters because of the subtraction.

EXAMPLE 1

Differentiate $y = (3x^2 - x)(x^3 + 5)$.

SOLUTION

Take $f = 3x^2 - x$ and $g = x^3 + 5$, so $f' = 6x - 1$ and $g' = 3x^2$.

$y' = (6x - 1)(x^3 + 5) + (3x^2 - x)(3x^2)$.

Expand: $(6x^4 - x^3 + 30x - 5) + (9x^4 - 3x^3) = 15x^4 - 4x^3 + 30x - 5$.

EXAMPLE 2

Differentiate $y = \dfrac{2x + 1}{x^2 + 3}$.

SOLUTION

Numerator $f = 2x + 1$, $f' = 2$; denominator $g = x^2 + 3$, $g' = 2x$.

$y' = \dfrac{2(x^2 + 3) - (2x + 1)(2x)}{(x^2 + 3)^2}$.

Simplify the numerator: $2x^2 + 6 - 4x^2 - 2x = -2x^2 - 2x + 6$. So $y' = \dfrac{-2x^2 - 2x + 6}{(x^2 + 3)^2}$.

EXAMPLE 3

Differentiate $y = (x^2 + 1)\sqrt{x}$ and write the result as a single fraction.

SOLUTION

Write $\sqrt{x} = x^{1/2}$, with derivative $\dfrac{1}{2\sqrt{x}}$.

$y' = 2x\sqrt{x} + \dfrac{x^2 + 1}{2\sqrt{x}}$.

Common denominator: $\dfrac{4x \cdot x + x^2 + 1}{2\sqrt{x}} = \dfrac{5x^2 + 1}{2\sqrt{x}}$.

x	1	2	3
$f(x)$	2	3	-1
$f'(x)$	4	-1	2
$g(x)$	5	4	1
$g'(x)$	-2	5	3

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EXAMPLE 4

Use the table above. Let $h(x) = f(x)g(x)$ and $k(x) = \dfrac{f(x)}{g(x)}$. Find $h'(2)$ and $k'(2)$.

SOLUTION

At $x = 2$: $f = 3$, $f' = -1$, $g = 4$, $g' = 5$.

$h'(2) = f'(2)g(2) + f(2)g'(2) = (-1)(4) + (3)(5) = 11.$

$k'(2) = \dfrac{f'(2)g(2) - f(2)g'(2)}{[g(2)]^2} = \dfrac{(-1)(4) - (3)(5)}{16} = -\dfrac{19}{16}.$

EXAMPLE 5

Find all points where the graph of $y = \dfrac{x}{x^2 + 1}$ has a horizontal tangent.

SOLUTION

$y' = \dfrac{1 \cdot (x^2 + 1) - x \cdot 2x}{(x^2 + 1)^2} = \dfrac{1 - x^2}{(x^2 + 1)^2}.$

Set $y' = 0$: the numerator is zero when $x = \pm 1$ (the denominator is never zero).

Points: $\left(1, \frac{1}{2}\right)$ and $\left(-1, -\frac{1}{2}\right)$.

Common mistakes

(1) $(fg)' \neq f'g'$ and $(f/g)' \neq f'/g'$. (2) In the quotient rule, the numerator is $f'g - fg'$, not $fg' - f'g$. (3) Do not use the quotient rule when simplifying first is easier: $\frac{x^2+4x}{x} = x + 4$ has derivative 1, and $\frac{3x+2}{\sqrt{x}} = 3x^{1/2} + 2x^{-1/2}$ can be done with the power rule alone.

Practice

SET A **Skills**

Differentiate or evaluate as directed. [No calculator]

1. Differentiate $y = (x^2 + 3)(x^3 - 2x)$.
2. Differentiate $f(x) = (4x - 1)(x^2 + x + 2)$.
3. Differentiate $g(x) = \dfrac{x}{x + 4}$.
4. Differentiate $y = \dfrac{x^2 - 1}{x^2 + 1}$.
5. Differentiate $h(t) = \sqrt{t}\,(t^2 - 3t + 1)$.
6. Differentiate $y = \dfrac{3x + 2}{\sqrt{x}}$. (Hint: rewrite first.)
7. Differentiate $f(x) = \dfrac{2x^3 - x}{x^2 - 5}$.
8. Use the product rule twice to differentiate $y = x(x + 1)(x - 4)$, then check by expanding first.
9. If $f(x) = \dfrac{x^2 + 2}{x + 1}$, find $f'(1)$.

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SET B Applications and Mixed

Answer each question. [No calculator]

10. The table gives values of differentiable functions f and g. Let $h = fg$, $k = \frac{f}{g}$ and $p = \frac{g}{f}$. Find (a) $h'(3)$, (b) $k'(3)$, (c) $p'(1)$.

x	1	2	3
$f(x)$	2	3	-1
$f'(x)$	4	-1	2
$g(x)$	5	4	1
$g'(x)$	-2	5	3

11. A rectangle has length $L(t) = 3t + 2$ centimeters and width $W(t) = t^2 + 1$ centimeters at time t seconds. Find the rate of change of its area at $t = 2$.

12. The concentration of a drug in the blood is $C(t) = \frac{5t}{t^2 + 4}$ milligrams per liter, t hours after an injection ($t \geq 0$). Find $C'(t)$, the time at which $C'(t) = 0$, and the concentration at that time.

13. Find an equation of the tangent line to $y = \frac{x + 1}{x - 1}$ at $x = 2$.

14. Find the points on the graph of $y = \frac{x^2}{x - 2}$ where the tangent line is horizontal.

15. The total cost of producing x units is $C(x) = 2x^2 + 30x + 200$ dollars, so the average cost per unit is $A(x) = \frac{C(x)}{x}$. Use the quotient rule to find $A'(x)$, and find the production level $x > 0$ at which $A'(x) = 0$.

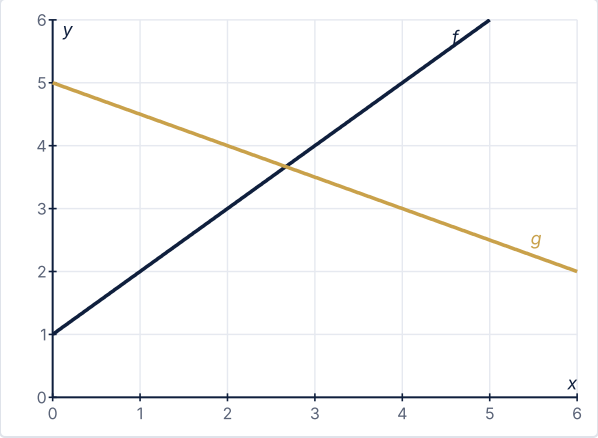
16. Let $f(x) = x^2 g(x)$ where $g(3) = 2$ and $g'(3) = -4$. Find $f'(3)$.

SET C Challenge

Answer each question. [No calculator]

17. Let $p(x) = f(x)g(x)h(x)$ with $f(1) = 2$, $f'(1) = 3$, $g(1) = -1$, $g'(1) = 4$, $h(1) = 5$, $h'(1) = -2$. Find $p'(1)$.

18. The graphs of the linear functions f and g are shown. Find (a) $(fg)'(2)$ and (b) $\left(\frac{f}{g}\right)'(2)$.



Graphs of the lines f and g.

19. Let $f(x) = x^2 g(x)$, where g has two derivatives. Show that $f''(x) = 2g(x) + 4x\,g'(x) + x^2 g''(x)$, and find $f''(2)$ if $g(2) = 3$, $g'(2) = -1$, $g''(2) = 5$.

20. The function $f(x) = \frac{ax + b}{x^2 + 1}$ satisfies $f(1) = 2$ and $f'(1) = -3$. Find a and b.

21. Let $f(x) = \frac{x^2 - 9}{x - 3}$ for $x \neq 3$. What is $f'(5)$?

(A) 0 **(B)** $\frac{1}{4}$ **(C)** 1 **(D)** 2 **(E)** 8

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2.7 The Chain Rule

Learn to differentiate composite functions, including nested ones, and to use the chain rule together with the product and quotient rules and with tables and graphs.

Many functions are built by plugging one function into another, like $\sqrt{x^2+1}$ or $(5x-2)^7$. The chain rule tells you how to differentiate such composites. It is the most heavily used derivative rule in calculus, and it combines with every other rule you know.

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then $y = f(g(x))$ is differentiable at x and

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Equivalent form: if $y = f(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$.

In words: differentiate the outer function (leaving the inside alone), then multiply by the derivative of the inside.

General Power Rule

For any differentiable function $u(x)$ and real number n,

$$\frac{d}{dx}\left[u^n\right] = n\,u^{n-1}\,u'.$$

For example, $\dfrac{d}{dx}\sqrt{u} = \dfrac{u'}{2\sqrt{u}}$ and $\dfrac{d}{dx}\dfrac{1}{u} = -\dfrac{u'}{u^2}$.

EXAMPLE 1

Differentiate $y = (x^3 - 4x)^6$.

SOLUTION

Outer: u^6. Inner: $u = x^3 - 4x$, with $u' = 3x^2 - 4$.

$y' = 6(x^3 - 4x)^5 \cdot (3x^2 - 4)$.

EXAMPLE 2

Differentiate $y = \dfrac{1}{(x^2+4)^3}$.

SOLUTION

Rewrite: $y = (x^2+4)^{-3}$.

$y' = -3(x^2+4)^{-4} \cdot 2x = -\dfrac{6x}{(x^2+4)^4}$.

EXAMPLE 3

Differentiate $y = \sqrt{1 + (2x-1)^3}$, a composite of three functions.

SOLUTION

Work from the outside in. Outer: $\sqrt{u}$ with $u = 1 + (2x-1)^3$, giving $\dfrac{u'}{2\sqrt{u}}$.

Now $u' = 3(2x-1)^2 \cdot 2 = 6(2x-1)^2$, using the chain rule again on $(2x-1)^3$.

$y' = \dfrac{6(2x-1)^2}{2\sqrt{1+(2x-1)^3}} = \dfrac{3(2x-1)^2}{\sqrt{1+(2x-1)^3}}$.

EXAMPLE 4

Differentiate $y = x^2(1-x)^4$ and factor the result.

SOLUTION

Product rule, with the chain rule on the second factor: $y' = 2x(1-x)^4 + x^2 \cdot 4(1-x)^3(-1)$.

Factor out $2x(1-x)^3$: $y' = 2x(1-x)^3\left[(1-x) - 2x\right] = 2x(1-x)^3(1-3x)$.

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x	1	2	3
$f(x)$	3	1	2
$f'(x)$	2	-3	5
$g(x)$	2	3	1
$g'(x)$	6	-1	4

EXAMPLE 5

Using the table above, let $F(x) = f(g(x))$, $G(x) = g(f(x))$ and $H(x) = f(f(x))$. Find $F'(2)$, $G'(1)$ and $H'(1)$.

SOLUTION

$F'(2) = f'(g(2)) \cdot g'(2) = f'(3) \cdot g'(2) = 5 \cdot (-1) = -5.$

$G'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot f'(1) = 4 \cdot 2 = 8.$

$H'(1) = f'(f(1)) \cdot f'(1) = f'(3) \cdot f'(1) = 5 \cdot 2 = 10.$

Common mistakes

(1) Forgetting the factor u': the derivative of $(x^2+1)^3$ is $3(x^2+1)^2 \cdot 2x$, not $3(x^2+1)^2$. (2) Evaluating f' at the wrong place: in $f(g(x))$ the derivative of the outer function is evaluated at $g(x)$, not at x. (3) Writing $f'(g(x))$ when you mean $\big(f(g(x))\big)'$: these differ by the factor $g'(x)$.

Practice

SET A Skills

Differentiate each function. [No calculator]

1. Differentiate $y = (5x-2)^7$.
2. Differentiate $f(x) = (x^2+3x)^4$.
3. Differentiate $y = \sqrt{4x^2+9}$.
4. Differentiate $g(x) = \dfrac{1}{(2x+5)^2}$.
5. Differentiate $y = (x+1)^3(x-2)^2$ and factor the result.
6. Differentiate $y = \left(\dfrac{x+1}{x-1}\right)^3$.
7. Differentiate $y = \sqrt[3]{x^2+8}$.
8. Differentiate $y = (1+\sqrt{x})^4$.
9. If $f(x) = (3x^2-2)^4$, find $f'(1)$.

SET B Applications and Mixed

Answer each question. Give decimals rounded to three places. [Calculator]

10. The radius of a spherical balloon is $r(t) = \sqrt{t+4}$ centimeters at time t seconds. Its volume is $V = \frac{4}{3}\pi r^3$. Find the rate of change of the volume at $t = 5$.
11. A cup of liquid cools so that its temperature is $T(t) = 20 + \dfrac{60}{(1+0.1t)^2}$ degrees Celsius, t minutes after it is poured. Find $T'(6)$ and interpret it.
12. A ship sails along the x-axis so that its position is $x(t) = 4t$ kilometers at time t hours. A port is at the point $(0, 3)$. The distance from the ship to the port is $d = \sqrt{x^2+9}$. Find the rate at which the distance is changing at $t = 2$.
13. Let $f(x) = (x^2+1)^{3/2}$. Write an equation of the tangent line at $x = 2$, with decimal coefficients rounded to three places.
14. The intensity of a signal at distance d meters from a source is $I(d) = \dfrac{100}{d^2+1}$ watts per square meter. Find $I'(1.5)$ and interpret it.
15. A company's revenue from selling x units is $R(x) = x\sqrt{100-x}$ thousand dollars for $0 \le x \le 100$. Find $R'(40)$ and interpret it.
16. Let $f(x) = \sqrt{x^3+2x+1}$. Find the value of $x > 0$ for which $f'(x) = 2$.

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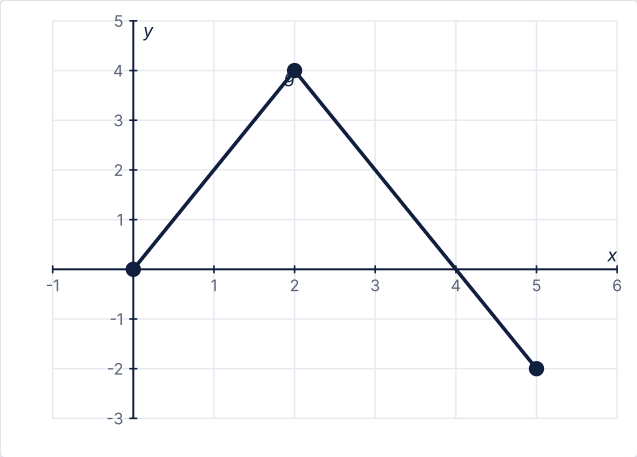
SET C **Challenge**

Answer each question. [No calculator]

17. The table gives values of differentiable functions f and g. Let $F(x) = f(g(x))$, $G(x) = g(f(x))$ and $H(x) = f(x)\,g(x)^2$. Find $F'(1)$, $G'(2)$ and $H'(3)$.

x	1	2	3
$f(x)$	3	1	2
$f'(x)$	2	-3	5
$g(x)$	2	3	1
$g'(x)$	6	-1	4

18. The graph of g is shown. It consists of two line segments with a corner at $x = 2$. Find (a) $h'(1)$ if $h(x) = [g(x)]^3$, (b) $k'(1.5)$ if $k(x) = g(g(x))$. (c) Explain why k is not differentiable at $x = 1$.



Graph of g, made of two line segments.

19. Let f be differentiable with $f'(4) = 3$, and let $h(x) = f(x^2)$. What is $h'(2)$?

(A) 0 **(B)** 3 **(C)** 6 **(D)** 9 **(E)** 12

20. Differentiate $y = \sqrt{x + \sqrt{x}}$.

21. Find y'' for $y = (3x - 1)^4$, then evaluate $y''(1)$.

2.8 Derivatives of Trigonometric Functions

Learn the derivatives of all six trigonometric functions, apply the chain rule to trig composites, and find tangent lines and rates of change for periodic models.

The sine and cosine functions model anything that repeats: waves, vibrations, alternating current, tides. In this section you learn the derivatives of all six trigonometric functions and how to use them with the product, quotient and chain rules. In calculus, angles are always measured in **radians**.

Derivatives of the Six Trigonometric Functions

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

Notice that the derivatives of cosine, cotangent and cosecant all carry a negative sign. The first two come from the limits $\lim_{h\to 0}\frac{\sin h}{h} = 1$ and $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$; the other four follow from the quotient rule.

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Chain Rule with Trigonometric Functions

If $u = u(x)$ is differentiable, then

$$\frac{d}{dx}\sin u = \cos u \cdot u', \qquad \frac{d}{dx}\cos u = -\sin u \cdot u', \qquad \frac{d}{dx}\tan u = \sec^2 u \cdot u',$$

and similarly for the other three.

Powers of trig functions: $\sin^3 x$ means $(\sin x)^3$, so $\frac{d}{dx}\sin^3 x = 3\sin^2 x \cos x$. Peel the layers from the outside in.

EXAMPLE 1

Differentiate $y = x^2 \sin x$.

SOLUTION

Product rule: $y' = 2x \sin x + x^2 \cos x$.

EXAMPLE 2

Differentiate $y = \dfrac{\cos x}{1 + \sin x}$ and simplify.

SOLUTION

Quotient rule: $y' = \dfrac{-\sin x(1 + \sin x) - \cos x \cdot \cos x}{(1 + \sin x)^2}$.

Numerator: $-\sin x - \sin^2 x - \cos^2 x = -\sin x - 1 = -(1 + \sin x)$.

So $y' = \dfrac{-(1 + \sin x)}{(1 + \sin x)^2} = -\dfrac{1}{1 + \sin x}$.

EXAMPLE 3

Differentiate (a) $y = \sin^3(2x)$ and (b) $y = \cos(x^2 + 1)$.

SOLUTION

(a) Outer power, then sine, then the inside $2x$: $y' = 3\sin^2(2x) \cdot \cos(2x) \cdot 2 = 6\sin^2(2x)\cos(2x)$.

(b) $y' = -\sin(x^2 + 1) \cdot 2x = -2x \sin(x^2 + 1)$.

EXAMPLE 4

Find an equation of the tangent line to $f(x) = 2\sin x - \tan x$ at $x = \dfrac{\pi}{4}$.

SOLUTION

$f\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} - 1 = \sqrt{2} - 1$.

$f'(x) = 2\cos x - \sec^2 x$, so $f'\left(\frac{\pi}{4}\right) = \sqrt{2} - 2$.

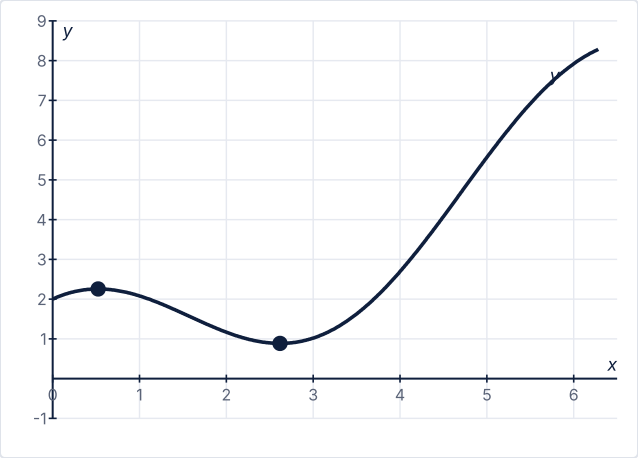
Tangent line: $y - (\sqrt{2} - 1) = (\sqrt{2} - 2)\left(x - \frac{\pi}{4}\right)$.

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EXAMPLE 5

Find the points on the graph of $y = x + 2\cos x$, $0 \le x \le 2\pi$, where the tangent line is horizontal.



Horizontal tangents at x = pi/6 and x = 5pi/6.

SOLUTION

$y' = 1 - 2\sin x$. Set it equal to zero: $\sin x = \frac{1}{2}$.

On $[0, 2\pi]$: $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$.

The y-values are $\frac{\pi}{6} + 2 \cdot \frac{\sqrt{3}}{2} = \frac{\pi}{6} + \sqrt{3}$ and $\frac{5\pi}{6} - \sqrt{3}$. Points: $\left(\frac{\pi}{6}, \frac{\pi}{6} + \sqrt{3}\right)$ and $\left(\frac{5\pi}{6}, \frac{5\pi}{6} - \sqrt{3}\right)$.

Common mistakes

(1) Forgetting the inside derivative: $\frac{d}{dx}\sin(3x) = 3\cos(3x)$, not $\cos(3x)$. (2) Sign errors on the derivatives of cosine, cotangent and cosecant. (3) Using degrees: the formulas $\frac{d}{dx}\sin x = \cos x$ hold only when x is in radians. (4) $\sin^2 x$ is $(\sin x)^2$, not $\sin(x^2)$; their derivatives are $2\sin x\cos x$ and $2x\cos(x^2)$.

Practice

SET A ### Skills

Differentiate or evaluate as directed. [No calculator]

1. Differentiate $y = 3\sin x - 4\cos x$.
2. Differentiate $f(x) = x\cos x$.
3. Differentiate $y = \tan x - \cot x$.
4. Differentiate $y = \cos(x^3)$.
5. Differentiate $y = \sec x \tan x$.
6. Differentiate $y = \dfrac{\sin x}{1 + \cos x}$ and simplify.
7. Differentiate $y = \cos^2(3x)$.
8. Differentiate $y = \csc(2x)$.
9. If $f(x) = \sin x \cos x$, find $f'\left(\dfrac{\pi}{3}\right)$.

SET B ### Applications and Mixed

Use radian mode and round decimal answers to three places. [Calculator]

10. A particle moves along a line with position $s(t) = 3\sin(2t) + t$ meters at time t seconds. Find the velocity and acceleration at $t = 1.5$.
11. The depth of water in a harbor is $D(t) = 12 + 3\sin\left(\dfrac{\pi t}{6}\right)$ meters, t hours after midnight. Find how fast the depth is changing at $t = 2$.
12. The current in a circuit is $I(t) = 5\sin(50t)$ amperes at time t seconds. Find $I'(0.02)$.
13. Find the value of x in the interval $[0, 2]$ at which the graph of $f(x) = x^2 - 4\sin x$ has a horizontal tangent line.
14. Write an equation of the tangent line to $y = x\tan x$ at $x = 1$. Use three decimal places.
15. A triangle has two sides of lengths 6 and 8 with angle θ between them, so its area is $A = 24\sin\theta$. If θ increases at 0.2 radians per second, how fast is the area changing when $\theta = 0.9$?
16. The angle (in radians) of a pendulum from the vertical is $\theta(t) = 0.3\cos(2t)$ at time t seconds. Find the angular velocity at $t = 1$.

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SET C **Challenge**

Answer each question. [No calculator]

17. Use the quotient rule or the chain rule to derive $\frac{d}{dx}\sec x = \sec x \tan x$.

18. Evaluate each limit by recognizing a derivative: (a) $\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{6}+h\right)-\frac{1}{2}}{h}$ (b) $\lim_{x\to \pi/4} \frac{\tan x - 1}{x - \frac{\pi}{4}}$.

19. What is the slope of the graph of $y = \cos(2x)$ at $x = \frac{\pi}{6}$?

(A) -2 (B) $-\sqrt{3}$ (C) -1 (D) $\sqrt{3}$ (E) 2

20. Find the points on the graph of $y = \sin x + \cos x$, $0 \le x \le 2\pi$, where the tangent line is horizontal.

21. Show that $y = 2\sin 3x - \cos 3x$ satisfies $y'' + 9y = 0$.

2.9 Derivatives of Exponential and Logarithmic Functions

Learn the derivatives of e^x, a^x, $\ln x$ and $\log_a x$, extend them with the chain rule, and use logarithmic differentiation for products, quotients and variable exponents.

Exponential functions describe growth and decay, and logarithms undo them. Their derivatives are among the most important in all of calculus, because e^x is the one exponential function whose rate of change equals its own value. In this section you learn all four derivative formulas, how the chain rule extends them, and a technique called logarithmic differentiation.

Exponential and Logarithmic Derivatives

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}a^x = a^x \ln a \ \ (a>0) \qquad \frac{d}{dx}\ln x = \frac{1}{x} \ \ (x>0) \qquad \frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

With the chain rule, for a differentiable $u(x)$:

$$\frac{d}{dx}e^u = e^u u', \qquad \frac{d}{dx}a^u = a^u \ln a \cdot u', \qquad \frac{d}{dx}\ln u = \frac{u'}{u} \ \ (u>0), \qquad \frac{d}{dx}\log_a u = \frac{u'}{u\ln a}.$$

Also $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for all $x \neq 0$. Note $\log_a x = \frac{\ln x}{\ln a}$, which is where the last formula comes from.

Logarithmic Differentiation

Use it when the variable appears in both a base and an exponent (like x^x), or when y is a long product or quotient.

Steps: (1) take $\ln$ of both sides and simplify with log laws; (2) differentiate implicitly, remembering $\frac{d}{dx}\ln y = \frac{y'}{y}$; (3) multiply by y and replace y by the original expression.

The method is valid where $y > 0$ (or work with $|y|$).

EXAMPLE 1

Differentiate $y = x^2 e^{-2x}$.

SOLUTION

Product rule, with the chain rule on e^{-2x}: $y' = 2xe^{-2x} + x^2 \cdot (-2)e^{-2x}$.

Factor: $y' = 2x(1-x)e^{-2x}$.

EXAMPLE 2

Differentiate (a) $y = 5^{x^2}$ and (b) $y = \log_3(2x+1)$.

SOLUTION

(a) $y' = 5^{x^2}\ln 5 \cdot 2x = 2x\ln 5 \cdot 5^{x^2}$.

(b) $y' = \frac{2}{(2x+1)\ln 3}$.

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EXAMPLE 3

Differentiate $y = x \ln x - x$ for $x > 0$.

SOLUTION

$y' = \ln x + x \cdot \frac{1}{x} - 1 = \ln x + 1 - 1 = \ln x.$

EXAMPLE 4

Use logarithmic differentiation to find y' for $y = x^x$, $x > 0$.

SOLUTION

Take logs: $\ln y = x \ln x$.

Differentiate: $\frac{y'}{y} = \ln x + x \cdot \frac{1}{x} = \ln x + 1$.

So $y' = x^x(\ln x + 1)$.

EXAMPLE 5

Find an equation of the tangent line to $y = \frac{e^x}{x}$ at $x = 2$.

SOLUTION

$y' = \frac{e^x x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$.

At $x = 2$: $y = \frac{e^2}{2}$ and $y' = \frac{e^2}{4}$.

Tangent line: $y - \frac{e^2}{2} = \frac{e^2}{4}(x - 2)$.

Common mistakes

(1) $\frac{d}{dx}a^x$ is **not** xa^{x-1}; that is the power rule, which needs a constant exponent. (2) $\frac{d}{dx}e^x$ has no extra factor, but $\frac{d}{dx}e^{3x} = 3e^{3x}$. (3) $\frac{d}{dx}\ln(x^2) = \frac{2x}{x^2} = \frac{2}{x}$, and $\ln(x^2) = 2\ln x$ gives the same result. Simplify with log laws before differentiating when you can. (4) $\ln u$ is only defined for $u > 0$.

Practice

SET A Skills

Differentiate each function or answer as directed. [No calculator]

1. Differentiate $y = e^{5x}$.
2. Differentiate $f(x) = x^3 e^x$.
3. Differentiate $y = \ln(3x + 7)$.
4. Differentiate $g(x) = \ln(x^2 + 4x)$.
5. Differentiate $y = 7^{3x}$.
6. Differentiate $y = \log_5(x^2 + 1)$.
7. Differentiate $y = \frac{e^x}{1 + e^x}$.
8. Differentiate $y = \ln\sqrt{x^2 + 1}$. (Hint: use a log law first.)
9. Use logarithmic differentiation to find y' for $y = (x^2 + 1)^x$.

SET B Applications and Mixed

Round decimal answers to three places. [Calculator]

10. A bacteria culture has $P(t) = 500e^{0.12t}$ cells at time t hours. Find $P'(5)$ and interpret it.
11. The temperature of a cooling object is $T(t) = 22 + 68e^{-0.05t}$ degrees Celsius at time t minutes. Find $T'(10)$ and interpret it.
12. The concentration of a medicine in the bloodstream is $C(t) = 8te^{-0.4t}$ milligrams per liter, t hours after it is taken. At what time is the concentration neither increasing nor decreasing, and what is the concentration then?
13. The sound level of a noise with relative intensity x is $L(x) = 10\log_{10} x$ decibels. Find $L'(50)$ and interpret it.
14. An account earning continuously compounded interest has balance $A(t) = 2000e^{0.05t}$ dollars after t years. How fast is the balance growing at $t = 10$?
15. Find the point on the graph of $g(x) = e^x - 4x$ where the tangent line is horizontal.

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16. Find an equation of the tangent line to $y = \ln(x^2 + 1)$ at $x = 2$, using decimals rounded to three places.

SET C Challenge

Answer each question. [No calculator]

17. Find $\frac{dy}{dx}$ for $y = (\sin x)^x$ on $0 < x < \pi$.

18. Differentiate $f(x) = \ln|\cos x|$ and simplify.

19. If $f(x) = \ln\left(e^{2x} + 1\right)$, then $f'(0) =$

(A) 1 (B) 0 (C) $\frac{1}{2}$ (D) 2 (E) $\ln 2$

20. Find the point on the graph of $y = \frac{\ln x}{x}$ at which the tangent line is horizontal.

21. The function $f(x) = ae^x + b\ln x$ satisfies $f(1) = 3e$ and $f'(1) = 3e - 2$. Find a and b.

2.10 Implicit Differentiation

Learn to find dy/dx for curves that are not given as y = f(x), write tangent and normal lines, locate horizontal and vertical tangents, and compute second derivatives implicitly.

Some curves, like a circle, cannot be written as a single function $y = f(x)$. Others can, but only with an expression that is painful to differentiate. **Implicit differentiation** finds the slope of such a curve without solving for y. The key idea is that near a point of the curve, y is a differentiable function of x, so the chain rule applies to every term containing y.

Implicit Differentiation: The Method

1. Differentiate both sides of the equation with respect to x. Any time you differentiate an expression in y, apply the chain rule and attach a factor $\frac{dy}{dx}$: for example $\frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}$ and $\frac{d}{dx}\sin y = \cos y\,\frac{dy}{dx}$.

2. Use the product and quotient rules where needed: $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$.

3. Collect all terms containing $\frac{dy}{dx}$ on one side, factor out $\frac{dy}{dx}$, and solve.

The answer usually involves both x and y. To find a slope, substitute the coordinates of a point that lies **on the curve**.

Tangent Lines and Second Derivatives

The tangent line to the curve at (x_0, y_0) is $y - y_0 = m(x - x_0)$ with $m = \left.\frac{dy}{dx}\right|_{(x_0,y_0)}$. The normal line has slope $-\frac{1}{m}$ (if $m \neq 0$).

The tangent is horizontal where $\frac{dy}{dx} = 0$ (numerator zero), and vertical where the denominator of $\frac{dy}{dx}$ is zero but the numerator is not.

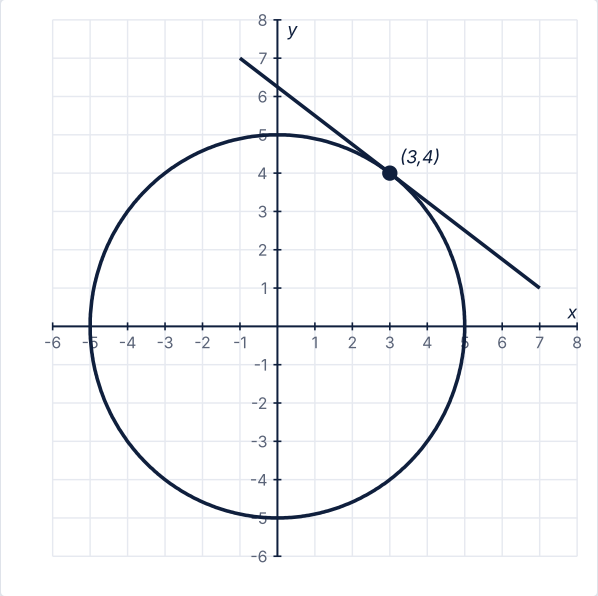
To find $\frac{d^2y}{dx^2}$, differentiate $\frac{dy}{dx}$ implicitly (again attaching $\frac{dy}{dx}$ to every y term), then substitute the known expression for $\frac{dy}{dx}$.

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EXAMPLE 1

Find $\frac{dy}{dx}$ for $x^2 + y^2 = 25$, then find the tangent line at $(3, 4)$.



The circle and its tangent line at (3, 4).

SOLUTION

Differentiate: $2x + 2y\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{x}{y}$.

At $(3, 4)$ the slope is $-\frac{3}{4}$.

Tangent line: $y - 4 = -\frac{3}{4}(x - 3)$.

EXAMPLE 2

Find the slope of the curve $x^2y + y^3 = 10$ at the point $(1, 2)$.

SOLUTION

Check: $1 \cdot 2 + 8 = 10$, so the point is on the curve.

Differentiate (product rule on the first term): $2xy + x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$.

Solve: $\frac{dy}{dx} = -\frac{2xy}{x^2 + 3y^2}$. At $(1, 2)$: $-\frac{4}{1 + 12} = -\frac{4}{13}$.

EXAMPLE 3

Find $\frac{dy}{dx}$ at $(1, 1)$ for the curve $xy + \ln y = 1$.

SOLUTION

Check: $1 + \ln 1 = 1$.

Differentiate: $y + x\frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx} = 0$.

Factor: $\frac{dy}{dx}\left(x + \frac{1}{y}\right) = -y$, so $\frac{dy}{dx} = -\frac{y^2}{xy + 1}$. At $(1, 1)$: $-\frac{1}{2}$.

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EXAMPLE 4

For $x^2 + y^2 = 25$, find $\dfrac{d^2y}{dx^2}$ and evaluate it at $(3, 4)$.

SOLUTION

From Example 1, $y' = -\dfrac{x}{y}$.

Differentiate with the quotient rule: $y'' = -\dfrac{y - x\,y'}{y^2} = -\dfrac{y + \frac{x^2}{y}}{y^2} = -\dfrac{y^2 + x^2}{y^3}$.

Since $x^2 + y^2 = 25$, $y'' = -\dfrac{25}{y^3}$. At $(3, 4)$: $-\dfrac{25}{64}$. (The curve is concave down at that point, as the top half of a circle should be.)

EXAMPLE 5

Find all points on $x^2 - xy + y^2 = 12$ where the tangent line is horizontal or vertical.

SOLUTION

Differentiate: $2x - y - x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = \dfrac{y - 2x}{2y - x}$.

Horizontal: $y = 2x$. Substitute: $x^2 - 2x^2 + 4x^2 = 3x^2 = 12$, so $x = \pm 2$. Points $(2, 4)$ and $(-2, -4)$.

Vertical: $x = 2y$. Substitute: $4y^2 - 2y^2 + y^2 = 3y^2 = 12$, so $y = \pm 2$. Points $(4, 2)$ and $(-4, -2)$.

Common mistakes

(1) Forgetting the factor $\frac{dy}{dx}$ on a y term: $\frac{d}{dx}y^2 = 2y\frac{dy}{dx}$, not $2y$. (2) Differentiating xy as y or as $x\frac{dy}{dx}$ alone; both terms of the product rule are needed. (3) Substituting a point that is not on the curve. (4) Leaving $\frac{dy}{dx}$ on both sides of the equation instead of solving for it.

Practice

SET A Skills

Find the indicated derivative or slope. [No calculator]

1. Find $\dfrac{dy}{dx}$ if $x^2 + y^2 = 36$.
2. For $x^3 + y^3 = 35$, find $\dfrac{dy}{dx}$ and its value at $(2, 3)$.
3. Find $\dfrac{dy}{dx}$ if $xy = 12$.
4. Find the slope of $y^2 = 4x$ at the point $(1, 2)$.
5. Find $\dfrac{dy}{dx}$ if $x^2 + 3xy - y^2 = 9$.
6. Find $\dfrac{dy}{dx}$ if $\sin y = x^2$.
7. Find $\dfrac{dy}{dx}$ if $e^y = x + y$.
8. Find an equation of the tangent line to $x^2 + y^2 = 20$ at $(2, 4)$.
9. For $xy = 4$, find $\dfrac{d^2y}{dx^2}$ in terms of x and y, and then in terms of x only.

SET B Applications and Mixed

Answer each question. [No calculator]

10. The circle $x^2 - 4x + y^2 + 6y = 12$ passes through $(5, 1)$. (a) Find the tangent line at $(5, 1)$. (b) Find the points on the circle where the tangent is horizontal.
11. Find an equation of the normal line to $x^2 + xy + y^2 = 7$ at $(2, 1)$.
12. For the ellipse $4x^2 + 9y^2 = 36$, find $\dfrac{dy}{dx}$ and the points where the tangent line is horizontal or vertical.
13. A gas is compressed so that its pressure P (kilopascals) and volume V (liters) satisfy $PV = 200$. Find $\dfrac{dP}{dV}$ implicitly and evaluate it when $V = 5$.
14. The price p (dollars) and quantity q (hundreds of units) of a product satisfy $pq + q^2 = 40$. Find $\dfrac{dp}{dq}$ when $q = 4$.
15. Show that $(0, \frac{1}{2})$ lies on the curve $e^{xy} = 2y$ and find the slope there.

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16. For $x^2 - y^2 = 4$, show that $\dfrac{d^2y}{dx^2} = -\dfrac{4}{y^3}$ and evaluate it at $(4, 2\sqrt{3})$.

SET C Challenge

Answer each question. [No calculator]

17. Find all points on $x^2 + xy + y^2 = 3$ where the tangent line is horizontal or vertical.

18. For the curve $y^2 = x^3$, find $\dfrac{d^2y}{dx^2}$ at the point $(1, 1)$.

19. On the curve $x^2 + xy = 6$, what is $\dfrac{dy}{dx}$ at the point $(2, 1)$?

(A) -4 (B) $-\dfrac{5}{2}$ (C) -2 (D) $\dfrac{5}{2}$ (E) 4

20. Show that the curves $x^2 - y^2 = 3$ and $xy = 2$ intersect at $(2, 1)$ and are perpendicular there.

21. The tangent lines to $x^2 + y^2 = 25$ at the two points with $x = 4$ intersect each other. Find the point of intersection.

2.11 Derivatives of Inverse Functions

Learn how the slope of an inverse function is the reciprocal of the original slope, and use it to differentiate arcsin, arccos and arctan.

If a function f takes an input x to an output y, its inverse $g = f^{-1}$ takes y back to x. Their graphs are mirror images across the line $y = x$, so a tangent line to f at (a, b) reflects into a tangent line to g at (b, a). Reflecting a line swaps rise and run, which means the slopes are reciprocals. That one geometric fact is the whole formula for the derivative of an inverse.

Derivative of an Inverse Function

Suppose f is differentiable and one-to-one on an interval, and $g = f^{-1}$. If $f(a) = b$ and $f'(a) \neq 0$, then g is differentiable at b and

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{f'(g(b))}.$$

In Leibniz form, $\dfrac{dx}{dy} = \dfrac{1}{dy/dx}$. If $f'(a) = 0$, the graph of g has a vertical tangent line at $x = b$, so $g'(b)$ does not exist.

Common mistake

The derivative of the inverse is NOT $\dfrac{1}{f'(b)}$. You must evaluate f' at the matching input $a = g(b)$, not at b. Always ask: which x does f send to the number I was given?

EXAMPLE 1

Let $f(x) = x^3 + x + 1$ and $g = f^{-1}$. Find $g'(3)$.

SOLUTION

Find the x with $f(x) = 3$: $x^3 + x + 1 = 3$ gives $x = 1$, so $g(3) = 1$.

$f'(x) = 3x^2 + 1$, so $f'(1) = 4$.

$g'(3) = \dfrac{1}{f'(g(3))} = \dfrac{1}{4}$.

x	1	2	3	4
$f(x)$	3	7	8	12
$f'(x)$	5	4	2	1

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EXAMPLE 2

The table above gives values of a differentiable, one-to-one function f and its derivative. Let $g = f^{-1}$. Find $g'(7)$ and an equation of the tangent line to g at $x = 7$.

SOLUTION

Read the table backward: $f(2) = 7$, so $g(7) = 2$.

$g'(7) = \dfrac{1}{f'(2)} = \dfrac{1}{4}$.

The tangent line passes through $(7, 2)$ with slope $\dfrac{1}{4}$: $y - 2 = \dfrac{1}{4}(x - 7)$.

Inverse trigonometric functions are the most important inverses in calculus. Each is the inverse of a restricted trig function: $\sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos x$ on $[0, \pi]$, and $\tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Differentiating $\sin y = x$ implicitly gives $\cos y \dfrac{dy}{dx} = 1$. On the restricted range $\cos y \geq 0$, so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$, and therefore $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$. The other two formulas come from the same idea.

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1), \qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1),$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \quad (\text{all real } x).$$

With the chain rule: $\dfrac{d}{dx}\arcsin u = \dfrac{u'}{\sqrt{1-u^2}}$ and $\dfrac{d}{dx}\arctan u = \dfrac{u'}{1+u^2}$, where u is a differentiable function of x.

EXAMPLE 3

Differentiate $y = \arcsin(3x)$ and state where the derivative exists.

SOLUTION

Here $u = 3x$ and $u' = 3$.

$y' = \dfrac{3}{\sqrt{1 - (3x)^2}} = \dfrac{3}{\sqrt{1 - 9x^2}}$.

This requires $1 - 9x^2 > 0$, so $-\dfrac{1}{3} < x < \dfrac{1}{3}$.

EXAMPLE 4

Find $\dfrac{d}{dx}\left[x^2 \arctan(x)\right]$.

SOLUTION

Use the product rule with $u = x^2$ and $v = \arctan x$.

$\dfrac{d}{dx}\left[x^2 \arctan x\right] = 2x \arctan x + x^2 \cdot \dfrac{1}{1 + x^2}$.

Final form: $2x \arctan x + \dfrac{x^2}{1 + x^2}$.

EXAMPLE 5

Let $f(x) = 2x + \sin x$ and $g = f^{-1}$. Find $g'(2\pi)$. Explain why f has an inverse.

SOLUTION

$f'(x) = 2 + \cos x \geq 1 > 0$ for every x, so f is strictly increasing and therefore one-to-one. Hence g exists.

Solve $f(x) = 2\pi$: $f(\pi) = 2\pi + 0 = 2\pi$, so $g(2\pi) = \pi$.

$f'(\pi) = 2 + \cos \pi = 1$, so $g'(2\pi) = \dfrac{1}{1} = 1$.

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AP justification

To justify that an inverse exists, show $f'(x) > 0$ for all x in the domain (or $f'(x) < 0$ for all x), so f is strictly monotonic and hence one-to-one. Then state the formula you used: 'Since $f(a) = b$ and $f'(a) \neq 0$, $g'(b) = \dfrac{1}{f'(a)}$.'

Practice

SET A Skills

Find the requested derivative or value. [No calculator]

1. Let $f(x) = 2x^3 + x$ and $g = f^{-1}$. Find $g'(3)$.
2. Let $f(x) = x^5 + x^3$ and $g = f^{-1}$. Find $g'(2)$.
3. Let $f(x) = e^x + x$ and $g = f^{-1}$. Find $g'(1)$.
4. Differentiate $y = \arcsin(5x)$.
5. Differentiate $y = \arctan(x^3)$.
6. Differentiate $y = x \arccos x$.
7. Differentiate $y = \arcsin(\sqrt{x})$ for $0 < x < 1$.
8. Differentiate $y = \arctan\left(\dfrac{1}{x}\right)$ for $x \neq 0$ and simplify.
9. Find the slope of the graph of $y = \arcsin\left(\dfrac{x}{2}\right)$ at $x = 1$.

SET B Applications and Mixed

Give decimal answers to three places where needed. [Calculator]

10. The table gives values of a differentiable, one-to-one function f. Let $g = f^{-1}$. Find $g'(7)$ and $g'(11)$ as decimals.

x	1	2	3
$f(x)$	3	7	11
$f'(x)$	5	4	8

11. The volume of water in a tank is $V(t) = t^3 + 2t$ liters at time t hours, for $t \geq 0$. Let $T(v)$ be the time at which the volume equals v liters. Find $T'(12)$ and give its units. Round to three decimal places.
12. Let $f(x) = x^3 + 4x$ and $g = f^{-1}$. Use a calculator to solve $f(x) = 7$, then find $g'(7)$ to three decimal places.
13. A ramp of length 10 meters rises to a height h meters above the ground. The angle with the ground is $\theta = \arcsin\left(\dfrac{h}{10}\right)$ radians. Find $\dfrac{d\theta}{dh}$ when $h = 6$, with units.
14. Find an equation of the tangent line to $y = \arctan(2x)$ at $x = \dfrac{1}{2}$. Write it as $y = mx + b$ with b rounded to three decimal places.
15. Find the slope of the tangent line to the graph of $y = \arcsin x$ at $x = 0.6$. Give a decimal answer.
16. The function $f(x) = x^2$ is not one-to-one on all real numbers, but it is on $x \geq 0$. Let g be the inverse of f restricted to $x \geq 0$. Use the inverse derivative formula to find $g'(9)$ as a decimal rounded to three places, and check it against the derivative of $\sqrt{x}$.

SET C Challenge

Show your reasoning. [No calculator]

17. Let f be differentiable and one-to-one with $f(2) = 5$ and $f'(2) = -4$, and let $g = f^{-1}$.
    (a) Find $g(5)$ and $g'(5)$.
    (b) Write an equation of the tangent line to $y = g(x)$ at $x = 5$.
    (c) Let $h(x) = (g(x))^2$. Find $h'(5)$.
18. Use implicit differentiation on $\cos y = x$, with $0 \leq y \leq \pi$, to derive the formula for $\dfrac{d}{dx} \arccos x$. State why you may take $\sin y = \sqrt{1 - x^2}$ rather than its negative.
19. (a) Show that $\dfrac{d}{dx}\left[\arcsin x + \arccos x\right] = 0$ for $-1 < x < 1$.
    (b) Use part (a) and the value at $x = 0$ to show that $\arcsin x + \arccos x = \dfrac{\pi}{2}$ on $-1 < x < 1$.
20. Let $y = \arctan\left(\dfrac{1+x}{1-x}\right)$ for $x < 1$. Find $\dfrac{dy}{dx}$ and simplify. What does your result tell you about how y differs from $\arctan x$?

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21. Let $f(x) = x^5 + x^3 + x$ and $g = f^{-1}$.
(a) Show that f has an inverse defined for all real numbers.
(b) Find $g'(3)$.
(c) Show that $g'(x) \le 1$ for all x, and find where equality holds.

2.12 Related Rates: Setting Up the Equation

Learn a six-step method for problems where several changing quantities are linked by one equation, using Pythagorean, area, volume and similar-triangle models.

In a related rates problem, several quantities change with time and are tied together by an equation from geometry or science. Differentiating that equation with respect to time t with the chain rule turns a known rate into an unknown one. For example, if $A = \pi r^2$ and both A and r depend on t, then $\dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt}$.

The Related Rates Method

1. Draw a picture and name every changing quantity with a variable (with units).
2. List what is given and what is asked, as rates such as $\dfrac{dx}{dt} = 3$ and $\dfrac{dy}{dt} = ?$.
3. Write an equation connecting the variables that holds at EVERY moment, not just the moment in question.
4. Differentiate both sides with respect to t, using the chain rule on every variable.
5. Only now substitute the numbers for the moment in question, then solve for the unknown rate.
6. State the answer with units and a sign: negative means decreasing.

Common mistake

Substituting a number for a changing quantity BEFORE differentiating. If a ladder is 8 m from a wall at the moment you care about, you cannot write $x = 8$ in $x^2 + y^2 = 100$ and then differentiate: a constant has derivative 0, and you would lose the term you need. Keep variables as variables until after step 4.

Models You Should Know

Circle: $A = \pi r^2$, $C = 2\pi r$. Sphere: $V = \dfrac{4}{3}\pi r^3$, $S = 4\pi r^2$. Cube: $V = s^3$, $S = 6s^2$. Cylinder: $V = \pi r^2 h$.

Right triangle with legs x, y and hypotenuse z: $x^2 + y^2 = z^2$, so $x\,x' + y\,y' = z\,z'$.

Similar triangles: corresponding sides are proportional, which lets you eliminate one variable before differentiating (for example, write r in terms of h).

EXAMPLE 1

Car A leaves an intersection heading east at 40 km/h. At the same moment car B leaves heading north at 30 km/h. How fast is the distance between the cars changing 2 hours later?

SOLUTION

Let x be A's distance east, y be B's distance north, z the distance apart. Then $x^2 + y^2 = z^2$, with $x' = 40$, $y' = 30$.

Differentiate: $2xx' + 2yy' = 2zz'$, so $z' = \dfrac{xx' + yy'}{z}$.

At $t = 2$: $x = 80$, $y = 60$, $z = \sqrt{80^2 + 60^2} = 100$.

$z' = \dfrac{80(40) + 60(30)}{100} = \dfrac{5000}{100} = 50$ km/h.

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EXAMPLE 2

A spherical balloon is inflated so that its volume increases at 100 cm^3/s. How fast is the radius increasing when $r = 5$ cm?

SOLUTION

Model: $V = \frac{4}{3}\pi r^3$. Given $\frac{dV}{dt} = 100$.

Differentiate: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

Substitute $r = 5$: $100 = 4\pi(25)\frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{100}{100\pi} = \frac{1}{\pi}$.

Units check: cm^3/s divided by cm^2 gives cm/s. The radius grows at $\frac{1}{\pi} \approx 0.318$ cm/s.

EXAMPLE 3

A rectangle has length $L = 10$ cm increasing at 3 cm/s and width $W = 8$ cm decreasing at 2 cm/s. Find the rate of change of (a) the area and (b) the diagonal at that instant.

SOLUTION

(a) $A = LW$, so $\frac{dA}{dt} = L'W + LW' = 3(8) + 10(-2) = 4$ cm^2/s. The area is increasing.

(b) $d^2 = L^2 + W^2$, so $d\,d' = LL' + WW'$. At this moment $d = \sqrt{100 + 64} = \sqrt{164} = 2\sqrt{41}$.

$d' = \frac{10(3) + 8(-2)}{2\sqrt{41}} = \frac{14}{2\sqrt{41}} = \frac{7}{\sqrt{41}} \approx 1.093$ cm/s.

EXAMPLE 4

A trough 4 m long has a cross-section that is an isosceles triangle, 2 m wide at the top and 1.5 m deep. Water is pumped in at 0.5 m ^{3}/min. How fast is the water level rising when it is 0.75 m deep?

SOLUTION

Let h be the water depth and w the width of the water surface. Similar triangles give $\frac{w}{h} = \frac{2}{1.5}$, so $w = \frac{4}{3}h$.

Cross-section area is $\frac{1}{2}wh = \frac{2}{3}h^2$, so $V = 4 \cdot \frac{2}{3}h^2 = \frac{8}{3}h^2$.

Differentiate: $\frac{dV}{dt} = \frac{16}{3}h\frac{dh}{dt}$.

With $h = 0.75$: $0.5 = \frac{16}{3}(0.75)\frac{dh}{dt} = 4\frac{dh}{dt}$, so $\frac{dh}{dt} = 0.125$ m/min.

Unit check

Before you finish, check that your answer has units of (quantity) per (time). In Example 4, $\frac{dV}{dt}$ is in m^3/min and equals (m^2) times (m/min), so $\frac{dh}{dt}$ must be in m/min. If the units do not work out, look for an error in your equation or in a substitution.

Practice

SET A **Skills**

Find the requested rate. Include units in your answer where they are given. [No calculator]

1. The radius of a circle increases at 3 cm/s. Find the rate at which the area is increasing when $r = 4$ cm.
2. The edge s of a cube increases at 2 cm/s. When $s = 5$ cm, find
   (a) the rate of change of the volume,
   (b) the rate of change of the total surface area.
3. A spherical droplet evaporates so that its volume decreases at 36π cm^3/min. Find $\frac{dr}{dt}$ when $r = 3$ cm.
4. A point (x, y) moves along the circle $x^2 + y^2 = 25$ in the first quadrant. When $x = 3$, $\frac{dx}{dt} = 3$. Find $\frac{dy}{dt}$ at that moment.

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5. A rectangle has length 12 cm increasing at 2 cm/s and width 5 cm decreasing at 1 cm/s. Is the area increasing or decreasing, and at what rate?
6. Boat A sails north at 5 km/h and boat B sails west at 12 km/h from the same harbor at the same time. How fast is the distance between them changing after 1 hour?
7. The area of a circular puddle increases at 10π cm^2/s. How fast is the radius increasing when the area is 25π cm^2?
8. A cylinder has fixed radius 3 cm and its height increases at 2 cm/s. How fast is the volume increasing?
9. Two positive quantities satisfy $xy = 12$. When $x = 3$, $\frac{dx}{dt} = 2$. Find $\frac{dy}{dt}$.

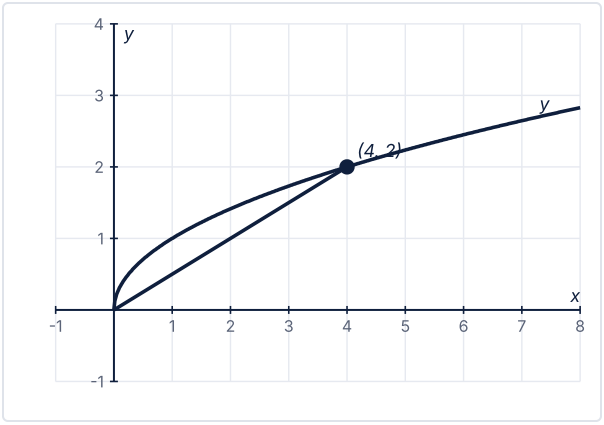
SET B Applications and Mixed

Round decimal answers to three places. [Calculator]

10. A spherical balloon is inflated at 15 cm^3/s. Find the rate at which the radius is increasing when $r = 6$ cm. Round to three decimal places.
11. An oil slick is a circle whose radius grows at 0.4 m/min. How fast is the area increasing when the radius is 50 m? Round to three decimal places.
12. A spherical ice ball melts so that its volume decreases at 2 cm^3/min. Find the rate of change of its surface area when $r = 3$ cm. Round to three decimal places.
13. A trough 5 m long has a triangular cross-section 1.2 m wide at the top and 0.9 m deep. Water is in the trough with depth h. If the depth is 0.6 m and rising at 0.02 m/min, at what rate is the volume of water changing? Round to three decimal places.
14. Car A is 3 km east of an intersection and driving east (away) at 60 km/h. Car B is 4 km south of the intersection and driving north (toward it) at 80 km/h. Find the rate of change of the distance between them at that moment, with units.
15. The table gives the radius $r(t)$, in cm, of a spherical bubble and its rate of change at selected times t seconds. Find $\frac{dV}{dt}$ at $t = 2$ to three decimal places.

t	0	2	4
$r(t)$	1	3	5
$r'(t)$	1.5	1.0	0.5

16. A point moves along the graph of $y = \sqrt{x}$ with $\frac{dx}{dt} = 3$ units per second. Let D be its distance from the origin. Find $\frac{dD}{dt}$ when the point is at $(4, 2)$, to three decimal places.



A point moving on y = sqrt(x).

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SET C Challenge

Show your setup and reasoning. [No calculator]

17. A spherical droplet grows so that $\frac{dV}{dt} = 0.3\,S$, where V is its volume in cm³, S is its surface area in cm², and t is in minutes. Show that the radius grows at a constant rate and find it.
18. A rectangle has sides x and y (cm). Side x increases at 2 cm/s while y changes so that the area stays 48 cm². At the moment $x = 6$:
    (a) Find $\frac{dy}{dt}$.
    (b) Is the perimeter increasing or decreasing, and at what rate?
    (c) Find the rate of change of the diagonal.
19. A particle moves along the parabola $y = x^2$ with $\frac{dx}{dt} = 3$ when it is at $(2, 4)$. Find $\frac{dy}{dt}$ and the rate of change of its distance from the origin at that moment.
20. A point (x, y) moves on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ in the first quadrant. At the instant $x = \frac{3\sqrt{3}}{2}$, $\frac{dx}{dt} = -2$. Find y and $\frac{dy}{dt}$ at that instant.
21. A spherical snowball melts so that its surface area decreases at a constant 0.5 cm²/min.
    (a) Find $\frac{dr}{dt}$ when $r = 2$ cm.
    (b) Find $\frac{dV}{dt}$ when $r = 2$ cm.
    (c) Explain why $\left|\frac{dr}{dt}\right|$ gets larger as the snowball gets smaller.

2.13 Related Rates: Angles and Harder Models

Use trigonometric relationships to find rates of change of angles and lengths in ladders, shadows, cones and camera problems, with all angles in radians.

Many related rates problems hide a right triangle with an angle θ that changes with time. You are either asked for the rate of change of that angle, $\frac{d\theta}{dt}$, or given it and asked for a length. The method from the previous section still applies, but now the chain rule brings in a trigonometric derivative. **Angles must be in radians**: $\frac{d}{d\theta}\sin\theta = \cos\theta$ is only true when θ is measured in radians.

Trig Relationships in Related Rates

$$\frac{d}{dt}\sin\theta = \cos\theta\,\frac{d\theta}{dt}, \qquad \frac{d}{dt}\cos\theta = -\sin\theta\,\frac{d\theta}{dt}, \qquad \frac{d}{dt}\tan\theta = \sec^2\theta\,\frac{d\theta}{dt}.$$

Pick the trig ratio that links the angle to the quantity you know, using a side that stays constant whenever possible (for example, $\tan\theta = \frac{h}{300}$ when the horizontal distance is fixed at 300).

After differentiating, find the remaining sides at the moment in question with the Pythagorean theorem, then read off $\cos\theta$, $\sin\theta$ or $\sec^2\theta$ from the triangle.

Eliminate Before You Differentiate

Cones, shadows and troughs have more variables than you need. Use similar triangles to rewrite everything in terms of one variable first. For a cone with radius r and height h in a fixed ratio $r = kh$:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi k^2 h^3.$$

Then $\frac{dV}{dt} = \pi k^2 h^2 \frac{dh}{dt}$. The ratio k comes from the fixed dimensions of the container, never from the changing water level.

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EXAMPLE 1

An observer stands 300 m from the point on the ground directly below a balloon that rises vertically at 5 m/s. How fast is the angle of elevation θ of the balloon changing when the balloon is 400 m high?

SOLUTION

Let h be the height. Then $\tan\theta = \dfrac{h}{300}$ with $h' = 5$.

Differentiate: $\sec^2\theta\,\theta' = \dfrac{h'}{300}$.

At $h = 400$ the line of sight is $\sqrt{300^2 + 400^2} = 500$, so $\cos\theta = \dfrac{3}{5}$ and $\sec^2\theta = \dfrac{25}{9}$.

$\theta' = \dfrac{5}{300}\cdot\dfrac{9}{25} = \dfrac{3}{500}$ rad/s $= 0.006$ rad/s.

EXAMPLE 2

A 15 ft ladder leans against a wall. Its base slides away from the wall at 2 ft/s. When the base is 9 ft from the wall, find (a) how fast the top is moving down the wall and (b) how fast the angle θ between the ladder and the ground is changing.

SOLUTION

Let x be the base distance and y the height of the top: $x^2 + y^2 = 225$. At $x = 9$, $y = 12$.

(a) $2xx' + 2yy' = 0$, so $y' = -\dfrac{xx'}{y} = -\dfrac{9\cdot 2}{12} = -\dfrac{3}{2}$ ft/s. The top slides down at 1.5 ft/s.

(b) Use $\cos\theta = \dfrac{x}{15}$. Then $-\sin\theta\,\theta' = \dfrac{x'}{15}$. Here $\sin\theta = \dfrac{12}{15}$, so $-\dfrac{12}{15}\theta' = \dfrac{2}{15}$ and $\theta' = -\dfrac{1}{6}$ rad/s.

The negative sign means the angle is decreasing, as expected when the ladder falls toward the ground.

EXAMPLE 3

A person 1.8 m tall walks away from a lamp on a pole 6 m tall at 1.2 m/s. At what rate is (a) the tip of the shadow moving and (b) the length of the shadow changing?

SOLUTION

Let x be the person's distance from the pole and L the shadow length. The triangles formed by the lamp and by the person are similar: $\dfrac{6}{x + L} = \dfrac{1.8}{L}$.

Cross-multiply: $6L = 1.8x + 1.8L$, so $4.2L = 1.8x$ and $L = \dfrac{3}{7}x$.

(b) $L' = \dfrac{3}{7}(1.2) = \dfrac{18}{35} \approx 0.514$ m/s.

(a) The tip is at distance $x + L = \dfrac{10}{7}x$ from the pole, so it moves at $\dfrac{10}{7}(1.2) = \dfrac{12}{7} \approx 1.714$ m/s. Both rates are constant, independent of x.

EXAMPLE 4

Water drains from a conical tank (vertex down) of radius 4 m and height 10 m at 2 m³/min. How fast is the water level falling when the depth is 5 m?

SOLUTION

Similar triangles: $\dfrac{r}{h} = \dfrac{4}{10}$, so $r = 0.4h$.

$V = \dfrac{1}{3}\pi(0.4h)^2 h = \dfrac{4\pi}{75}h^3$, so $\dfrac{dV}{dt} = \dfrac{4\pi}{25}h^2\dfrac{dh}{dt}$.

Draining means $\dfrac{dV}{dt} = -2$. At $h = 5$: $-2 = \dfrac{4\pi}{25}(25)\dfrac{dh}{dt} = 4\pi\dfrac{dh}{dt}$.

$\dfrac{dh}{dt} = -\dfrac{1}{2\pi} \approx -0.159$ m/min. The level falls at about 0.159 m/min.

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EXAMPLE 5

A lighthouse is 2 km from a straight shore. Its beam rotates at 0.5 rad/min. How fast is the spot of light moving along the shore when it is 1.5 km from the point of the shore nearest the lighthouse?

SOLUTION

Let x be the spot's distance from the nearest point and θ the angle of the beam from the perpendicular to the shore: $\tan\theta = \dfrac{x}{2}$, so $x = 2\tan\theta$.

$x' = 2\sec^2\theta\,\theta'$. With $x = 1.5$, $\tan\theta = \dfrac{3}{4}$ and $\sec^2\theta = 1 + \dfrac{9}{16} = \dfrac{25}{16}$.

$x' = 2 \cdot \dfrac{25}{16} \cdot 0.5 = \dfrac{25}{16} = 1.5625$ km/min.

Common mistakes

(1) Working in degrees: if the problem gives degrees per second, convert to radians before using calculus. (2) Treating the hypotenuse as constant when it is not: a ladder has a fixed length, a line of sight does not. (3) Dropping a sign: decide in advance whether each variable is increasing (positive rate) or decreasing (negative rate).

Practice

SET A Skills

Find the requested rate. Angles are in radians. [No calculator]

1. An observer is 12 m from the base of a vertical pole. A climber rises along the pole at 3 m/s. If θ is the angle of elevation of the climber from the observer, find $\dfrac{d\theta}{dt}$ when the climber is 16 m high.
2. A 10 ft ladder slides down a wall. Let θ be the angle between the ladder and the ground. When the top is 6 ft high it is moving down at 1 ft/s. Find $\dfrac{d\theta}{dt}$ at that moment.
3. A point moves along the x-axis at 4 units/s. An observer at $(0, 3)$ watches it, and θ is the angle between the line of sight and the y-axis. Find $\dfrac{d\theta}{dt}$ when the point is at $x = 4$.
4. Sand falls at 12π m^3/min and forms a conical pile whose height always equals its radius. How fast is the height rising when the pile is 6 m high?
5. A person 2 m tall walks away from a lamp 5 m tall at 1.5 m/s. Find the speed of the tip of the shadow and the rate at which the shadow is lengthening.
6. A 25 ft ladder leans on a wall. The top slides down at 2 ft/s. How fast is the base moving away from the wall when the top is 24 ft high?
7. A beam of light from a tower 4 m from a long straight wall rotates at 2 rad/s. How fast is the spot of light moving along the wall when it is 3 m from the point of the wall nearest the tower?
8. An angle θ in the first quadrant satisfies $\sin\theta = \frac{3}{5}$ at the instant $\dfrac{d\theta}{dt} = 0.2$ rad/s. Find at that instant
   (a) $\dfrac{d}{dt}(\cos\theta)$,
   (b) $\dfrac{d}{dt}(\tan\theta)$.
9. Two sides of a triangle are each 6 cm long and the angle θ between them increases at 0.1 rad/s. Find the rate of change of the area when $\theta = \dfrac{\pi}{3}$.

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SET B Applications and Mixed

Round decimal answers to three places. [Calculator]

10. A camera 800 m from a launch pad tracks a rocket rising straight up. When the rocket is 600 m high it rises at 120 m/s. Find the rate of change of the camera's angle of elevation
    (a) in radians per second,
    (b) in degrees per second, to three decimal places.
11. A plane flies horizontally at 3000 m altitude and 240 m/s directly toward a radar station on the ground. Find the rate of change of the angle of elevation of the plane, as seen from the station, when the horizontal distance to the station is 4000 m. Round to three decimal places and give the sign.
12. A conical tank, vertex down, has radius 3 m and height 9 m. Water is pumped in at 1.5 m^3/min. How fast is the water level rising when the depth is 4 m? Round to three decimal places.
13. A lamp is 5.5 m above the ground. A person 1.7 m tall walks toward the pole at 1.4 m/s. Find, to three decimal places,
    (a) the rate of change of the length of the shadow,
    (b) the rate of change of the distance from the pole to the tip of the shadow.
14. A 6.5 m ladder leans against a wall and its base is pulled away at 0.8 m/s. When the top is 5.2 m above the ground, find (a) the rate of change of the height of the top and (b) the rate of change of the angle between the ladder and the ground. Round to three decimal places.
15. Water is poured into a conical cup (vertex down) whose radius is always 0.75 times the water depth h. The table gives the depth in cm and its rate of change at several times (seconds). Find $\frac{dV}{dt}$ at $t = 3$, to three decimal places.

t	1	3	5
$h(t)$	4	8	10
$h'(t)$	0.5	0.25	0.2

16. A lighthouse is 1.5 km from a straight shore and its beam turns at 0.8 rad/min. How fast does the spot move along the shore when it is 2 km from the nearest point? Round to three decimal places.

SET C Challenge

Show your setup and reasoning. [No calculator]

17. Grain pours at 8 m^3/min onto a conical pile whose height is always twice its base radius. When the pile is 4 m high, find (a) the rate at which the radius is increasing and (b) the rate at which the area of the circular base is increasing.
18. A ladder of length 10 m slides down a wall with its base on level ground. At what height of the top is the speed of the top equal to the speed of the base? Then explain what happens to the speed of the top (relative to the speed of the base) as the top approaches the ground, and why this makes the model unrealistic near the end.
19. Two straight roads meet at an angle of 60°. Car A is 3 km from the junction driving away at 50 km/h on one road; car B is 4 km from the junction driving away at 40 km/h on the other. Use the law of cosines to find the rate at which the distance between the cars is changing.
20. A vertical pole 12 m tall casts a shadow whose length s increases at 0.5 m/min as the sun lowers. Let θ be the sun's angle of elevation. Find $\frac{d\theta}{dt}$ when $s = 16$ m.
21. An inverted conical tank has radius 6 m and depth 12 m. Water drains at a rate numerically equal to 0.25 times the area of the water surface (that is, $\frac{dV}{dt} = -0.25\,A$, with A in m^2 and V in m^3/min).
    (a) Show that the water level falls at a constant rate and find it.
    (b) How long does it take to drain a full tank?

2.14 Higher-Order Derivatives and Mixed Differentiation

Learn to take second and third derivatives, connect them to acceleration and concavity, and practice choosing the right rule in mixed problems.

The derivative of a derivative is called the **second derivative**. You can keep going: third, fourth, and so on. Higher derivatives are not just an exercise in repeating a rule. The second derivative tells you how fast the slope is changing, which means concavity for a graph and acceleration for a moving object.

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Higher-Order Derivatives

$$f''(x) = \frac{d}{dx}\big(f'(x)\big) = \frac{d^2y}{dx^2}, \qquad f'''(x) = \frac{d}{dx}\big(f''(x)\big), \qquad f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

Be careful with notation: $f''(x)$ is the second derivative, while $(f'(x))^2$ is the square of the first derivative. They are different functions.

Useful patterns: a polynomial of degree n has $f^{(n+1)}(x) = 0$; $\dfrac{d^n}{dx^n}e^{kx} = k^n e^{kx}$; derivatives of $\sin x$ repeat in a cycle of length 4: $\cos x,\ -\sin x,\ -\cos x,\ \sin x$.

Acceleration and Speed

For a particle on a line with position $s(t)$: velocity $v(t) = s'(t)$, acceleration $a(t) = v'(t) = s''(t)$, speed $|v(t)|$.

The particle is **speeding up** when v and a have the SAME sign, and **slowing down** when they have OPPOSITE signs. (If $a > 0$ but $v < 0$, the particle moves left and is slowing down.)

EXAMPLE 1

Let $f(x) = x^4 - 3x^2 + 2x$. Find $f'(x)$, $f''(x)$, $f'''(x)$ and $f''(1)$.

SOLUTION

$f'(x) = 4x^3 - 6x + 2.$

$f''(x) = 12x^2 - 6.$

$f'''(x) = 24x.$

$f''(1) = 12 - 6 = 6.$

EXAMPLE 2

Find $\dfrac{d^2y}{dx^2}$ for $y = xe^{2x}$.

SOLUTION

Product rule: $y' = e^{2x} + x \cdot 2e^{2x} = e^{2x}(1 + 2x)$.

Product rule again: $y'' = 2e^{2x}(1 + 2x) + e^{2x} \cdot 2 = e^{2x}(2 + 4x + 2)$.

$y'' = 4(x + 1)e^{2x}$.

EXAMPLE 3

A particle moves on a line with position $s(t) = t^3 - 9t^2 + 24t$ for $t \geq 0$ (meters, seconds). Find $v(t)$, $a(t)$ and the time intervals on which the particle is speeding up.

SOLUTION

$v(t) = 3t^2 - 18t + 24 = 3(t - 2)(t - 4)$ and $a(t) = 6t - 18 = 6(t - 3)$.

Signs: $v > 0$ on $(0, 2)$, $v < 0$ on $(2, 4)$, $v > 0$ for $t > 4$. $a < 0$ for $t < 3$ and $a > 0$ for $t > 3$.

Same sign: $(2, 3)$ with both negative, and $t > 4$ with both positive. So the particle speeds up on $(2, 3)$ and $(4, \infty)$.

Justification: on $(2, 3)$, $v(t) < 0$ and $a(t) < 0$, so the speed is increasing. On $(0, 2)$ and $(3, 4)$ the signs are opposite and it is slowing down.

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EXAMPLE 4

Find $\frac{d^2y}{dx^2}$ for the curve $x^2 + y^2 = 25$ and evaluate it at $(3, 4)$.

SOLUTION

First derivative: $2x + 2yy' = 0$, so $y' = -\frac{x}{y}$.

Differentiate again with the quotient rule, remembering y': $y'' = -\frac{y - xy'}{y^2}$.

Substitute $y' = -\frac{x}{y}$: $y'' = -\frac{y + \frac{x^2}{y}}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{25}{y^3}$.

At $(3, 4)$: $y'' = -\frac{25}{64}$. The circle is concave down there, as expected for the upper half.

EXAMPLE 5

Show that $h(x) = e^x \cos x$ satisfies $h'' - 2h' + 2h = 0$.

SOLUTION

$h' = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x)$.

$h'' = e^x(\cos x - \sin x) + e^x(-\sin x - \cos x) = -2e^x \sin x$.

$h'' - 2h' + 2h = -2e^x \sin x - 2e^x(\cos x - \sin x) + 2e^x \cos x = -2e^x \sin x - 2e^x \cos x + 2e^x \sin x + 2e^x \cos x = 0$.

Which Rule? A Mixed-Differentiation Checklist

Look at the OUTERMOST operation first. A sum or difference: differentiate term by term. A product: product rule. A quotient: quotient rule. A function inside another function: chain rule (outer first, then multiply by the derivative of the inner).

Rewrite when it helps: $\frac{1}{x^3} = x^{-3}$, $\sqrt[3]{x} = x^{1/3}$. Simplify f' before differentiating again, since factoring out e^{kx} makes the second derivative far easier.

Common mistakes

(1) Writing $y'' = (y')^2$ or $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$. (2) Forgetting the chain rule factor again on the second derivative, as in $\frac{d^2}{dx^2}\sin(3x) = -9\sin(3x)$, not $-\sin(3x)$. (3) In implicit second derivatives, leaving y' in the answer instead of substituting it. (4) Deciding speeding up or slowing down from the sign of a alone, instead of comparing the signs of v and a.

Practice

SET A Skills

Find the requested derivative or value. [No calculator]

1. Find $f''(x)$ for $f(x) = x^5 - 4x^3 + x$.
2. Find y''' for $y = 3x^4 - 2x^2 + 7$.
3. Find $\frac{d^2y}{dx^2}$ for $y = \sin(3x)$.
4. Find $f''''(x)$ for $f(x) = e^{-2x}$.
5. For $f(x) = \ln x$, find
   (a) $f''(x)$,
   (b) $f''''(x)$.
6. Find y'' for $y = x^2 e^x$.
7. Find $f''(4)$ for $f(x) = \sqrt{x}$.
8. A particle has position $s(t) = t^3 - 6t^2 + 5$. Find its velocity and acceleration at $t = 3$.
9. Find $\frac{d^2y}{dx^2}$ in terms of x for the curve $xy = 1$ (use implicit differentiation).

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SET B Applications and Mixed

Round decimal answers to three places. [Calculator]

10. A particle's position is $s(t) = t^2 e^{-t}$ for $t \geq 0$. Find $v(1.5)$ and $a(1.5)$ to three decimal places, and decide whether the particle is speeding up or slowing down at $t = 1.5$.
11. A mass on a spring has position $x(t) = 0.4\cos(3t)$ meters. Find the acceleration at $t = 0.5$ seconds to three decimal places, and show that $x'' = -9x$.
12. The table gives the velocity $v(t)$ (m/s) of a particle at selected times. Use the nearest values to approximate $a(3)$ and $a(7)$, and find the average acceleration on $[0, 8]$. Round to three decimal places.

t	0	2	4	6	8
$v(t)$	3	7	8	5	1

13. The concentration of a drug is $C(t) = 50e^{-0.3t}$ mg/L after t hours. Find $C'(2)$ and $C''(2)$ to three decimal places and interpret both in context.
14. A particle has velocity $v(t) = t\cos t$ for $0 \leq t \leq 3$. Find the time intervals (endpoints to three decimal places) on which the particle is speeding up.
15. The curve $x^2 + xy + y^2 = 7$ passes through $(2, 1)$. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at that point. Give the second derivative as a decimal to three places.
16. Let $f(x) = xe^{-x}$. Find $f''(x)$, the x-coordinate where the graph changes concavity, and the y-coordinate of that point to three decimal places.

SET C Challenge

Show your reasoning. [No calculator]

17. Let $f(x) = x^2 \ln x$ for $x > 0$. Find $f'(x)$, $f''(x)$ and $f'''\left(\dfrac{1}{2}\right)$.
18. Let $y = e^{-x}\sin(2x)$. Show that $y'' + 2y' + 5y = 0$.
19. Find $\dfrac{d^2y}{dx^2}$ at $(1, 2)$ on the curve $x^3 + y^3 = 9$.
20. Let $f(x) = \begin{cases} x^3, & x \leq 1 \\ ax^2 + bx + c, & x > 1 \end{cases}$.
    (a) Find a, b, c so that f, f' and f'' are all continuous at $x = 1$.
    (b) With these values, does $f'''(1)$ exist? Explain.
21. A particle moves on a line with velocity $v(t) = 4t - t^2$ for $0 \leq t \leq 6$. Find $a(t)$ and the intervals where the particle is speeding up and slowing down. At what time does the particle change direction?

2.15 Multiple-Choice Practice on Derivatives

Thirty AP-style multiple-choice questions covering every topic in the chapter: rules, implicit and inverse derivatives, related rates and higher-order derivatives.

Work Part A without a calculator and Part B with one. Calculator answers are rounded to three decimal places. Check your reasoning against the explanations at the end of the chapter.

SET A Part A [No calculator]

Choose the best answer for each question. [No calculator]

1. What is $\lim\limits_{h \to 0} \dfrac{(2+h)^3 - 8}{h}$?
   (A) 6 **(B)** 8 **(C)** 12
   (D) 24 **(E)** The limit does not exist
2. If $f(x) = (3x^2 + 1)^4$, then $f'(1) =$
   (A) 1536 **(B)** 256 **(C)** 384 **(D)** 768 **(E)** 1024
3. If $y = \dfrac{x}{x^2 + 4}$, then $\dfrac{dy}{dx}$ at $x = 1$ is
   (A) $\frac{1}{5}$ **(B)** $-\frac{3}{25}$ **(C)** $\frac{1}{25}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{3}{25}$

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4. If $f(x) = \tan(2x)$, then $f'\left(\frac{\pi}{8}\right) =$

(A) 1 **(B)** 4 **(C)** 2 **(D)** 8 **(E)** $2\sqrt{2}$

5. The table gives values of differentiable functions f and g. If $h(x) = f(g(x))$, what is $h'(2)$?

x	1	2	3
$f(x)$	5	2	6
$f'(x)$	2	7	4
$g(x)$	2	3	1
$g'(x)$	2	-1	5

(A) -7 **(B)** -1 **(C)** 3 **(D)** -4 **(E)** 4

6. For the curve $x^2 + xy = 6$, $\frac{dy}{dx}$ at the point $(2, 1)$ is

(A) $\frac{5}{2}$ **(B)** $-\frac{5}{2}$ **(C)** -2 **(D)** $-\frac{3}{2}$ **(E)** -4

7. If $f(x) = \ln(x^2 + 1)$, then $f'(1) =$

(A) 1 **(B)** $\frac{1}{2}$ **(C)** 2 **(D)** $\ln 2$ **(E)** 0

8. If $f(x) = e^{2x}\cos x$, then $f'(0) =$

(A) 0 **(B)** 1 **(C)** 3 **(D)** 2 **(E)** -2

9. Let $f(x) = x^3 + 2x + 1$ and let g be the inverse of f. What is $g'(4)$?

(A) 5 **(B)** $\frac{1}{50}$ **(C)** $\frac{1}{5}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{3}$

10. $\frac{d}{dx}\left[\arctan(x^2)\right]$ at $x = 1$ equals

(A) $\frac{1}{2}$ **(B)** 2 **(C)** $\frac{\pi}{4}$ **(D)** $\frac{\pi}{2}$ **(E)** 1

11. The side of a square increases at 3 cm/s. At what rate is the length of its diagonal increasing?

(A) 3 cm/s **(B)** $3\sqrt{2}$ cm/s **(C)** 6 cm/s **(D)** $6\sqrt{2}$ cm/s **(E)** $\frac{3}{\sqrt{2}}$ cm/s

12. A spherical balloon is inflated so that its volume increases at 32π cm^3/s. How fast is the radius increasing when $r = 2$ cm?

(A) 1 cm/s **(B)** 4 cm/s **(C)** 2 cm/s **(D)** 8 cm/s **(E)** $\frac{1}{2}$ cm/s

13. If $f(x) = \sin(2x)$, then $f''\left(\frac{\pi}{4}\right) =$

(A) -4 **(B)** -2 **(C)** 0 **(D)** 2 **(E)** 4

14. A particle moves along a line with position $s(t) = t^3 - 6t^2 + 9t$ for $t \geq 0$. On which interval(s) is the particle speeding up?

(A) $(0, 1)$ only **(B)** $(1, 2)$ only **(C)** $(2, 3)$ only
(D) $(1, 2)$ and $(3, \infty)$ **(E)** $(0, 1)$ and $(2, 3)$

15. Let $f(x) = \begin{cases} x^2, & x \leq 1 \\ ax + b, & x > 1 \end{cases}$. If f is differentiable at $x = 1$, what is $a + b$?

(A) 0 **(B)** 2 **(C)** 3 **(D)** -1 **(E)** 1

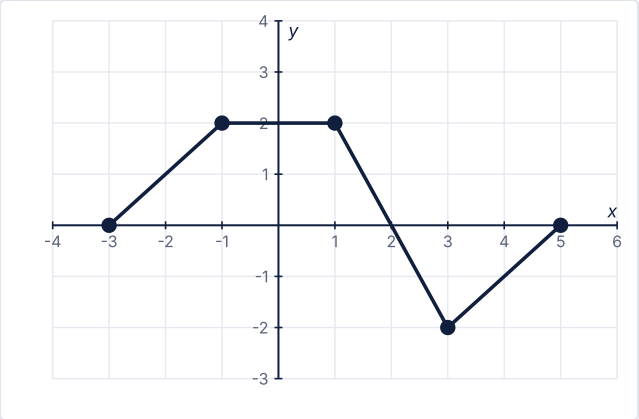
16. An equation of the line tangent to $y = x\ln x$ at $x = e$ is

(A) $y = 2x - e$ **(B)** $y = 2x + e$ **(C)** $y = x$ **(D)** $y = x + e$ **(E)** $y = 2x - 2e$

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17. The graph of f on $[-3, 5]$ consists of line segments joining the points shown. For how many values of x in $(-3, 5)$ is f not differentiable?



Graph of f.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

18. If $f'(x) = x^2 e^{-x}$, then $f''(1) =$

(A) 0 **(B)** e **(C)** $\frac{1}{e}$ **(D)** 1 **(E)** $\frac{2}{e}$

19. $\frac{d}{dx}\left[\arcsin\left(\frac{x}{2}\right)\right]$ at $x = 1$ equals

(A) $\frac{1}{2}$ **(B)** $\frac{2}{\sqrt{3}}$ **(C)** $\sqrt{3}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{\sqrt{3}}$

20. $\lim_{h \to 0} \frac{\sin\left(\frac{\pi}{6} + h\right) - \frac{1}{2}}{h} =$

(A) $\frac{1}{2}$ **(B)** $\frac{\sqrt{3}}{2}$ **(C)** $-\frac{1}{2}$ **(D)** 1 **(E)** 0

SET B Part B [Calculator]

Choose the best answer for each question. [Calculator]

21. If $f(x) = e^{-x^2}$, then $f'(0.5) =$

(A) -0.389 **(B)** 0.779 **(C)** -0.607 **(D)** -0.779 **(E)** -0.882

22. If $f(x) = \ln(3 + \sin x)$, then $f'(2) \approx$

(A) -0.106 **(B)** 0.106 **(C)** -0.416 **(D)** -0.139 **(E)** 0.139

23. Water is poured at 0.8 m^3/min into a conical tank, vertex down, with radius 2.5 m and height 10 m. How fast is the water level rising when the depth is 5 m?

(A) 0.653 m/min **(B)** 0.326 m/min **(C)** 0.082 m/min
(D) 0.041 m/min **(E)** 0.163 m/min

24. A camera 500 m from a launch pad follows a rocket rising vertically at 60 m/s. How fast is the camera's angle of elevation changing (in rad/s) when the rocket is 1200 m high?

(A) 0.102 **(B)** 0.120 **(C)** 0.018 **(D)** 0.050 **(E)** 0.008

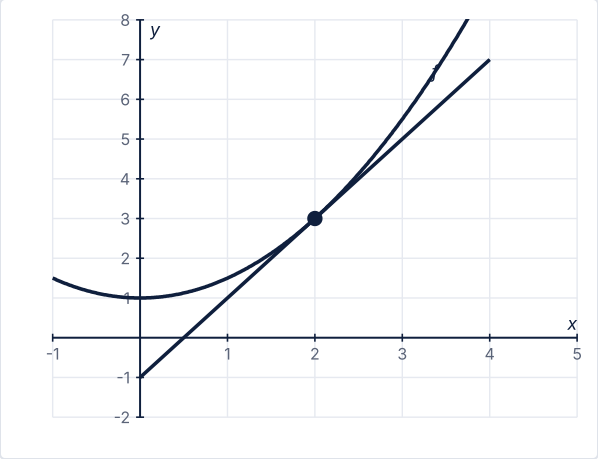
25. Let $f(x) = x^5 + x - 3$, which is one-to-one, and let g be its inverse. What is $g'(0)$?

(A) 0.167 **(B)** 0.108 **(C)** 9.239 **(D)** 0.500 **(E)** 0.003

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26. The graph of f is shown together with its tangent line at $x = 2$, which passes through $(0, -1)$ and $(4, 7)$. If $g(x) = x\,f(x)$, then $g'(2) =$



Graph of f and its tangent line at x = 2.

(A) 6 **(B)** 5 **(C)** 4 **(D)** 12 **(E)** 7

27. The curve $x \sin y = 1$ passes through $\left(2, \frac{\pi}{6}\right)$. The slope of the curve at that point is approximately

(A) 0.289 **(B)** -0.577 **(C)** -0.289 **(D)** -0.500 **(E)** 0.250

28. The table gives the radius r (cm) of a circular ripple and its rate of change. At $t = 4$, how fast is the area of the ripple increasing?

t (s)	2	4	6
$r(t)$ (cm)	6	10	13
$r'(t)$ (cm/s)	1.2	0.8	0.5

(A) 50.265 cm^2/s **(B)** 25.133 cm^2/s **(C)** 100.531 cm^2/s
(D) 502.655 cm^2/s **(E)** 16.000 cm^2/s

29. If $f(x) = \sqrt{x}\cos x$, then $f''(1) \approx$

(A) -0.571 **(B)** 1.517 **(C)** -0.842 **(D)** -1.517 **(E)** -2.088

30. $\frac{d}{dx}\left[x \arcsin x\right]$ at $x = 0.6$ is approximately

(A) 0.643 **(B)** 1.394 **(C)** 0.750 **(D)** 1.286 **(E)** 2.037

2.16 Free-Response Practice on Derivatives

Four multi-part free-response questions on derivatives, with a table, a graph, a related rates setting and an implicit curve. Write full justifications.

Each question is worth 9 points. Justify every conclusion with a theorem or computation. Answers include a point breakdown so you can score yourself.

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QUESTION 1 Temperature of a Mixture

A calculator is allowed. Show your work.

1. The temperature $T(t)$, in degrees Celsius, of a chemical mixture is a differentiable function of time t in minutes. The temperature is measured at selected times, as shown in the table.

   (a) Use the data in the table to approximate $T'(7)$. Show the computation that leads to your answer, and use correct units to interpret the meaning of $T'(7)$ in the context of the problem.

   (b) Is there a time t with $0 < t < 12$ at which $T'(t) = \frac{23}{12}$? Justify your answer.

   (c) The temperature is modeled by $W(t) = 45 - 25e^{-0.18t}$. Find $W'(7)$, and compare it with your approximation in part (a).

   (d) Based on the model W, find the time t at which the temperature is rising at 1 degree Celsius per minute, and determine whether the rate of temperature increase is itself increasing or decreasing. Justify your answer. [Calculator allowed]

t (min)	0	2	5	9	12
$T(t)$ (degrees C)	20	26	35	41	43

QUESTION 2 Graph of a Piecewise Linear Function

No calculator. Show your work.

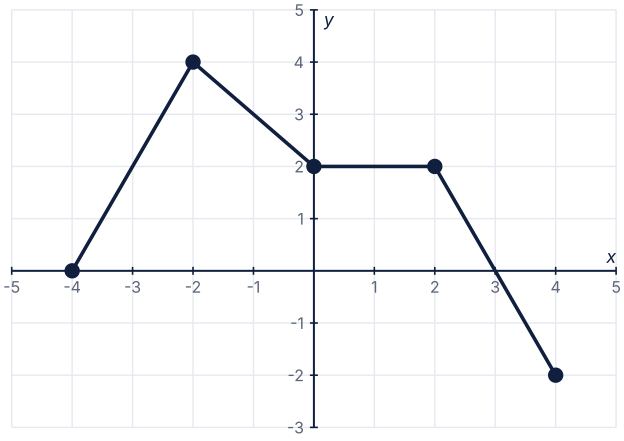
2. The graph of the continuous function f on $[-4, 4]$ consists of four line segments, as shown. Let $h(x) = x\,f(x)$ and $p(x) = f(2x - 1)$.

   (a) Find $f'(-3)$ and $f'(1)$. List every value of x in $(-4, 4)$ at which f is not differentiable, and give a reason.

   (b) Find $h'(-1)$.

   (c) Find $p'(-1)$.

   (d) On the interval $[-4, -2]$ the function f is one-to-one. Let m be the inverse of f on that interval. Find $m'(2)$. [No calculator]



Graph of f on [-4, 4].

QUESTION 3 Draining Filter

A calculator is allowed. Show your work.

3. A conical filter (vertex down) has radius 6 cm and height 12 cm. Liquid drains out of the vertex so that the volume V of liquid, in cm ^{3}, decreases according to $\frac{dV}{dt} = -0.8\sqrt{h}$, where h is the depth of the liquid in cm and t is in seconds.

   (a) Show that $V = \frac{\pi}{12}h^3$ for the liquid in the filter.

   (b) Find $\frac{dh}{dt}$ when $h = 9$. Give units. Round to three decimal places.

   (c) At the instant $h = 9$, find the rate of change of the area of the circular surface of the liquid. Round to three decimal places and say whether it is increasing or decreasing.

   (d) Find $\frac{dh}{dt}$ as a function of h. Explain what happens to the speed at which the level falls as the filter nearly empties. [Calculator allowed]

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QUESTION 4 **Implicit Curve**

No calculator. Show your work.

4. Consider the curve $x^2 + xy + y^2 = 7$.

   (a) Show that $\dfrac{dy}{dx} = -\dfrac{2x + y}{x + 2y}$.

   (b) Find an equation of the tangent line to the curve at the point $(2, 1)$.

   (c) Find the coordinates of all points on the curve at which the tangent line is horizontal.

   (d) Find $\dfrac{d^2y}{dx^2}$ at the point $(2, 1)$. [No calculator]

Answers Chapter 2: Answers and Explanations

Answers for every practice problem, numbered to match. Read the explanation even when your answer is correct.

2.1 Exponent Rules and the Power Rule

SET A · SKILLS

1. $f'(x) = 28x^3 - 6x + 5$. Each term: $7 \cdot 4 = 28$, $3 \cdot 2 = 6$, $5 \cdot 1 = 5$, and the constant vanishes.
2. Rewrite $y = 4x^{-3}$. Then $y' = -12x^{-4} = -\dfrac{12}{x^4}$.
3. $g = 5x^{1/2}$, so $g'(x) = \dfrac{5}{2}x^{-1/2} = \dfrac{5}{2\sqrt{x}}$.
4. $y = x^{2/3}$, so $y' = \dfrac{2}{3}x^{-1/3} = \dfrac{2}{3\sqrt[3]{x}}$.
5. Split: $h(x) = x - 4x^{-1}$. Then $h'(x) = 1 + 4x^{-2} = 1 + \dfrac{4}{x^2}$.
6. $y = \frac{3}{2}x^{-1/2}$, so $y' = \frac{3}{2} \cdot \left(-\frac{1}{2}\right) x^{-3/2} = -\dfrac{3}{4x^{3/2}} = -\dfrac{3}{4x\sqrt{x}}$.
7. $f(x) = 4x^2 + 4x + 1$, so $f'(x) = 8x + 4$.
8. $y = 3x^3 - 5x^2$, so $y' = 9x^2 - 10x$.
9. π and π^2 are constants. $f'(t) = 3\pi t^2$ (the constant π^2 has derivative 0).

SET B · APPLICATIONS AND MIXED PROBLEMS

10. $v(t) = 3t^2 - 12t + 9$. $v(2) = 12 - 24 + 9 = -3$ m/s. At rest when $3(t - 1)(t - 3) = 0$, so $t = 1$ s and $t = 3$ s.
11. $A'(r) = 2\pi r$. $A'(5) = 10\pi \approx 31.416$ square centimeters per centimeter: the area grows by about 31.416 cm^2 for each extra centimeter of radius near $r = 5$.
12. $V'(r) = 4\pi r^2$, which equals the surface area. $V'(3) = 36\pi \approx 113.097$.
13. $y' = 3x^2 - 3 = 9$ gives $x^2 = 4$, so $x = \pm 2$. Points: $(2, 2)$ and $(-2, -2)$.
14. $C'(q) = 4 + 0.02q$. $C'(120) = 4 + 2.4 = 6.4$. At a production level of 120 units, producing one more unit costs about 6.40 dollars.
15. $h'(t) = 80 - 10t$, so $h'(3) = 50$ m/s (rising). $h'(t) = 0$ at $t = 8$ s: the moment the projectile reaches its greatest height, where it is momentarily not rising or falling.
16. $f(1) = a + b = 4$ and $f'(x) = 2ax + b$ so $f'(1) = 2a + b = 7$. Subtract: $a = 3$, then $b = 1$.

SET C · CHALLENGE AND AP-STYLE

17. $p'(x) = 3x^2 + 2ax + b$ must be zero at $x = 1$ and $x = 3$, so $p'(x) = 3(x - 1)(x - 3) = 3x^2 - 12x + 9$. Match coefficients: $2a = -12$, so $a = -6$; $b = 9$.
18. (a) $f(x) = x + 4x^{-1}$, $f'(x) = 1 - 4x^{-2} = 1 - \dfrac{4}{x^2} = 0$ gives $x^2 = 4$, so $x = \pm 2$.

    (b) $f'(x) > 0$ when $\dfrac{4}{x^2} < 1$, that is $x^2 > 4$, so $x < -2$ or $x > 2$. (At $x = 0$, f is undefined.)
19. The student differentiated only the denominator, which is not allowed. Rewrite $\frac{1}{3x^2} = \frac{1}{3}x^{-2}$. The correct derivative is $\frac{1}{3}(-2)x^{-3} = -\dfrac{2}{3x^3}$.
20. $f'(x) = \frac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$, defined for $x \neq 0$. As $x \to 0$, $f'(x) \to +\infty$, so the graph of f is continuous at 0 but has a vertical tangent line there (slope grows without bound). f is not differentiable at $x = 0$.
21. (a) The graph turns around near $x = -1$ and $x = 1$, so horizontal tangents there.

    (b) $f'(x) = x^2 - 1 = (x - 1)(x + 1) = 0$ gives $x = \pm 1$, matching the graph.

    (c) $f'(2) = 4 - 1 = 3$.

2.2 The Derivative by Definition

SET A · SKILLS

1. $\dfrac{5(x + h) - 2 - (5x - 2)}{h} = \dfrac{5h}{h} = 5$. So $f'(x) = 5$ (the slope of a line).
2. $\dfrac{(x + h)^2 + 4(x + h) - x^2 - 4x}{h} = \dfrac{2xh + h^2 + 4h}{h} = 2x + h + 4 \to 2x + 4$. $f'(x) = 2x + 4$.
3. $\dfrac{3(2xh + h^2) - h}{h} = 6x + 3h - 1 \to 6x - 1$. $f'(x) = 6x - 1$.
4. $\dfrac{(x + h)^3 - x^3}{h} = \dfrac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + 3xh + h^2 \to 3x^2$. $f'(x) = 3x^2$.
5. $\dfrac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h} = \dfrac{(x + 2) - (x + h + 2)}{h(x + h + 2)(x + 2)} = \dfrac{-1}{(x + h + 2)(x + 2)} \to -\dfrac{1}{(x + 2)^2}$.
6. Multiply by the conjugate: $\dfrac{(x + h + 3) - (x + 3)}{h\left(\sqrt{x + h + 3} + \sqrt{x + 3}\right)} = \dfrac{1}{\sqrt{x + h + 3} + \sqrt{x + 3}} \to \dfrac{1}{2\sqrt{x + 3}}$.

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7. $\dfrac{x^2-2x-3}{x-3} = \dfrac{(x-3)(x+1)}{x-3} = x+1 \to 4$. $f'(3) = 4$.
8. $\dfrac{\frac{2}{x}-2}{x-1} = \dfrac{2(1-x)}{x(x-1)} = -\dfrac{2}{x} \to -2$. $f'(1) = -2$.
9. This is $f'(2)$ for $f(x) = x^5$, since $f(2) = 32$. By the power rule, $f'(2) = 5 \cdot 2^4 = 80$.

SET B · APPLICATIONS AND MIXED PROBLEMS

10. $\dfrac{(t+h)^2+3(t+h)-t^2-3t}{h} = 2t+h+3 \to 2t+3$. $s'(t) = 2t+3$ and $s'(2) = 7$ meters per second.
11. $T'(5) \approx \dfrac{T(6)-T(4)}{6-4} = \dfrac{70.8-61.2}{2} = 4.8$. At $t = 5$ minutes the temperature is rising at about 4.8 degrees Celsius per minute.
12. For $h = 0.001$: $\frac{2^{1.001}-2}{0.001} \approx 1.387$. For $h = -0.001$: $\frac{2^{0.999}-2}{-0.001} \approx 1.386$. The two agree to about 1.39, so $f'(1) \approx 1.39$.
13. $\dfrac{\pi(r+h)^2-\pi r^2}{h} = \pi(2r+h) \to 2\pi r$. $A'(4) = 8\pi \approx 25.133$ square centimeters per centimeter: near $r = 4$ the area grows by about 25.133 cm^2 per extra centimeter of radius.
14. $C(51) = 0.5(2601) + 510 = 1810.5$ and $C(50) = 1250 + 500 = 1750$, so $C(51) - C(50) = 60.5$. For the exact value, $C'(q) = q + 10$, so $C'(50) = 60$. The cost of the 51st unit (60.50 dollars) is close to the marginal cost 60 dollars per unit.
15. (a) $f'(1) > 0$ (rising), $f'(2) = 0$ (top of the parabola), $f'(3.5) < 0$ (falling).
    (b) $\frac{-(x+h)^2+4(x+h)+x^2-4x}{h} = -2x-h+4 \to -2x+4$. So $f'(x) = 4-2x$: $f'(1) = 2$, $f'(2) = 0$, $f'(3.5) = -3$.
16. Right: $h > 0$, $\frac{|h|}{h} = 1$. Left: $h < 0$, $\frac{|h|}{h} = -1$. The difference quotient is $\frac{|2+h-2|-0}{h} = \frac{|h|}{h}$, so the one-sided limits are 1 and -1 and $f'(2)$ does not exist (a corner).

SET C · CHALLENGE AND AP-STYLE

17. $f'(0) = \lim_{h\to 0} \dfrac{h|h|-0}{h} = \lim_{h\to 0} |h| = 0$. Yes, f is differentiable at 0 and $f'(0) = 0$.
18. Continuity at 1 requires $a + b = 1^2 = 1$. The left slope is $2x = 2$ at $x = 1$, so the right slope must be $a = 2$. Then $b = -1$. (Both pieces then have value 1 and slope 2 at $x = 1$.)
19. This is $f'(8)$ for $f(x) = x^{1/3}$, since $f(8) = 2$. $f'(x) = \frac{1}{3}x^{-2/3}$, so $f'(8) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$.
20. $\dfrac{f(x)-7}{x^2-9} = \dfrac{f(x)-f(3)}{x-3} \cdot \dfrac{1}{x+3} \to f'(3) \cdot \dfrac{1}{6} = \dfrac{4}{6} = \dfrac{2}{3}$.
21. (a) $g'(2) \approx \dfrac{g(2.5)-g(1.5)}{1} = \dfrac{6.1-4.9}{1} = 1.2$.
    (b) The slopes are $\frac{0.9}{0.5} = 1.8$, $\frac{0.7}{0.5} = 1.4$, $\frac{0.5}{0.5} = 1.0$, $\frac{0.3}{0.5} = 0.6$. They decrease, so g' is positive but decreasing: g is increasing at a slowing rate.

2.3 Tangent and Normal Lines

SET A · SKILLS

1. $f(1) = 4$, $f'(x) = 2x + 3$ so $f'(1) = 5$. $y - 4 = 5(x-1)$, so $y = 5x - 1$.
2. $f(9) = 3$, $f'(x) = \frac{1}{2\sqrt{x}}$ so $f'(9) = \frac{1}{6}$. $y - 3 = \frac{1}{6}(x-9)$, so $y = \frac{1}{6}x + \frac{3}{2}$.
3. $y = 4x^{-1}$, $y' = -\frac{4}{x^2}$. At $x = 2$: point $(2, 2)$, slope -1. $y - 2 = -(x-2)$, so $y = -x + 4$.
4. $f(2) = 6$, $f'(x) = 3x^2 - 1$, so $f'(2) = 11$. Normal slope $-\frac{1}{11}$: $y - 6 = -\frac{1}{11}(x-2)$.
5. $y' = 4x - 4$, so the tangent slope at $x = 3$ is 8. Normal slope $-\frac{1}{8}$: $y - 7 = -\frac{1}{8}(x-3)$.
6. $y' = 3x^2 - 12 = 3(x-2)(x+2) = 0$ gives $x = \pm 2$. $y(2) = 8 - 24 = -16$, $y(-2) = 16$. Points: $(2, -16)$ and $(-2, 16)$.
7. The slope of $y = 4x + 1$ is 4. $y' = 2x - 6 = 4$ gives $x = 5$, and $y(5) = 25 - 30 + 5 = 0$. The point is $(5, 0)$.
8. $y' = \frac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$, which is undefined at $x = 0$ and tends to $+\infty$ there, while y is continuous. The vertical tangent is at $(0, 0)$ and its equation is $x = 0$.
9. $y(-1) = -3$, $y' = 3x^2 + 2$ so $y'(-1) = 5$. $y + 3 = 5(x+1)$, so $y = 5x + 2$.

SET B · APPLICATIONS AND MIXED PROBLEMS

10. $s(4) = 16$, $s'(t) = 3\sqrt{t}$, $s'(4) = 6$. Tangent: $y = 16 + 6(t-4)$. $s(4.2) \approx 16 + 6(0.2) = 17.2$. True value: $s(4.2) = 2(4.2)^{3/2} \approx 17.215$.
11. Take $a = 49$: $f(49) = 7$, $f'(49) = \frac{1}{14}$. $\sqrt{50} \approx 7 + \frac{1}{14} \approx 7.071$ (the true value is also 7.071 to three places).
12. $f(2) = 8$, $f'(2) = 12$, $L(x) = 8 + 12(x-2)$. $L(1.97) = 8 - 0.36 = 7.640$. Exact: $1.97^3 \approx 7.645$.
13. The tangent at $x = a$ is $y = (a^2 - 4a + 7) + (2a-4)(x-a)$. At $x = 2$ it must give 2: $a^2 - 4a + 7 + (2a-4)(2-a) = -a^2 + 4a - 1 = 2$, so $a^2 - 4a + 3 = 0$ and $a = 1$ or $a = 3$. For $a = 1$: point $(1, 4)$, slope -2, line $y = -2x + 6$. For $a = 3$: point $(3, 4)$, slope 2, line $y = 2x - 2$.
14. Tangent slope at $(1, 1)$ is 2, so the normal slope is $-\frac{1}{2}$: $y - 1 = -\frac{1}{2}(x-1)$. Set $x^2 = 1 - \frac{1}{2}(x-1)$, so $2x^2 + x - 3 = 0$, $(2x+3)(x-1) = 0$. The new point is $x = -\frac{3}{2}$, $y = \frac{9}{4}$: $(-\frac{3}{2}, \frac{9}{4})$.
15. $p(16) = 30$, $p'(q) = -60q^{-3/2}$, $p'(16) = -\frac{60}{64} = -0.9375$. $y = 30 - 0.9375(q - 16)$. $p(18) \approx 30 - 1.875 = 28.125$. True value $\frac{120}{\sqrt{18}} \approx 28.284$.
16. The line has slope -2, so the tangent must have slope $\frac{1}{2}$. $\frac{1}{2\sqrt{x}} = \frac{1}{2}$ gives $x = 1$. The point is $(1, 1)$.

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SET C · CHALLENGE AND AP-STYLE

17. The tangent at (a, a^3) is $y - a^3 = 3a^2(x - a)$. Through $(0, -2)$: $-2 - a^3 = -3a^3$, so $2a^3 = 2$, $a = 1$. The tangent line is $y - 1 = 3(x - 1)$, that is $y = 3x - 2$.
18. $y - k = 2k(x - 1)$. At $(0, -1)$: $-1 - k = -2k$, so $k = 1$. (Check: $y = x^2$, tangent $y = 2x - 1$ passes through $(0, -1)$.)
19. The tangent at $(a, a^2 + 1)$ is $y - (a^2 + 1) = 2a(x - a)$. Through $(0, 0)$: $-(a^2 + 1) = -2a^2$, so $a^2 = 1$. Two lines: for $a = 1$, $y = 2x$; for $a = -1$, $y = -2x$.
20. (a) $f(2) = 2$. The tangent line passes through $(0, 0)$ and $(4, 4)$, so its slope is 1: $f'(2) = 1$.
    (b) $y - 2 = 1(x - 2)$, i.e. $y = x$, so $f(2.2) \approx 2.2$.
    (c) The curve lies above its tangent line near A (concave up), so the estimate 2.2 is too low.
21. $f'(x) = 3x^2 + 1 = 4$ gives $x = \pm 1$. At $x = 1$: $f = 2$, line $y = 4x - 2$. At $x = -1$: $f = -2$, line $y = 4x + 2$. Neither is the line $y = 4x - 7$.

2.4 Trigonometry Refresher

SET A · SKILLS

1. (a) $-\frac{1}{2}$ (reference angle $\frac{\pi}{6}$, Quadrant III).
   (b) $\frac{1}{2}$ (reference angle $\frac{\pi}{3}$, Quadrant IV, cosine positive).
   (c) -1 (Quadrant II).
   (d) $\sin\left(-\frac{\pi}{2}\right) = -1$, so $\csc = -1$.
2. (a) $150 \cdot \frac{\pi}{180} = \frac{5\pi}{6}$.
   (b) $\frac{7\pi}{4} \cdot \frac{180}{\pi} = 315^\circ$.
   (c) $s = r\theta = 6 \cdot \frac{\pi}{3} = 2\pi$.
3. $\sin^2\theta = 1 - \frac{25}{169} = \frac{144}{169}$, and sine is positive in Quadrant II: $\sin\theta = \frac{12}{13}$. $\tan\theta = \frac{12/13}{-5/13} = -\frac{12}{5}$. $\sin 2\theta = 2 \cdot \frac{12}{13} \cdot \left(-\frac{5}{13}\right) = -\frac{120}{169}$.
4. $\sin^2 x = 1 - \cos^2 x = (1 - \cos x)(1 + \cos x)$. The quotient is $1 + \cos x$.
5. $1 + \tan^2 x = \sec^2 x$, so the product is $\sec^2 x \cos^2 x = 1$.
6. $\cos x = \frac{\sqrt{3}}{2}$, so $x = \frac{\pi}{6}$ or $x = \frac{11\pi}{6}$.
7. Where $\cos x \neq 0$, $\tan x = 1$. So $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$ (cosine is not 0 at either).
8. $(2\sin x - 1)(\sin x + 1) = 0$. $\sin x = \frac{1}{2}$: $x = \frac{\pi}{6}, \frac{5\pi}{6}$. $\sin x = -1$: $x = \frac{3\pi}{2}$.
9. $2\sin x\cos x - \sin x = \sin x\,(2\cos x - 1) = 0$. $\sin x = 0$: $x = 0, \pi$. $\cos x = \frac{1}{2}$: $x = \frac{\pi}{3}, \frac{5\pi}{3}$. Solutions: $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

SET B · APPLICATIONS AND MIXED PROBLEMS

10. Period $= \frac{2\pi}{\pi/15} = 30$ s. Maximum $12 + 10 = 22$ m, minimum $12 - 10 = 2$ m. Solve $12 - 10\cos\theta = 17$, $\cos\theta = -\frac{1}{2}$, $\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$, where $\theta = \frac{\pi t}{15}$. So $t = 10$ s and $t = 20$ s.
11. $12\sin(\cdot) = 6$, so $\sin\theta = \frac{1}{2}$ with $\theta = \frac{2\pi(t-100)}{365}$. In the year, $\theta = \frac{\pi}{6}$ gives $t = 100 + \frac{365}{12} \approx 130.417$ and $\theta = \frac{5\pi}{6}$ gives $t = 100 + \frac{5 \cdot 365}{12} \approx 252.083$. Other angles with the same sine fall outside $0 \le t \le 365$.
12. $\cos x = \frac{1}{3}$. $x = \arccos\frac{1}{3} \approx 1.231$ and $x = 2\pi - 1.231 \approx 5.052$.
13. $x = \arctan 2.5 \approx 1.190$ (Quadrant I) and $x = 1.190 + \pi \approx 4.332$ (Quadrant III).
14. $(3\sin x + 1)(\sin x - 1) = 0$. $\sin x = 1$: $x = \frac{\pi}{2} \approx 1.571$. $\sin x = -\frac{1}{3}$: $x = \pi + 0.340 \approx 3.481$ and $x = 2\pi - 0.340 \approx 5.943$.
15. (a) 0.305 (b) 2.498 (c) 1.373.
16. The maximum is 2 and the minimum is -4, so $a = \frac{2-(-4)}{2} = 3$ and $d = \frac{2+(-4)}{2} = -1$. The maximum occurs at $x = 0$ and the next at $x = 4$, so the period is 4 and $b = \frac{2\pi}{4} = \frac{\pi}{2}$. Equation: $y = 3\cos\left(\frac{\pi}{2}x\right) - 1$. Range $[-4, 2]$.

SET C · CHALLENGE AND AP-STYLE

17. Combine over the common denominator $\sin x\,(1 + \cos x)$: numerator $\sin^2 x + (1 + \cos x)^2 = \sin^2 x + 1 + 2\cos x + \cos^2 x = 2 + 2\cos x = 2(1 + \cos x)$. So the sum is $\dfrac{2(1 + \cos x)}{\sin x(1 + \cos x)} = \dfrac{2}{\sin x}$.
18. Replace $\cos^2 x = 1 - \sin^2 x$: $2 - 2\sin^2 x + 3\sin x = 3$, so $2\sin^2 x - 3\sin x + 1 = 0$, $(2\sin x - 1)(\sin x - 1) = 0$. $\sin x = \frac{1}{2}$: $\frac{\pi}{6}, \frac{5\pi}{6}$. $\sin x = 1$: $\frac{\pi}{2}$. Solutions: $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.
19. $2\cos^2 x - 1 = \cos x$ gives $2\cos^2 x - \cos x - 1 = 0$, $(2\cos x + 1)(\cos x - 1) = 0$. $\cos x = 1$: $x = 0$. $\cos x = -\frac{1}{2}$: $x = \frac{2\pi}{3}, \frac{4\pi}{3}$.
20. Let $u = 2x$ with $0 \le u \le 2\pi$. $\sin u = -\frac{\sqrt{3}}{2}$ gives $u = \frac{4\pi}{3}$ or $\frac{5\pi}{3}$. So $x = \frac{2\pi}{3}$ or $\frac{5\pi}{6}$.
21. Amplitude $A = \frac{7-(-1)}{2} = 4$ and midline $D = 3$. A maximum to the next minimum is half a period: $\frac{5\pi}{6} - \frac{\pi}{3} = \frac{\pi}{2}$, so the period is π and $B = 2$. Starting the cosine at its maximum at $x = \frac{\pi}{3}$: $f(x) = 4\cos\left(2\left(x - \frac{\pi}{3}\right)\right) + 3$. Check: $f\left(\frac{5\pi}{6}\right) = 4\cos\pi + 3 = -1$.

2.5 Differentiability and Continuity

SET A · SKILLS

1. No. f is continuous at -3, but the left-hand derivative is -1 and the right-hand derivative is 1. They differ, so there is a corner and $f'(-3)$ does not exist.
2. (a) Yes: $\lim_{x\to 2^-} x^2 = 4$, $\lim_{x\to 2^+}(4x - 4) = 4$, and $f(2) = 4$.
   (b) Left derivative $2x = 4$, right derivative 4. They agree, so f is differentiable at 2 and $f'(2) = 4$.
3. No. $\lim_{x\to 1^-} g(x) = 1$ but $g(1) = 2$, so g has a jump discontinuity at 1. A function that is not continuous at a point cannot be differentiable there.
4. $h'(x) = \dfrac{1}{3x^{2/3}}$. As $x \to 0$ from either side, $h'(x) \to +\infty$. The graph has a vertical tangent at the origin, so $h'(0)$ does not exist.
5. $k = 2$. Need $3k + 2 = 3^2 - 1 = 8$, so $3k = 6$.
6. False. Counterexample: $f(x) = |x|$ is continuous at 0 but has a corner there, so it is not differentiable at 0.
7. $f'(x) = \dfrac{2}{5x^{3/5}}$. As $x \to 0^-$, $f'(x) \to -\infty$; as $x \to 0^+$, $f'(x) \to +\infty$. The graph has a cusp at the origin and $f'(0)$ does not exist.
8. $x = -2$ and $x = 2$. The inside $x^2 - 4$ changes sign at these points with nonzero slope (± 4), so the graph has a corner at each.

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9. The set is $x > 4$. Here $f'(x) = \dfrac{1}{2\sqrt{x-4}}$, which grows without bound as $x \to 4^+$; the graph has a vertical tangent at $(4, 0)$. Also f is not defined for $x < 4$.

SET B · APPLICATIONS AND MIXED

10. $x = -2$: not differentiable, corner (slopes -1 and 1).
$x = 0$: not differentiable, corner (slopes 1 and 0).
$x = 2$: not differentiable, jump discontinuity.
$x = 4$: differentiable, the graph is a smooth line there.

11. $a = 4,\ b = -3$. Continuity: $a + b = 2 - 1 = 1$. Slopes: $2a + b = 6 - 1 = 5$. Subtract: $a = 4$, so $b = -3$.

12. $\dfrac{|h|^3}{h} = h|h|$, and $\lim_{h\to 0} h|h| = 0$. So $f'(0) = 0$.

13. $t^2 - 4t + 3 = (t-1)(t-3)$, so s has corners at $t = 1$ and $t = 3$ and the velocity is undefined there. On $1 < t < 3$, $s = -(t^2 - 4t + 3)$, so $v = -2t + 4$ and $v(2) = 0$.

14. No. $g(3) = 0$ and $\dfrac{g(3+h)}{h} = \dfrac{(3+h)|h|}{h}$, which tends to 3 as $h \to 0^+$ and to -3 as $h \to 0^-$. The one-sided derivatives differ, so there is a corner.

15. $f'(4) = -3$ by definition. Since f is differentiable at 4, it is continuous at 4, so $\lim_{x\to 4} f(x) = f(4) = 7$.

16. Left value: $6 - 4 = 2$; left slope: $3 - 2x = -1$. So $m = -1$ and $-2 + c = 2$ gives $c = 4$. The tangent line is $y = -x + 4$.

SET C · CHALLENGE

17. $\dfrac{f(h) - 0}{h} = h \sin\left(\dfrac{1}{h}\right)$. Since $-|h| \le h\sin(1/h) \le |h|$, the Squeeze Theorem gives limit 0. So $f'(0) = 0$.

18. At $x = 1$: $a + b = 1$ and $a = 2$, so $b = -1$. Check $x = 3$: $3a + b = 5$ and the last piece has value $-9 + 24 - 10 = 5$; slopes $a = 2$ and $-6 + 8 = 2$. All match. $a = 2,\ b = -1$.

19. Rewrite as $2 \cdot \dfrac{f(3+2h) - f(3)}{2h}$. As $h \to 0$, $2h \to 0$, so the fraction tends to $f'(3)$. The limit is $2(-4) = -8$.

20. $\dfrac{f(h) - f(0)}{h} = \operatorname{sgn}(h)|h|^{p-1}$. For $p > 1$ this tends to 0, so $f'(0) = 0$. For $p = 1$ the one-sided limits are ± 1 (corner). For $0 < p < 1$ the quotient is unbounded (cusp). So $p > 1$.

21. D. $x|x|$ has difference quotient $|h| \to 0$. The others fail: $|x|$ and $|x| + x$ have corners, $x^{2/3}$ has a cusp, $x^{1/3}$ has a vertical tangent.

2.6 The Product and Quotient Rules

SET A · SKILLS

1. $y' = 2x(x^3 - 2x) + (x^2 + 3)(3x^2 - 2) = 5x^4 + 3x^2 - 6$.

2. $f'(x) = 4(x^2 + x + 2) + (4x - 1)(2x + 1) = 12x^2 + 6x + 7$.

3. $g'(x) = \dfrac{(x+4) - x}{(x+4)^2} = \dfrac{4}{(x+4)^2}$.

4. $y' = \dfrac{2x(x^2+1) - (x^2-1)(2x)}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$.

5. $h'(t) = \dfrac{t^2 - 3t + 1}{2\sqrt{t}} + \sqrt{t}\,(2t - 3) = \dfrac{5t^2 - 9t + 1}{2\sqrt{t}}$.

6. $y = 3x^{1/2} + 2x^{-1/2}$, so $y' = \dfrac{3}{2}x^{-1/2} - x^{-3/2} = \dfrac{3x - 2}{2x^{3/2}}$.

7. $f'(x) = \dfrac{(6x^2-1)(x^2-5) - (2x^3 - x)(2x)}{(x^2-5)^2} = \dfrac{2x^4 - 29x^2 + 5}{(x^2-5)^2}$.

8. $y' = (x+1)(x-4) + x(x-4) + x(x+1) = 3x^2 - 6x - 4$. Expanding gives $y = x^3 - 3x^2 - 4x$, which has the same derivative.

9. $f'(x) = \dfrac{2x(x+1) - (x^2+2)}{(x+1)^2} = \dfrac{x^2 + 2x - 2}{(x+1)^2}$. So $f'(1) = \dfrac{1}{4}$.

SET B · APPLICATIONS AND MIXED

10. (a) $h'(3) = f'(3)g(3) + f(3)g'(3) = (2)(1) + (-1)(3) = -1$.
(b) $k'(3) = \dfrac{(2)(1) - (-1)(3)}{1^2} = 5$.
(c) $p'(1) = \dfrac{g'(1)f(1) - g(1)f'(1)}{[f(1)]^2} = \dfrac{(-2)(2) - (5)(4)}{4} = -6$.

11. $A = LW$, so $A' = L'W + LW' = 3(t^2+1) + (3t+2)(2t)$. At $t = 2$: $3(5) + (8)(4) = 47$. The area is increasing at 47 square centimeters per second.

12. $C'(t) = \dfrac{5(t^2+4) - 5t(2t)}{(t^2+4)^2} = \dfrac{5(4 - t^2)}{(t^2+4)^2}$. It is zero at $t = 2$ hours, where $C(2) = \dfrac{10}{8} = 1.25$ milligrams per liter.

13. $y' = \dfrac{(x-1) - (x+1)}{(x-1)^2} = \dfrac{-2}{(x-1)^2}$. At $x = 2$: $y = 3$, $y' = -2$. Tangent line: $y - 3 = -2(x - 2)$, i.e. $y = -2x + 7$.

14. $y' = \dfrac{2x(x-2) - x^2}{(x-2)^2} = \dfrac{x(x-4)}{(x-2)^2}$, which is 0 at $x = 0$ and $x = 4$. Points: $(0, 0)$ and $(4, 8)$.

15. $A'(x) = \dfrac{(4x+30)x - (2x^2 + 30x + 200)}{x^2} = \dfrac{2x^2 - 200}{x^2}$. Setting $A'(x) = 0$ gives $x^2 = 100$, so $x = 10$ units.

16. $f'(x) = 2x\,g(x) + x^2 g'(x)$, so $f'(3) = 2(3)(2) + 9(-4) = 12 - 36 = -24$.

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SET C · CHALLENGE

17. $p' = f'gh + fg'h + fgh' = (3)(-1)(5) + (2)(4)(5) + (2)(-1)(-2) = -15 + 40 + 4 = 29$.

18. Reading the lines: $f(x) = x + 1$, $g(x) = 5 - \frac{x}{2}$. At $x = 2$: $f = 3$, $f' = 1$, $g = 4$, $g' = -\frac{1}{2}$.
(a) $(fg)'(2) = (1)(4) + (3)(-\frac{1}{2}) = \frac{5}{2}$.
(b) $\left(\frac{f}{g}\right)'(2) = \frac{(1)(4) - (3)(-\frac{1}{2})}{16} = \frac{11}{32}$.

19. $f' = 2xg + x^2g'$. Differentiating again: $f'' = 2g + 2xg' + 2xg' + x^2g'' = 2g + 4xg' + x^2g''$. At $x = 2$: $6 + 8(-1) + 4(5) = 18$.

20. $f'(x) = \frac{-ax^2 - 2bx + a}{(x^2+1)^2}$, so $f'(1) = \frac{-2b}{4} = -3$ gives $b = 6$. Then $f(1) = \frac{a+6}{2} = 2$ gives $a = -2$.

21. C. The quotient rule gives $f'(5) = \frac{(10)(2) - (16)(1)}{4} = 1$. (Simplifying first, $f(x) = x + 3$, confirms it.)

2.7 The Chain Rule

SET A · SKILLS

1. $y' = 7(5x-2)^6 \cdot 5 = 35(5x-2)^6$.
2. $f'(x) = 4(x^2+3x)^3(2x+3)$.
3. $y' = \frac{8x}{2\sqrt{4x^2+9}} = \frac{4x}{\sqrt{4x^2+9}}$.
4. $g(x) = (2x+5)^{-2}$, so $g'(x) = -2(2x+5)^{-3} \cdot 2 = -\frac{4}{(2x+5)^3}$.
5. $y' = 3(x+1)^2(x-2)^2 + 2(x+1)^3(x-2) = (x+1)^2(x-2)\big[3(x-2) + 2(x+1)\big] = (x+1)^2(x-2)(5x-4)$.
6. $y' = 3\left(\frac{x+1}{x-1}\right)^2 \cdot \frac{-2}{(x-1)^2} = -\frac{6(x+1)^2}{(x-1)^4}$.
7. $y = (x^2+8)^{1/3}$, so $y' = \frac{1}{3}(x^2+8)^{-2/3} \cdot 2x = \frac{2x}{3(x^2+8)^{2/3}}$.
8. $y' = 4\left(1+\sqrt{x}\right)^3 \cdot \frac{1}{2\sqrt{x}} = \frac{2\left(1+\sqrt{x}\right)^3}{\sqrt{x}}$.
9. $f'(x) = 4(3x^2-2)^3(6x)$. At $x = 1$: $4(1)^3(6) = 24$.

SET B · APPLICATIONS AND MIXED

10. $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi(t+4) \cdot \frac{1}{2\sqrt{t+4}} = 2\pi\sqrt{t+4}$. At $t = 5$: $6\pi \approx 18.850$ cubic centimeters per second.
11. $T'(t) = -120(1+0.1t)^{-3}(0.1) = -\frac{12}{(1+0.1t)^3}$. $T'(6) = -\frac{12}{1.6^3} \approx -2.930$. At $t = 6$ minutes the temperature is falling at about 2.930 degrees Celsius per minute.
12. $d(t) = \sqrt{16t^2+9}$, so $d'(t) = \frac{32t}{2\sqrt{16t^2+9}} = \frac{16t}{\sqrt{16t^2+9}}$. At $t = 2$: $\frac{32}{\sqrt{73}} \approx 3.745$ kilometers per hour.
13. $f'(x) = \frac{3}{2}(x^2+1)^{1/2}(2x) = 3x\sqrt{x^2+1}$. $f(2) = 5^{3/2} \approx 11.180$ and $f'(2) = 6\sqrt{5} \approx 13.416$. Tangent line: $y - 11.180 = 13.416(x-2)$.
14. $I(d) = 100(d^2+1)^{-1}$, so $I'(d) = -\frac{200d}{(d^2+1)^2}$. $I'(1.5) = -\frac{300}{3.25^2} \approx -28.402$. At 1.5 meters the intensity is dropping by about 28.402 watts per square meter for each extra meter.
15. $R'(x) = \sqrt{100-x} - \frac{x}{2\sqrt{100-x}} = \frac{200-3x}{2\sqrt{100-x}}$. $R'(40) = \frac{80}{2\sqrt{60}} \approx 5.164$. Revenue is rising at about 5.164 thousand dollars per additional unit sold.
16. $f'(x) = \frac{3x^2+2}{2\sqrt{x^3+2x+1}}$. Solving $f'(x) = 2$ numerically (on $x > 0$ the only solution) gives $x \approx 2.096$.

SET C · CHALLENGE

17. $F'(1) = f'(g(1))g'(1) = f'(2) \cdot 6 = -18$.
$G'(2) = g'(f(2))f'(2) = g'(1) \cdot (-3) = -18$.
$H'(3) = f'(3)g(3)^2 + f(3) \cdot 2g(3)g'(3) = 5(1) + 2 \cdot 2 \cdot 1 \cdot 4 = 21$.

18. On $[0,2]$, $g(x) = 2x$; on $[2,5]$, $g(x) = 8 - 2x$.
(a) $h'(1) = 3g(1)^2g'(1) = 3(4)(2) = 24$.
(b) $g(1.5) = 3$ and $g(3) = 2$ with $g'(3) = -2$, so $k'(1.5) = g'(3)\,g'(1.5) = (-2)(2) = -4$.
(c) Near $x = 1$, $g(x) = 2x$, so $k(x) = g(2x)$, and $2x$ passes through 2 at $x = 1$, where g has a corner. The one-sided derivatives of k at 1 are 4 and -4.

19. E. $h'(x) = f'(x^2) \cdot 2x$, so $h'(2) = f'(4) \cdot 4 = 12$.

20. $y' = \frac{1}{2\sqrt{x+\sqrt{x}}}\left(1 + \frac{1}{2\sqrt{x}}\right) = \frac{2\sqrt{x}+1}{4\sqrt{x}\,\sqrt{x+\sqrt{x}}}$.

21. $y' = 12(3x-1)^3$, $y'' = 36(3x-1)^2 \cdot 3 = 108(3x-1)^2$. So $y''(1) = 108 \cdot 4 = 432$.

2.8 Derivatives of Trigonometric Functions

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1. $y' = 3\cos x + 4\sin x$.
2. $f'(x) = \cos x - x\sin x$.
3. $y' = \sec^2 x + \csc^2 x$.
4. $y' = -\sin(x^3)\cdot 3x^2 = -3x^2\sin(x^3)$.
5. $y' = \sec x\tan x\cdot\tan x + \sec x\cdot\sec^2 x = \sec x\tan^2 x + \sec^3 x$.
6. $y' = \dfrac{\cos x(1+\cos x)+\sin^2 x}{(1+\cos x)^2} = \dfrac{\cos x+1}{(1+\cos x)^2} = \dfrac{1}{1+\cos x}$.
7. $y' = 2\cos(3x)\cdot(-\sin 3x)\cdot 3 = -6\sin(3x)\cos(3x)$.
8. $y' = -\csc(2x)\cot(2x)\cdot 2 = -2\csc(2x)\cot(2x)$.
9. $f'(x) = \cos^2 x - \sin^2 x$. At $\frac{\pi}{3}$: $\frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$.

SET B · APPLICATIONS AND MIXED

10. $v(t) = 6\cos(2t) + 1$ and $a(t) = -12\sin(2t)$. At $t = 1.5$: $v = 6\cos 3 + 1 \approx -4.940$ meters per second and $a = -12\sin 3 \approx -1.693$ meters per second squared.
11. $D'(t) = \dfrac{\pi}{2}\cos\left(\dfrac{\pi t}{6}\right)$. $D'(2) = \dfrac{\pi}{2}\cos\dfrac{\pi}{3} = \dfrac{\pi}{4} \approx 0.785$ meters per hour (rising).
12. $I'(t) = 250\cos(50t)$. $I'(0.02) = 250\cos 1 \approx 135.076$ amperes per second.
13. $f'(x) = 2x - 4\cos x = 0$. Solving numerically on $[0, 2]$ gives $x \approx 1.030$.
14. $y' = \tan x + x\sec^2 x$. $y(1) = \tan 1 \approx 1.557$ and $y'(1) \approx 1.557 + 3.426 = 4.983$. Tangent line: $y - 1.557 = 4.983(x - 1)$.
15. $\dfrac{dA}{dt} = 24\cos\theta\cdot\dfrac{d\theta}{dt} = 24\cos(0.9)(0.2) \approx 2.984$ square units per second.
16. $\theta'(t) = -0.6\sin(2t)$. $\theta'(1) = -0.6\sin 2 \approx -0.546$ radians per second.

SET C · CHALLENGE

17. Write $\sec x = (\cos x)^{-1}$. Then $\dfrac{d}{dx} = -(\cos x)^{-2}(-\sin x) = \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x} = \sec x\tan x$.
18. (a) This is $f'\left(\frac{\pi}{6}\right)$ for $f = \sin x$: $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.
(b) This is $f'\left(\frac{\pi}{4}\right)$ for $f = \tan x$: $\sec^2\frac{\pi}{4} = 2$.
19. B. $y' = -2\sin(2x)$, so at $x = \frac{\pi}{6}$ the slope is $-2\sin\frac{\pi}{3} = -\sqrt{3}$.
20. $y' = \cos x - \sin x = 0$ gives $\tan x = 1$, so $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. Points: $\left(\frac{\pi}{4}, \sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$.
21. $y' = 6\cos 3x + 3\sin 3x$ and $y'' = -18\sin 3x + 9\cos 3x$. Then $9y = 18\sin 3x - 9\cos 3x$, so $y'' + 9y = 0$.

2.9 Derivatives of Exponential and Logarithmic Functions

SET A · SKILLS

1. $y' = 5e^{5x}$.
2. $f'(x) = 3x^2e^x + x^3e^x = x^2e^x(x+3)$.
3. $y' = \dfrac{3}{3x+7}$.
4. $g'(x) = \dfrac{2x+4}{x^2+4x}$.
5. $y' = 7^{3x}\ln 7\cdot 3 = 3\ln 7\cdot 7^{3x}$.
6. $y' = \dfrac{2x}{(x^2+1)\ln 5}$.
7. $y' = \dfrac{e^x(1+e^x) - e^x\cdot e^x}{(1+e^x)^2} = \dfrac{e^x}{(1+e^x)^2}$.
8. $y = \frac{1}{2}\ln(x^2+1)$, so $y' = \frac{1}{2}\cdot\dfrac{2x}{x^2+1} = \dfrac{x}{x^2+1}$.
9. $\ln y = x\ln(x^2+1)$, so $\dfrac{y'}{y} = \ln(x^2+1) + \dfrac{2x^2}{x^2+1}$. Then $y' = (x^2+1)^x\left[\ln(x^2+1) + \dfrac{2x^2}{x^2+1}\right]$.

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10. $P'(t) = 60e^{0.12t}$. $P'(5) = 60e^{0.6} \approx 109.327$. At $t = 5$ hours the population is growing at about 109.327 cells per hour.
11. $T'(t) = -3.4e^{-0.05t}$. $T'(10) = -3.4e^{-0.5} \approx -2.062$. At 10 minutes the object is cooling at about 2.062 degrees Celsius per minute.
12. $C'(t) = 8e^{-0.4t}(1 - 0.4t) = 0$ when $t = 2.5$ hours. $C(2.5) = 20e^{-1} \approx 7.358$ milligrams per liter.
13. $L'(x) = \dfrac{10}{x\ln 10}$. $L'(50) = \dfrac{10}{50\ln 10} \approx 0.087$. At relative intensity 50 the sound level rises by about 0.087 decibels per unit of relative intensity.
14. $A'(t) = 100e^{0.05t}$, so $A'(10) = 100e^{0.5} \approx 164.872$ dollars per year.
15. $g'(x) = e^x - 4 = 0$ gives $x = \ln 4 \approx 1.386$. Then $g(\ln 4) = 4 - 4\ln 4 \approx -1.545$. Point: $(1.386,\ -1.545)$.
16. $y' = \dfrac{2x}{x^2+1}$, so $y'(2) = 0.8$. Also $y(2) = \ln 5 \approx 1.609$. Tangent line: $y - 1.609 = 0.8(x - 2)$.

SET C · CHALLENGE

17. $\ln y = x\ln(\sin x)$. Then $\dfrac{y'}{y} = \ln(\sin x) + x\cot x$, so $y' = (\sin x)^x\left[\ln(\sin x) + x\cot x\right]$.
18. $f'(x) = \dfrac{-\sin x}{\cos x} = -\tan x$.
19. A. $f'(x) = \dfrac{2e^{2x}}{e^{2x}+1}$, so $f'(0) = \dfrac{2}{2} = 1$.
20. $y' = \dfrac{1-\ln x}{x^2} = 0$ gives $\ln x = 1$, so $x = e$. The point is $\left(e, \frac{1}{e}\right)$.
21. $f(1) = ae = 3e$ gives $a = 3$. $f'(x) = ae^x + \frac{b}{x}$, so $f'(1) = 3e + b = 3e - 2$ gives $b = -2$.

2.10 Implicit Differentiation

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1. $2x + 2y\,y' = 0$, so $y' = -\dfrac{x}{y}$.
2. $3x^2 + 3y^2 y' = 0$ gives $y' = -\dfrac{x^2}{y^2}$. At $(2,3)$: $-\dfrac{4}{9}$.
3. $y + xy' = 0$, so $y' = -\dfrac{y}{x}$.
4. $2y\,y' = 4$, so $y' = \dfrac{2}{y}$. At $(1,2)$ the slope is 1.
5. $2x + 3y + 3x\,y' - 2y\,y' = 0$, so $y' = -\dfrac{2x+3y}{3x-2y}$.
6. $\cos y \cdot y' = 2x$, so $y' = \dfrac{2x}{\cos y}$.
7. $e^y y' = 1 + y'$. So $y'(e^y - 1) = 1$ and $y' = \dfrac{1}{e^y - 1}$.
8. $y' = -\frac{x}{y} = -\frac{1}{2}$ at $(2,4)$. Tangent line: $y - 4 = -\frac{1}{2}(x-2)$.
9. $y' = -\dfrac{y}{x}$. Then $y'' = -\dfrac{y'x - y}{x^2} = -\dfrac{-y - y}{x^2} = \dfrac{2y}{x^2}$. Since $y = \frac{4}{x}$, $y'' = \dfrac{8}{x^3}$.

SET B · APPLICATIONS AND MIXED

10. (a) $2x - 4 + 2yy' + 6y' = 0$, so $y' = \dfrac{4-2x}{2y+6}$. At $(5,1)$: $\dfrac{-6}{8} = -\dfrac{3}{4}$. Tangent: $y - 1 = -\frac{3}{4}(x-5)$.
(b) Horizontal where $x = 2$. Then $4 - 8 + y^2 + 6y = 12$, so $y^2 + 6y - 16 = 0$ and $y = 2$ or $y = -8$. Points $(2,2)$ and $(2,-8)$.
11. $2x + y + xy' + 2yy' = 0$ gives $y' = -\dfrac{2x+y}{x+2y}$; at $(2,1)$ the slope is $-\frac{5}{4}$. The normal slope is $\frac{4}{5}$: $y - 1 = \frac{4}{5}(x-2)$.
12. $8x + 18yy' = 0$, so $y' = -\dfrac{4x}{9y}$. Horizontal: $x = 0$, giving $(0,2)$ and $(0,-2)$. Vertical: $y = 0$, giving $(3,0)$ and $(-3,0)$.
13. $P + V\dfrac{dP}{dV} = 0$, so $\dfrac{dP}{dV} = -\dfrac{P}{V}$. When $V = 5$, $P = 40$, so $\dfrac{dP}{dV} = -8$ kilopascals per liter.
14. At $q = 4$: $4p + 16 = 40$, so $p = 6$. Differentiating: $p + q\dfrac{dp}{dq} + 2q = 0$, so $\dfrac{dp}{dq} = -\dfrac{p+2q}{q} = -\dfrac{14}{4} = -\dfrac{7}{2}$ dollars per hundred units.
15. $e^0 = 1 = 2 \cdot \frac{1}{2}$. Differentiate: $e^{xy}(y + xy') = 2y'$. At $(0, \frac{1}{2})$: $\frac{1}{2} = 2y'$, so $y' = \dfrac{1}{4}$.
16. $y' = \dfrac{x}{y}$, so $y'' = \dfrac{y - xy'}{y^2} = \dfrac{y^2 - x^2}{y^3} = -\dfrac{4}{y^3}$. At $y = 2\sqrt{3}$: $-\dfrac{4}{24\sqrt{3}} = -\dfrac{\sqrt{3}}{18}$.

SET C · CHALLENGE

17. $y' = -\dfrac{2x+y}{x+2y}$. Horizontal: $y = -2x$, so $3x^2 = 3$ and the points are $(1,-2)$, $(-1,2)$. Vertical: $x = -2y$, so $3y^2 = 3$ and the points are $(-2,1)$, $(2,-1)$.
18. $2yy' = 3x^2$, so $y' = \dfrac{3x^2}{2y}$, which is $\frac{3}{2}$ at $(1,1)$. Then $y'' = \dfrac{6x \cdot 2y - 3x^2 \cdot 2y'}{4y^2} = \dfrac{12 - 9}{4} = \dfrac{3}{4}$.
19. B. $2x + y + xy' = 0$ gives $y' = -\dfrac{2x+y}{x} = -\dfrac{5}{2}$ at $(2,1)$.
20. Both hold at $(2,1)$: $4 - 1 = 3$ and $2 \cdot 1 = 2$. For the first, $y' = \dfrac{x}{y} = 2$. For the second, $y' = -\dfrac{y}{x} = -\dfrac{1}{2}$. Since $2 \cdot (-\frac{1}{2}) = -1$, the tangent lines are perpendicular.
21. Points $(4,3)$ and $(4,-3)$. Slopes $-\frac{x}{y}$: $-\frac{4}{3}$ and $\frac{4}{3}$. Lines: $y - 3 = -\frac{4}{3}(x-4)$ and $y + 3 = \frac{4}{3}(x-4)$. Both give $y = 0$ at $x = \frac{25}{4}$, so they meet at $\left(\frac{25}{4}, 0\right)$.

2.11 Derivatives of Inverse Functions

SET A · SKILLS

1. $g'(3) = \dfrac{1}{7}$. Since $f(1) = 3$ and $f'(x) = 6x^2 + 1$, $f'(1) = 7$, so $g'(3) = \frac{1}{7}$.
2. $g'(2) = \dfrac{1}{8}$. $f(1) = 2$ and $f'(x) = 5x^4 + 3x^2$ gives $f'(1) = 8$, so $g'(2) = \frac{1}{8}$.
3. $g'(1) = \dfrac{1}{2}$. $f(0) = 1$ and $f'(x) = e^x + 1$ gives $f'(0) = 2$, so $g'(1) = \frac{1}{2}$.
4. $y' = \dfrac{5}{\sqrt{1 - 25x^2}}$. Use $\dfrac{u'}{\sqrt{1-u^2}}$ with $u = 5x$.
5. $y' = \dfrac{3x^2}{1 + x^6}$. Use $\dfrac{u'}{1+u^2}$ with $u = x^3$, $u' = 3x^2$.
6. $y' = \arccos x - \dfrac{x}{\sqrt{1-x^2}}$. Product rule: $1 \cdot \arccos x + x \cdot \left(-\dfrac{1}{\sqrt{1-x^2}}\right)$.
7. $y' = \dfrac{1}{2\sqrt{x}\sqrt{1-x}} = \dfrac{1}{2\sqrt{x(1-x)}}$. With $u = \sqrt{x}$, $u' = \dfrac{1}{2\sqrt{x}}$ and $1 - u^2 = 1 - x$.
8. $y' = -\dfrac{1}{1+x^2}$. With $u = x^{-1}$, $u' = -x^{-2}$ and $1 + u^2 = \dfrac{x^2+1}{x^2}$, so $y' = \dfrac{-1/x^2}{(x^2+1)/x^2} = -\dfrac{1}{1+x^2}$.
9. $\dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$. $y' = \dfrac{1/2}{\sqrt{1 - x^2/4}}$; at $x = 1$ this is $\dfrac{1/2}{\sqrt{3}/2} = \dfrac{1}{\sqrt{3}}$.

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10. $g'(7) = 0.250$ and $g'(11) = 0.125$. $f(2) = 7$ so $g'(7) = \frac{1}{f'(2)} = \frac{1}{4}$; $f(3) = 11$ so $g'(11) = \frac{1}{f'(3)} = \frac{1}{8}$.
11. $T'(12) \approx 0.071$ hours per liter. $V(2) = 12$, so $T(12) = 2$. $V'(t) = 3t^2 + 2$, $V'(2) = 14$, so $T'(12) = \frac{1}{14} \approx 0.071$. It takes about 0.071 more hours per extra liter at that moment.
12. $x \approx 1.255$ and $g'(7) \approx 0.115$. $f'(x) = 3x^2 + 4$, so $f'(1.2554) \approx 8.728$ and $g'(7) = \frac{1}{8.728} \approx 0.115$.
13. 0.125 radians per meter. $\dfrac{d\theta}{dh} = \dfrac{1/10}{\sqrt{1 - h^2/100}}$; at $h = 6$ this is $\dfrac{0.1}{0.8} = 0.125$.
14. $y = x + 0.285$. At $x = \frac{1}{2}$, $y = \arctan 1 = \frac{\pi}{4}$ and $y' = \frac{2}{1+4x^2} = 1$. So $y - \frac{\pi}{4} = x - \frac{1}{2}$, giving $b = \frac{\pi}{4} - \frac{1}{2} \approx 0.285$.
15. 1.250. $y' = \dfrac{1}{\sqrt{1 - 0.36}} = \dfrac{1}{0.8} = 1.25$.
16. $g'(9) \approx 0.167$. $g(9) = 3$ and $f'(3) = 6$, so $g'(9) = \frac{1}{6}$. Direct: $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} = \frac{1}{6}$ at $x = 9$. They agree.

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SET C · CHALLENGE

17. (a) $g(5) = 2$, $g'(5) = -\frac{1}{4}$.
(b) $y - 2 = -\frac{1}{4}(x - 5)$.
(c) $h'(5) = -1$.
Steps: $f(2) = 5$ gives $g(5) = 2$ and $g'(5) = \frac{1}{f'(2)} = -\frac{1}{4}$. The line goes through $(5, 2)$. $h'(x) = 2g(x)g'(x)$, so $h'(5) = 2(2)(-\frac{1}{4}) = -1$.

18. Differentiate $\cos y = x$: $-\sin y \dfrac{dy}{dx} = 1$, so $\dfrac{dy}{dx} = -\dfrac{1}{\sin y}$. Since $0 \le y \le \pi$, $\sin y \ge 0$, so $\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$. Therefore $\dfrac{d}{dx} \arccos x = -\dfrac{1}{\sqrt{1 - x^2}}$.

19. (a) $\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$.
(b) A function with derivative 0 on an interval is constant, so $\arcsin x + \arccos x = C$. At $x = 0$: $0 + \frac{\pi}{2} = \frac{\pi}{2}$, so $C = \frac{\pi}{2}$.

20. $\dfrac{dy}{dx} = \dfrac{1}{1 + x^2}$. With $u = \dfrac{1 + x}{1 - x}$, $u' = \dfrac{2}{(1 - x)^2}$ and $1 + u^2 = \dfrac{2 + 2x^2}{(1 - x)^2}$, so $y' = \dfrac{2}{2 + 2x^2} = \dfrac{1}{1 + x^2}$. This is the derivative of $\arctan x$, so y and $\arctan x$ differ by a constant on $x < 1$ (the constant is $\frac{\pi}{4}$).

21. (a) $f'(x) = 5x^4 + 3x^2 + 1 \ge 1 > 0$ for all x, so f is strictly increasing on all reals, hence one-to-one, and it is continuous with range all reals.
(b) $f(1) = 3$, $f'(1) = 9$, so $g'(3) = \frac{1}{9}$.
(c) $g'(x) = \frac{1}{f'(g(x))}$ and $f' \ge 1$, so $g'(x) \le 1$. Equality needs $f'(g(x)) = 1$, i.e. $g(x) = 0$, i.e. $x = 0$.

2.12 Related Rates: Setting Up the Equation

SET A · SKILLS

1. 24π cm^2/s. $A = \pi r^2$, $A' = 2\pi r r' = 2\pi(4)(3)$.

2. (a) 150 cm^3/s. (b) 120 cm^2/s.
$V = s^3$ gives $V' = 3s^2 s' = 3(25)(2) = 150$. $S = 6s^2$ gives $S' = 12ss' = 12(5)(2) = 120$.

3. -1 cm/min. $V' = 4\pi r^2 r'$ gives $-36\pi = 4\pi(9)r'$, so $r' = -1$. The radius is shrinking.

4. $-\dfrac{9}{4}$. Differentiate $2xx' + 2yy' = 0$, so $y' = -\dfrac{xx'}{y}$. At $x = 3$, $y = 4$, so $y' = -\dfrac{3 \cdot 3}{4} = -\dfrac{9}{4}$.

5. Decreasing at 2 cm^2/s. $A = LW$, $A' = L'W + LW' = 2(5) + 12(-1) = -2$.

6. 13 km/h. $z^2 = x^2 + y^2$; after 1 hour $x = 12$, $y = 5$, $z = 13$. $z' = \dfrac{xx' + yy'}{z} = \dfrac{12(12) + 5(5)}{13} = \dfrac{169}{13} = 13$.

7. 1 cm/s. $A = \pi r^2$ gives $A' = 2\pi r r'$. When $A = 25\pi$, $r = 5$, so $10\pi = 2\pi(5)r'$ and $r' = 1$.

8. 18π cm^3/s. $V = \pi(3)^2 h = 9\pi h$, so $V' = 9\pi h' = 9\pi(2)$.

9. $-\dfrac{8}{3}$. Differentiate $xy = 12$: $x'y + xy' = 0$. At $x = 3$, $y = 4$, so $2(4) + 3y' = 0$ and $y' = -\dfrac{8}{3}$.

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10. 0.033 cm/s. $V' = 4\pi r^2 r'$, so $r' = \dfrac{15}{4\pi(36)} = \dfrac{15}{144\pi} \approx 0.033$.

11. 125.664 m^2/min. $A' = 2\pi r r' = 2\pi(50)(0.4) = 40\pi \approx 125.664$.

12. -1.333 cm^2/min. $V' = 4\pi r^2 r'$ gives $r' = \dfrac{-2}{36\pi}$. $S' = 8\pi r r' = 8\pi(3) \cdot \dfrac{-2}{36\pi} = -\dfrac{4}{3} \approx -1.333$.

13. 0.080 m^3/min. Similar triangles: $w = \dfrac{1.2}{0.9}h = \dfrac{4}{3}h$. $V = 5 \cdot \dfrac{1}{2}wh = \dfrac{10}{3}h^2$, $V' = \dfrac{20}{3}hh' = \dfrac{20}{3}(0.6)(0.02) = 0.08$. The volume is increasing.

14. -28 km/h (the distance is decreasing). $z = \sqrt{3^2 + 4^2} = 5$. $x' = 60$, $y' = -80$, so $z' = \dfrac{xx' + yy'}{z} = \dfrac{3(60) + 4(-80)}{5} = \dfrac{-140}{5} = -28$.

15. 113.097 cm^3/s. $V = \dfrac{4}{3}\pi r^3$, $V' = 4\pi r^2 r'$. At $t = 2$: $4\pi(9)(1.0) = 36\pi \approx 113.097$.

16. 3.019 units/s. $D^2 = x^2 + y^2$, $y = \sqrt{x}$, $y' = \dfrac{x'}{2\sqrt{x}} = \dfrac{3}{4}$. $D = \sqrt{20}$. $D' = \dfrac{xx' + yy'}{D} = \dfrac{4(3) + 2(0.75)}{\sqrt{20}} = \dfrac{13.5}{\sqrt{20}} \approx 3.019$.

SET C · CHALLENGE

17. $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$ and $S = 4\pi r^2$, so $4\pi r^2 \dfrac{dr}{dt} = 0.3(4\pi r^2)$. Divide by $4\pi r^2 \ne 0$: $\dfrac{dr}{dt} = 0.3$ cm/min, a constant.

18. (a) $-\frac{8}{3}$ cm/s. (b) Decreasing at $\frac{4}{3}$ cm/s. (c) $-\frac{14}{15}$ cm/s.
$xy = 48$ gives $y = 8$ and $x'y + xy' = 0$, so $y' = -\frac{8(2)}{6} = -\frac{8}{3}$. $P = 2(x + y)$, $P' = 2(2 - \frac{8}{3}) = -\frac{4}{3}$. $d^2 = x^2 + y^2$ with $d = 10$: $d' = \frac{6(2) + 8(-8/3)}{10} = \frac{-28/3}{10} = -\frac{14}{15}$.

19. $\dfrac{dy}{dt} = 12$ and $\dfrac{dD}{dt} = \dfrac{27\sqrt{5}}{5} \approx 12.075$. $y' = 2xx' = 2(2)(3) = 12$. $D = \sqrt{x^2 + y^2} = \sqrt{20}$, $D' = \dfrac{xx' + yy'}{D} = \dfrac{2(3) + 4(12)}{2\sqrt{5}} = \dfrac{54}{2\sqrt{5}} = \dfrac{27}{\sqrt{5}} = \dfrac{27\sqrt{5}}{5}$.

20. $y = 1$ and $\dfrac{dy}{dt} = \dfrac{4\sqrt{3}}{3}$. From $\dfrac{27/4}{9} = \dfrac{3}{4}$ we get $\dfrac{y^2}{4} = \dfrac{1}{4}$, so $y = 1$. Differentiate: $\dfrac{2xx'}{9} + \dfrac{2yy'}{4} = 0$, so $y' = -\dfrac{4xx'}{9y} = -\dfrac{4}{9} \cdot \dfrac{3\sqrt{3}}{2}(-2) = \dfrac{4\sqrt{3}}{3}$.

21. (a) $-\frac{1}{32\pi}$ cm/min. (b) $-\frac{1}{2}$ cm^3/min. (c) See below.
$S = 4\pi r^2$, $S' = 8\pi r r' = -0.5$, so $r' = -\frac{1}{16\pi r}$; at $r = 2$ this is $-\frac{1}{32\pi}$. Then $V' = 4\pi r^2 r' = 4\pi(4)\left(-\frac{1}{32\pi}\right) = -\frac{1}{2}$. (c) Since $r' = -\frac{1}{16\pi r}$, a smaller r makes the denominator smaller, so $|r'|$ grows: the same loss of area must come off a shorter radius.

2.13 Related Rates: Angles and Harder Models

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1. $\frac{9}{100}$ rad/s. $\tan\theta = \frac{h}{12}$, so $\sec^2\theta\,\theta' = \frac{3}{12}$. At $h = 16$ the sight line is 20, $\sec^2\theta = \frac{25}{9}$, and $\theta' = \frac{1}{4}\cdot\frac{9}{25} = \frac{9}{100}$.
2. $-\frac{1}{8}$ rad/s. $\sin\theta = \frac{y}{10}$ gives $\cos\theta\,\theta' = \frac{y'}{10}$. At $y = 6$, $x = 8$ and $\cos\theta = \frac{4}{5}$, so $\theta' = \frac{-1/10}{4/5} = -\frac{1}{8}$.
3. $\frac{12}{25}$ rad/s. $\tan\theta = \frac{x}{3}$, so $\sec^2\theta\,\theta' = \frac{x'}{3}$. At $x = 4$, $\sec^2\theta = 1 + \frac{16}{9} = \frac{25}{9}$. $\theta' = \frac{4}{3}\cdot\frac{9}{25} = \frac{12}{25}$.
4. $\frac{1}{3}$ m/min. $V = \frac{1}{3}\pi h^3$ (since $r = h$), so $V' = \pi h^2 h'$. Then $12\pi = \pi(36)h'$ and $h' = \frac{1}{3}$.
5. Tip: 2.5 m/s. Shadow length: 1 m/s. Similar triangles: $\frac{5}{x+L} = \frac{2}{L}$ gives $5L = 2x + 2L$, $L = \frac{2}{3}x$. So $L' = \frac{2}{3}(1.5) = 1$ and the tip speed is $(x+L)' = \frac{5}{3}(1.5) = 2.5$.
6. $\frac{48}{7}$ ft/s. $x^2 + y^2 = 625$ with $y = 24$, $x = 7$. $x' = -\frac{yy'}{x} = -\frac{24(-2)}{7} = \frac{48}{7}$.
7. 12.5 m/s. $x = 4\tan\theta$, $x' = 4\sec^2\theta\,\theta'$. At $x = 3$, $\tan\theta = \frac{3}{4}$ and $\sec^2\theta = \frac{25}{16}$. $x' = 4\cdot\frac{25}{16}\cdot 2 = \frac{25}{2}$.
8. (a) $-\frac{3}{25}$ (units per second). (b) $\frac{5}{16}$.
   Since $\sin\theta = \frac{3}{5}$ in the first quadrant, $\cos\theta = \frac{4}{5}$ and $\tan\theta = \frac{3}{4}$. (a) $\frac{d}{dt}\cos\theta = -\sin\theta\,\theta' = -\frac{3}{5}(0.2) = -\frac{3}{25}$. (b) $\sec^2\theta = \frac{25}{16}$, so $\frac{d}{dt}\tan\theta = \frac{25}{16}(0.2) = \frac{5}{16}$.
9. 0.9 cm²/s. $A = \frac{1}{2}(6)(6)\sin\theta = 18\sin\theta$, $A' = 18\cos\theta\,\theta' = 18\cdot\frac{1}{2}\cdot 0.1 = 0.9$.

SET B · APPLICATIONS AND MIXED

10. (a) 0.096 rad/s. (b) 5.500 degrees/s.
    $\tan\theta = \frac{h}{800}$, $\sec^2\theta\,\theta' = \frac{h'}{800}$. At $h = 600$ the sight line is 1000, $\cos^2\theta = 0.64$, so $\theta' = \frac{120}{800}(0.64) = 0.096$. Multiply by $\frac{180}{\pi}$ to get 5.500.
11. 0.029 rad/s (the angle is increasing). $\tan\theta = \frac{3000}{x}$, so $\sec^2\theta\,\theta' = -\frac{3000x'}{x^2}$. With $x' = -240$, $x = 4000$: $\theta' = \frac{3000(240)}{4000^2}\cos^2\theta = 0.045(0.64) = 0.0288 \approx 0.029$.
12. 0.269 m/min. $r = \frac{h}{3}$, $V = \frac{\pi}{27}h^3$, $V' = \frac{\pi}{9}h^2h'$. So $1.5 = \frac{\pi}{9}(16)h'$ and $h' = \frac{13.5}{16\pi} \approx 0.269$.
13. (a) -0.626 m/s. (b) -2.026 m/s.
    Similar triangles: $\frac{5.5}{x+L} = \frac{1.7}{L}$ gives $3.8L = 1.7x$, $L = \frac{1.7}{3.8}x$. With $x' = -1.4$: $L' = \frac{1.7}{3.8}(-1.4) \approx -0.626$. The tip is at $x + L = \frac{5.5}{3.8}x$, which changes at $\frac{5.5}{3.8}(-1.4) \approx -2.026$.
14. (a) -0.600 m/s. (b) -0.154 rad/s.
    With ladder 6.5 and $y = 5.2$: $x = \sqrt{42.25 - 27.04} = 3.9$. (a) $y' = -\frac{xx'}{y} = -\frac{3.9(0.8)}{5.2} = -0.6$. (b) $\cos\theta = \frac{x}{6.5}$, $-\sin\theta\,\theta' = \frac{x'}{6.5}$, $\sin\theta = \frac{5.2}{6.5}$, so $\theta' = -\frac{0.8}{5.2} \approx -0.154$.
15. 28.274 cm³/s. $V = \frac{1}{3}\pi(0.75h)^2 h = 0.1875\pi h^3$, $V' = 0.5625\pi h^2 h'$. At $t = 3$: $0.5625\pi(64)(0.25) = 9\pi \approx 28.274$.
16. 3.333 km/min. $x = 1.5\tan\theta$, $x' = 1.5\sec^2\theta\,\theta'$. At $x = 2$: $\tan\theta = \frac{4}{3}$, $\sec^2\theta = \frac{25}{9}$. $x' = 1.5\cdot\frac{25}{9}\cdot 0.8 = \frac{10}{3} \approx 3.333$.

SET C · CHALLENGE

17. (a) $\frac{1}{\pi}$ m/min. (b) 4 m²/min.
    $h = 2r$, so $V = \frac{1}{3}\pi r^2(2r) = \frac{2}{3}\pi r^3$ and $V' = 2\pi r^2 r'$. At $h = 4$, $r = 2$: $8 = 2\pi(4)r'$, so $r' = \frac{1}{\pi}$. Base area $A = \pi r^2$, $A' = 2\pi r r' = 2\pi(2)\frac{1}{\pi} = 4$.
18. Height $5\sqrt{2} \approx 7.07$ m. From $x^2 + y^2 = 100$: $y' = -\frac{x}{y}x'$. The speeds are equal when $\frac{x}{y} = 1$, so $x = y = 5\sqrt{2}$. As $y \to 0$, $\frac{x}{y} \to \infty$, so the top's speed grows without bound (it would exceed any speed, even the speed of light). In reality the ladder loses contact with the wall earlier, so the model fails near the end.
19. $\frac{150}{\sqrt{13}} = \frac{150\sqrt{13}}{13} \approx 41.6$ km/h. $z^2 = x^2 + y^2 - 2xy\cos 60^\circ = x^2 + y^2 - xy$. At the moment $z^2 = 9 + 16 - 12 = 13$. Differentiate: $2zz' = 2xx' + 2yy' - x'y - xy' = 300 + 320 - 200 - 120 = 300$. So $z' = \frac{300}{2\sqrt{13}} = \frac{150}{\sqrt{13}}$.
20. $-\frac{3}{200}$ rad/min. $\tan\theta = \frac{12}{s}$, so $\sec^2\theta\,\theta' = -\frac{12s'}{s^2}$. At $s = 16$ the sight line is 20, $\cos^2\theta = \frac{16}{25}$. $\theta' = -\frac{12(0.5)}{256}\cdot\frac{16}{25} = -\frac{3}{200}$. The sun's elevation is decreasing.
21. (a) The level falls at 0.25 m/min. (b) 48 minutes.
    (a) $r = \frac{h}{2}$, $V = \frac{\pi}{12}h^3$, $V' = \frac{\pi}{4}h^2h'$. The surface area is $A = \pi r^2 = \frac{\pi}{4}h^2$. So $\frac{\pi}{4}h^2h' = -0.25\cdot\frac{\pi}{4}h^2$, giving $h' = -0.25$ (constant, for $h > 0$). (b) Constant rate: $12 \div 0.25 = 48$ minutes.

2.14 Higher-Order Derivatives and Mixed Differentiation

SET A · SKILLS

1. $f''(x) = 20x^3 - 24x$. $f'(x) = 5x^4 - 12x^2 + 1$, then differentiate again.
2. $y''' = 72x$. $y' = 12x^3 - 4x$, $y'' = 36x^2 - 4$, $y''' = 72x$.
3. $-9\sin(3x)$. $y' = 3\cos(3x)$, $y'' = -9\sin(3x)$.
4. $f'''(x) = -8e^{-2x}$. Each derivative multiplies by -2: $(-2)^3 = -8$.
5. (a) $f''(x) = -\frac{1}{x^2}$. (b) $f'''(x) = \frac{2}{x^3}$. Start from $f'(x) = x^{-1}$.
6. $y'' = (x^2 + 4x + 2)e^x$. $y' = (2x + x^2)e^x$, $y'' = (2 + 2x)e^x + (2x + x^2)e^x$.
7. $-\frac{1}{32}$. $f' = \frac{1}{2}x^{-1/2}$, $f'' = -\frac{1}{4}x^{-3/2}$, so $f''(4) = -\frac{1}{4\cdot 8}$.
8. $v(3) = -9$ and $a(3) = 6$. $v = 3t^2 - 12t$, $a = 6t - 12$.

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9. $\frac{d^2y}{dx^2} = \frac{2}{x^3}$. $y + xy' = 0$ gives $y' = -\frac{y}{x}$. Then $y'' = -\frac{y'x - y}{x^2} = \frac{2y}{x^2}$, and $y = \frac{1}{x}$ gives $\frac{2}{x^3}$.

SET B · APPLICATIONS AND MIXED

10. $v(1.5) \approx 0.167$, $a(1.5) \approx -0.390$; slowing down. $v = (2t - t^2)e^{-t}$ and $a = (t^2 - 4t + 2)e^{-t}$. Since $v > 0$ and $a < 0$ (opposite signs), the particle is slowing down.
11. $a(0.5) \approx -0.255$ m/s^2. $x' = -1.2\sin 3t$, $x'' = -3.6\cos 3t = -9(0.4\cos 3t) = -9x$. At $t = 0.5$: $-3.6\cos 1.5 \approx -0.255$.
12. $a(3) \approx 0.500$ m/s^2, $a(7) \approx -2.000$ m/s^2, average acceleration -0.250 m/s^2.
    Using the nearest table values: $a(3) \approx \frac{8-7}{4-2} = 0.5$ and $a(7) \approx \frac{1-5}{8-6} = -2$. The average acceleration is $\frac{v(8)-v(0)}{8} = \frac{1-3}{8} = -0.25$.
13. $C'(2) \approx -8.232$ mg/L per hour and $C''(2) \approx 2.470$ mg/L per hour squared. $C' = -15e^{-0.3t}$, $C'' = 4.5e^{-0.3t}$. The concentration is decreasing at about 8.232 mg/L per hour at $t = 2$, and since $C'' > 0$ the decrease is slowing down.
14. Speeding up on $(0, 0.860)$ and $(1.571, 3)$. $a = \cos t - t\sin t$, which equals 0 when $t\tan t = 1$, at $t \approx 0.860$. On $(0, 0.860)$: $v > 0$, $a > 0$. On $(0.860, 1.571)$: $v > 0$, $a < 0$. On $(1.571, 3)$: $v < 0$ (since $\cos t < 0$) and $a < 0$, so speeding up.
15. $\frac{dy}{dx} = -\frac{5}{4} = -1.25$ and $\frac{d^2y}{dx^2} = -\frac{21}{32} \approx -0.656$. $y' = -\frac{2x + y}{x + 2y}$. Differentiating again and substituting $x = 2$, $y = 1$, $y' = -\frac{5}{4}$ gives $y'' = -\frac{21}{32}$.
16. $f''(x) = (x - 2)e^{-x}$. Concavity changes at $x = 2$ (from down to up) at the point $(2, 0.271)$. $f' = (1 - x)e^{-x}$, $f'' = (x - 2)e^{-x}$; f'' changes sign at $x = 2$ and $f(2) = 2e^{-2} \approx 0.271$.

SET C · CHALLENGE

17. $f'(x) = 2x\ln x + x$, $f''(x) = 2\ln x + 3$, $f'''(1/2) = 4$. $f'''(x) = \frac{2}{x}$, so $f'''\left(\frac{1}{2}\right) = 4$.
18. $y' = e^{-x}(2\cos 2x - \sin 2x)$, $y'' = -e^{-x}(2\cos 2x - \sin 2x) + e^{-x}(-4\sin 2x - 2\cos 2x) = e^{-x}(-4\cos 2x - 3\sin 2x)$. Then $y'' + 2y' + 5y = e^{-x}(-4\cos 2x - 3\sin 2x + 4\cos 2x - 2\sin 2x + 5\sin 2x) = 0$.
19. $-\frac{9}{16}$. $3x^2 + 3y^2y' = 0$ gives $y' = -\frac{x^2}{y^2} = -\frac{1}{4}$ at $(1, 2)$. Quotient rule: $y'' = -\frac{2xy^2 - x^2 \cdot 2yy'}{y^4}$. At $(1, 2)$ the numerator is $8 - 4\left(-\frac{1}{4}\right) = 9$, so $y'' = -\frac{9}{16}$.
20. (a) $a = 3$, $b = -3$, $c = 1$. (b) No.
    (a) At $x = 1$: $f(1) = 1$, $f'(1) = 3$, $f''(1) = 6$. Match $a + b + c = 1$, $2a + b = 3$, $2a = 6$: $a = 3$, $b = -3$, $c = 1$. (b) $f'''(x) = 6$ for $x < 1$ and $f'''(x) = 0$ for $x > 1$. The one-sided limits differ, so $f'''(1)$ does not exist.
21. $a(t) = 4 - 2t$. Speeding up on $(0, 2)$ and $(4, 6)$; slowing down on $(2, 4)$; the particle changes direction at $t = 4$. $v = t(4 - t) > 0$ on $(0, 4)$ and < 0 on $(4, 6)$. $a > 0$ for $t < 2$ and $a < 0$ for $t > 2$. Same signs on $(0, 2)$ and $(4, 6)$, opposite on $(2, 4)$. At $t = 4$, v changes from positive to negative.

2.15 Multiple-Choice Practice on Derivatives

SET A · PART A [NO CALCULATOR]

1. C. This is the derivative of x^3 at $x = 2$: $3(2)^2 = 12$.
2. A. $f'(x) = 4(3x^2 + 1)^3 \cdot 6x$, so $f'(1) = 4(64)(6) = 1536$.
3. E. Quotient rule: $y' = \frac{(x^2 + 4) - x(2x)}{(x^2 + 4)^2} = \frac{4 - x^2}{(x^2 + 4)^2}$, which is $\frac{3}{25}$ at $x = 1$.
4. B. $f'(x) = 2\sec^2(2x)$. At $x = \frac{\pi}{8}$, $\sec^2\frac{\pi}{4} = 2$, so $f' = 4$.
5. D. Chain rule: $h'(2) = f'(g(2))\,g'(2) = f'(3) \cdot (-1) = 4(-1) = -4$.
6. B. Differentiate: $2x + y + xy' = 0$, so $y' = -\frac{2x + y}{x} = -\frac{5}{2}$ at $(2, 1)$.
7. A. $f'(x) = \frac{2x}{x^2 + 1}$, so $f'(1) = \frac{2}{2} = 1$.
8. D. Product rule: $f'(x) = 2e^{2x}\cos x - e^{2x}\sin x$. At $x = 0$: $2(1)(1) - 0 = 2$.
9. C. $f(1) = 4$, so $g(4) = 1$. $f'(x) = 3x^2 + 2$, $f'(1) = 5$, and $g'(4) = \frac{1}{f'(1)} = \frac{1}{5}$.
10. E. $\frac{d}{dx}\arctan u = \frac{u'}{1 + u^2} = \frac{2x}{1 + x^4}$. At $x = 1$ this is $\frac{2}{2} = 1$.
11. B. The diagonal is $d = s\sqrt{2}$, so $\frac{dd}{dt} = \sqrt{2}\,\frac{ds}{dt} = 3\sqrt{2}$ cm/s.
12. C. $V' = 4\pi r^2 r'$, so $32\pi = 4\pi(4)r' = 16\pi r'$ and $r' = 2$.
13. A. $f' = 2\cos 2x$ and $f'' = -4\sin 2x$. At $x = \frac{\pi}{4}$: $-4\sin\frac{\pi}{2} = -4$.
14. D. $v = 3(t - 1)(t - 3)$ and $a = 6(t - 2)$. Speeding up needs v and a of the same sign: both negative on $(1, 2)$, both positive for $t > 3$.
15. E. Continuity gives $a + b = 1$ (it must match $f(1) = 1$). Differentiability gives $a = 2$, so $b = -1$ and $a + b = 1$.
16. A. $y' = \ln x + 1 = 2$ at $x = e$, and $y(e) = e$. So $y - e = 2(x - e)$, i.e. $y = 2x - e$.
17. D. The slopes of the four segments are $1, 0, -2, 1$. The slope changes at $x = -1$, 1 and 3, giving three corners.
18. C. $f''(x) = (2x - x^2)e^{-x}$ by the product rule. At $x = 1$: $(2 - 1)e^{-1} = \frac{1}{e}$.
19. E. $\frac{u'}{\sqrt{1 - u^2}} = \frac{1/2}{\sqrt{1 - x^2/4}}$. At $x = 1$: $\frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$.
20. B. This is the derivative of $\sin x$ at $x = \frac{\pi}{6}$, which is $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

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SET B · PART B [CALCULATOR]

21. D. $f'(x) = -2xe^{-x^2}$, so $f'(0.5) = -e^{-0.25} \approx -0.779$.

22. A. $f'(x) = \dfrac{\cos x}{3 + \sin x}$. At $x = 2$: $\dfrac{-0.416}{3.909} \approx -0.106$.

23. E. $r = \dfrac{h}{4}$, $V = \dfrac{\pi}{48}h^3$, $V' = \dfrac{\pi}{16}h^2 h'$. So $0.8 = \dfrac{\pi}{16}(25)h'$ and $h' = \dfrac{12.8}{25\pi} \approx 0.163$.

24. C. $\tan\theta = \dfrac{h}{500}$, so $\theta' = \dfrac{h'}{500}\cos^2\theta$. The sight line is 1300, so $\cos^2\theta = \dfrac{25}{169}$ and $\theta' = 0.12 \cdot \dfrac{25}{169} \approx 0.018$.

25. B. Solve $f(x) = 0$: $x \approx 1.133$, so $g(0) \approx 1.133$. $f'(x) = 5x^4 + 1 \approx 9.239$, so $g'(0) = \dfrac{1}{9.239} \approx 0.108$.

26. E. The point $(2, 3)$ is on both curves, so $f(2) = 3$, and the tangent slope is $\dfrac{7 - (-1)}{4 - 0} = 2 = f'(2)$. Product rule: $g'(2) = f(2) + 2f'(2) = 3 + 4 = 7$.

27. C. Differentiate: $\sin y + x\cos y\, y' = 0$, so $y' = -\dfrac{\sin y}{x\cos y} = -\dfrac{\tan y}{x} = -\dfrac{0.577}{2} \approx -0.289$.

28. A. $A = \pi r^2$, $A' = 2\pi r r' = 2\pi(10)(0.8) = 16\pi \approx 50.265$.

29. D. $f' = \dfrac{\cos x}{2\sqrt{x}} - \sqrt{x}\sin x$ and $f'' = -\dfrac{\cos x}{4x^{3/2}} - \dfrac{\sin x}{\sqrt{x}} - \sqrt{x}\cos x$. At $x = 1$: $-0.135 - 0.841 - 0.540 \approx -1.517$.

30. B. Product rule: $\arcsin x + \dfrac{x}{\sqrt{1 - x^2}} = 0.6435 + \dfrac{0.6}{0.8} = 1.3935 \approx 1.394$.

2.16 Free-Response Practice on Derivatives

QUESTION 1 · TEMPERATURE OF A MIXTURE

1. (a) $T'(7) \approx \dfrac{T(9) - T(5)}{9 - 5} = \dfrac{41 - 35}{4} = 1.5$ degrees Celsius per minute. At $t = 7$ minutes, the temperature of the mixture is increasing at approximately 1.5 degrees Celsius per minute.

(b) The average rate of change on $[0, 12]$ is $\dfrac{T(12) - T(0)}{12 - 0} = \dfrac{43 - 20}{12} = \dfrac{23}{12}$. T is differentiable on $(0, 12)$ and therefore continuous on $[0, 12]$. By the Mean Value Theorem, there is a time c with $0 < c < 12$ such that $T'(c) = \dfrac{23}{12}$. So yes.

(c) $W'(t) = 25(0.18)e^{-0.18t} = 4.5e^{-0.18t}$, so $W'(7) = 4.5e^{-1.26} \approx 1.276$ degrees Celsius per minute. This is a little smaller than the table estimate of 1.5, which is plausible because the table estimate is an average over $[5, 9]$ and the model is only an approximation.

(d) Solve $4.5e^{-0.18t} = 1$: $e^{-0.18t} = \dfrac{1}{4.5}$, so $t = \dfrac{\ln 4.5}{0.18} \approx 8.356$ minutes. The rate of increase is $W'(t) = 4.5e^{-0.18t}$, and $W''(t) = -0.81e^{-0.18t} < 0$ for all t, so W' is decreasing: the temperature keeps rising, but more and more slowly.

Scoring: 9 points. (a) 2 points: 1 for the difference quotient with correct values, 1 for the interpretation with units. (b) 3 points: 1 for the average rate $\frac{23}{12}$, 1 for stating continuity and differentiability, 1 for the conclusion citing the Mean Value Theorem. (c) 2 points: 1 for $W'(t)$, 1 for the value 1.276 and a comparison. (d) 2 points: 1 for the time 8.356, 1 for $W'' < 0$ with the conclusion.

QUESTION 2 · GRAPH OF A PIECEWISE LINEAR FUNCTION

2. (a) The segment from $(-4, 0)$ to $(-2, 4)$ has slope $\frac{4-0}{-2-(-4)} = 2$, so $f'(-3) = 2$. The segment from $(0, 2)$ to $(2, 2)$ is horizontal, so $f'(1) = 0$. The function is not differentiable at $x = -2$, $x = 0$ and $x = 2$, because the slopes of the two adjacent segments differ at each of these points (slopes $2, -1, 0, -2$), so the graph has a corner there.

(b) On $[-2, 0]$ the segment runs from $(-2, 4)$ to $(0, 2)$, so $f(x) = 2 - x$ there. So $f(-1) = 3$ and $f'(-1) = -1$. Product rule: $h'(-1) = f(-1) + (-1)f'(-1) = 3 + 1 = 4$.

(c) Chain rule: $p'(x) = 2f'(2x - 1)$. At $x = -1$, $2x - 1 = -3$, and $f'(-3) = 2$, so $p'(-1) = 2 \cdot 2 = 4$.

(d) $f(-3) = 2$ (the midpoint of the first segment), so $m(2) = -3$. By the inverse function theorem, $m'(2) = \dfrac{1}{f'(m(2))} = \dfrac{1}{f'(-3)} = \dfrac{1}{2}$.

Scoring: 9 points. (a) 3 points: 1 for $f'(-3) = 2$, 1 for $f'(1) = 0$, 1 for the three x-values with the corner reason. (b) 2 points: 1 for the product rule, 1 for the value 4. (c) 2 points: 1 for the chain rule form $2f'(2x - 1)$, 1 for the value 4. (d) 2 points: 1 for $m(2) = -3$, 1 for $m'(2) = \frac{1}{2}$.

QUESTION 3 · DRAINING FILTER

3. (a) By similar triangles, $\dfrac{r}{h} = \dfrac{6}{12}$, so $r = \dfrac{h}{2}$. Then $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi\dfrac{h^2}{4}h = \dfrac{\pi}{12}h^3$.

(b) $\dfrac{dV}{dt} = \dfrac{\pi}{4}h^2\dfrac{dh}{dt}$. Then $-0.8\sqrt{9} = -2.4 = \dfrac{\pi}{4}(81)\dfrac{dh}{dt}$, so $\dfrac{dh}{dt} = \dfrac{-9.6}{81\pi} \approx -0.038$ cm/s. The depth is decreasing at about 0.038 cm per second.

(c) The surface area is $A = \pi r^2 = \dfrac{\pi}{4}h^2$, so $\dfrac{dA}{dt} = \dfrac{\pi}{2}h\dfrac{dh}{dt} = \dfrac{\pi}{2}(9)\left(-\dfrac{9.6}{81\pi}\right) = -\dfrac{8}{15} \approx -0.533$ cm²/s. The surface area is decreasing.

(d) $\dfrac{\pi}{4}h^2\dfrac{dh}{dt} = -0.8h^{1/2}$ gives $\dfrac{dh}{dt} = -\dfrac{3.2}{\pi h^{3/2}}$. As $h \to 0^+$, $h^{3/2} \to 0$, so $\left|\dfrac{dh}{dt}\right|$ increases without bound: the level falls faster and faster, because the same volume loss comes from a thinner and thinner cone.

Scoring: 9 points. (a) 2 points: 1 for the similar triangles $r = \frac{h}{2}$, 1 for the volume formula. (b) 3 points: 1 for differentiating with respect to t, 1 for the substitution, 1 for the value with units. (c) 2 points: 1 for $\frac{dA}{dt}$ in terms of h and $\frac{dh}{dt}$, 1 for the value and its sign. (d) 2 points: 1 for the formula, 1 for the explanation.

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QUESTION 4 · IMPLICIT CURVE

4. (a) Differentiate implicitly: $2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$. Then $(x + 2y)\frac{dy}{dx} = -(2x + y)$, so $\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$.

(b) At $(2, 1)$: $\frac{dy}{dx} = -\frac{5}{4}$. Tangent line: $y - 1 = -\frac{5}{4}(x - 2)$.

(c) A horizontal tangent needs $\frac{dy}{dx} = 0$, so $2x + y = 0$ and $y = -2x$. Substitute into the curve: $x^2 - 2x^2 + 4x^2 = 3x^2 = 7$, so $x = \pm\sqrt{\frac{7}{3}} = \pm\frac{\sqrt{21}}{3}$. The points are $\left(\frac{\sqrt{21}}{3}, -\frac{2\sqrt{21}}{3}\right)$ and $\left(-\frac{\sqrt{21}}{3}, \frac{2\sqrt{21}}{3}\right)$. At both points $x + 2y = -3x \neq 0$, so the slope formula is defined there.

(d) Quotient rule on $y' = -\frac{2x + y}{x + 2y}$: $y'' = -\frac{(2 + y')(x + 2y) - (2x + y)(1 + 2y')}{(x + 2y)^2}$. At $(2, 1)$ with $y' = -\frac{5}{4}$: $(2 - \frac{5}{4})(4) = 3$ and $(5)(1 - \frac{5}{2}) = -\frac{15}{2}$. So $y'' = -\frac{3 + \frac{15}{2}}{16} = -\frac{21}{32}$.

Scoring: 9 points. (a) 2 points: 1 for the implicit differentiation with all terms, 1 for solving. (b) 2 points: 1 for the slope $-\frac{5}{4}$, 1 for the line. (c) 3 points: 1 for setting $2x + y = 0$, 1 for substituting, 1 for both points. (d) 2 points: 1 for a correct expression for y'' using the quotient rule with y', 1 for the value $-\frac{21}{32}$.

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Chapter 3: Applications of Derivatives.

13 sections with explanations and worked examples, then 265 practice problems. Answers and full explanations are at the end of the chapter.

3.1 The Extreme Value Theorem and Critical Points

You will learn to locate absolute and relative extrema, identify critical numbers, and use the closed interval method, the dependable tool for finding the largest and smallest values of a function.

Many of the most useful questions in calculus ask for a largest or smallest value: the greatest height of a projectile, the lowest cost, the peak concentration of a drug. This section gives you the vocabulary and the reliable method for answering them.

Absolute and Relative Extrema

Let f be defined on an interval I containing c. Then $f(c)$ is the **absolute maximum** of f on I if $f(c) \geq f(x)$ for every x in I, and the **absolute minimum** if $f(c) \leq f(x)$ for every x in I.

Then $f(c)$ is a **relative (local) maximum** if $f(c) \geq f(x)$ for all x in some open interval around c, and a **relative minimum** if $f(c) \leq f(x)$ for all x in some open interval around c.

Extrema is the plural of extremum (a maximum or a minimum). The value $f(c)$ is the extreme value; the number c is where it occurs.

The Extreme Value Theorem

If f is **continuous on a closed interval** $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.

Both hypotheses matter: the function must be continuous at every point of the interval, and the interval must be closed and bounded. If either fails, the theorem promises nothing (extremes may or may not exist).

Critical Numbers and Fermat's Theorem

A number c in the domain of f is a **critical number** (and $(c, f(c))$ a critical point) if $f'(c) = 0$ or $f'(c)$ does not exist.

Fermat's Theorem. If f has a relative extremum at an interior point c, then c is a critical number of f.

The converse is false: a critical number may give a maximum, a minimum, or neither. Endpoints of a closed interval are not critical numbers, but they must still be checked.

1. **Confirm the hypotheses.** Check that f is continuous on the closed interval $[a, b]$.
2. **Find critical numbers.** Compute $f'(x)$ and solve $f'(x) = 0$; also find where f' does not exist. Keep only those numbers inside (a, b).
3. **Evaluate.** Compute f at each critical number in (a, b) and at both endpoints a and b.
4. **Compare.** The largest value is the absolute maximum and the smallest is the absolute minimum. State both the value and where it occurs.

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EXAMPLE 1

Find the absolute extrema of $f(x) = x^3 - 6x^2 + 9x + 1$ on $[0, 5]$.

SOLUTION

f is a polynomial, so it is continuous on $[0, 5]$ and the Extreme Value Theorem applies.

$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$, so the critical numbers are $x = 1$ and $x = 3$, both in $(0, 5)$.

$f(0) = 1$, $f(1) = 5$, $f(3) = 1$, $f(5) = 21$.

The absolute maximum is 21 at $x = 5$. The absolute minimum is 1, attained at both $x = 0$ and $x = 3$.

EXAMPLE 2

Find the absolute extrema of $f(x) = x^{5/3} - 5x^{2/3}$ on $[-1, 4]$. (Use the real cube root.)

SOLUTION

f is continuous on $[-1, 4]$.

$f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5(x-2)}{3x^{1/3}}$.

$f'(x) = 0$ at $x = 2$, and $f'(x)$ does not exist at $x = 0$, which is in the domain of f. Both are critical numbers.

$f(-1) = -1 - 5 = -6$, $f(0) = 0$, $f(2) = -3 \cdot 2^{2/3} \approx -4.762$, $f(4) = -4^{2/3} \approx -2.520$.

Absolute maximum 0 at $x = 0$; absolute minimum -6 at $x = -1$. Note the maximum occurs at a point where the derivative does not exist.

EXAMPLE 3

Does the Extreme Value Theorem guarantee extrema for (a) $g(x) = \dfrac{1}{(x-1)^2}$ on $[0, 3]$? (b) $h(x) = x^2$ on $(0, 3)$? Do extrema exist?

SOLUTION

(a) g is not continuous at $x = 1$, which lies in $[0, 3]$, so the theorem does not apply. Indeed $g(x) \to \infty$ as $x \to 1$, so g has no absolute maximum on $[0, 3]$ (it does have an absolute minimum, $g(3) = \frac{1}{4}$).

(b) The interval $(0, 3)$ is open, so the theorem does not apply. The values of h come arbitrarily close to 0 and 9 but never reach them, so h has no absolute maximum and no absolute minimum on $(0, 3)$.

EXAMPLE 4 [CALCULATOR]

The concentration of a chemical in a reaction vessel is $C(t) = 20te^{-t/2}$ grams per liter, for $0 \le t \le 6$ hours. Find the greatest and least concentrations.

SOLUTION

C is continuous on $[0, 6]$. $C'(t) = 20e^{-t/2}\left(1 - \frac{t}{2}\right)$, which is 0 only at $t = 2$.

$C(0) = 0$, $C(2) = 40e^{-1} \approx 14.715$, $C(6) = 120e^{-3} \approx 5.974$.

The greatest concentration is about 14.715 grams per liter at $t = 2$ hours; the least is 0 at $t = 0$.

Common mistakes

(1) Do not forget the endpoints: many absolute extrema occur there. (2) $f'(c) = 0$ does not guarantee an extremum: $f(x) = x^3$ has $f'(0) = 0$ but no extremum at 0. (3) A critical number must be in the domain of f: for $f(x) = 1/x$, f' does not exist at 0, but 0 is not a critical number. (4) Report the extreme value (a y-value) and where it occurs (an x-value).

Practice

SET A **Skills**

Find what is asked. [No calculator]

1. Find all critical numbers of $f(x) = x^3 - 3x^2 - 9x + 4$.
2. Find all critical numbers of $f(x) = x^4 - 8x^2$.
3. Find all critical numbers of $f(x) = \dfrac{x}{x^2 + 1}$.

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4. Find all critical numbers of $f(x) = x^{4/3} - 4x^{1/3}$.
5. Find the absolute maximum and minimum of $f(x) = 2\sin x + x$ on $[0, 2\pi]$.
6. Find the absolute maximum and minimum of $f(x) = x^3 - 12x$ on $[-3, 5]$.
7. Find the absolute maximum and minimum of $g(x) = xe^{-x}$ on $[0, 3]$.
8. Find the absolute maximum and minimum of $f(x) = |x^2 - 4|$ on $[-3, 3]$.
9. Let $f(x) = \begin{cases} x + 1 & 0 \le x < 2 \\ x - 4 & 2 \le x \le 4. \end{cases}$ Explain why the Extreme Value Theorem does not apply on $[0, 4]$, and show that f has no absolute maximum on $[0, 4]$.

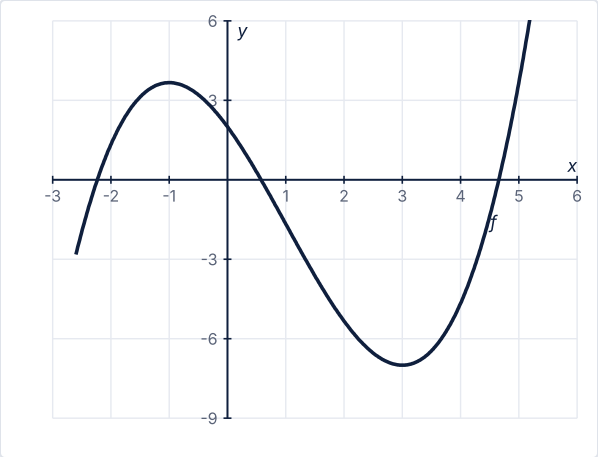
SET B Applications and Mixed

Solve each problem. [Calculator] Give decimal answers to three decimal places where they are not exact.

10. The concentration of a medication in the bloodstream is $C(t) = 8te^{-0.4t}$ milligrams per liter, $0 \le t \le 10$ hours. Find the maximum and minimum concentration on this interval and when each occurs.
11. The temperature in a greenhouse h hours after sunrise is $T(h) = \frac{1}{2}h^3 - 6h^2 + 18h + 10$ degrees, for $0 \le h \le 8$. Find the absolute maximum and minimum temperatures and when they occur.
12. The function f is continuous on $[0, 6]$ and differentiable on $(0, 6)$. The table gives some values, and $f'(x) = 0$ only at $x = 1, 3, 5$. Find the absolute maximum and minimum values of f on $[0, 6]$.

x	0	1	3	5	6
$f(x)$	3	7	-2	4	1

13. The graph shows $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 2$. Find the absolute maximum and minimum values of f on $[-2, 5]$ and the x-values where they occur.



Graph of f.

14. A ball is thrown upward from a platform so that its height in meters after t seconds is $h(t) = -4.9t^2 + 19.6t + 2$, for $0 \le t \le 4$. Find the greatest and least heights on this interval.
15. How many critical numbers does $f(x) = x^4 - 4x^3$ have?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

16. A company's net gain, in thousands of dollars, from producing x hundred items is $R(x) = -x^3 + 9x^2 - 15x + 4$ for $0 \le x \le 6$. Find the production level that gives the greatest net gain and the level that gives the least, with those values.

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SET C Challenge and AP-Style

Justify your answers completely. [No calculator]

17. The function $f(x) = x^3 + ax^2 + bx$ has critical numbers $x = 1$ and $x = 3$. Find a and b, then find the absolute extrema of f on $[0, 4]$.

18. Let $f(x) = x^2 - 2ax$ on $[0, 3]$, where $a > 0$ is a constant. For which values of a does the absolute minimum occur at an interior point of the interval? Find that minimum value.

19. The function f is continuous on $[-2, 6]$ and differentiable except at $x = 2$. Also $f'(x) = 0$ only at $x = 0$ and $x = 4$. Using the table, find the absolute maximum and minimum of f and say where each occurs.

x	-2	0	2	4	6
$f(x)$	1	3	-1	6	5

20. Show that $f(x) = x^3 + 3x + 1$ has no critical numbers. Then find its absolute extrema on $[-2, 2]$ and explain how the location of the extrema follows from your first result.

21. Let $f(x) = \begin{cases} x + 2 & x \neq 2 \\ 7 & x = 2. \end{cases}$ Find the absolute maximum and minimum of f on $[0, 4]$. Is f continuous on $[0, 4]$? Does this contradict the Extreme Value Theorem?

3.2 Rolle's Theorem and the Mean Value Theorem

You will state Rolle's Theorem and the Mean Value Theorem with their hypotheses, find the guaranteed values of c, and use the theorems to justify claims about speed, rates, and inequalities.

The Mean Value Theorem links the average rate of change of a function over an interval to its instantaneous rate at some interior moment. It is the logical engine behind nearly every later result about increasing, decreasing, and extrema.

Rolle's Theorem

Suppose f satisfies: (1) f is continuous on the closed interval $[a, b]$; (2) f is differentiable on the open interval (a, b); (3) $f(a) = f(b)$.
Then there is at least one number c in (a, b) such that $f'(c) = 0$.
Picture: if the graph starts and ends at the same height, somewhere between it has a horizontal tangent.

The Mean Value Theorem (MVT)

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there is at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Picture: somewhere between a and b the tangent line is parallel to the chord joining the endpoints. In motion language: at some moment the instantaneous velocity equals the average velocity.

Two Consequences

If $f'(x) = 0$ for every x in an interval, then f is constant on that interval.
If $f'(x) = g'(x)$ for every x in an interval, then $f(x) = g(x) + C$ for some constant C on that interval.
Both follow from the MVT: any two points of the interval give an average rate of change equal to a value of the derivative.

1. **Check the hypotheses first.** State that f is continuous on $[a, b]$ and differentiable on (a, b), and say why (for example, it is a polynomial, or it is a quotient whose denominator is nonzero on the interval).
2. **Compute the average rate.** Find $\frac{f(b) - f(a)}{b - a}$.
3. **Solve for c.** Solve $f'(c) = \frac{f(b) - f(a)}{b - a}$ and keep only solutions strictly between a and b.
4. **Write the conclusion.** In AP style: "Because f is continuous on [a,b] and differentiable on (a,b), the Mean Value Theorem guarantees a number c in (a,b) with f'(c) equal to the average rate of change."

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EXAMPLE 1

Verify Rolle's Theorem for $f(x) = x^3 - 4x$ on $[-2, 2]$ and find all values of c.

SOLUTION

f is a polynomial, so it is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$.

$f(-2) = -8 + 8 = 0$ and $f(2) = 8 - 8 = 0$, so $f(-2) = f(2)$.

$f'(x) = 3x^2 - 4 = 0$ gives $x = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$. Both lie in $(-2, 2)$, so there are two values of c.

EXAMPLE 2

Find the value of c guaranteed by the Mean Value Theorem for $f(x) = \sqrt{x+5}$ on $[-1, 4]$.

SOLUTION

f is continuous for $x \geq -5$ and differentiable for $x > -5$, so both hypotheses hold on $[-1, 4]$.

Average rate: $\dfrac{f(4) - f(-1)}{4 - (-1)} = \dfrac{3 - 2}{5} = \dfrac{1}{5}$.

$f'(x) = \dfrac{1}{2\sqrt{x+5}} = \dfrac{1}{5}$ gives $\sqrt{x+5} = \frac{5}{2}$, so $x + 5 = \frac{25}{4}$ and $c = \frac{5}{4}$, which is in $(-1, 4)$.

EXAMPLE 3

Explain why the Mean Value Theorem cannot be applied to $f(x) = |x|$ on $[-1, 3]$, and show that its conclusion fails.

SOLUTION

f is continuous on $[-1, 3]$, but f is not differentiable at $x = 0$, which is in $(-1, 3)$. Hypothesis (2) fails.

Average rate: $\frac{3-1}{3-(-1)} = \frac{1}{2}$. But $f'(x) = -1$ for $x < 0$ and $f'(x) = 1$ for $x > 0$, so f' is never $\frac{1}{2}$. The conclusion fails, which shows the differentiability hypothesis cannot be dropped.

EXAMPLE 4

A truck travels 120 kilometers in 1.5 hours along a straight road. Show that at some moment its speed is exactly 80 kilometers per hour.

SOLUTION

Let $s(t)$ be the position (in kilometers) at time t hours. Position is a differentiable function of time, so s is continuous on $[0, 1.5]$ and differentiable on $(0, 1.5)$.

Average velocity: $\dfrac{s(1.5) - s(0)}{1.5 - 0} = \dfrac{120}{1.5} = 80$ km/h.

By the Mean Value Theorem there is a time c in $(0, 1.5)$ with $s'(c) = 80$, so the speed is exactly 80 km/h at that moment.

EXAMPLE 5

Use the Mean Value Theorem to show that $e^x > 1 + x$ for all $x > 0$.

SOLUTION

Fix $x > 0$ and let $f(t) = e^t$ on $[0, x]$. It is continuous on $[0, x]$ and differentiable on $(0, x)$.

There is c in $(0, x)$ with $e^c = \dfrac{e^x - e^0}{x - 0} = \dfrac{e^x - 1}{x}$, so $e^x - 1 = x\,e^c$.

Since $c > 0$, $e^c > 1$, and since $x > 0$, $x\,e^c > x$. Therefore $e^x - 1 > x$, that is, $e^x > 1 + x$.

Common mistakes

(1) Skipping the hypotheses: always verify continuity on the closed interval and differentiability on the open interval. (2) Keeping a solution for c that is not strictly inside (a, b). (3) Believing the theorem tells you how many values of c exist or what they are: it only guarantees at least one. (4) Confusing the converse: failing a hypothesis does not prove the conclusion is false, only that the theorem gives no guarantee.

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Practice

SET A Skills

Answer each question, checking hypotheses where appropriate. [No calculator]

1. Verify that $f(x) = x^2 - 6x + 5$ satisfies the hypotheses of Rolle's Theorem on $[1, 5]$ and find c.
2. Verify the hypotheses of Rolle's Theorem for $f(x) = x^3 - 3x^2$ on $[0, 3]$ and find all values of c in $(0, 3)$.
3. Verify Rolle's Theorem for $f(x) = \sin(2x)$ on $\left[0, \frac{\pi}{2}\right]$ and find all values of c.
4. Find the value of c guaranteed by the Mean Value Theorem for $f(x) = x^2 + 3x$ on $[1, 4]$.
5. Find the value of c guaranteed by the Mean Value Theorem for $f(x) = x^3 - 2x$ on $[0, 2]$.
6. Find the value of c guaranteed by the Mean Value Theorem for $f(x) = \ln x$ on $[1, e]$.
7. Find the value of c guaranteed by the Mean Value Theorem for $f(x) = \dfrac{1}{x}$ on $[1, 4]$.
8. Let $f(x) = \dfrac{1}{x^2}$. Then $f(-1) = f(1)$, but $f'(x) = -\dfrac{2}{x^3}$ is never 0. Why does this not contradict Rolle's Theorem on $[-1, 1]$?
9. Let $f(x) = |x - 2|$ on $[0, 4]$. Show that $f(0) = f(4)$ but there is no c in $(0, 4)$ with $f'(c) = 0$. Which hypothesis of Rolle's Theorem fails?

SET B Applications and Mixed

Solve each problem. Justify your use of any theorem. [Calculator]

10. A car passes mile marker 12 at 2:00 pm and mile marker 57 at 2:30 pm on a straight highway. Show that at some time between 2:00 and 2:30 its speed was exactly 90 miles per hour.
11. The temperature $T(t)$ of a liquid, in degrees Celsius, is a differentiable function of time t in minutes. Some readings are shown.

    (a) Explain why $T'(c) = 6$ for some c in $(0, 3)$.

    (b) Find an interval on which $T'(c) = -3$ must occur for some c.

    (c) Must there be a time in $(0, 10)$ when $T'(t) = 1$? Explain.

t (min)	0	3	7	10
$T(t)$	12	30	18	22

12. Use the Mean Value Theorem to show that $\ln x < x - 1$ for all $x > 1$.
13. [Calculator] Find the value of c guaranteed by the Mean Value Theorem for $f(x) = x \ln x$ on $[1, 5]$. Round to three decimal places.
14. [Calculator] For $f(x) = e^x - x^2$ on $[0, 2]$, find all values of c in $(0, 2)$ that satisfy the conclusion of the Mean Value Theorem. Round to three decimal places.
15. Show that $g(x) = \arctan x + \arctan \dfrac{1}{x}$ is constant on $(0, \infty)$ and find the constant.
16. Which value of c satisfies the conclusion of the Mean Value Theorem for $f(x) = x^3$ on $[0, 3]$?

    (A) $\frac{3}{2}$ **(B)** $\sqrt{3}$ **(C)** 2 **(D)** $\frac{9}{4}$ **(E)** 3

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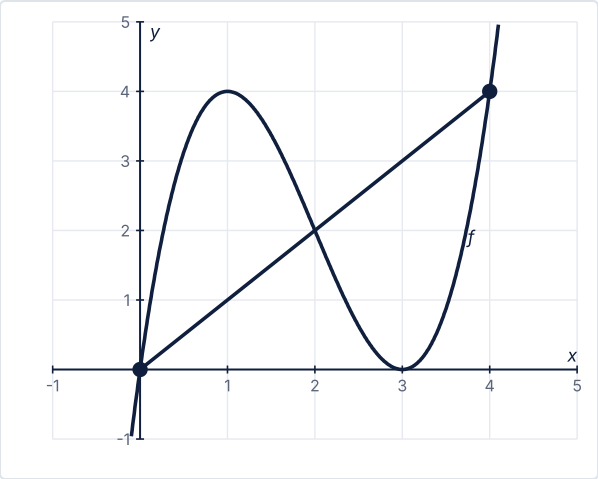
SET C Challenge and AP-Style

Give complete justifications. [No calculator]

17. The function $f(x) = x^2 + bx + 3$ satisfies the hypotheses of Rolle's Theorem on $[1, 3]$. Find b and the value of c guaranteed by the theorem.

18. Let $f(x) = \begin{cases} x^2 & x \le 1 \\ ax + b & x > 1 \end{cases}$ be differentiable for all x. Find a and b. Then find all values of c in $(0, 3)$ guaranteed by the Mean Value Theorem on $[0, 3]$.

19. The graph shows $f(x) = x^3 - 6x^2 + 9x$ on $[0, 4]$ and the chord joining $(0, 0)$ and $(4, 4)$. Find, in exact form, all values of c in $(0, 4)$ at which the tangent line is parallel to the chord.



Graph of f with the chord from x = 0 to x = 4.

20. Use the Mean Value Theorem to prove that $|\sin x - \sin y| \le |x - y|$ for all real numbers x and y.

21. Show that for $x > 0$, $\dfrac{x}{1+x} < \ln(1+x) < x$.

3.3 Increasing, Decreasing, and the First Derivative Test

You will use the sign of f' to find where a function increases and decreases, classify critical numbers with the First Derivative Test, and write the justifications that earn full credit.

The sign of the derivative tells you which way a function is heading. By reading where f' is positive or negative, you can describe exactly where f rises and falls and decide whether each critical number is a maximum, a minimum, or neither.

Increasing and Decreasing Test

Let f be continuous on an interval I and differentiable at every interior point of I.

If $f'(x) > 0$ for all interior x in I, then f is **increasing** on I. If $f'(x) < 0$ for all interior x in I, then f is **decreasing** on I.

Increasing means $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I. The test is a consequence of the Mean Value Theorem.

The First Derivative Test

Let c be a critical number of f with f continuous at c.

If f' changes from positive to negative at c, then $f(c)$ is a relative maximum. If f' changes from negative to positive at c, then $f(c)$ is a relative minimum. If f' does not change sign at c, then $f(c)$ is not a relative extremum.

Only a sign change of f' matters, not the value of $f'(c)$.

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1. **Find the domain and f'.** Note any numbers excluded from the domain of f.
2. **Locate the dividing points.** Solve $f'(x) = 0$ and find where f' does not exist. Also include any numbers where f itself is undefined. Place these on a number line.
3. **Test each interval.** Choose a convenient number in each interval, or use the factored form of f', to find the sign of f'.
4. **Read the chart.** Where $f' > 0$, f is increasing; where $f' < 0$, decreasing. Classify each critical number by the change of sign, and compute the y-values.
5. **Justify in words.** Write a sentence that names the function, the sign change, and the conclusion.

AP-style justification

"f has a relative maximum at x = c because f'(x) changes from positive to negative at x = c." Always name f' (not f), state the change of sign, and say where it happens. A table of signs alone, with no sentence, is usually not enough. To justify an absolute extremum on an interval, you may add: "f' is negative for x < c and positive for x > c, so f(c) is the absolute minimum."

EXAMPLE 1

Find the intervals on which $f(x) = 2x^3 - 9x^2 + 12x + 1$ is increasing or decreasing, and find all relative extrema.

SOLUTION

$f'(x) = 6x^2 - 18x + 12 = 6(x - 1)(x - 2)$, defined everywhere, with zeros $x = 1$ and $x = 2$.

Sign of f': positive for $x < 1$, negative for $1 < x < 2$, positive for $x > 2$.

f is increasing on $(-\infty, 1)$ and $(2, \infty)$, and decreasing on $(1, 2)$.

f' changes from positive to negative at $x = 1$: relative maximum $f(1) = 6$. f' changes from negative to positive at $x = 2$: relative minimum $f(2) = 5$.

EXAMPLE 2

The derivative of f is $f'(x) = (x + 2)(x - 1)^2(x - 4)$. Classify each critical number of f.

SOLUTION

Critical numbers: $x = -2$, 1, 4 (f' is a polynomial, so it exists everywhere).

For $x < -2$: $(-)(+)(-) = +$. For $-2 < x < 1$: $(+)(+)(-) = -$. For $1 < x < 4$: $(+)(+)(-) = -$. For $x > 4$: $(+)(+)(+) = +$.

At $x = -2$, f' changes from $+$ to $-$: relative maximum. At $x = 4$, f' changes from $-$ to $+$: relative minimum.

At $x = 1$, the factor $(x - 1)^2$ is squared, so f' does not change sign (negative on both sides): there is no relative extremum at $x = 1$.

EXAMPLE 3

Find the relative extrema of $f(x) = x^{4/3} + 4x^{1/3}$.

SOLUTION

$f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4(x + 1)}{3x^{2/3}}$.

Critical numbers: $x = -1$ ($f' = 0$) and $x = 0$ (f' does not exist, but 0 is in the domain of f).

Since $x^{2/3} > 0$ for $x \neq 0$, the sign of f' is the sign of $x + 1$: negative for $x < -1$, positive for $-1 < x < 0$, positive for $x > 0$.

Relative minimum $f(-1) = 1 - 4 = -3$ at $x = -1$. At $x = 0$ the sign does not change, so there is no extremum there (the graph has a vertical tangent).

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EXAMPLE 4

Analyze $f(x) = \dfrac{x^2}{x-2}$: intervals of increase and decrease and relative extrema.

SOLUTION

The domain is all $x \neq 2$. $f'(x) = \dfrac{2x(x-2) - x^2}{(x-2)^2} = \dfrac{x(x-4)}{(x-2)^2}$.

Dividing points: $x = 0$, $x = 4$ (zeros of f') and $x = 2$ (not in the domain).

Signs: $x < 0$: $(-)(-)/(+) = +$. $0 < x < 2$: $(+)(-)/(+) = -$. $2 < x < 4$: $-$. $x > 4$: $+$.

Increasing on $(-\infty, 0)$ and $(4, \infty)$; decreasing on $(0, 2)$ and $(2, 4)$. Relative maximum $f(0) = 0$, relative minimum $f(4) = 8$. Do not combine $(0, 2)$ and $(2, 4)$ into one interval, since f is undefined at 2.

EXAMPLE 5

Show that $f(x) = x + \dfrac{4}{x}$ has an absolute minimum on $(0, \infty)$ and find it.

SOLUTION

$f'(x) = 1 - \dfrac{4}{x^2} = \dfrac{(x-2)(x+2)}{x^2}$. On $(0, \infty)$ the only critical number is $x = 2$.

For $0 < x < 2$, $f'(x) < 0$; for $x > 2$, $f'(x) > 0$.

Because f' changes from negative to positive at the only critical number, and f decreases before it and increases after it, $f(2) = 4$ is the absolute minimum value of f on $(0, \infty)$.

Common mistakes

(1) Concluding an extremum from $f'(c) = 0$ without checking for a sign change. (2) Testing the sign of f instead of f'. (3) Forgetting numbers where f' does not exist, or numbers where f is undefined. (4) Calling an interval "increasing" when it contains a point outside the domain. (5) Stating only the x-value of an extremum when the question asks for the extreme value.

Practice

SET A **Skills**

Answer each question and justify your conclusions. [No calculator]

1. For $f(x) = x^3 - 3x^2 - 24x + 5$, find the intervals of increase and decrease and all relative extrema.
2. For $f(x) = x^4 - 4x^3$, find the intervals on which f is increasing or decreasing and classify each critical number.
3. For $f(x) = \dfrac{x}{x^2 + 1}$, find where f is increasing and decreasing and find the relative extrema.
4. For $f(x) = x^2 e^x$, find the intervals of increase and decrease and the relative extrema.
5. For $f(x) = x - 2\ln x$ (with $x > 0$), find the intervals of increase and decrease and the relative extremum.
6. For $f(x) = x - 2\sin x$ on $(0, 2\pi)$, find where f is increasing and decreasing and the x-values of the relative extrema.
7. The derivative of f is $f'(x) = x(x-2)^2(x+1)$. Find the x-values where f has a relative maximum and where it has a relative minimum. Justify.
8. The function f is continuous for all x. The table gives information about f'. Find the x-values of all relative extrema of f and classify each. Assume that on each open interval between consecutive tabled x-values, f' keeps a single sign, matching the nonzero tabled values at its ends.

x	-5	-4	-1	1	2	3	4
$f'(x)$	-2	0	3	DNE	1	0	-5

9. Find the x-values of the relative extrema of $f(x) = x^3 - 3x$ and write an AP-style justification for each.

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SET B Applications and Mixed

Solve each problem. [Calculator] Give decimals to three places where not exact.

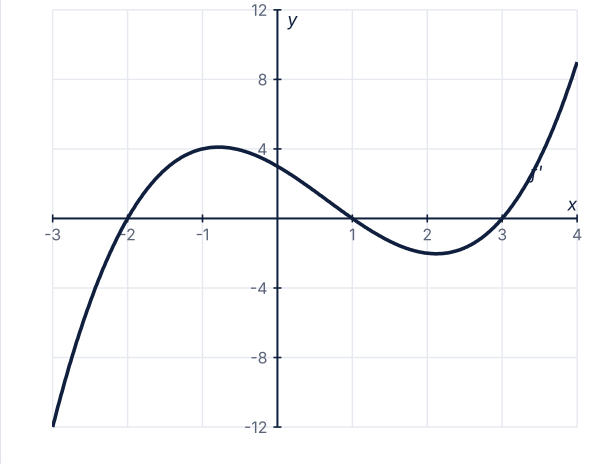
10. A particle moves along a line with position $x(t) = t^3 - 9t^2 + 24t$ meters for $t \geq 0$ seconds. Find when it moves right and left, and its positions at the times it turns around.

11. The concentration of a drug in the blood is $C(t) = \dfrac{12t}{t^2 + 4}$ milligrams per liter, $t \geq 0$ hours. Show that the concentration increases and then decreases, and find the maximum concentration and when it occurs.

12. The average cost per unit of producing x units is $A(x) = \dfrac{x}{50} + \dfrac{200}{x}$ dollars, for $x > 0$. Use the First Derivative Test to show that A has an absolute minimum, and find where it occurs and its value.

13. [Calculator] For $f(x) = x^3 e^{-x}$, find the intervals on which f is increasing or decreasing, and the relative extrema. Give decimals to three places.

14. The graph shows $f'(x) = \frac{1}{2}(x+2)(x-1)(x-3)$, the derivative of a function f. Find the intervals where f is increasing and decreasing, and the x-values of its relative maxima and minima.



Graph of f'.

15. The derivative of f is $f'(x) = (x-2)(x+1)^2$. At which x-values does f have a relative minimum?

(A) $x = -1$ only **(B)** $x = 2$ only
(C) $x = -1$ and $x = 2$ **(D)** $x = 0$ only
(E) f has no relative minimum

16. On which interval is $f(x) = x^3 - 6x^2 + 9x + 2$ decreasing?

(A) $(-\infty, 1)$ **(B)** $(0, 3)$ **(C)** $(3, \infty)$ **(D)** $(1, 3)$ **(E)** $(-\infty, 3)$

SET C Challenge and AP-Style

Give complete justifications. [No calculator]

17. The function $f(x) = x^3 + ax^2 + bx + c$ has a relative maximum at $x = -1$ with value 7 and a relative minimum at $x = 3$. Find a, b, c and the relative minimum value.

18. The function f is continuous on $(-\infty, \infty)$. It is given that $f'(x) > 0$ for $x < -2$; $f'(-2)$ does not exist; $f'(x) < 0$ for $-2 < x < 1$; $f'(1) = 0$; $f'(x) < 0$ for $1 < x < 4$; $f'(4) = 0$; and $f'(x) > 0$ for $x > 4$. Classify each critical number of f and justify.

19. Show that $f(x) = 2x + \sin x$ is increasing on the entire real line. Explain why $f(x) = 0$ has exactly one real solution.

20. Let $h(x) = \dfrac{\ln x}{x}$ for $x > 0$. Show that h has an absolute maximum at $x = e$, and use this to prove that $\ln x \leq \dfrac{x}{e}$ for all $x > 0$.

21. For which values of k does $f(x) = x^3 - 3x + k$ have exactly three real zeros? Use relative extrema in your argument.

3.4 Concavity and the Second Derivative Test

You will use the second derivative to describe how a graph bends, locate points of inflection, and classify critical numbers with the Second Derivative Test, always with a clear justification.

The first derivative tells you whether a graph rises or falls; the second derivative tells you how it bends. Concavity explains why a rising graph can speed up or slow down, and it gives a second way to classify critical numbers.

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Concavity

Let f be differentiable on an interval I. The graph of f is **concave up** on I if f' is increasing on I, and **concave down** on I if f' is decreasing on I.

Concavity Test. If $f''(x) > 0$ for all x in I, the graph is concave up on I. If $f''(x) < 0$ for all x in I, it is concave down on I.

Concave up: the graph lies above its tangent lines and looks like a cup. Concave down: it lies below its tangent lines and looks like a cap.

Points of Inflection

A point $(c, f(c))$ is a **point of inflection** of f if f is continuous at c and the concavity of the graph changes at c (that is, f'' changes sign at c).

If $(c, f(c))$ is a point of inflection and $f''(c)$ exists, then $f''(c) = 0$. The converse is false: $f''(c) = 0$ does not guarantee an inflection point.

Always confirm a sign change of f''. For $f(x) = x^4$, $f''(0) = 0$, but $f''(x) \geq 0$ on both sides, so there is no inflection point.

The Second Derivative Test

Suppose $f'(c) = 0$ and f'' exists near c and is continuous at c.

If $f''(c) > 0$, then $f(c)$ is a relative minimum. If $f''(c) < 0$, then $f(c)$ is a relative maximum. If $f''(c) = 0$, the test is **inconclusive**: use the First Derivative Test (or another argument).

It applies only at numbers where $f'(c) = 0$, not where f' fails to exist.

1. **Compute f''.** Differentiate twice and simplify. Note the domain.
2. **Find dividing points.** Solve $f''(x) = 0$ and find where f'' does not exist (in the domain of f).
3. **Make a sign chart for f''.** Test each interval. Positive means concave up; negative means concave down.
4. **Identify inflection points.** A dividing point is an inflection point only if the sign of f'' changes there and f is continuous. Give both coordinates.
5. **Justify.** "The graph of f has a point of inflection at x = c because f'' changes sign from negative to positive at x = c."

EXAMPLE 1

Find the concavity and inflection points of $f(x) = x^3 - 6x^2 + 9x + 1$.

SOLUTION

$f'(x) = 3x^2 - 12x + 9$, so $f''(x) = 6x - 12 = 6(x - 2)$.

$f''(x) < 0$ for $x < 2$ and $f''(x) > 0$ for $x > 2$.

Concave down on $(-\infty, 2)$, concave up on $(2, \infty)$. Since f'' changes sign at $x = 2$, there is an inflection point at $(2, f(2)) = (2, 3)$.

EXAMPLE 2

Find the concavity and inflection points of $f(x) = x^4 - 4x^3$.

SOLUTION

$f'(x) = 4x^3 - 12x^2$, $f''(x) = 12x^2 - 24x = 12x(x - 2)$, zero at $x = 0$ and $x = 2$.

Signs of f'': positive for $x < 0$, negative for $0 < x < 2$, positive for $x > 2$.

Concave up on $(-\infty, 0)$ and $(2, \infty)$; concave down on $(0, 2)$. The sign changes at both zeros, so the inflection points are $(0, 0)$ and $(2, -16)$.

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EXAMPLE 3

Use the Second Derivative Test to classify the critical numbers of $f(x) = 2x^3 - 3x^2 - 12x$.

SOLUTION

$f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$, so the critical numbers are $x = -1$ and $x = 2$.

$f''(x) = 12x - 6$.

$f''(-1) = -18 < 0$: relative maximum, $f(-1) = 7$. $f''(2) = 18 > 0$: relative minimum, $f(2) = -20$.

EXAMPLE 4

For each of $f(x) = x^4$, $g(x) = -x^4$ and $h(x) = x^3$, the derivative is 0 at $x = 0$ and the second derivative is 0 there. Classify $x = 0$ for each.

SOLUTION

The Second Derivative Test is inconclusive in all three cases, so we use the First Derivative Test.

$f'(x) = 4x^3$ changes from negative to positive at 0: relative minimum. $g'(x) = -4x^3$ changes from positive to negative: relative maximum. $h'(x) = 3x^2$ is positive on both sides: neither.

The three functions show that $f''(c) = 0$ can accompany a minimum, a maximum, or no extremum at all.

EXAMPLE 5

Let $f(x) = xe^{-x}$. Find the inflection point, and use the Second Derivative Test to classify the critical number. Write the justifications.

SOLUTION

$f'(x) = e^{-x}(1-x)$, zero only at $x = 1$. $f''(x) = e^{-x}(x-2)$.

$f''(1) = -e^{-1} < 0$, so f has a relative maximum at $x = 1$: "f'(1)=0 and f''(1)<0." The value is $f(1) = \frac{1}{e}$.

$f''(x) = e^{-x}(x-2)$ is negative for $x < 2$ and positive for $x > 2$ (since $e^{-x} > 0$). The graph is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$, with an inflection point at $\left(2, \frac{2}{e^2}\right)$ because f'' changes sign at $x = 2$.

Common mistakes

(1) Calling every zero of f'' an inflection point without checking for a sign change. (2) Using the Second Derivative Test where $f'(c) \neq 0$. (3) Treating an inconclusive result ($f''(c) = 0$) as "no extremum": it means no information. (4) Mixing up the two derivatives: increasing/decreasing is about f', concavity is about f''. (5) Giving only the x-coordinate of an inflection point when the point is requested.

Practice

SET A **Skills**

Answer each question and justify your conclusions. [No calculator]

1. Find the intervals of concavity and the inflection points of $f(x) = x^4 - 6x^2$.
2. Find the intervals of concavity and the inflection point of $f(x) = 2x^3 - 9x^2 + 12x$.
3. Find the inflection point of $f(x) = xe^x$ and the intervals of concavity.
4. Find the concavity and inflection points of $f(x) = \ln(x^2 + 1)$.
5. Find the concavity of $f(x) = \sin x$ on $(0, 2\pi)$ and the inflection point.
6. Use the Second Derivative Test to find the relative extrema of $f(x) = x^3 - 3x^2 - 24x$.
7. Use the Second Derivative Test to classify the critical numbers of $f(x) = x + \dfrac{4}{x}$.
8. The Second Derivative Test fails for both $f(x) = x^4$ and $g(x) = x^3$ at $x = 0$. For each, state whether $(0, 0)$ is a relative extremum and whether it is a point of inflection, with reasons.
9. Show that $f(x) = x^{1/3}$ has a point of inflection at $x = 0$ even though $f''(0)$ does not exist.

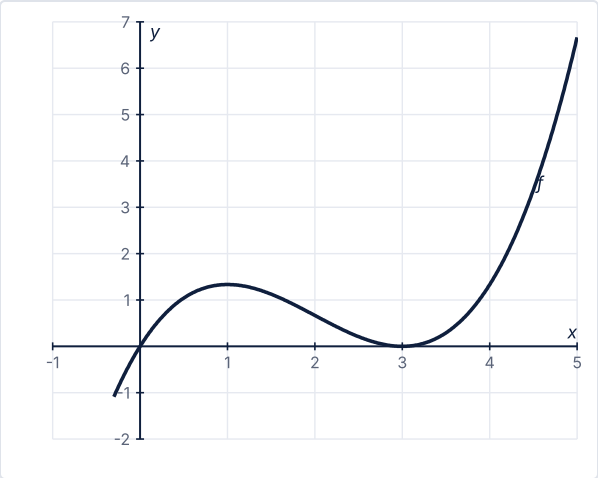
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SET B Applications and Mixed

Solve each problem. [Calculator] Give decimals to three places where not exact.

10. The graph shows $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x$. Find the interval(s) on which the graph is concave up, concave down, and the coordinates of the inflection point.



Graph of f.

11. [Calculator] A population of bacteria is modeled by $P(t) = \dfrac{1000}{1 + 9e^{-0.5t}}$ for $t \geq 0$ hours. Show that P has an inflection point, find t (to three decimal places) and the population there, and explain what the inflection point means for the growth rate.

12. The total output of a process is $Q(h) = -h^3 + 12h^2 + 20h$ units after h hours, $0 \leq h \leq 10$. At what time is the rate of production $Q'(h)$ greatest? Justify with Q'' and give the maximum rate.

13. [Calculator] A cooling liquid has temperature $T(t) = 20 + 60e^{-0.3t}$ degrees Celsius after t minutes. Show that the graph of T is decreasing and concave up for all $t \geq 0$. Find $T'(2)$ to three decimal places, and explain in words what concavity says about the cooling.

14. The second derivative of f is $f''(x) = x(x-2)^2$. At which x-values does the graph of f have a point of inflection?

(A) $x = 2$ only **(B)** $x = 0$ and $x = 2$

(C) $x = -2$ and $x = 0$ **(D)** $x = 0$ only

(E) f has no points of inflection

15. A function f has $f'(3) = 0$ and $f''(3) = 0$. Which statement must be true?

(A) f has a relative maximum at $x = 3$

(B) f has a relative minimum at $x = 3$

(C) f has a point of inflection at $x = 3$

(D) f has neither a relative maximum nor a relative minimum at $x = 3$

(E) The Second Derivative Test gives no conclusion about $x = 3$

16. If $f'(x) = x^3 - 3x$, on which interval is the graph of f concave down?

(A) $(-\infty, -1)$ **(B)** $(-\infty, 0)$ **(C)** $(-1, 1)$ **(D)** $(0, \infty)$ **(E)** $(1, \infty)$

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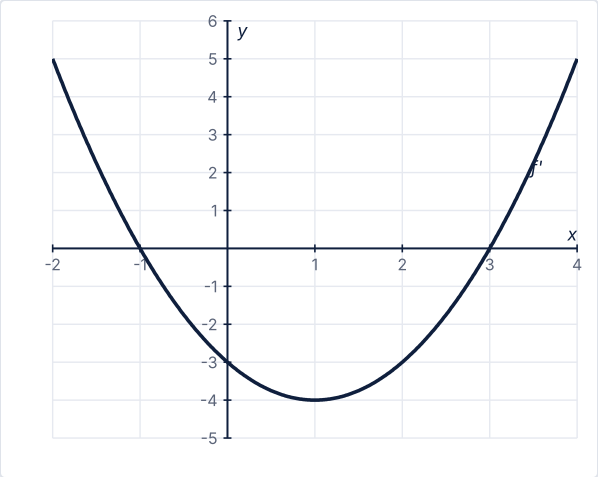
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SET C Challenge and AP-Style

Give complete justifications. [No calculator]

17. The graph of $f(x) = ax^3 + bx^2$ has a point of inflection at $(1, 6)$. Find a and b and verify that the concavity changes there.

18. The graph shows $f'(x) = x^2 - 2x - 3$, the derivative of a function f.
    (a) On what intervals is f increasing? Where are its relative extrema?
    (b) On what intervals is the graph of f concave up and concave down?
    (c) Does f have a point of inflection? Justify.



Graph of f'.

19. Show that every cubic $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$ has exactly one point of inflection, and find its x-coordinate.

20. Let $f(x) = x - k\ln x$ for $x > 0$, where $k > 0$. Show that f has an absolute minimum at $x = k$ and find the value of k for which that minimum value equals 0.

21. The function f is twice differentiable. Some values are shown. Classify each of $x = 1$ and $x = 3$ as a relative maximum, relative minimum, or not determined, and explain what can be said about $x = 2$.

x	1	2	3
$f'(x)$	0	-1	0
$f''(x)$	-2	0	4

3.5 Reading the Graphs of f, f', and f"

You will translate between the graphs of a function and its first two derivatives, reading increasing, extrema, concavity, and inflection points from whichever graph you are given.

AP questions often show you the graph of one of f, f', or f'' and ask about another. The skill is translation: the sign of a derivative describes the behavior of the function one level down, and the slope of a derivative graph describes the sign of the next derivative.

If the graph of ... shows	then the graph of the function one level below ...
$f' > 0$ **(graph of f' above the x-axis)**	f is increasing
$f' < 0$ **(below the x-axis)**	f is decreasing
f' **crosses the x-axis from above to below**	f has a relative maximum
f' **crosses the x-axis from below to above**	f has a relative minimum
f' **is increasing (positive slope), i.e. $f'' > 0$**	f is concave up
f' **is decreasing (negative slope), i.e. $f'' < 0$**	f is concave down
f' **has a relative maximum or minimum (its slope changes sign)**	f has a point of inflection
f'' **changes sign**	f has a point of inflection

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Reading the Graph of f'

The **zeros** of f' (where it crosses the x-axis) locate the relative extrema of f. The **sign** of f' tells where f rises or falls. The **peaks and valleys** of f' locate the inflection points of f. The **steepness** of f' is f''.

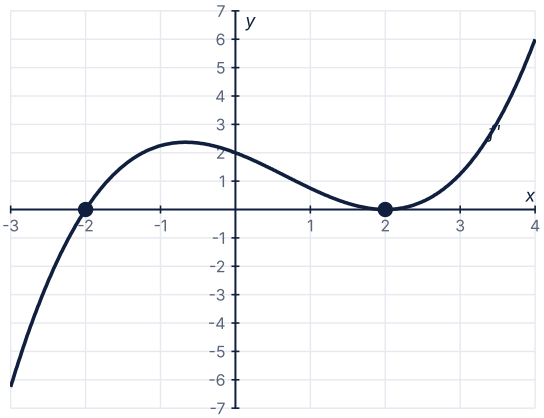
A common trap: a point where the graph of f' merely touches the x-axis and turns back (a double zero) is a critical number of f but not an extremum.

Reading the Graph of f

Where the graph of f rises, $f' > 0$; where it falls, $f' < 0$; at a smooth peak or valley, $f' = 0$. Where the graph is cup-shaped, $f'' > 0$; where it is cap-shaped, $f'' < 0$; where the bend switches, f'' changes sign.

At a corner or cusp, f' does not exist. A vertical tangent also gives a derivative that does not exist.

1. **Identify what the graph shows.** Look at the label: is it f, f', or f''? Everything you say must be translated from that graph.
2. **Mark the key x-values.** Locate zeros and sign changes for the function shown, and its peaks, valleys, and corners.
3. **Translate level by level.** Use the table above to turn each feature into a statement about the target function.
4. **Write the reason.** Name the graph you used: "f has a relative maximum at x = 3 because f' changes from positive to negative at x = 3."



Graph of f' (Example 1).

EXAMPLE 1

The graph shows f' for a function f. Describe where f increases and decreases, its relative extrema, and its concavity and inflection points.

SOLUTION

For a precise reading, note that $f'(x) = \frac{1}{4}(x+2)(x-2)^2$.

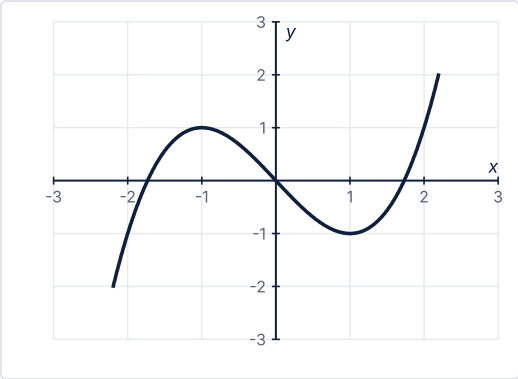
$f' < 0$ for $x < -2$ and $f' > 0$ for $-2 < x < 2$ and $x > 2$. So f is decreasing on $(-\infty, -2)$ and increasing on $(-2, \infty)$. At $x = -2$, f' changes from negative to positive: relative minimum. At $x = 2$, f' touches zero but stays positive: no extremum.

The graph of f' rises up to a peak near $x = -\frac{2}{3}$, falls to the x-axis at $x = 2$, then rises again. So f' is increasing on $(-\infty, -\frac{2}{3})$ and $(2, \infty)$, decreasing on $(-\frac{2}{3}, 2)$.

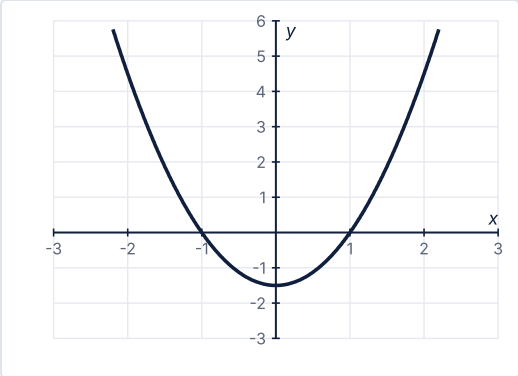
Therefore f is concave up on $(-\infty, -\frac{2}{3})$ and $(2, \infty)$ and concave down on $(-\frac{2}{3}, 2)$, with inflection points at $x = -\frac{2}{3}$ and $x = 2$.

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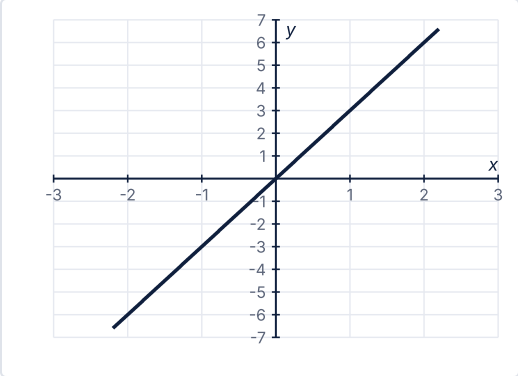
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Graph I



Graph II



Graph III

EXAMPLE 2

Graphs I, II and III above are the graphs of a function f, its derivative f' and its second derivative f'', in some order. Decide which is which and justify.

SOLUTION

Graph III is a line. A line can only be a derivative of a quadratic, which suggests that III is f'' and II (the parabola) is f', which in turn is the derivative of the cubic-looking Graph I, which must be f.

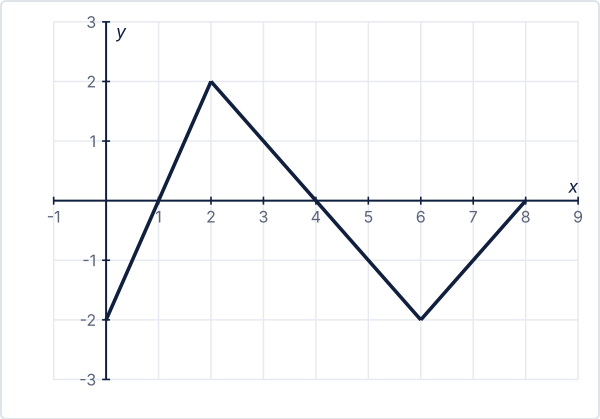
Check with zeros and signs: Graph I has a peak near $x = -1$ and a valley near $x = 1$, and Graph II is zero at exactly $x = -1$ and $x = 1$ (positive outside, negative between), matching f' for a function that rises, falls, then rises.

Graph II has its lowest point at $x = 0$, and Graph III is zero at $x = 0$ (negative left, positive right), matching f'' for a f' that decreases then increases. Graph I changes from cap-shaped to cup-shaped at $x = 0$, matching the sign change of Graph III.

Conclusion: f is Graph I, f' is Graph II, and f'' is Graph III.

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Graph of f' (Example 3).

EXAMPLE 3

A function f has the derivative graph shown (made of three line segments), on $[0, 8]$. Find the x-values of the relative extrema of f and the intervals where f is concave up and concave down.

SOLUTION

The segments join $(0, -2)$, $(2, 2)$, $(6, -2)$ and $(8, 0)$. The graph crosses the x-axis at $x = 1$ (negative to positive) and $x = 4$ (positive to negative).

So f has a relative minimum at $x = 1$ and a relative maximum at $x = 4$. (The right endpoint $x = 8$ is not a relative extremum.)

The slope of f' is 2 on $(0, 2)$, -1 on $(2, 6)$ and 1 on $(6, 8)$. So $f'' > 0$ on $(0, 2)$ and $(6, 8)$ (concave up), and $f'' < 0$ on $(2, 6)$ (concave down). The concavity changes at $x = 2$ and $x = 6$, which are inflection points, even though f'' does not exist at these corners of the graph of f'.

Common mistakes

(1) Reading a rising graph of f' as "f is increasing": a rising f' means f is concave up; f increases when f' is positive. (2) Calling every zero of f' an extremum of f (a double zero is not). (3) Locating inflection points at zeros of f': they are at the peaks and valleys of f' (zeros of f'' with a sign change). (4) Forgetting to say which graph supports your reason.

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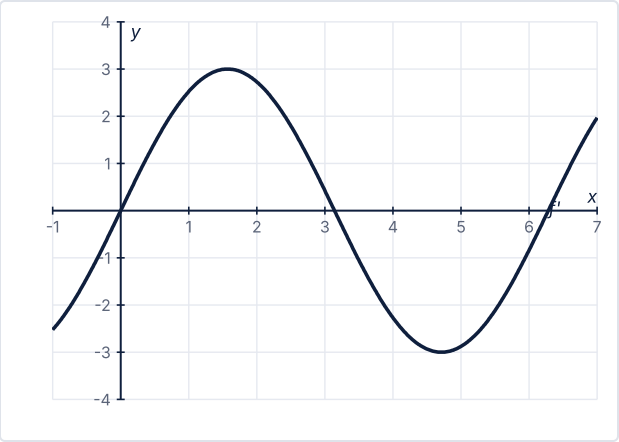
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Practice

SET A Skills

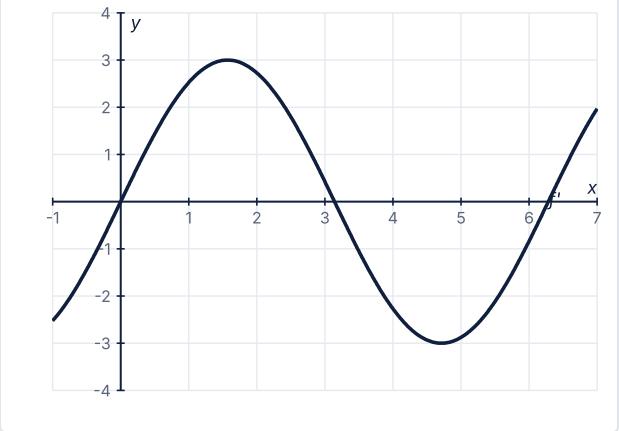
Answer each question using the information given. [No calculator]

1. The graph shows $f'(x) = 3\sin x$ for $-1 \le x \le 7$. Find the x-values of all relative extrema of f on the open interval $(-1, 7)$ and classify each.



Graph of f'.

2. Using the same graph of $f'(x) = 3\sin x$ on $(-1, 7)$, find the x-values of all points of inflection of f and give a reason.

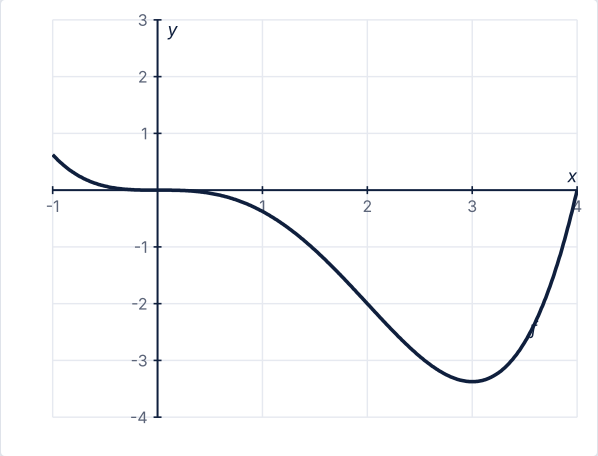


Graph of f'.

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3. The graph of $f(x) = \frac{1}{8}\left(x^4 - 4x^3\right)$ is shown. State whether f' and f'' are positive, negative, or zero at each of $x = -0.5$, $x = 1$, and $x = 3$.



Graph of f.

4. A function f is increasing and concave down on the interval $(0, 5)$. Which must be true on $(0, 5)$?

(A) $f' > 0$ and $f'' < 0$ **(B)** $f' > 0$ and $f'' > 0$ **(C)** $f' < 0$ and $f'' > 0$
(D) $f' < 0$ and $f'' < 0$ **(E)** $f' < 0$ and $f'' = 0$

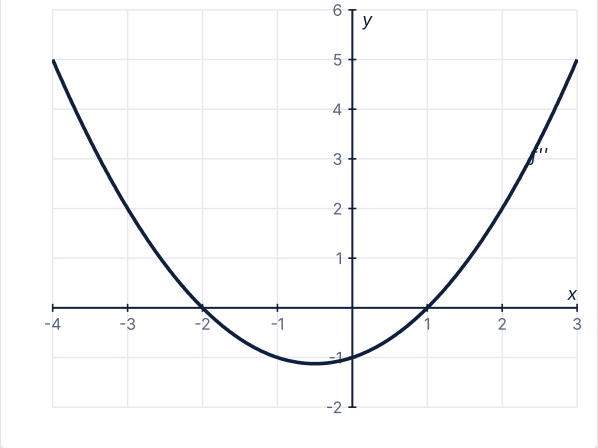
5. A twice-differentiable function f has the following values of f'. (a) Explain why f has a relative maximum and a relative minimum somewhere on $(0, 5)$, and locate each between consecutive table values. (b) Why must $f'(x) = 0$ for some x in $(1, 2)$?

x	0	1	2	3	4	5
$f'(x)$	3	1	-2	-4	-1	2

6. The derivative of f is $f'(x) = x^2 - 4x + 3$. Describe f completely: where it is increasing and decreasing, relative extrema, concavity and inflection points.

7. The derivative of f is $f'(x) = xe^{-x}$. Find where f is increasing and decreasing, the relative extrema, the intervals of concavity, and the inflection point's x-value.

8. The graph shows $f''(x) = \frac{1}{2}(x + 2)(x - 1)$, the second derivative of f. Find the intervals on which the graph of f is concave up and concave down, and the x-values of its inflection points. Is there enough information to find any relative extrema of f?



Graph of f''.

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9. The graph of f' is a straight line with positive slope that crosses the x-axis at $x = 2$. Which statement about f is true?

(A) relative maximum at $x = 2$; concave down everywhere

(B) relative minimum at $x = 2$; concave down everywhere

(C) inflection point at $x = 2$ and no relative extrema

(D) relative maximum at $x = 2$; concave up everywhere

(E) relative minimum at $x = 2$; concave up everywhere

SET B Applications and Mixed

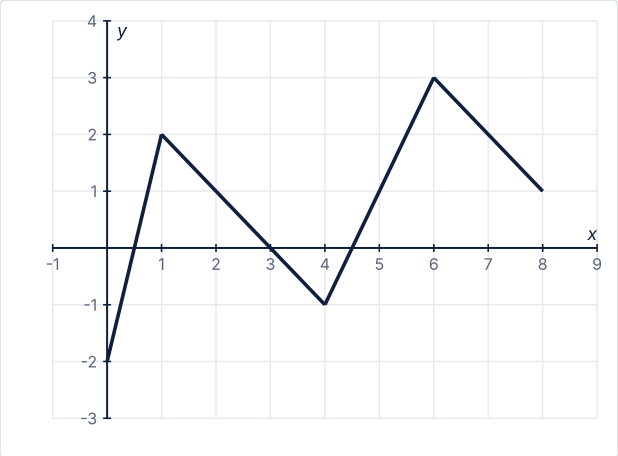
Solve each problem. [Calculator] Give decimals to three places where not exact.

10. The graph shows f' on $[0, 8]$ (four line segments joining $(0, -2)$, $(1, 2)$, $(4, -1)$, $(6, 3)$, $(8, 1)$).

(a) Find the x-values of the relative minima and maxima of f.

(b) Find the x-values of the inflection points of f.

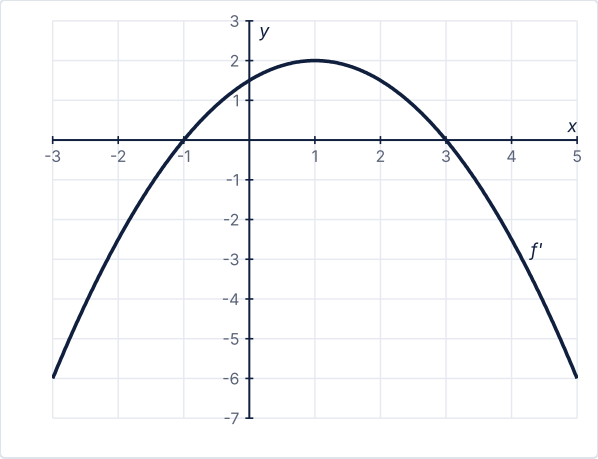
(c) On what interval is f both decreasing and concave down?



Graph of f'.

11. Water in a tank changes at a rate of $R(t) = -t^2 + 10t - 16$ liters per minute for $0 \le t \le 10$ minutes (negative means water is leaving). Let $W(t)$ be the amount of water in the tank. (a) When does W have a relative minimum and a relative maximum? (b) When is W increasing most rapidly, and what is that rate? (c) Over which interval is the graph of W concave down?

12. The graph of f' is shown, with $f'(x) = -\frac{1}{2}(x + 1)(x - 3)$. On which interval is f both increasing and concave down?



Graph of f'.

(A) $(-1, 1)$ **(B)** $(-1, 3)$ **(C)** $(1, 3)$ **(D)** $(3, \infty)$ **(E)** $(-\infty, -1)$

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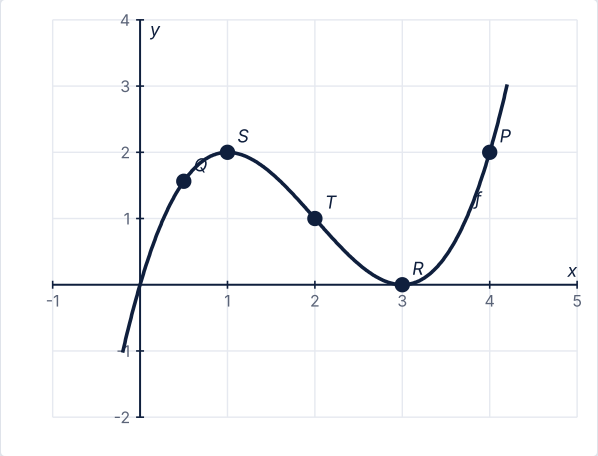
13. A twice-differentiable function f satisfies $f'(2) = 0$ and $f''(x) < 0$ for all x in $(0, 5)$. Which must be true?

(A) f has a relative minimum at $x = 2$ **(B)** f has a point of inflection at $x = 2$

(C) f is increasing on $(0, 5)$ **(D)** $f(2) > f(x)$ for all other x in $(0, 5)$

(E) $f'(x) > 0$ for all x in $(0, 5)$

14. The graph of $f(x) = \frac{1}{2}\left(x^3 - 6x^2 + 9x\right)$ is shown with five labeled points. At which point are both $f'(x) > 0$ and $f''(x) < 0$?



Graph of f.

(A) P **(B)** Q **(C)** R **(D)** S **(E)** T

15. [Calculator] For $f(x) = xe^{-x/2}$ with $x \geq 0$, use f' and f'' to describe f: the absolute maximum value and where it occurs, the intervals of concavity, and the inflection point. Give decimals to three places.

16. The height $H(t)$ in meters of a sapling is measured every two years. Compute the average rate of change over each interval. What do the results suggest about the signs of H' and H'' and about the growth?

t (years)	0	2	4	6	8
$H(t)$	1.0	2.6	3.8	4.6	5.0

SET C Challenge and AP-Style

Give complete justifications. [No calculator]

17. The derivative of f is $f'(x) = x(x-4)e^{-x}$. Find the x-values of the relative extrema and the inflection points of f (exact values), with justification.

18. Suppose $f''(x) > 0$ for all real x, $f(1) = 3$ and $f'(1) = 2$. Show that $f(x) > 3 + 2(x-1)$ for all $x \neq 1$. (The graph lies above its tangent line.)

19. The function $f(x) = x^4 + ax^3 + bx^2$ has points of inflection at $x = 1$ and $x = 2$. Find a and b, then find and classify the critical numbers of f.

20. A twice-differentiable function f has $f'(2) = 0$, $f''(x) > 0$ for $x < 2$ and $f''(x) < 0$ for $x > 2$. Show that f has an inflection point at $x = 2$ but no relative extremum there. Give an example.

21. The function f is twice differentiable and concave down on $(0, 5)$. Put $f'(1)$, $f'(3)$ and $\dfrac{f(3) - f(1)}{2}$ in increasing order. Then apply your conclusion to $f(x) = \ln x$.

3.6 Curve Sketching

Learn to combine domain, intercepts, asymptotes, and both derivatives into a complete picture of a function. This is how you turn formulas into graphs and graphs back into facts.

A good sketch is not a table of values. It is a **summary of what the derivatives tell you**: where the graph lives (domain), where it touches the axes, what it does far away (asymptotes), where it rises and falls, where it turns, and how it bends. The first derivative controls direction; the second derivative controls bending. Together with the limits at the edges, they fix the shape.

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Curve-Sketching Checklist

1. **Domain** and symmetry (even: $f(-x) = f(x)$; odd: $f(-x) = -f(x)$).
2. **Intercepts**: y-intercept $f(0)$; x-intercepts where $f(x) = 0$ (when solvable).
3. **Asymptotes**: vertical where $f(x) \to \pm\infty$ at a finite x; horizontal from $\lim_{x\to\pm\infty} f(x)$; slant when $f(x) - (mx + b) \to 0$.
4. **First derivative**: critical numbers (where $f' = 0$ or f' is undefined but f is defined); sign chart for increasing and decreasing; local extrema.
5. **Second derivative**: where $f'' = 0$ or is undefined; sign chart for concavity; inflection points.
6. **Assemble**: plot the key points, respect the asymptotes, and connect with the correct direction and bend.

Asymptotes of a Rational Function

Let $f(x) = \dfrac{p(x)}{q(x)}$ in lowest terms. A zero of q is a **vertical asymptote**. A zero of q that cancels completely with a factor of p is a **hole**.

If $\deg p < \deg q$ the horizontal asymptote is $y = 0$. If $\deg p = \deg q$ it is $y =$ (ratio of leading coefficients). If $\deg p = \deg q + 1$ there is a **slant asymptote**, found by polynomial division.

Reading Shape from the Derivatives

On an interval: $f' > 0 \Rightarrow f$ increasing; $f' < 0 \Rightarrow f$ decreasing; $f'' > 0 \Rightarrow f$ concave up; $f'' < 0 \Rightarrow f$ concave down.

A **local extremum** occurs at c where f' changes sign (or where $f'(c) = 0$ and $f''(c) \neq 0$). An **inflection point** occurs at c where f is defined and f'' changes sign. Neither $f'(c) = 0$ nor $f''(c) = 0$ alone is enough.

EXAMPLE 1

Analyze $f(x) = x^3 - 6x^2 + 9x + 1$: find intervals of increase and decrease, local extrema, concavity, and inflection points, then describe the graph.

SOLUTION

Domain: all real numbers. $f(0) = 1$. As $x \to \infty$, $f \to \infty$; as $x \to -\infty$, $f \to -\infty$.

$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$, so the critical numbers are $x = 1$ and $x = 3$. $f' > 0$ on $(-\infty, 1)$ and $(3, \infty)$; $f' < 0$ on $(1, 3)$.

Since f' changes from positive to negative at $x = 1$, there is a local maximum $f(1) = 5$. Since f' changes from negative to positive at $x = 3$, there is a local minimum $f(3) = 1$.

$f''(x) = 6x - 12 = 6(x - 2)$. $f'' < 0$ for $x < 2$ (concave down) and $f'' > 0$ for $x > 2$ (concave up). The sign changes, so $(2, f(2)) = (2, 3)$ is an inflection point.

Description: the graph rises to the peak $(1, 5)$, falls through the inflection point $(2, 3)$ to the valley $(3, 1)$, then rises forever.

EXAMPLE 2

Analyze $f(x) = \dfrac{x^2}{x - 2}$: asymptotes, extrema, and concavity.

SOLUTION

Domain: $x \neq 2$. The only intercept is $(0, 0)$. At $x = 2$ the numerator is $4 \neq 0$, so $x = 2$ is a vertical asymptote: $f \to -\infty$ as $x \to 2^-$ and $f \to +\infty$ as $x \to 2^+$.

Divide: $\dfrac{x^2}{x - 2} = x + 2 + \dfrac{4}{x - 2}$. The remainder term tends to 0, so $y = x + 2$ is a slant asymptote.

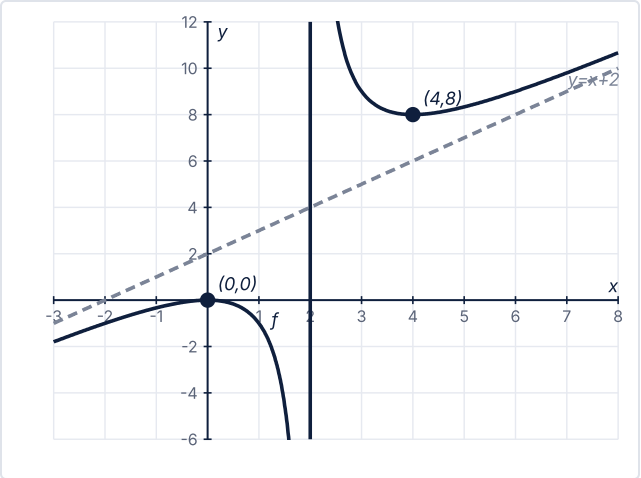
$f'(x) = \dfrac{2x(x - 2) - x^2}{(x - 2)^2} = \dfrac{x(x - 4)}{(x - 2)^2}$. Critical numbers: $x = 0$ and $x = 4$ ($x = 2$ is not in the domain). $f' > 0$ on $(-\infty, 0)$ and $(4, \infty)$; $f' < 0$ on $(0, 2)$ and $(2, 4)$.

Local maximum at $(0, 0)$; local minimum at $(4, 8)$.

$f''(x) = \dfrac{8}{(x - 2)^3}$ is never 0. It is negative for $x < 2$ and positive for $x > 2$: concave down on $(-\infty, 2)$, concave up on $(2, \infty)$. There is no inflection point, because $x = 2$ is not in the domain.

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Graph of Example 2 with its vertical asymptote x=2 and slant asymptote y=x+2.

EXAMPLE 3

Analyze $f(x) = xe^{-x}$ and sketch its main features.

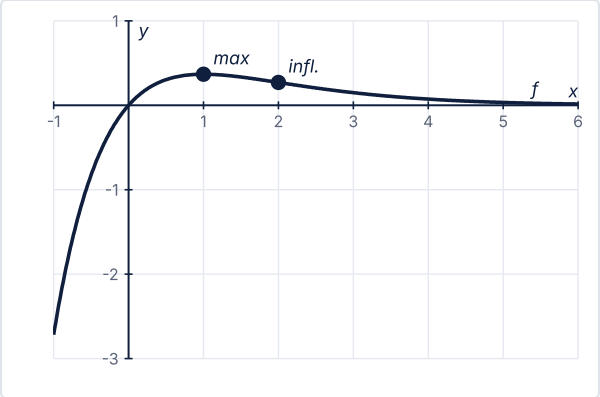
SOLUTION

Domain: all real numbers. $f(x) = 0$ only at $x = 0$. As $x \to \infty$, $xe^{-x} \to 0$, so $y = 0$ is a horizontal asymptote on the right. As $x \to -\infty$, $f(x) \to -\infty$.

$f'(x) = e^{-x} - xe^{-x} = (1 - x)e^{-x}$. Since $e^{-x} > 0$, the sign of f' is the sign of $1 - x$: increasing on $(-\infty, 1)$, decreasing on $(1, \infty)$. Local (and absolute) maximum $f(1) = \frac{1}{e}$.

$f''(x) = -e^{-x} - (1 - x)e^{-x} = (x - 2)e^{-x}$. Concave down on $(-\infty, 2)$, concave up on $(2, \infty)$. Inflection point $\left(2, \frac{2}{e^2}\right)$.

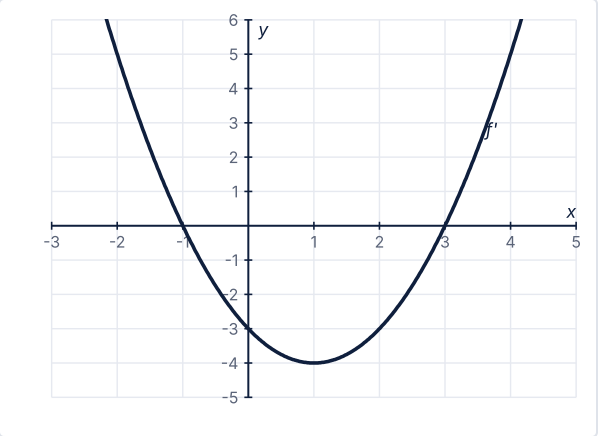
Description: the curve climbs from far below on the left to a peak at $(1, 1/e)$, bends upward at $x = 2$, and then flattens toward the x-axis from above.



Graph of f(x)=x e^(-x).

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The graph of f', the derivative of a function f.

EXAMPLE 4

The graph above is the graph of f', the derivative of a continuous function f. Using only this graph, find where f is increasing, where it has local extrema, and where it is concave up. Justify.

SOLUTION

The graph is $f'(x) = (x+1)(x-3)$. It is above the x-axis for $x < -1$ and $x > 3$, and below it on $(-1, 3)$. So f is increasing on $(-\infty, -1)$ and $(3, \infty)$, decreasing on $(-1, 3)$.

Local extrema of f: f' changes from positive to negative at $x = -1$, so f has a local maximum there; f' changes from negative to positive at $x = 3$, so f has a local minimum there.

Concavity of f comes from the slope of the graph of f'. The graph of f' is decreasing for $x < 1$ and increasing for $x > 1$, so $f'' < 0$ for $x < 1$ and $f'' > 0$ for $x > 1$. Thus f is concave up on $(1, \infty)$ and has an inflection point at $x = 1$, where f' has its minimum value -4.

AP-style sentence: "f has a local minimum at $x = 3$ because f' changes from negative to positive at $x = 3$."

Common mistakes

(1) Concluding there is an extremum just because $f'(c) = 0$. For $f(x) = x^3$, $f'(0) = 0$ but f keeps increasing. (2) Calling $f''(c) = 0$ an inflection point without checking that f'' changes sign: $f(x) = x^4$ has $f''(0) = 0$ but is concave up on both sides. (3) Forgetting that a critical number or inflection point must lie in the domain of f. (4) Cancelling a common factor and forgetting the hole it leaves. (5) Reading a graph of f' as if it were the graph of f: the sign of f' gives increase or decrease, the slope of f' gives concavity.

Practice

SET A **Skills**

Answer each question. [No calculator]

1. Let $f(x) = \dfrac{x-1}{x^2 - x - 6}$. Find the domain, all intercepts, and all asymptotes of f.
2. For $f(x) = x^3 - 3x^2 - 9x + 2$, find the intervals on which f is increasing or decreasing and all local extrema.
3. Find the intervals of concavity and all inflection points of $f(x) = x^4 - 8x^2$.
4. Find all vertical and slant asymptotes of $f(x) = \dfrac{x^2 + 3x + 1}{x + 1}$.
5. Find $\lim\limits_{x \to \infty} \dfrac{6x}{\sqrt{4x^2 + 1}}$ and $\lim\limits_{x \to -\infty} \dfrac{6x}{\sqrt{4x^2 + 1}}$, and state the horizontal asymptotes of the function.
6. Let $f(x) = x \ln x$. State the domain, find where f is decreasing and increasing, locate any extremum, and determine the concavity.
7. Use the Second Derivative Test to classify the critical numbers of $f(x) = x^3 - 12x + 5$.

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8. The table gives the signs of f' and f'' for a continuous function f. (a) Find the x-values of all local extrema and classify them.
(b) Find the x-value of any inflection point.
(c) On which interval is f both increasing and concave up?

Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, 5)$	$(5, \infty)$
Sign of f'	+	-	-	+
Sign of f''	-	-	+	+

9. Let $f(x) = \dfrac{x^2 - x - 6}{x^2 - 9}$. Find the vertical asymptote(s), any hole, and the horizontal asymptote.

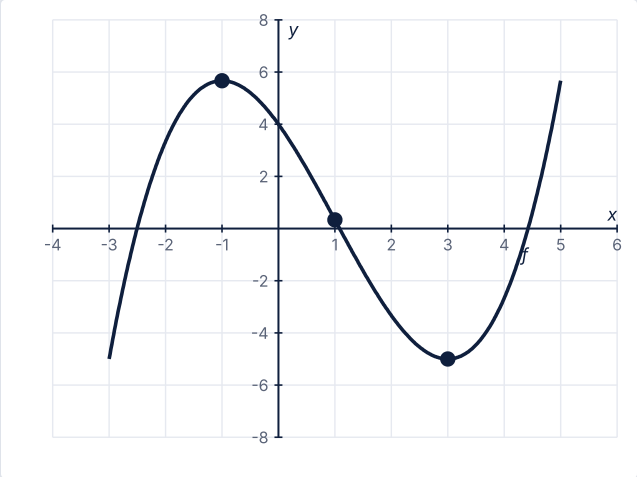
SET B Applications and Mixed Problems

Solve each problem. Show your reasoning. [No calculator]

10. The derivative of a function f is $f'(x) = (x-1)^2(x+3)$. (a) Find the critical numbers of f.
(b) At which critical numbers does f have a local extremum? Justify.
(c) Find the x-coordinates of the inflection points of f.

11. Let $f(x) = x^3 + ax^2 + bx$. Find the constants a and b so that f has a local maximum at $x = -1$ and an inflection point at $x = 1$. Verify your answer.

12. The graph of a function f is shown. (a) On what interval is f decreasing?
(b) On what interval is f concave up?
(c) At what value of x does f' have its minimum value? Explain.



Graph of f.

13. The cost of producing x hundred units of a product is $C(x) = x^3 - 12x^2 + 60x + 100$ thousand dollars. The marginal cost is $C'(x)$. Show that the cost always increases with $x > 0$, and find the production level at which the marginal cost is smallest. What is that smallest marginal cost?

14. The concentration of a medication in the blood is $C(t) = \dfrac{20t}{t^2 + 4}$ mg/L, $t \geq 0$ hours after a dose. (a) Find when the concentration is greatest, and the greatest value. Justify.
(b) What happens to $C(t)$ as $t \to \infty$?
(c) Given $C''(t) = \dfrac{40t(t^2 - 12)}{(t^2+4)^3}$, at what time is the concentration decreasing most rapidly, and at what rate?

15. For $f(x) = \ln(x^2 - 1)$, find the domain, intercepts, vertical asymptotes, intervals of increase and decrease, extrema, and concavity.

16. Let $f(x) = x^{5/3} - 5x^{2/3}$. (a) Find the critical numbers of f.
(b) Classify each as a local maximum, local minimum, or neither.
(c) Given $f''(x) = \dfrac{10(x+1)}{9x^{4/3}}$, find the inflection point(s).

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SET C Challenge and AP-Style

Give decimal answers to three places where needed. [Calculator]

17. Let $f(x) = x^2 e^{-x/2}$ for all real x. Find the local extrema, the inflection points, and the horizontal asymptote. Give decimals to three places.

18. A bacteria population is $P(t) = \dfrac{1000}{1 + 9e^{-0.5t}}$, t in hours. (a) Find $P(0)$ and $\lim_{t\to\infty} P(t)$.

(b) Show P is increasing for all t.

(c) When is the population growing fastest, what is the population then, and how fast is it growing?

19. Let $f(x) = x^3 - 6x^2 + 9x - 2$. (a) Find the local extrema.

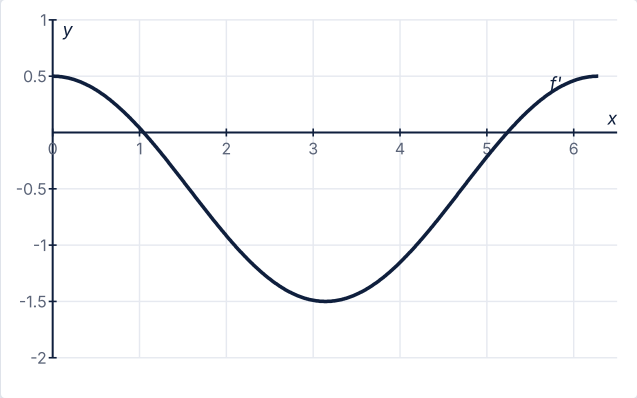
(b) Explain why f has exactly three real zeros.

(c) Find the smallest zero to three decimal places.

20. The graph of f' for $0 \le x \le 2\pi$ is shown, where $f'(x) = \cos x - 0.5$ and $f(0) = 1$. (a) Find the x-values of the local extrema of f on $(0, 2\pi)$ and classify them.

(b) Find the x-value of the inflection point of f.

(c) Find the local maximum value of f to three decimal places.



Graph of f'.

21. Let $f(x) = x + 2\cos x$ on $[0, 2\pi]$. (a) Find the x-values of the local extrema on $(0, 2\pi)$.

(b) Find the inflection points.

(c) Find the absolute maximum and minimum values on $[0, 2\pi]$, to three decimals.

3.7 Particle Motion Along a Line

Use position, velocity, and acceleration to say where a particle is, which way it moves, and when it speeds up or slows down. This is the most common application of derivatives on the AP exam.

A **particle** moving along a line (say the x-axis or a vertical axis) is described by its position $s(t)$ at time t. Derivatives turn that single function into a complete story: how fast it moves, which way, and whether it is picking up speed or losing it. The key is to read **signs** carefully.

Position, Velocity, Acceleration, Speed

$$v(t) = s'(t), \qquad a(t) = v'(t) = s''(t), \qquad \text{speed} = |v(t)|.$$

Velocity is the rate of change of position (units: length per time). Acceleration is the rate of change of velocity. Speed is the magnitude of velocity and is never negative.

If $v > 0$ the particle moves in the positive direction (right or up); if $v < 0$ it moves in the negative direction (left or down); if $v = 0$ it is momentarily at rest.

Direction Changes, Speeding Up, and Slowing Down

The particle **changes direction** at time c when $v(c) = 0$ **and** v changes sign at c. (A zero of v without a sign change is only a pause.)

The particle is **speeding up** when v and a have the **same** sign, and **slowing down** when they have **opposite** signs.

If $v(c) = 0$ and $a(c) \neq 0$, the velocity changes sign at c, so the particle turns around at c.

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1. **Find $v(t)$ and $a(t)$.** Differentiate once and twice.
2. **Solve $v(t) = 0$.** These are the candidate turning times.
3. **Make a sign chart for v and for a** on the same time axis.
4. **Compare signs.** Same signs: speeding up. Opposite signs: slowing down. A sign change in v: direction change.
5. **Compute positions** at the endpoints and at each turning time to get displacement and distance.

Displacement versus distance. Net displacement over $[a, b]$ is $s(b) - s(a)$ and can be positive, negative, or zero. **Total distance** counts every bit of travel, so it adds the sizes of the displacements between consecutive direction changes. If the particle turns around at times $t_1 < t_2$ inside $[a, b]$, then

$$\text{total distance} = |s(t_1) - s(a)| + |s(t_2) - s(t_1)| + |s(b) - s(t_2)|.$$

Later, with integrals, this will become $\int_a^b |v(t)|\,dt$.

EXAMPLE 1

A particle moves along a line with position $s(t) = t^3 - 9t^2 + 24t$ meters for $t \geq 0$ seconds. Find when it changes direction, describe when it is speeding up or slowing down, and find its net displacement and total distance for $0 \leq t \leq 5$.

SOLUTION

$v(t) = 3t^2 - 18t + 24 = 3(t-2)(t-4)$ and $a(t) = 6t - 18 = 6(t-3)$.

$v > 0$ on $[0, 2)$ and $(4, \infty)$; $v < 0$ on $(2, 4)$. So the particle changes direction at $t = 2$ and $t = 4$. Also $a < 0$ for $t < 3$ and $a > 0$ for $t > 3$.

$(0, 2)$: $v > 0, a < 0$, slowing down. $(2, 3)$: $v < 0, a < 0$, speeding up. $(3, 4)$: $v < 0, a > 0$, slowing down. $(4, 5)$: $v > 0, a > 0$, speeding up.

Positions: $s(0) = 0$, $s(2) = 20$, $s(4) = 16$, $s(5) = 20$. Net displacement $= 20 - 0 = 20$ m. Total distance $= |20 - 0| + |16 - 20| + |20 - 16| = 28$ m.

t	0	2	4	6	8
$v(t)$	6	2	-3	-1	5

EXAMPLE 2

The velocity $v(t)$ (in cm/s) of a particle is differentiable, with values in the table. (a) Find the average acceleration on $0 \leq t \leq 8$. (b) Must the particle change direction? Justify. (c) Show that $a(c) = -\frac{5}{2}$ for some c in $(2, 4)$.

SOLUTION

(a) $\dfrac{v(8) - v(0)}{8 - 0} = \dfrac{5 - 6}{8} = -\dfrac{1}{8}$ cm/s^2.

(b) v is continuous (it is differentiable). Since $v(2) = 2 > 0 > v(4) = -3$, the Intermediate Value Theorem gives a time in $(2, 4)$ where $v = 0$, and likewise in $(6, 8)$ since $v(6) = -1 < 0 < v(8) = 5$. The velocity changes sign, so the particle changes direction at least twice.

(c) v is continuous on $[2, 4]$ and differentiable on $(2, 4)$. By the Mean Value Theorem there is c in $(2, 4)$ with $a(c) = v'(c) = \frac{v(4) - v(2)}{4 - 2} = \frac{-3 - 2}{2} = -\frac{5}{2}$.

EXAMPLE 3

A particle has velocity $v(t) = t^2 - 6t + 8$ for $t \geq 0$ and starts at $s(0) = 5$. (a) Find $s(t)$. (b) Determine whether the particle is speeding up or slowing down at $t = 1$ and at $t = 5$. (c) Find the total distance traveled on $0 \leq t \leq 3$.

SOLUTION

(a) Position is an antiderivative of velocity: $s(t) = \frac{t^3}{3} - 3t^2 + 8t + C$. Since $s(0) = 5$, $C = 5$.

(b) $a(t) = 2t - 6$. At $t = 1$: $v = 3 > 0$ and $a = -4 < 0$, opposite signs, so slowing down. At $t = 5$: $v = 3 > 0$ and $a = 4 > 0$, same signs, so speeding up.

(c) $v = (t-2)(t-4)$ changes sign at $t = 2$ (in the interval). $s(0) = 5$, $s(2) = \frac{35}{3}$, $s(3) = 11$. Distance $= \left|\frac{35}{3} - 5\right| + \left|11 - \frac{35}{3}\right| = \frac{20}{3} + \frac{2}{3} = \frac{22}{3}$.

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EXAMPLE 4

A stone is thrown upward from a cliff. Its height above the ground is $h(t) = -5t^2 + 30t + 35$ meters, t in seconds. Find its maximum height, when it lands, and its speed at impact.

SOLUTION

$v(t) = h'(t) = -10t + 30$, which is zero at $t = 3$. Since $v > 0$ before and $v < 0$ after, h has its maximum at $t = 3$: $h(3) = -45 + 90 + 35 = 80$ m.

It lands when $h = 0$: $-5(t^2 - 6t - 7) = -5(t - 7)(t + 1) = 0$, so $t = 7$ s (reject $t = -1$).

$v(7) = -70 + 30 = -40$ m/s, so the speed at impact is 40 m/s. The acceleration is $a(t) = -10$ m/s^2 throughout, constant and downward.

Common mistakes

(1) Saying a particle slows down whenever $a < 0$. Slowing down depends on the signs of v and a together: with $v < 0$ and $a < 0$ it is speeding up. (2) Assuming $v(c) = 0$ always means a turn: for $v(t) = (t - 1)^2$ the particle pauses at $t = 1$ and keeps moving forward. (3) Mixing up velocity and speed: a velocity of -4 means speed 4 moving in the negative direction. (4) Using net displacement when total distance is asked. (5) Forgetting units, which are length per time for v and length per time squared for a. AP-style justification: "The particle is speeding up at $t = 2$ because $v(2)$ and $a(2)$ are both negative."

Practice

SET A ### Skills

Answer each question. [No calculator]

1. A particle has position $s(t) = t^2 - 6t + 5$. Find $v(t)$ and $a(t)$, the time when it is at rest, and its position then.
2. For $s(t) = t^3 - 12t$ ($t \geq 0$), find the time intervals when the particle moves left and when it moves right.
3. A particle has velocity $v(t) = t^2 - 4t$ for $t \geq 0$. Decide whether it is speeding up or slowing down at $t = 1$, $t = 3$, and $t = 5$.
4. A particle has position $s(t) = 2t^3 - 15t^2 + 36t$, $t \geq 0$. On which intervals is it speeding up?
5. A particle has position $s(t) = t^2 - 4t$ on $0 \leq t \leq 5$. Find the net displacement and the total distance traveled.
6. A particle has velocity $v(t) = t^2 - t - 2$ and $s(0) = 1$. Find $s(t)$, then the net displacement and total distance on $0 \leq t \leq 3$.
7. A ball is thrown straight up with height $s(t) = -16t^2 + 64t$ feet. Find the maximum height, the time it returns to the ground, and its speed on landing.
8. A particle has velocity $v(t) = t^2 - 6t + 5$. When is its velocity least, and what is the least velocity? Justify.
9. The table gives the velocity $v(t)$ of a particle moving along a line; v is continuous. (a) Find the average acceleration on $0 \leq t \leq 5$. (b) Approximate $a(2)$ using the data.
   (c) Explain why $v(t) = 0$ for some t in $(1, 3)$.

t	0	1	3	5
$v(t)$	4	2	-1	-3

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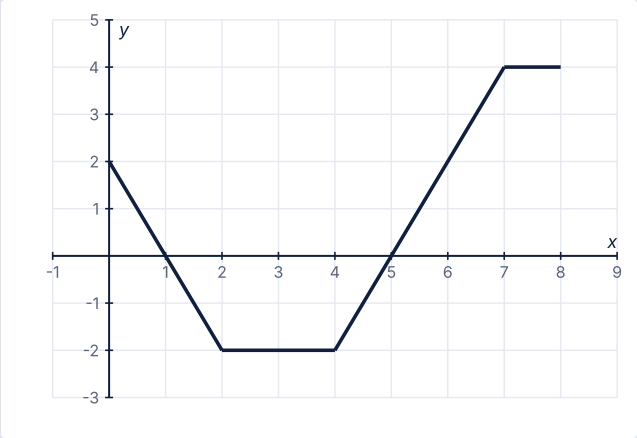
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SET B Applications and Mixed Problems

Solve each problem. Show your reasoning. [No calculator]

10. A particle has position $x(t) = te^{-t}$ for $t \geq 0$. (a) When is it at rest, and where?
(b) On which intervals is it speeding up?
(c) What is its greatest speed for $t \geq 0$?

11. The graph shows the velocity $v(t)$ of a particle on $0 \leq t \leq 8$; it consists of line segments. (a) At what times does the particle change direction?
(b) On which intervals is the particle speeding up?
(c) Use areas of triangles and rectangles to find the total distance traveled on $0 \leq t \leq 5$.



Velocity v(t).

12. A particle has velocity $v(t) = 2\sin t - 1$ for $0 \leq t \leq 2\pi$. Find the times when it changes direction and the intervals on which it is speeding up.

13. A rock is thrown upward from the top of a building. Its height is $h(t) = -5t^2 + 20t + 60$ meters. Find its maximum height, the time it hits the ground, and its velocity on impact.

14. The position of a piston is $s(t) = 5 + 3\sin(2t)$ centimeters. (a) Find $v(t)$ and $a(t)$.
(b) Show that $a(t) = -4\,(s(t) - 5)$.
(c) At $t = \frac{\pi}{3}$, is the piston speeding up or slowing down?

15. Two particles move on a number line with positions $x_A(t) = t^2 - 6t + 8$ and $x_B(t) = 2t - 4$, $t \geq 0$. (a) When do they meet?
(b) What are their velocities at each meeting? (c) When they meet at $t = 6$, which is moving faster, and are they moving in the same direction?

16. A particle has acceleration $a(t) = 6t - 12$, $v(0) = 9$, and $s(0) = 2$. Find $v(t)$ and $s(t)$, the times it is at rest, and its total distance on $0 \leq t \leq 4$.

SET C Challenge and AP-Style

Give decimal answers to three places. [Calculator]

17. A particle has position $s(t) = t^3 - 7t^2 + 10t + 3$ for $t \geq 0$. (a) Find the times it is at rest and its positions then.
(b) Is the particle speeding up or slowing down at $t = 1$?
(c) Find the total distance traveled on $0 \leq t \leq 5$.

18. A particle has velocity $v(t) = t\cos t$ for $0 \leq t \leq 6$, with $s(0) = 0$, so $s(t) = t\sin t + \cos t - 1$. (a) Find the times when it changes direction.
(b) Find its position at those times and at $t = 6$.
(c) Find the total distance traveled on $[0, 6]$.

19. A particle has position $s(t) = \dfrac{\ln t}{t}$ for $t > 0$, with $v(t) = \dfrac{1 - \ln t}{t^2}$ and $a(t) = \dfrac{2\ln t - 3}{t^3}$. (a) When is it at rest, and what is the position?
(b) On which interval is it speeding up? Give endpoints to three decimals.

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20. A damped oscillator has position $x(t) = e^{-t/2}\sin(3t)$ for $t \geq 0$. (a) Show $x'(t) = \frac{1}{2}e^{-t/2}(6\cos 3t - \sin 3t)$.
(b) Find the first two times it is at rest and its positions then.
(c) At $t = 1$ is it speeding up or slowing down? Given $x''(1) = 1.052$.

21. A particle moves along a line; its velocity $v(t)$ (m/s) is continuous and recorded in the table. (a) Explain why the particle must change direction at least twice on $0 < t < 8$.
(b) Find the average acceleration on $[0, 8]$.
(c) Use a left sum with four equal subintervals to estimate the net displacement.
(d) Use the same left sum idea on $|v|$ to estimate total distance.

t	0	2	4	6	8
$v(t)$	5	3	-2	-4	1

3.8 Optimization I

Learn to translate a word problem into a one-variable function and to find, with justification, its largest or smallest value. Boxes, fences, sums, products, and distances to a curve are the core cases.

Optimization asks: of all the ways to do something under a constraint, which one gives the largest or smallest value of a quantity? Calculus answers by turning the situation into a function of **one variable** on a stated domain and finding its absolute maximum or minimum. The real work is the setup; the calculus is usually short.

1. **Understand and draw.** Sketch the situation and label every quantity with a letter.
2. **Name the objective.** Write the quantity to maximize or minimize (area, volume, cost, distance) as a formula.
3. **Use the constraint.** Solve the constraint equation for one variable and substitute, so the objective depends on a single variable.
4. **State the domain.** What values make physical sense (lengths positive, pieces that fit)?
5. **Find critical numbers** of the objective by solving the derivative equal to 0 (and where it is undefined).
6. **Justify the extremum** with one of the tests below, then answer the question that was asked, with units.

Justifying an Absolute Extremum

Closed interval (Candidates Test). If the objective f is continuous on $[a, b]$, evaluate f at the endpoints and at every critical number in (a, b). The largest value is the absolute maximum and the smallest is the absolute minimum (Extreme Value Theorem).

Open or unbounded interval. If f is continuous on an interval with exactly one critical number c, and f' changes from positive to negative at c (or $f'' < 0$ on the whole interval), then $f(c)$ is the absolute maximum. Changing from negative to positive (or $f'' > 0$ on the whole interval) gives the absolute minimum.

Useful Geometry and Algebra

Rectangle: $A = xy$, $P = 2x + 2y$. Box: $V = \ell wh$. Open-top box with a square base x and height h: surface $S = x^2 + 4xh$. Distance between (x, y) and (a, b): $D = \sqrt{(x-a)^2 + (y-b)^2}$.

Squared-distance trick. D is smallest exactly when D^2 is smallest (since $D \geq 0$), so minimize D^2 and avoid the square root.

EXAMPLE 1

A farmer has 120 m of fencing and will enclose a rectangular pen against a long straight wall. No fence is needed along the wall. What dimensions give the greatest area?

SOLUTION

Let x be the length of each of the two sides perpendicular to the wall and y the side parallel to it. Constraint: $2x + y = 120$, so $y = 120 - 2x$. Domain: $0 < x < 60$.

Objective: $A(x) = x(120 - 2x) = 120x - 2x^2$.

$A'(x) = 120 - 4x = 0$ gives $x = 30$. $A''(x) = -4 < 0$ for all x, so A is concave down and $x = 30$ gives the absolute maximum.

Dimensions: 30 m by 60 m ($y = 120 - 60 = 60$), with maximum area $1800\ \text{m}^2$.

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EXAMPLE 2

An open-top box is made from an 18 cm by 18 cm square sheet of cardboard by cutting equal squares of side x from the corners and folding up the sides. Find x for the greatest volume, and the volume.

SOLUTION

The base is $(18-2x)$ by $(18-2x)$ and the height is x. Domain: $0<x<9$. Objective: $V(x)=x(18-2x)^2$.

$V'(x)=(18-2x)^2+x\cdot 2(18-2x)(-2)=(18-2x)(18-2x-4x)=(18-2x)(18-6x)$.

Critical numbers in the domain: $x=3$ (since $x=9$ is an endpoint). $V'>0$ on $(0,3)$ and $V'<0$ on $(3,9)$, so V has an absolute maximum at $x=3$ by the First Derivative Test.

$V(3)=3\cdot 12^2=432\text{ cm}^3$.

EXAMPLE 3

Nonnegative numbers x and y satisfy $x+2y=30$. Find the values that maximize $P=x^2y$.

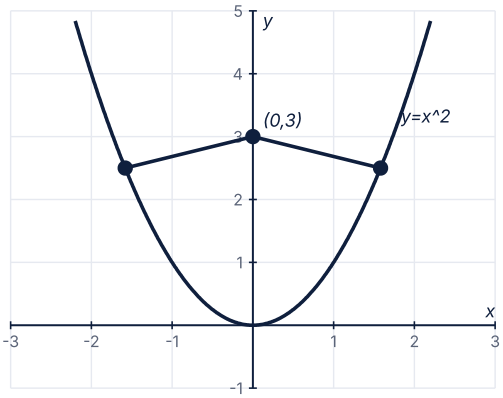
SOLUTION

From the constraint, $y=\frac{30-x}{2}$. Since $y\geq 0$, the domain is $[0,30]$. Then $P(x)=\dfrac{x^2(30-x)}{2}=15x^2-\dfrac{x^3}{2}$.

$P'(x)=30x-\dfrac{3x^2}{2}=\dfrac{3x(20-x)}{2}$. Critical numbers in $(0,30)$: $x=20$.

Candidates Test on the closed interval: $P(0)=0$, $P(20)=15\cdot 400-4000=2000$, $P(30)=0$.

The absolute maximum is 2000, at $x=20$ and $y=5$.



Points on y=x^2 closest to (0,3).

EXAMPLE 4

Find the points on the parabola $y=x^2$ that are closest to the point $(0,3)$.

SOLUTION

A point on the curve is (x,x^2). Minimize the squared distance $D^2=x^2+(x^2-3)^2$.

Let $u=x^2\geq 0$. Then $D^2=u+(u-3)^2=u^2-5u+9$, with $\dfrac{d(D^2)}{du}=2u-5=0$ at $u=\frac{5}{2}$. The second derivative is $2>0$, so this is the absolute minimum over $u\geq 0$ (and $u=0$ gives the larger value 9).

$x^2=\frac{5}{2}$ gives $x=\pm\frac{\sqrt{10}}{2}$ and $y=\frac{5}{2}$. Minimum $D^2=\frac{5}{2}+\frac{1}{4}=\frac{11}{4}$, so the shortest distance is $\frac{\sqrt{11}}{2}$.

The closest points are $\left(\pm\frac{\sqrt{10}}{2},\frac{5}{2}\right)$.

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Common mistakes

(1) Skipping the justification: always say why the critical number gives a maximum or minimum. (2) Leaving two variables in the objective. (3) Ignoring the domain, which can make an endpoint the true answer. (4) Answering with x when the question asks for the maximum area, or the reverse. (5) Dropping units. (6) Minimizing D and making the algebra harder when minimizing D^2 gives the same x.

Practice

SET A Skills

Solve each problem and justify that your answer is a maximum or minimum. [No calculator]

1. Two numbers have sum 40. What are they if their product is as large as possible? What is the product?
2. Positive numbers x and y satisfy $xy = 12$. Find the minimum value of $3x + y$.
3. A rectangle has perimeter 36. Find its dimensions for maximum area and the area.
4. A rectangular field of area $800\ \text{m}^2$ is fenced on three sides and the fourth side is a river (no fence). Find the dimensions that use the least fencing.
5. An open-top box with a square base must hold $32\ \text{m}^3$. Find the base side and height that minimize the surface area, and the minimum area.
6. A 12 cm by 12 cm sheet has squares of side x cut from each corner to make an open box. Find x that maximizes volume, and the maximum volume.
7. Nonnegative numbers satisfy $x + y = 12$. Find the maximum of xy^2.
8. Find the point on the line $y = 2x + 1$ closest to the origin, and the distance.
9. Find the point on the curve $y = \sqrt{x}$ closest to $(4, 0)$, and that shortest distance.

SET B Applications and Mixed Problems

Solve each problem. Give exact answers. [No calculator]

10. A poster has area $600\ \text{cm}^2$ with margins of 3 cm at top and bottom and 2 cm on each side. Find the poster dimensions that maximize the printed area, and that area.
11. A rancher has 600 m of fencing for a rectangular pen split into two equal parts by a fence parallel to one pair of sides. Find the overall dimensions with the greatest total area.
12. An open-top storage container with a square base must hold $27\ \text{m}^3$. The base costs 10 dollars per square meter and the sides cost 5 dollars per square meter. Find the dimensions of least cost and the cost.
13. Positive numbers x and y satisfy $x^2y = 32$. Find the minimum value of $x + y$. Justify that it is a minimum.
14. A rectangle has its base on the x-axis and its two upper vertices on the parabola $y = 12 - x^2$. Find the largest possible area and the dimensions.
15. A rectangle has diagonal 10. Find the dimensions of the rectangle of greatest area, and the area.
16. A window is a rectangle topped by a semicircle (the diameter of the semicircle equals the width of the rectangle). Its outer perimeter is $(4 + \pi)$ meters. Find the radius of the semicircle that makes the area largest, and that area.

SET C Challenge and AP-Style

Give decimal answers to three places. [Calculator]

17. An open box is made from a 30 cm by 20 cm sheet by cutting squares of side x from the corners. Find x for maximum volume, to three decimal places, the dimensions of the box, and the volume.
18. A company sells x units per week at the price $p(x) = 80 - 0.02x$ dollars per unit, and the cost of producing x units is $C(x) = 1000 + 10x + 0.00004x^3$ dollars, for $x \geq 0$. Find the production level that maximizes weekly profit, the price at that level, and the maximum profit, to three decimal places.
19. Find the point on the curve $y = \sqrt{x}$ closest to $(3, 1)$, and the shortest distance, to three decimal places.
20. A 100 cm wire is cut into two pieces. One is bent into a square and the other into a circle. Where should the cut be to (a) minimize and (b) maximize the total area? Give lengths and areas to three decimals.

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21. Find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ with sides parallel to the axes. Give decimals to three places.

3.9 Optimization II

Extend optimization to cylinders, cones in spheres, light and travel time, and cost models, and learn to justify with the Candidates Test or the derivative tests that your answer is truly a maximum or minimum.

The problems in this section add geometry (cylinders, cones, spheres), motion at different speeds (light and travel time), and cost models. The method is the same as before: one variable, a domain, critical numbers, and a **justification** that the critical number really gives the absolute extremum. On the AP exam the justification is worth its own point, so you must write it.

Three Ways to Justify an Absolute Extremum

1. Candidates Test (closed interval). Compare the values of the objective at the endpoints and at the critical numbers inside. This is the safest method when the domain is $[a, b]$.

2. First Derivative Test (global version). If f is continuous on an interval and c is the only critical number there, a sign change of f' from $+$ to $-$ at c gives the absolute maximum, and from $-$ to $+$ gives the absolute minimum.

3. Second Derivative Test (global version). If $f'(c) = 0$ and $f''(x) > 0$ for all x in the interval, then $f(c)$ is the absolute minimum; if $f''(x) < 0$ for all x in the interval, it is the absolute maximum. A bare $f''(c) > 0$ only proves a local minimum.

Formulas Used Here

Cylinder: $V = \pi r^2 h$, closed surface $S = 2\pi r^2 + 2\pi rh$, open top $S = \pi r^2 + 2\pi rh$. Cone: $V = \frac{1}{3}\pi r^2 h$. Sphere of radius R: a cross-section at distance d from the center is a circle of radius $\sqrt{R^2 - d^2}$.

Travel time $= \dfrac{\text{distance}}{\text{speed}}$. A straight-line path from $(0, a)$ to $(x, 0)$ has length $\sqrt{x^2 + a^2}$. Total time or total cost is the **sum** of the parts, each written in terms of the one variable x.

EXAMPLE 1

A closed cylindrical can must hold 250π cm^3. Find the radius and height that minimize the surface area, and the minimum area.

SOLUTION

Constraint: $\pi r^2 h = 250\pi$, so $h = \dfrac{250}{r^2}$. Domain: $r > 0$.

Objective: $S(r) = 2\pi r^2 + 2\pi r \cdot \dfrac{250}{r^2} = 2\pi r^2 + \dfrac{500\pi}{r}$.

$S'(r) = 4\pi r - \dfrac{500\pi}{r^2} = 0$ gives $r^3 = 125$, so $r = 5$. $S''(r) = 4\pi + \dfrac{1000\pi}{r^3} > 0$ for all $r > 0$, so $r = 5$ gives the absolute minimum.

$h = \frac{250}{25} = 10$ cm (height equals the diameter). $S(5) = 50\pi + 100\pi = 150\pi$ cm^2.

EXAMPLE 2

Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 6.

SOLUTION

Let the cone have height h and base radius r, with its apex at the top of the sphere. A cross-section through the axis shows $r^2 + (h - 6)^2 = 36$, so $r^2 = 12h - h^2 = h(12 - h)$. Domain: $0 < h < 12$.

$V(h) = \dfrac{1}{3}\pi r^2 h = \dfrac{\pi}{3} h^2(12 - h) = \dfrac{\pi}{3}(12h^2 - h^3)$.

$V'(h) = \dfrac{\pi}{3}(24h - 3h^2) = \pi h(8 - h)$. The only critical number in the domain is $h = 8$, with $V' > 0$ on $(0, 8)$ and $V' < 0$ on $(8, 12)$. By the First Derivative Test, $h = 8$ gives the absolute maximum.

$r^2 = 8 \cdot 4 = 32$ and $V = \dfrac{\pi}{3} \cdot 64 \cdot 4 = \dfrac{256\pi}{3}$. The largest cone has height 8, base radius $4\sqrt{2}$, and volume $\dfrac{256\pi}{3}$.

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EXAMPLE 3

Light travels from point $A = (0, 3)$ in one medium (speed 4) to a point $(x, 0)$ on the boundary $y = 0$, then to point $B = (7, -4)$ in a second medium (speed 3). Find the x that minimizes the travel time, and the minimum time.

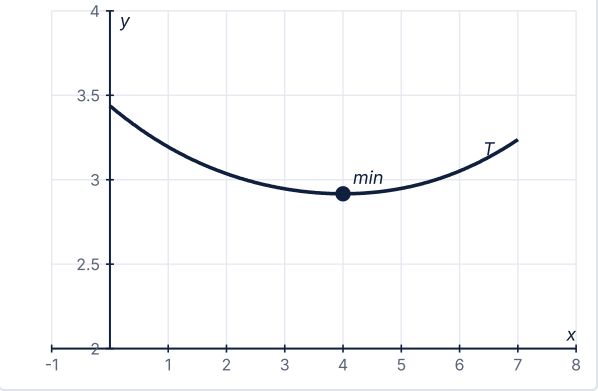
SOLUTION

Domain: $0 \le x \le 7$. Time $=$ distance/speed: $T(x) = \dfrac{\sqrt{x^2+9}}{4} + \dfrac{\sqrt{(7-x)^2+16}}{3}$.

$T'(x) = \dfrac{x}{4\sqrt{x^2+9}} - \dfrac{7-x}{3\sqrt{(7-x)^2+16}}$. Try $x = 4$: $\dfrac{4}{4 \cdot 5} - \dfrac{3}{3 \cdot 5} = \dfrac{1}{5} - \dfrac{1}{5} = 0$.

Uniqueness: $T''(x) = \dfrac{9}{4(x^2+9)^{3/2}} + \dfrac{16}{3\left((7-x)^2+16\right)^{3/2}} > 0$, so T' is increasing and $x = 4$ is the only critical number, with T' changing from negative to positive.

$T(4) = \dfrac{5}{4} + \dfrac{5}{3} = \dfrac{35}{12}$. Check the endpoints: $T(0) \approx 3.437$ and $T(7) \approx 3.237$ are larger. The minimum time is $\frac{35}{12}$ at $x = 4$. (At the minimum, $\dfrac{\sin\theta_1}{4} = \dfrac{\sin\theta_2}{3}$, where $\sin\theta_1 = \frac{4}{5}$ and $\sin\theta_2 = \frac{3}{5}$ are measured from the vertical; this is the law of refraction.)



Travel time T(x) for Example 3.

EXAMPLE 4

A pipeline must run from point P on one bank of a river 2 km wide to a point Q on the opposite bank, 6 km downstream from the point directly across from P. Laying pipe under water costs 5000 dollars per km and laying it along the land costs 3000 dollars per km. The pipe crosses the river diagonally to a point x km downstream of the point across from P, then follows the bank to Q. Find the least cost.

SOLUTION

Domain: $0 \le x \le 6$. Underwater length $\sqrt{4+x^2}$, land length $6 - x$. $C(x) = 5000\sqrt{4+x^2} + 3000(6-x)$.

$C'(x) = \dfrac{5000x}{\sqrt{4+x^2}} - 3000 = 0$ gives $5x = 3\sqrt{4+x^2}$, so $25x^2 = 36 + 9x^2$, $x^2 = \frac{9}{4}$, $x = \frac{3}{2}$.

Candidates Test: $C(0) = 10000 + 18000 = 28000$; $C\left(\frac{3}{2}\right) = 5000(2.5) + 3000(4.5) = 26000$; $C(6) = 5000\sqrt{40} \approx 31623$.

The least cost is 26000 dollars, when the pipe reaches the far bank 1.5 km downstream.

Common mistakes

(1) Writing the objective in terms of two variables, or substituting the constraint incorrectly (check by plugging in the final numbers). (2) Using $f''(c) > 0$ at a single point as proof of an absolute minimum; say $f'' > 0$ on the whole domain, or use a sign chart for f'. (3) Forgetting to compare endpoints when the domain is closed. (4) Solving the derivative equation and stopping: after $5x = 3\sqrt{4+x^2}$ you must square, solve, and check the root is in the domain. (5) Leaving out the value asked for (volume, cost, time) after finding x. AP-style sentence: "Since $V'(h) > 0$ for $0 < h < 8$ and $V'(h) < 0$ for $8 < h < 12$, V has an absolute maximum at $h = 8$."

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Practice

SET A Skills

Solve each problem and justify that your answer is a maximum or minimum. [No calculator]

1. A closed cylindrical can holds 54π cm^3. Find the radius and height that minimize the surface area, and the minimum surface area.
2. An open-top cylindrical tank (base but no lid) holds 64π m^3. Find r and h that minimize the material (surface area), and the minimum area.
3. Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius 3.
4. A cone has slant height 10. Find the height that gives the greatest volume, and that volume.
5. A ray of light starts at $A = (0, 2)$, reflects off the x-axis at $(x, 0)$, and reaches $B = (6, 4)$. Find the x that makes the total path length least, and that length. (Write $D(x)$ and use calculus.)
6. A closed box with a square base has volume 1000 cm^3. Find the dimensions that minimize surface area, and the minimum area.
7. A closed cylindrical can holds 32π cm^3. The top and bottom cost 2 cents per cm^2 and the side costs 1 cent per cm^2. Find the radius and height of least cost, and the cost.
8. The cost in dollars of making x items is $C(x) = 2x^2 + 50$, $x > 0$. The average cost per item is $A(x) = \frac{C(x)}{x}$. Find the production level that minimizes A, justify, and compare A with C' there.
9. A rectangle has one vertex at the origin, sides along the positive axes, and the opposite vertex on the curve $y = e^{-x}$. Find the maximum area.

SET B Applications and Mixed Problems

Solve each problem. Give exact answers. [No calculator]

10. The strength of a rectangular beam is proportional to wh^2, where w is its width and h its depth. A beam is cut from a circular log of radius 6 dm, so its cross-section has diagonal 12 dm. Find w and h for the strongest beam. Justify.
11. A right circular cone has base radius 6 and height 12. A cylinder is inscribed in it with its base on the base of the cone. Find the dimensions and volume of the cylinder of greatest volume.
12. A canoeist is 2 km from a straight shore and wants to reach a point on the shore 6 km along the shore from the point nearest her. She paddles at 3 km/h and walks at 5 km/h. She lands at a point x km along the shore from the nearest point. Find x that minimizes the total time and the minimum time.
13. A cable must run from a point on one bank of a river 3 km wide to a point 8 km downstream on the opposite bank. It costs 5000 dollars per km under water and 4000 dollars per km on land. It crosses diagonally to a point x km downstream, then runs along the bank. Find the least cost.
14. A rectangle is inscribed in a semicircle of radius 5, with one side on the diameter. Find the largest possible area and the dimensions.
15. A closed box with a square base has total surface area 96 cm^2. Find the dimensions that maximize the volume, and the volume.
16. A triangle has two sides of length 10 with included angle θ, $0 < \theta < \pi$. Find the θ that maximizes the area, and the maximum area.

SET C Challenge and AP-Style

Give decimal answers to three places. [Calculator]

17. A closed cylindrical can holds 1000 cm^3. Find the radius and height that minimize surface area, and the minimum area, to three decimals. Justify.
18. A closed can holds 500 cm^3. The top and bottom cost 0.04 dollars per cm^2 and the side costs 0.02 dollars per cm^2. Find the radius, height and minimum cost to three decimals.
19. Find the dimensions and the volume of the largest right circular cylinder that fits inside a sphere of radius 10 cm. Give decimals to three places.
20. A lifeguard stands on sand 30 m from the water's edge. A swimmer is in the water 50 m from the edge, and the horizontal distance between them is 100 m. The lifeguard runs at 5 m/s on sand and swims at 1.5 m/s. She runs in a straight line to the edge, entering the water x m (measured along the shoreline) from the point nearest her, then swims straight to the swimmer. Find the x that minimizes the total time, and the time, to three decimals.
21. A pipeline crosses a river 1.2 km wide diagonally and then runs along the bank to a point 4 km downstream of the point directly across. Underwater cost is 80 thousand dollars per km and land cost is 50 thousand dollars per km. Find the crossing point and least cost, to three decimals.

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3.10 Linearization and Differentials

Use tangent lines to approximate function values, decide from concavity whether the approximation is too large or too small, and use differentials to estimate the effect of measurement error.

Close to the point of tangency, a smooth curve is almost indistinguishable from its tangent line. That simple fact gives a fast way to estimate hard values such as $\sqrt{26}$ or $e^{0.1}$ using only arithmetic, and it explains how a small error in a measurement turns into an error in a calculated quantity.

Linearization (Tangent Line Approximation)

If f is differentiable at $x = a$, the **linearization** of f at a is

$$L(x) = f(a) + f'(a)(x - a).$$

For x near a, $f(x) \approx L(x)$. The approximation is usually good only close to a, so choose a as the nearest point where $f(a)$ and $f'(a)$ are easy to compute.

Over- or Underestimate: Use Concavity

If $f'' > 0$ on an interval containing a and x (concave up), the tangent line lies **below** the graph, so $L(x) \le f(x)$: the approximation is an **underestimate**.

If $f'' < 0$ on that interval (concave down), the tangent line lies **above** the graph, so $L(x) \ge f(x)$: the approximation is an **overestimate**. This holds on both sides of a.

Differentials and Error Estimates

For $y = f(x)$, the **differential** is $dy = f'(x)\,dx$. If x changes by $\Delta x = dx$, the actual change is $\Delta y = f(x + dx) - f(x) \approx dy$.

If a measurement x has error dx, the resulting error in $y = f(x)$ is approximately dy. The **relative error** is $\dfrac{dy}{y}$, and the **percent error** is $100 \cdot \dfrac{dy}{y}$.

EXAMPLE 1

Find the linearization of $f(x) = \sqrt{3x + 1}$ at $a = 1$ and use it to approximate $f(1.04)$. Is the estimate too large or too small?

SOLUTION

$f(1) = \sqrt{4} = 2$. $f'(x) = \dfrac{3}{2\sqrt{3x+1}}$, so $f'(1) = \dfrac{3}{4}$.

$L(x) = 2 + \dfrac{3}{4}(x - 1)$.

$f(1.04) \approx L(1.04) = 2 + \dfrac{3}{4}(0.04) = 2.03$.

$f''(x) = -\dfrac{9}{4(3x+1)^{3/2}} < 0$, so the graph is concave down and the tangent line is above it: 2.03 is an **overestimate**. (The true value is about 2.02978.)

EXAMPLE 2

Let $f(x) = e^{2x}$. Use the tangent line at $x = 0$ to approximate $f(0.05) = e^{0.1}$ and $f(-0.05) = e^{-0.1}$. Decide whether each result is an over- or underestimate.

SOLUTION

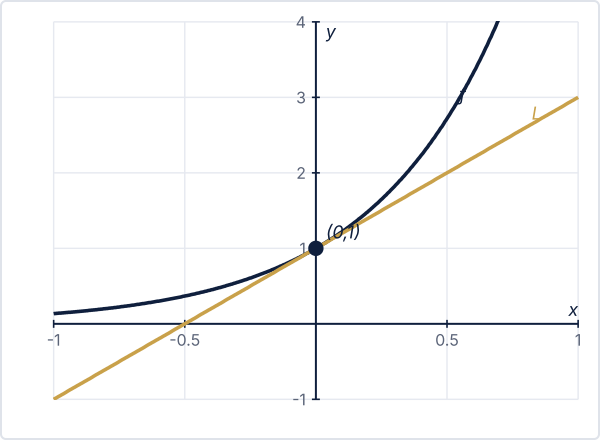
$f(0) = 1$ and $f'(x) = 2e^{2x}$, so $f'(0) = 2$. $L(x) = 1 + 2x$.

$f(0.05) = e^{0.1} \approx L(0.05) = 1.1$ and $f(-0.05) = e^{-0.1} \approx L(-0.05) = 0.9$.

$f''(x) = 4e^{2x} > 0$ for all x: concave up, so the tangent line lies below the curve on both sides. Both 1.1 and 0.9 are **underestimates**. (True values: 1.10517 and 0.90484.)

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The graph of f(x)=e^(2x) lies above its tangent line L(x)=1+2x.

EXAMPLE 3

A function f is twice differentiable with $f(2) = 5$ and $f'(2) = -3$, and $f''(x) < 0$ for $1 \le x \le 3$. Approximate $f(2.1)$ and $f(1.9)$. Are the approximations over- or underestimates? Justify.

SOLUTION

$L(x) = 5 - 3(x - 2)$. So $f(2.1) \approx 5 - 0.3 = 4.7$ and $f(1.9) \approx 5 + 0.3 = 5.3$.

Since $f'' < 0$ on $[1, 3]$, the graph of f is concave down there, so the tangent line at $x = 2$ lies above the graph on this whole interval. Therefore $L(2.1) \ge f(2.1)$ and $L(1.9) \ge f(1.9)$: both approximations are **overestimates**.

AP-style sentence: "The tangent line lies above the graph of f because f is concave down on $[1, 3]$, so each approximation is greater than the actual value."

EXAMPLE 4

For $y = x^3 - 2x$, find dy when $x = 2$ and $dx = 0.05$. Compare with the exact change Δy.

SOLUTION

$dy = (3x^2 - 2)\,dx$. At $x = 2$: $dy = (12 - 2)(0.05) = 10(0.05) = 0.5$.

$\Delta y = f(2.05) - f(2) = (8.615125 - 4.1) - (8 - 4) = 0.515125$.

The differential 0.5 is close to the exact change 0.515; the difference is small because dx is small.

EXAMPLE 5

The radius of a sphere is measured as 10 cm with a possible error of ± 0.1 cm. Estimate the resulting error in the volume and the percent error.

SOLUTION

$V = \frac{4}{3}\pi r^3$, so $dV = 4\pi r^2\,dr$. With $r = 10$ and $dr = 0.1$: $dV = 4\pi(100)(0.1) = 40\pi \approx 125.7\ \text{cm}^3$.

$V = \frac{4}{3}\pi(1000) = \frac{4000\pi}{3}$. Relative error $\dfrac{dV}{V} = \dfrac{40\pi}{4000\pi/3} = 0.03$. Equivalent shortcut: $\dfrac{dV}{V} = 3\dfrac{dr}{r} = 3(0.01)$.

The volume is uncertain by about $\pm 40\pi\ \text{cm}^3$, which is a 3% error: a 1% error in the radius becomes about a 3% error in volume.

Common mistakes

(1) Using $f(x)$ in place of $f(a)$, or forgetting the factor $(x - a)$. (2) Choosing a base point a far from x, which makes the approximation poor. (3) Deciding over or under from the sign of f' instead of f''. (4) Writing $\Delta y = dy$: the differential is only an approximation of Δy. (5) Mixing up relative error $\frac{dy}{y}$ with the absolute error dy. (6) Using degrees in a derivative of a trigonometric function: dx must be in radians.

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Practice

SET A Skills

Answer each question. [No calculator]

1. Find the linearization of $f(x) = x^3 + 2x$ at $a = 1$.
2. Find the linearization of $g(x) = \sqrt{x}$ at $a = 16$ and use it to approximate $\sqrt{17}$.
3. Let $h(x) = \dfrac{1}{x}$. Find its linearization at $a = 2$, approximate $\dfrac{1}{2.1}$, and decide whether the estimate is too large or too small.
4. Find the linearization of $f(x) = \tan x$ at $a = \frac{\pi}{4}$, and use it to approximate $\tan\left(\frac{\pi}{4} + 0.05\right)$. Over or under?
5. Use the tangent line to $f(x) = \ln x$ at $a = 1$ to approximate $\ln(1.2)$. Is this estimate too large or too small? Explain.
6. Use a linearization of $f(x) = \sqrt[3]{x}$ at $a = 8$ to approximate $\sqrt[3]{8.06}$.
7. For $y = x^3 - 4x$, find dy at $x = 2$ with $dx = -0.1$, and compare with Δy.
8. A twice-differentiable function f satisfies the values in the table, and $f''(x) < 0$ on $(2, 4)$. Estimate $f(3.2)$ and $f(2.9)$ using the tangent line at $x = 3$, and state whether each is an over- or underestimate.

x	3
$f(x)$	7
$f'(x)$	2

9. The edge of a cube is measured as 5 cm with an error of at most 0.02 cm. Estimate the maximum error in the volume and the relative error.

SET B Applications and Mixed Problems

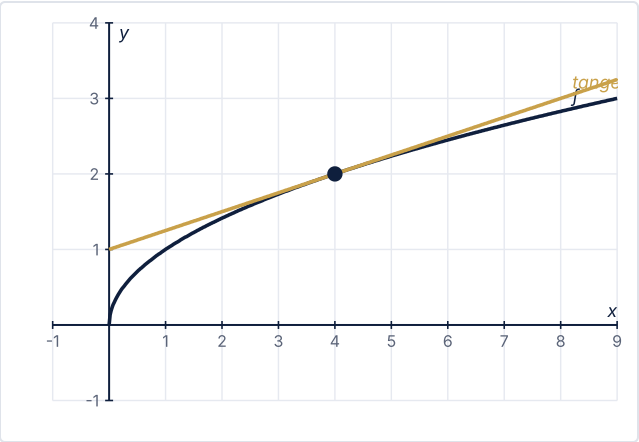
Solve each problem. Give exact answers or simple decimals. [No calculator]

10. The radius of a sphere increases from 6 cm by 0.05 cm. Use a differential to estimate the increase in volume.
11. The period of a pendulum of length L meters is $T = 2\pi\sqrt{\frac{L}{10}}$ seconds. Use a differential to estimate the change in T when L increases from 2.5 m to 2.6 m.
12. A resistor is measured as 20 ohms with error ± 0.5 ohm. It carries a 10 volt supply, so the current is $I = \frac{10}{R}$ amperes. Estimate the error in I and the relative error.
13. The amount of a substance is $N(t) = 800e^{-0.05t}$ grams after t days. Use the tangent line at $t = 0$ to estimate $N(2)$. Is the estimate an over- or underestimate? Justify.
14. The radius of a circle is measured as 10 cm with error ± 0.2 cm. Estimate the error in the area and the percent error.
15. Let g be twice differentiable with $g(1) = 4$, $g'(1) = 3$ and $g''(x) > 0$ for $0 \le x \le 2$. (a) Write the linearization of g at $x = 1$.
(b) Estimate $g(1.2)$ and say whether it is over or under.
(c) Find the x-intercept of the tangent line.

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16. The graph shows $f(x) = \sqrt{x}$ and its tangent line at $x = 4$. (a) Use the graph to find $f(4)$ and the slope of the tangent line.
(b) Approximate $f(4.4)$.
(c) Is your approximation too large or too small? Explain.



y=sqrt(x) and its tangent line at x=4.

SET C Challenge and AP-Style

Give decimal answers to three places. [Calculator]

17. The radius of a sphere is measured as 12.0 cm with error at most 0.1 cm. Estimate the maximum error in (a) the volume and (b) the surface area $S = 4\pi r^2$, and the percent error of each, to three decimals.

18. Use the tangent line to $f(x) = \sqrt[3]{x}$ at $a = 27$ to approximate $\sqrt[3]{30}$. Compare with the calculator value and state whether the estimate is too large or too small. Give three decimals.

19. Let $f(x) = \arctan x$. Find the linearization at $a = 1$, approximate $\arctan(1.1)$, and compare with the calculator value, to three decimals. Explain the sign of the error using $f''(x) = \frac{-2x}{(1+x^2)^2}$.

20. A spherical tumor has diameter measured as 2.4 cm with an error of at most 0.1 cm. Its volume is $V = \frac{\pi}{6}d^3$. Estimate the maximum error in the volume and the percent error, to three decimals.

21. A cooling liquid has temperature $T(t)$ in degrees Celsius with $T(10) = 60$, $T'(10) = -1.5$ degrees per minute, and $T'(t) < 0$ and $T''(t) > 0$ for $t \geq 0$. (a) Estimate $T(12)$ and $T(9)$.
(b) Are these over- or underestimates?
(c) Use the tangent line to estimate when $T = 50$. Is the actual time earlier or later than this estimate? Give decimals to three places.

3.11 Rates of Change in Context

Interpret derivatives with units in science, economics, and geometry, and use marginal analysis. Learn the sentences that earn full credit when a problem asks what a derivative means.

Every derivative is a **rate**: it says how fast one quantity changes with respect to another. In applications the number alone is not enough. You must attach units and say what it means in the context. This section collects the interpretations you need for science, economics, and geometry, and the habits that earn full credit on the AP exam.

Interpreting a Derivative in Context

$f'(a)$ is the **instantaneous rate of change** of f with respect to x at $x = a$. Its units are (units of f) per (units of x).

A good sentence has four parts: the quantity, the value of the variable, the rate with units, and the direction. Example: "At $t = 5$ minutes, the temperature is decreasing at 3 degrees Celsius per minute."

The **average rate of change** on $[a, b]$ is $\dfrac{f(b) - f(a)}{b - a}$ (a slope of a secant line). For x increasing by one unit, $f(x + 1) - f(x) \approx f'(x)$.

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Marginal Analysis in Economics

For cost $C(x)$, revenue $R(x)$ and profit $P(x) = R(x) - C(x)$ with x items: **marginal cost** $C'(x)$, **marginal revenue** $R'(x)$, and **marginal profit** $P'(x) = R'(x) - C'(x)$. Each approximates the change from producing and selling one more item.

Profit is maximized where $R'(x) = C'(x)$ (marginal revenue equals marginal cost) and $P'' < 0$, or at an endpoint of the allowed range.

Average cost $A(x) = \dfrac{C(x)}{x}$ is minimized where $A'(x) = 0$, which happens when $C'(x) = A(x)$.

EXAMPLE 1

The temperature (in degrees Fahrenheit) of a cup of coffee is $T(t) = 70 + 90e^{-0.1t}$, where t is in minutes. Find and interpret $T'(10)$ and the average rate of change on $0 \le t \le 10$.

SOLUTION

$T'(t) = -9e^{-0.1t}$, so $T'(10) = -9e^{-1} = -\dfrac{9}{e} \approx -3.311$.

Interpretation: "At $t = 10$ minutes the temperature of the coffee is decreasing at about 3.311 degrees Fahrenheit per minute."

Average rate: $\dfrac{T(10) - T(0)}{10} = \dfrac{\left(70 + \frac{90}{e}\right) - 160}{10} = \dfrac{9}{e} - 9 \approx -5.689$. "From $t = 0$ to $t = 10$ minutes the temperature decreases by an average of about 5.689 degrees per minute." The cooling slows down, which is why the instantaneous rate at 10 is smaller in size than the average.

t (hours)	0	2	5	9	12
$W(t)$ (gallons)	40	52	61	70	64

EXAMPLE 2

The amount of water $W(t)$, in gallons, in a tank is differentiable and given in the table. Estimate $W'(7)$, interpret it, and find the average rate of change on $0 \le t \le 12$.

SOLUTION

Use the nearest table values on each side of $t = 7$: $W'(7) \approx \dfrac{W(9) - W(5)}{9 - 5} = \dfrac{70 - 61}{4} = 2.25$ gallons per hour.

Interpretation: "At $t = 7$ hours the amount of water in the tank is increasing at about 2.25 gallons per hour."

Average rate: $\dfrac{W(12) - W(0)}{12 - 0} = \dfrac{64 - 40}{12} = 2$ gallons per hour.

EXAMPLE 3

Making x items costs $C(x) = 500 + 12x + 0.03x^2$ dollars, and selling x items brings in revenue $R(x) = 40x - 0.05x^2$ dollars. (a) Find the marginal cost at $x = 100$ and compare with the actual cost of the 101st item. (b) Find the production level that maximizes profit and the maximum profit.

SOLUTION

(a) $C'(x) = 12 + 0.06x$, so $C'(100) = 18$ dollars per item. Actual: $C(101) - C(100) = 12 + 0.03(201) = 18.03$ dollars, very close.

(b) $P(x) = R(x) - C(x) = 28x - 0.08x^2 - 500$. $P'(x) = 28 - 0.16x = 0$ at $x = 175$. Equivalently $R'(x) = C'(x)$: $40 - 0.1x = 12 + 0.06x$.

$P''(x) = -0.16 < 0$, so $x = 175$ gives the absolute maximum. $P(175) = 4900 - 2450 - 500 = 1950$ dollars.

EXAMPLE 4

A spherical balloon has volume $V = \frac{4}{3}\pi r^3$ cubic centimeters when its radius is r centimeters. Find $\dfrac{dV}{dr}$ at $r = 3$ and interpret it.

SOLUTION

$\dfrac{dV}{dr} = 4\pi r^2$, so at $r = 3$ the value is $36\pi \approx 113.1$ cubic centimeters per centimeter.

Interpretation: when the radius is 3 cm, the volume increases by about 36π cm^3 for each additional centimeter of radius.

Notice that $4\pi r^2$ is the surface area of the sphere: adding a thin shell of thickness dr adds volume $\approx (\text{area})\,dr$.

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EXAMPLE 5

The concentration of a chemical is $c(t) = \dfrac{2}{1+t}$ moles per liter, t in seconds. Find $c'(1)$, interpret it, and find the time in $(0, 3)$ when the instantaneous rate equals the average rate on $[0, 3]$.

SOLUTION

$c'(t) = -\dfrac{2}{(1+t)^2}$, so $c'(1) = -\dfrac{1}{2}$. "At $t = 1$ second the concentration is decreasing at 0.5 moles per liter per second."

Average rate on $[0, 3]$: $\dfrac{c(3) - c(0)}{3} = \dfrac{0.5 - 2}{3} = -0.5$.

Solve $-\dfrac{2}{(1+t)^2} = -\dfrac{1}{2}$: $(1+t)^2 = 4$, $t = 1$. This is the time guaranteed by the Mean Value Theorem, since c is continuous on $[0, 3]$ and differentiable on $(0, 3)$.

Common mistakes

(1) Giving a derivative interpretation without units or without the time or value of the variable. (2) Saying "the function is -3" instead of "the quantity is decreasing at 3 units per unit". (3) Confusing average rate (a difference quotient) with instantaneous rate (a derivative). (4) Writing the units of f' as the units of f only. (5) Treating marginal cost as the exact cost of the next item; it is an approximation. (6) Forgetting that a negative derivative means decreasing, and that the magnitude is the speed of that change.

Practice

SET A Skills

Answer each question and include units. [No calculator]

1. The area of an oil slick is $A(t)$ square meters, t in minutes. Interpret $A'(20) = 15$.
2. The cost of making x chairs is $C(x)$ dollars, and $C'(50) = 34$. Interpret this and estimate $C(51) - C(50)$.
3. The ocean temperature is $T(d)$ degrees Celsius at depth d meters, and $T'(100) = -0.04$. Interpret this.
4. For a circle, find $\dfrac{dA}{dr}$ at $r = 4$ cm with units, and explain what it measures.
5. A cube has edge s cm and volume $V = s^3$. Find $\dfrac{dV}{ds}$ at $s = 6$ with units. Compare with the surface area $6s^2$.
6. The revenue from selling x items is $R(x) = 60x - 0.2x^2$ dollars. Find the marginal revenue at $x = 50$, and compare it with the exact revenue from the 51st item.
7. The profit from making x items is $P(x) = -2x^2 + 80x - 350$ dollars. Find the number of items that maximizes profit and the maximum profit.
8. The cost of making x items is $C(x) = 0.5x^2 + 8x + 98$ dollars. Find the production level that minimizes the average cost $A(x) = \frac{C(x)}{x}$, and show that the marginal cost equals the average cost there.
9. The table gives the amount $Q(t)$ of liquid, in liters, in a tank at time t minutes. (a) Estimate $Q'(15)$ with units.
   (b) Find the average rate of change on $0 \le t \le 30$.
   (c) The slopes of consecutive intervals are decreasing. What does this suggest about Q''?

t	0	10	20	30
$Q(t)$	100	140	165	180

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SET B Applications and Mixed Problems

Solve each problem and include units in your interpretations. [No calculator]

10. A drug amount in the body is $A(t) = 40e^{-0.2t}$ mg after t hours. (a) Find $A'(0)$ and interpret it.
(b) Show that $A'(t) = -0.2A(t)$ and explain what this says.
(c) At what time is $A'(t) = -2$?

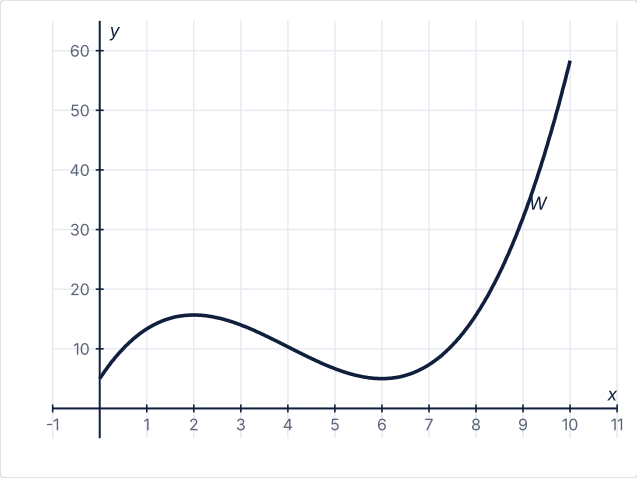
11. The height of a seedling is $H(t) = 20 + 6t - 0.15t^2$ centimeters after t days, for $0 \le t \le 20$. (a) Find $H'(10)$ and interpret it with units.
(b) Find the average rate of change of H on $[0, 20]$.
(c) At what time is the seedling growing at 1.5 centimeters per day?

12. The price of x items is $p = 100 - 0.5x$ dollars each, and the cost is $C(x) = 20x + 500$ dollars. (a) Write the revenue and find the marginal revenue.
(b) Find the number of items that maximizes profit, the price, and the profit.

13. For a sphere, show that $\frac{dV}{dr}$ equals the surface area. For $r = 5$ cm, give $\frac{dV}{dr}$ and estimate the increase in volume when the radius grows by 0.02 cm.

14. The pressure of a gas at constant temperature is $P(V) = \frac{24}{V}$ atmospheres when the volume is V liters. (a) Find $P'(4)$ and interpret it.
(b) Find the average rate of change on $[2, 4]$.
(c) Find the volume in $(2, 4)$ where the instantaneous rate equals that average rate, and name the theorem.

15. The graph shows $W(t)$, the amount of water in liters in a tank, t hours after midnight, for $0 \le t \le 10$. (a) During which time interval is water leaving the tank?
(b) At what time is water leaving the tank at the greatest rate? (Hint: the slope is most negative at the inflection point.)
(c) At $t = 1$, is W' increasing or decreasing?



Graph of W(t).

16. The area of an equilateral triangle with side s cm is $A = \frac{\sqrt{3}}{4}s^2$. Find $\frac{dA}{ds}$ at $s = 6$ with units and interpret.

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SET C Challenge and AP-Style

Give decimal answers to three places. [Calculator]

17. A hot object has temperature $T(t) = 22 + 68e^{-0.07t}$ degrees Celsius after t minutes. (a) Find $T'(5)$ with units and interpret it.
(b) Find the average rate of change on $[0, 10]$.
(c) Find the time when $T'(t) = -2$. Give decimals to three places.

18. The table gives the number of bacteria $P(t)$ in thousands, t days after the start of an experiment; P is differentiable. (a) Estimate $P'(5)$ and $P'(14)$ with units.
(b) Find the average rate of change on $[0, 16]$.
(c) Use linear interpolation between the two nearest data points to estimate when the population reaches 230 thousand. Give decimals to three places.

t	0	3	7	12	16
$P(t)$	120	150	200	248	270

19. The revenue from selling x items is $R(x) = 50xe^{-0.02x}$ dollars and the cost is $C(x) = 300 + 8x$. Find the number of items that maximizes profit, the maximum profit, and the revenue there, to three decimals. Justify.

20. Water is poured into a cone-shaped container with its vertex down; the radius of the water surface is half the depth h (meters), so $V = \frac{\pi}{12}h^3$. (a) Find $\dfrac{dV}{dh}$ at $h = 6$ and interpret it.
(b) Estimate the volume added when the depth increases from 6 m by 0.05 m. Give three decimals.

21. A chemical concentration is $C(t) = 5e^{-0.3t} + 1$ moles per liter, t in seconds. Find the average rate of change on $[0, 4]$ and the time in $(0, 4)$ when the instantaneous rate equals it. Name the theorem. Give three decimals.

3.12 Multiple-Choice Practice on Applications of Derivatives

Thirty multiple-choice questions on extrema, the Mean Value Theorem, concavity, linearization, related rates, and optimization, in the style of the AP exam.

Work Part A without a calculator, then Part B with one. Aim for about two minutes per question. Check your answers and read every explanation, including the ones you got right.

SET A Part A [No calculator]

Part A: 20 questions. No calculator is allowed. Choose the best answer for each question.

1. Let $f(x) = x^3 - 6x^2 + 9x + 2$. On which interval is f decreasing?
(A) $(-\infty, 1)$ **(B)** $(1, 3)$ **(C)** $(3, \infty)$ **(D)** $(0, 2)$ **(E)** $(-\infty, 3)$

2. What is the absolute maximum value of $f(x) = x^3 - 12x$ on the interval $[-3, 5]$?
(A) 16 **(B)** 9 **(C)** -16 **(D)** 65 **(E)** 53

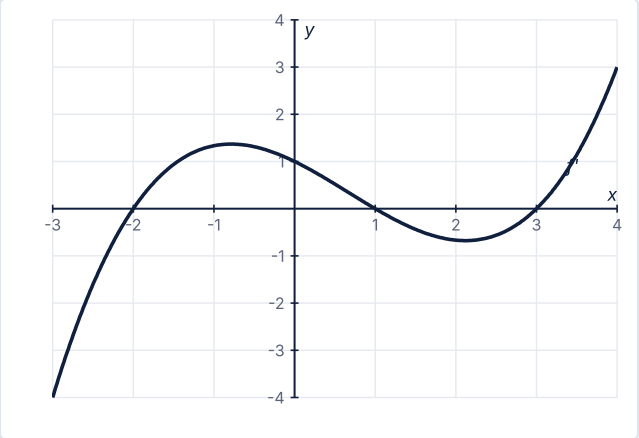
3. Let $f(x) = \sqrt{x + 1}$ on $[0, 8]$. The value of c guaranteed by the Mean Value Theorem is
(A) 3 **(B)** 2 **(C)** 4 **(D)** $\dfrac{8}{3}$ **(E)** $\dfrac{1}{4}$

4. The second derivative of f is $f''(x) = x(x - 2)^2(x + 1)$. At how many values of x does the graph of f have a point of inflection?
(A) Three, at $x = -1$, 0, and 2 **(B)** Two, at $x = 0$ and $x = 2$
(C) Two, at $x = -1$ and $x = 0$ only **(D)** One, at $x = -1$ only
(E) One, at $x = 2$ only

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5. The graph of f', the derivative of f, is shown. At which values of x does f have a local minimum?



Graph of f'.

(A) $x = 1$ only **(B)** $x = -2$ only **(C)** $x = -2$, $x = 1$, and $x = 3$
(D) $x = 3$ only **(E)** $x = -2$ and $x = 3$ only

6. A 17-foot ladder leans against a vertical wall. The bottom slides away from the wall at 3 feet per second. When the bottom is 8 feet from the wall, how fast is the top of the ladder sliding down the wall?

(A) $\frac{45}{8}$ ft/s **(B)** $\frac{5}{8}$ ft/s **(C)** 3 ft/s **(D)** $\frac{24}{17}$ ft/s **(E)** $\frac{8}{5}$ ft/s

7. The tangent line to $f(x) = x^{1/3}$ at $x = 64$ is used to approximate $\sqrt[3]{65}$. The approximation is

(A) $\frac{49}{12}$ **(B)** $\frac{191}{48}$ **(C)** $\frac{193}{48}$ **(D)** $\frac{13}{3}$ **(E)** $\frac{769}{192}$

8. An open-top box has a square base and a volume of 32 cubic meters. What is the minimum possible surface area of the box (base plus four sides)?

(A) 40 m^2 **(B)** 48 m^2 **(C)** 56 m^2 **(D)** 64 m^2 **(E)** 96 m^2

9. $\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x^2} =$

(A) 2 **(B)** 0 **(C)** 1
(D) 4 **(E)** The limit does not exist

10. The function f is twice differentiable. Values of f' and f'' at three points are given in the table. Which statement must be true?

x	1	3	5
$f'(x)$	0	0	0
$f''(x)$	4	0	-2

(A) f has a local maximum at $x = 1$
(B) f has a local extremum at $x = 3$
(C) f has an inflection point at $x = 3$
(D) f has a local minimum at $x = 1$ and a local maximum at $x = 5$
(E) f has a local minimum at $x = 5$

11. A particle moves along a line with position $s(t) = t^3 - 9t^2 + 15t$ for $t \geq 0$. At $t = 2$ the particle is

(A) moving left and speeding up **(B)** moving right and speeding up
(C) moving right and slowing down **(D)** moving left and slowing down
(E) at rest

12. How many critical points does $f(x) = x^{2/3}(x - 5)$ have?

(A) 0 **(B)** 1 **(C)** 3 **(D)** 2 **(E)** 4

13. On the interval $[-1, 1]$, the hypotheses of the Mean Value Theorem are satisfied by which function?

(A) $f(x) = |x|$ **(B)** $f(x) = \frac{1}{x}$ **(C)** $f(x) = x^{2/3}$
(D) $f(x) = \ln(x + 1)$ **(E)** $f(x) = x^{5/3}$

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14. Let $f'(x) = x^2(x-3)(x+1)$. At what value of x does f have a local maximum?

(A) 0 **(B)** -1 **(C)** 3 **(D)** 1 **(E)** -3

15. The graph of $f(x) = xe^{-x}$ has a point of inflection at $x =$

(A) 0 **(B)** 1 **(C)** 2 **(D)** e **(E)** -2

16. What is the x-coordinate of the point on the curve $y = \sqrt{x}$ that is closest to the point $(5, 0)$?

(A) 4 **(B)** 2 **(C)** $\frac{9}{2}$ **(D)** 3 **(E)** $\frac{7}{2}$

17. The radius of a cylinder is increasing at 2 mm/s while its height is decreasing at 3 mm/s. At the instant when $r = 4$ mm and $h = 10$ mm, the volume is changing at a rate of

(A) 160π mm ^{3} /s, increasing **(B)** 48π mm ^{3} /s, decreasing **(C)** 208π mm ^{3} /s, increasing
(D) 112π mm ^{3} /s, decreasing **(E)** 112π mm ^{3} /s, increasing

18. The function f is differentiable with $f(3) = 2$, $f'(3) = 4$, and $f''(x) > 0$ for $3 \le x \le 3.5$. The tangent line at $x = 3$ is used to approximate $f(3.2)$. The approximation is

(A) 2.8, and it is greater than $f(3.2)$ **(B)** 2.8, and it is less than $f(3.2)$
(C) 0.8, and it is less than $f(3.2)$ **(D)** 0.8, and it is greater than $f(3.2)$
(E) 2.2, and it is greater than $f(3.2)$

19. The function f is differentiable on $[1, 5]$ with $f(1) = 3$ and $f(5) = 11$. Which statement must be true?

(A) $f'(x) = 2$ for all x in $[1, 5]$ **(B)** $f'(c) = 8$ for some c in $(1, 5)$
(C) f is increasing on $[1, 5]$ **(D)** $f'(c) = 2$ for some c in $(1, 5)$
(E) $f(c) = 0$ for some c in $[1, 5]$

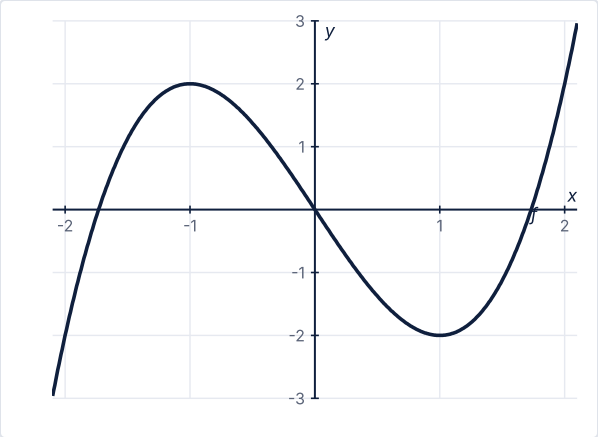
20. What is the absolute maximum value of $f(x) = xe^{-x^2}$ on $[0, \infty)$?

(A) $\frac{1}{\sqrt{2e}}$ **(B)** $\frac{1}{\sqrt{2}}$ **(C)** $\frac{1}{2e}$ **(D)** $\frac{1}{\sqrt{e}}$ **(E)** $\frac{1}{e}$

SET B Part B [Calculator]

Part B: 10 questions. A calculator is allowed. Round decimals to three places. Choose the best answer.

21. The function $f(x) = x^3 - 3x$ is graphed. On which interval is $f'(x) < 0$ and $f''(x) > 0$?



Graph of f.

(A) $(0, 1)$ **(B)** $(-2, -1)$ **(C)** $(-1, 0)$ **(D)** $(1, 2)$ **(E)** $(-1, 1)$

22. What is the absolute minimum value of $f(x) = e^x - 3x^2$ on $[0, 2]$? [Calculator]

(A) 1.101 **(B)** -4.611 **(C)** 1.000 **(D)** 0.204 **(E)** -3.611

23. Let $f(x) = \ln x + x$ on $[1, 4]$. The value of c guaranteed by the Mean Value Theorem is approximately [Calculator]

(A) 1.386 **(B)** 2.500 **(C)** 2.164 **(D)** 3.000 **(E)** 1.462

24. Water is poured into a conical tank, vertex down, with height equal to twice the radius of the top, at 5 cubic feet per minute. How fast is the water level rising when the depth is 3 feet? [Calculator]

(A) 0.442 ft/min **(B)** 1.061 ft/min **(C)** 1.592 ft/min
(D) 0.707 ft/min **(E)** 0.354 ft/min

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25. A rectangle is inscribed with its base on the x-axis and its upper vertices on the curve $y = 3\cos x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, symmetric about the y-axis. The maximum possible area is approximately [Calculator]

(A) 2.598 **(B)** 3.000 **(C)** 4.712 **(D)** 2.472 **(E)** 3.367

26. For $0 \le x \le 3$, let $f''(x) = e^x - 4x$. At what values of x does the graph of f have points of inflection? [Calculator]

(A) $x \approx 0.357$ only **(B)** $x \approx 2.153$ only **(C)** $x \approx 0.357$, 2.153, and 3
(D) $x \approx 1.386$ only **(E)** $x \approx 0.357$ and $x \approx 2.153$

27. The tangent line to $f(x) = \ln x$ at $x = 4$ is used to approximate $\ln(4.3)$. The approximation is [Calculator]

(A) less than $\ln(4.3)$ by about 0.003 **(B)** greater than $\ln(4.3)$ by about 0.075
(C) less than $\ln(4.3)$ by about 0.075 **(D)** greater than $\ln(4.3)$ by about 0.003
(E) equal to $\ln(4.3)$ to three decimal places

28. A particle moves along a line with velocity $v(t) = t^2 e^{-t}$ for $0 \le t \le 5$. The greatest velocity attained is [Calculator]

(A) 0.168 **(B)** 0.271 **(C)** 0.541 **(D)** 1.472 **(E)** 0.135

29. A closed cylindrical can must hold 500 cubic centimeters. The smallest possible total surface area is approximately [Calculator]

(A) $300.000\ \text{cm}^2$ **(B)** $348.734\ \text{cm}^2$ **(C)** $372.500\ \text{cm}^2$
(D) $250.000\ \text{cm}^2$ **(E)** $412.317\ \text{cm}^2$

30. A plane flies horizontally at a constant altitude of 3 km at 400 km/h on a straight path that passes directly over a radar station. When the plane is 2 km horizontally away from the point above the station and moving away from it, how fast is the distance from the plane to the station increasing? [Calculator]

(A) 221.880 km/h **(B)** 400.000 km/h **(C)** 277.350 km/h
(D) 110.940 km/h **(E)** 153.846 km/h

3.13 Free-Response Practice on Applications of Derivatives

Four multi-part free-response questions on motion, related rates, optimization, and graph analysis, with full solutions and point breakdowns.

Each question is worth 9 points. Write complete justifications: name the theorem or test, check its hypotheses, and state the conclusion in a sentence. Allow about 15 minutes per question.

QUESTION 1 Question 1

Show all work. Justify your answers with complete sentences. Decimal answers should be given to three decimal places.

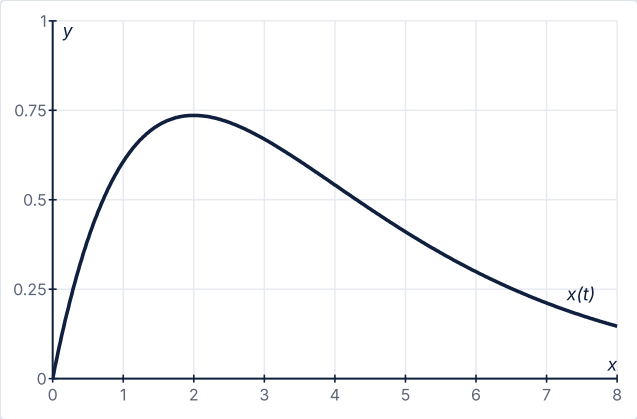
1. [Calculator] A particle moves along the x-axis. For $0 \le t \le 8$, its position is $x(t) = te^{-t/2}$, where x is in centimeters and t is in seconds. The graph of x is shown.

(a) Find $v(3)$ and $a(3)$. Is the speed of the particle increasing or decreasing at $t = 3$? Give a reason.

(b) At what time t does the particle change direction? Justify your answer.

(c) Find the greatest position the particle reaches for $0 \le t \le 8$. Justify that it is the greatest.

(d) Find the average velocity of the particle over $0 \le t \le 8$. Explain why the Mean Value Theorem guarantees a time c in $(0, 8)$ at which $v(c)$ equals this value, and find the value of c in $(0, 2)$.



Graph of the position x(t), with t on the horizontal axis.

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QUESTION 2

Question 2

Show all work. Justify your answers with complete sentences. Decimal answers should be given to three decimal places.

2. [Calculator] Gas is pumped into a spherical balloon. At time t seconds, $0 \le t \le 10$, the volume V in cubic centimeters is increasing at the rate $\frac{dV}{dt} = \frac{40}{1+t}$. At $t = 3$ the radius of the balloon is 5 cm. (The volume of a sphere is $V = \frac{4}{3}\pi r^3$ and its surface area is $S = 4\pi r^2$.)
   (a) Find $\frac{dr}{dt}$ at $t = 3$. Include units.
   (b) Find the rate at which the surface area is changing at $t = 3$. Include units.
   (c) Find $\frac{d^2r}{dt^2}$ at $t = 3$. Is the radius growing faster or slower at $t = 3$ than a moment earlier? Give a reason.
   (d) Use the tangent line to $r(t)$ at $t = 3$ to approximate $r(3.2)$. Assuming r is concave down on $3 \le t \le 3.2$, is your approximation an overestimate or an underestimate? Explain.

QUESTION 3

Question 3

Show all work. Justify your answers with complete sentences. Decimal answers should be given to three decimal places.

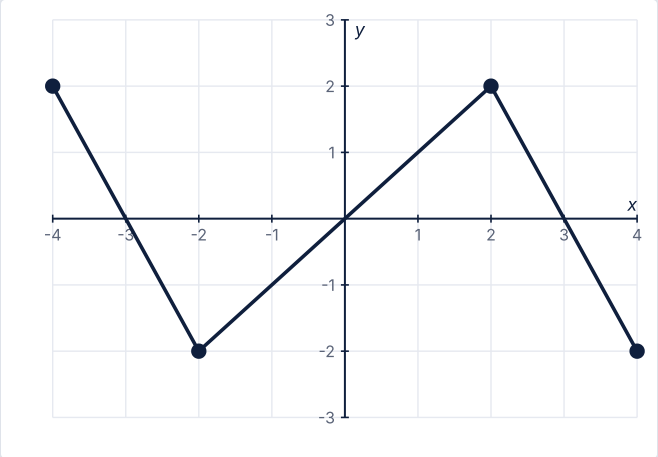
3. [No calculator] A power company must run a cable from a point P on an island to a town at point B on a straight shoreline. Point A on the shore is 3 km from P (the segment PA is perpendicular to the shore), and B is 8 km from A along the shore. The cable runs underwater from P to a point Q on the shore that is x km from A ($0 \le x \le 8$), and then along the shore from Q to B. Underwater cable costs 5 thousand dollars per km and shore cable costs 3 thousand dollars per km.
   (a) Write an expression for the total cost $C(x)$ in thousands of dollars.
   (b) Find the value of x for which $C'(x) = 0$.
   (c) Use the second derivative to justify that your value of x gives a local minimum of C.
   (d) Find the absolute minimum cost. Justify that it is smaller than the cost at each endpoint, $x = 0$ and $x = 8$, without using a calculator.

QUESTION 4

Question 4

Show all work. Justify your answers with complete sentences. Decimal answers should be given to three decimal places.

4. [No calculator] The function f is continuous on $[-4, 4]$. The graph of f', the derivative of f, consists of three line segments and is shown. It passes through the points $(-4, 2)$, $(-2, -2)$, $(2, 2)$, and $(4, -2)$. Also, $f(1) = 4$.
   (a) Find the x-coordinate of each local maximum and local minimum of f on $(-4, 4)$. Justify your answers.
   (b) On what intervals is the graph of f concave up? Find the x-coordinates of all points of inflection of f and justify.
   (c) Write an equation for the tangent line to the graph of f at $x = 1$ and use it to approximate $f(1.5)$. Is this an overestimate or an underestimate? Explain.
   (d) Show that f is both increasing and concave down on the interval $(-4, -3)$.



Graph of f'.

Answers

Chapter 3: Answers and Explanations

Answers for every practice problem, numbered to match. Read the explanation even when your answer is correct.

3.1 The Extreme Value Theorem and Critical Points

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SET A · SKILLS

1. $x=-1$ and $x=3$. $f'(x)=3x^2-6x-9=3(x-3)(x+1)$, defined everywhere.
2. $x=-2,\ 0,\ 2$. $f'(x)=4x^3-16x=4x(x-2)(x+2)$.
3. $x=-1$ and $x=1$. $f'(x)=\dfrac{1-x^2}{(x^2+1)^2}$ is zero when $x^2=1$ and defined for all x.
4. $x=0$ and $x=1$. $f'(x)=\frac{4}{3}x^{-2/3}(x-1)$: zero at $x=1$, undefined at $x=0$, and 0 is in the domain of f.
5. Absolute maximum 2π at $x=2\pi$; absolute minimum 0 at $x=0$. Critical numbers solve $2\cos x+1=0$: $x=\frac{2\pi}{3},\frac{4\pi}{3}$, giving $f\approx 3.826$ and 2.457. Endpoints give 0 and $2\pi\approx 6.283$.
6. Absolute maximum 65 at $x=5$; absolute minimum -16 at $x=2$. $f'(x)=3x^2-12=0$ at $x=\pm 2$. Values: $f(-3)=9$, $f(-2)=16$, $f(2)=-16$, $f(5)=65$.
7. Absolute maximum $\frac{1}{e}$ at $x=1$; absolute minimum 0 at $x=0$. $g'(x)=e^{-x}(1-x)$, zero at $x=1$. Values: $g(0)=0$, $g(1)=\frac{1}{e}$, $g(3)=3e^{-3}\approx 0.149$.
8. Absolute maximum 5 at $x=\pm 3$; absolute minimum 0 at $x=\pm 2$. Critical numbers: $x=0$ (where $f'(x)=-2x=0$) and $x=\pm 2$ (corners, f' does not exist). Values: $f(\pm 3)=5$, $f(\pm 2)=0$, $f(0)=4$.
9. f is not continuous at $x=2$: $\lim_{x\to 2^-}f(x)=3$ but $f(2)=-2$. On $[0,2)$ the values $x+1$ get arbitrarily close to 3 without reaching it, and on $[2,4]$ the values are at most 0. So no value of f equals the least upper bound 3: there is no absolute maximum.

SET B · APPLICATIONS AND MIXED

10. Maximum about 7.358 mg/L at $t=2.5$ hours; minimum 0 at $t=0$. $C'(t)=8e^{-0.4t}(1-0.4t)=0$ at $t=2.5$. $C(0)=0$, $C(2.5)=20e^{-1}\approx 7.358$, $C(10)=80e^{-4}\approx 1.465$.
11. Maximum 26 degrees at $h=2$ and $h=8$; minimum 10 degrees at $h=0$ and $h=6$. $T'(h)=1.5(h-2)(h-6)$. Values: $T(0)=10$, $T(2)=26$, $T(6)=10$, $T(8)=26$.
12. Maximum 7 (at $x=1$) and minimum -2 (at $x=3$). By the Extreme Value Theorem extremes exist, and they occur at endpoints or at critical numbers. The candidates are $x=0,1,3,5,6$ with values $3,7,-2,4,1$.
13. Absolute maximum $\frac{11}{3}$ at both $x=-1$ and $x=5$; absolute minimum -7 at $x=3$. $f'(x)=(x-3)(x+1)$. Values: $f(-2)=\frac{4}{3}$, $f(-1)=\frac{11}{3}$, $f(3)=-7$, $f(5)=\frac{11}{3}$.
14. Greatest height 21.6 m at $t=2$ s; least height 2 m, at $t=0$ and $t=4$. $h'(t)=-9.8t+19.6=0$ at $t=2$. Values: $h(0)=2$, $h(2)=21.6$, $h(4)=2$.
15. C. $f'(x)=4x^3-12x^2=4x^2(x-3)$ is zero at $x=0$ and $x=3$, and f' exists everywhere. Two critical numbers.
16. Greatest net gain: 29 thousand dollars at $x=5$ (500 items). Least: -3 thousand dollars at $x=1$ (100 items). $R'(x)=-3(x-1)(x-5)$. Values: $R(0)=4$, $R(1)=-3$, $R(5)=29$, $R(6)=22$.

SET C · CHALLENGE AND AP-STYLE

17. $a=-6$, $b=9$. $f'(x)=3x^2+2ax+b$ must equal $3(x-1)(x-3)=3x^2-12x+9$. Then $f(x)=x^3-6x^2+9x$: $f(0)=0$, $f(1)=4$, $f(3)=0$, $f(4)=4$. Absolute maximum 4 at $x=1$ and $x=4$; absolute minimum 0 at $x=0$ and $x=3$.
18. $0<a<3$; the minimum value is $-a^2$ at $x=a$. $f'(x)=2x-2a=0$ at $x=a$, which is interior exactly when $0<a<3$. Then $f(a)=-a^2$, which is less than $f(0)=0$ and $f(3)=9-6a$ (since $9-6a-(-a^2)=(a-3)^2>0$). For $a\geq 3$ the minimum is at the endpoint $x=3$.
19. Absolute maximum 6 at $x=4$; absolute minimum -1 at $x=2$. Candidates are the endpoints -2, 6 and the critical numbers 0, 2 (where f' does not exist), 4. Values: $1,3,-1,6,5$.
20. $f'(x)=3x^2+3\geq 3>0$ for all x, so f' is never 0 and always exists: no critical numbers. On $[-2,2]$: $f(-2)=-13$, $f(2)=15$. Absolute maximum 15 at $x=2$, minimum -13 at $x=-2$. With no interior critical numbers (Fermat's Theorem), extrema can only occur at endpoints.
21. Absolute maximum 7 at $x=2$; absolute minimum 2 at $x=0$. For $x\neq 2$, $f(x)=x+2$ takes values from 2 at $x=0$ up to 6 at $x=4$ (all less than 7), and $f(2)=7$. f is not continuous at 2 (the limit is 4 but $f(2)=7$). There is no contradiction: continuity is sufficient for extrema to exist, not necessary.

3.2 Rolle's Theorem and the Mean Value Theorem

SET A · SKILLS

1. $c=3$. f is a polynomial (continuous and differentiable everywhere) and $f(1)=f(5)=0$. $f'(x)=2x-6=0$ gives $x=3$, in $(1,5)$.
2. $c=2$. Polynomial, and $f(0)=f(3)=0$. $f'(x)=3x(x-2)=0$ at $x=0,2$; only $x=2$ is in the open interval $(0,3)$.
3. $c=\frac{\pi}{4}$. f is continuous and differentiable everywhere, $f(0)=0=f\left(\frac{\pi}{2}\right)$. $f'(x)=2\cos 2x=0$ when $2x=\frac{\pi}{2}$, so $x=\frac{\pi}{4}$ (the next solution $\frac{3\pi}{4}$ is outside the interval).
4. $c=\frac{5}{2}$. Average rate $=\frac{28-4}{3}=8$. $f'(c)=2c+3=8$ gives $c=\frac{5}{2}$.
5. $c=\dfrac{2\sqrt{3}}{3}$. Average rate $=\frac{4-0}{2}=2$. $3c^2-2=2$ gives $c^2=\frac{4}{3}$; the positive root $\frac{2}{\sqrt{3}}\approx 1.155$ is in $(0,2)$.
6. $c=e-1$. Average rate $=\frac{1-0}{e-1}$. $f'(c)=\frac{1}{c}=\frac{1}{e-1}$ gives $c=e-1\approx 1.718$, in $(1,e)$.
7. $c=2$. Average rate $=\frac{\frac{1}{4}-1}{3}=-\frac{1}{4}$. $f'(c)=-\frac{1}{c^2}=-\frac{1}{4}$ gives $c=\pm 2$; only $c=2$ is in $(1,4)$. (f is continuous and differentiable on $[1,4]$ since 0 is not in the interval.)
8. f is not defined at $x=0$, so it is not continuous on $[-1,1]$. Hypothesis (1) fails, so Rolle's Theorem does not apply and no contradiction arises.
9. $f(0)=2=f(4)$. For $x<2$, $f'(x)=-1$; for $x>2$, $f'(x)=1$; f' is never 0. The function has a corner at $x=2$, so hypothesis (2), differentiability on $(0,4)$, fails.

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SET B · APPLICATIONS AND MIXED

10. Let $s(t)$ be the car's position in miles, t in hours after 2:00 pm, on $[0, 0.5]$. s is differentiable, hence continuous. Average velocity $= \frac{57-12}{0.5} = 90$ mph. By the Mean Value Theorem, $s'(c) = 90$ for some c in $(0, 0.5)$.
11. (a) T is differentiable, hence continuous, on $[0, 3]$, and $\frac{T(3)-T(0)}{3-0} = \frac{18}{3} = 6$. By the MVT, $T'(c) = 6$ for some c in $(0, 3)$.
    (b) On $[3, 7]$: $\frac{18-30}{4} = -3$, so $T'(c) = -3$ for some c in $(3, 7)$.
    (c) Yes. On $[0, 10]$: $\frac{22-12}{10} = 1$, so by the MVT $T'(c) = 1$ for some c in $(0, 10)$.
12. Let $f(t) = \ln t$ on $[1, x]$ for $x > 1$. It is continuous on $[1, x]$ and differentiable on $(1, x)$. By the MVT, $\ln x - \ln 1 = \frac{1}{c}(x - 1)$ for some c in $(1, x)$. Since $c > 1$, $\frac{1}{c} < 1$, so $\ln x < x - 1$ (as $x - 1 > 0$).
13. $c \approx 2.751$. f is continuous and differentiable on the interval. Average rate $= \frac{5 \ln 5 - 0}{4} \approx 2.012$. $f'(x) = \ln x + 1 = \frac{5}{4} \ln 5$ gives $c = 5^{5/4}/e \approx 2.751$, in $(1, 5)$.
14. $c \approx 1.370$ (only one value). Average rate $= \frac{(e^2-4)-1}{2} \approx 1.195$. $f'(x) = e^x - 2x$ has minimum value about 0.614 at $x = \ln 2$, equals 1 at $x = 0$, and increases for $x > \ln 2$, so $e^x - 2x = 1.195$ has exactly one solution in $(0, 2)$.
15. $g'(x) = \frac{1}{1+x^2} + \frac{-1/x^2}{1+1/x^2} = \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0$ on $(0, \infty)$, so g is constant there. At $x = 1$: $g(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$. The constant is $\frac{\pi}{2}$.
16. B. Average rate $= \frac{27-0}{3} = 9$. $f'(c) = 3c^2 = 9$ gives $c = \sqrt{3}$ (positive root, in $(0, 3)$).

SET C · CHALLENGE AND AP-STYLE

17. $b = -4$ and $c = 2$. Rolle requires $f(1) = f(3)$: $4 + b = 12 + 3b$, so $b = -4$. Then $f'(x) = 2x - 4 = 0$ gives $c = 2$, in $(1, 3)$.
18. $a = 2$, $b = -1$, and $c = \frac{5}{6}$. Matching values at $x = 1$: $a + b = 1$. Matching derivatives: $2 = a$. So $b = -1$. On $[0, 3]$: $f(3) = 5$, $f(0) = 0$, average rate $\frac{5}{3}$. For $x < 1$, $f'(c) = 2c = \frac{5}{3}$ gives $c = \frac{5}{6}$ (in $(0, 1)$). For $x > 1$, $f'(x) = 2 \neq \frac{5}{3}$. At $x = 1$, $f'(1) = 2 \neq \frac{5}{3}$. So only $c = \frac{5}{6}$.
19. $c = 2 \pm \dfrac{2\sqrt{3}}{3}$ (about 0.845 and 3.155). The chord has slope $\frac{4-0}{4-0} = 1$. $f'(x) = 3x^2 - 12x + 9 = 1$ gives $3x^2 - 12x + 8 = 0$, so $x = \frac{12 \pm \sqrt{48}}{6} = 2 \pm \frac{2\sqrt{3}}{3}$. Both are in $(0, 4)$.
20. If $x = y$ the inequality is $0 \leq 0$. Otherwise, $\sin$ is continuous and differentiable everywhere, so by the MVT on the interval between x and y there is c with $\sin x - \sin y = \cos c\,(x - y)$. Since $|\cos c| \leq 1$, $|\sin x - \sin y| \leq |x - y|$.
21. Apply the MVT to $f(t) = \ln t$ on $[1, 1 + x]$: $\ln(1 + x) = \frac{1}{c} \cdot x$ for some c in $(1, 1 + x)$. Since $1 < c < 1 + x$, $\frac{1}{1+x} < \frac{1}{c} < 1$. Multiplying by $x > 0$: $\frac{x}{1+x} < \ln(1 + x) < x$.

3.3 Increasing, Decreasing, and the First Derivative Test

SET A · SKILLS

1. Increasing on $(-\infty, -2)$ and $(4, \infty)$; decreasing on $(-2, 4)$. Relative maximum $f(-2) = 33$; relative minimum $f(4) = -75$. $f'(x) = 3(x - 4)(x + 2)$: positive, negative, positive.
2. Decreasing on $(-\infty, 3)$, increasing on $(3, \infty)$. Relative minimum $f(3) = -27$ at $x = 3$; no extremum at $x = 0$. $f'(x) = 4x^2(x - 3)$ is negative on both sides of 0.
3. Decreasing on $(-\infty, -1)$ and $(1, \infty)$; increasing on $(-1, 1)$. Relative minimum $f(-1) = -\frac{1}{2}$, relative maximum $f(1) = \frac{1}{2}$. $f'(x) = \frac{(1-x)(1+x)}{(x^2+1)^2}$.
4. Increasing on $(-\infty, -2)$ and $(0, \infty)$; decreasing on $(-2, 0)$. Relative maximum $f(-2) = 4e^{-2}$; relative minimum $f(0) = 0$. $f'(x) = xe^x(x + 2)$ and $e^x > 0$.
5. Decreasing on $(0, 2)$, increasing on $(2, \infty)$. Relative (and absolute) minimum $f(2) = 2 - 2 \ln 2$. $f'(x) = 1 - \frac{2}{x} = \frac{x-2}{x}$.
6. Increasing on $(\frac{\pi}{3}, \frac{5\pi}{3})$; decreasing on $(0, \frac{\pi}{3})$ and $(\frac{5\pi}{3}, 2\pi)$. Relative minimum at $x = \frac{\pi}{3}$, relative maximum at $x = \frac{5\pi}{3}$. $f'(x) = 1 - 2\cos x$ is zero when $\cos x = \frac{1}{2}$.
7. Relative minimum at $x = 0$ and relative maximum at $x = -1$; none at $x = 2$. Signs of f': $x < -1$: $(-)(+)(-) = +$; $-1 < x < 0$: $(-)(+)(+) = -$; $0 < x < 2$: $+$; $x > 2$: $+$. So f' changes $+$ to $-$ at -1 (maximum) and $-$ to $+$ at 0 (minimum); it does not change sign at 2.
8. Relative minimum at $x = -4$ (f' goes from negative to positive); relative maximum at $x = 3$ (f' goes from positive to negative); no extremum at $x = 1$, a critical number where f' does not exist, since f' is positive on both sides (no sign change).
9. $f'(x) = 3(x - 1)(x + 1)$. Relative maximum at $x = -1$ because f' changes from positive to negative at $x = -1$; relative minimum at $x = 1$ because f' changes from negative to positive at $x = 1$. (Values: $f(-1) = 2$, $f(1) = -2$.)

SET B · APPLICATIONS AND MIXED

10. Moves right on $(0, 2)$ and $(4, \infty)$; moves left on $(2, 4)$. $x'(t) = 3t^2 - 18t + 24 = 3(t - 2)(t - 4)$. It turns around at $t = 2$ (position $x(2) = 20$ m) and $t = 4$ (position $x(4) = 16$ m).
11. $C'(t) = \dfrac{12(4 - t^2)}{(t^2 + 4)^2}$, positive on $(0, 2)$ and negative on $(2, \infty)$. So C increases then decreases; C' changes from positive to negative at $t = 2$, giving the maximum $C(2) = \frac{24}{8} = 3$ mg/L at $t = 2$ hours.
12. $A'(x) = \frac{1}{50} - \frac{200}{x^2} = 0$ gives $x = 100$ (for $x > 0$). $A' < 0$ for $0 < x < 100$ and $A' > 0$ for $x > 100$, so A decreases then increases: the absolute minimum is $A(100) = 2 + 2 = 4$ dollars per unit, at $x = 100$ units.
13. $f'(x) = x^2e^{-x}(3 - x)$. Increasing on $(-\infty, 3)$ and decreasing on $(3, \infty)$. Relative maximum $f(3) = 27e^{-3} \approx 1.344$ at $x = 3$. At $x = 0$ ($f' = 0$) there is no sign change, so no extremum there.
14. Decreasing on $(-\infty, -2)$ and $(1, 3)$; increasing on $(-2, 1)$ and $(3, \infty)$. Relative minima at $x = -2$ and $x = 3$; relative maximum at $x = 1$. f' goes negative to positive at -2 and 3, positive to negative at 1.
15. B. $f'(x)$ is negative for $x < 2$ (except $x = -1$, where it is 0) and positive for $x > 2$. So f' changes from negative to positive only at $x = 2$. At $x = -1$ the squared factor means no sign change.

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16. D. $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$ is negative exactly on $(1, 3)$.

SET C · CHALLENGE AND AP-STYLE

17. $a = -3$, $b = -9$, $c = 2$, relative minimum value -25. A relative extremum at -1 and 3 means $f'(x) = 3(x+1)(x-3) = 3x^2 - 6x - 9$, so $2a = -6$ and $b = -9$. Then $f(-1) = -1 + a - b + c = -1 - 3 + 9 + c = 5 + c = 7$, so $c = 2$. $f(3) = 27 - 27 - 27 + 2 = -25$.
18. At $x = -2$: relative maximum, since f is continuous there and f' changes from positive to negative. At $x = 1$: neither, since f' is negative on both sides (no sign change). At $x = 4$: relative minimum, since f' changes from negative to positive.
19. $f'(x) = 2 + \cos x \geq 2 - 1 = 1 > 0$ for all x, so f is increasing on $(-\infty, \infty)$. An increasing function takes each value at most once, and $f(0) = 0$, so $x = 0$ is the only solution of $f(x) = 0$.
20. $h'(x) = \dfrac{1 - \ln x}{x^2}$, zero only at $x = e$. For $0 < x < e$, $\ln x < 1$ so $h' > 0$; for $x > e$, $h' < 0$. So h increases then decreases, and $h(e) = \frac{1}{e}$ is the absolute maximum. Hence $\frac{\ln x}{x} \leq \frac{1}{e}$, and multiplying by $x > 0$ gives $\ln x \leq \frac{x}{e}$.
21. $-2 < k < 2$. $f'(x) = 3(x-1)(x+1)$: relative maximum at $x = -1$ with value $k + 2$; relative minimum at $x = 1$ with value $k - 2$. Three zeros require the maximum positive and the minimum negative: $k + 2 > 0$ and $k - 2 < 0$. (As $x \to \pm\infty$, $f \to \pm\infty$, so the graph crosses the axis once on each of the three monotonic pieces.)

3.4 Concavity and the Second Derivative Test

SET A · SKILLS

1. Concave up on $(-\infty, -1)$ and $(1, \infty)$; concave down on $(-1, 1)$. Inflection points $(-1, -5)$ and $(1, -5)$. $f''(x) = 12x^2 - 12 = 12(x-1)(x+1)$.
2. Concave down on $\left(-\infty, \frac{3}{2}\right)$, concave up on $\left(\frac{3}{2}, \infty\right)$. Inflection point $\left(\frac{3}{2}, \frac{9}{2}\right)$. $f''(x) = 12x - 18$.
3. $f''(x) = (x+2)e^x$. Concave down on $(-\infty, -2)$, concave up on $(-2, \infty)$. Inflection point $\left(-2, -2e^{-2}\right)$.
4. $f''(x) = \dfrac{2(1-x^2)}{(x^2+1)^2}$. Concave up on $(-1, 1)$; concave down on $(-\infty, -1)$ and $(1, \infty)$. Inflection points $(-1, \ln 2)$ and $(1, \ln 2)$.
5. $f''(x) = -\sin x$: negative on $(0, \pi)$, positive on $(\pi, 2\pi)$. Concave down on $(0, \pi)$, concave up on $(\pi, 2\pi)$, inflection point $(\pi, 0)$.
6. $f'(x) = 3(x-4)(x+2)$, critical numbers 4 and -2. $f''(x) = 6x - 6$. $f''(4) = 18 > 0$: relative minimum $f(4) = -80$. $f''(-2) = -18 < 0$: relative maximum $f(-2) = 28$.
7. $f'(x) = 1 - \frac{4}{x^2} = 0$ at $x = \pm 2$. $f''(x) = \frac{8}{x^3}$. $f''(2) = 1 > 0$: relative minimum $f(2) = 4$. $f''(-2) = -1 < 0$: relative maximum $f(-2) = -4$.
8. $f(x) = x^4$: $(0, 0)$ is a relative (and absolute) minimum since $f'(x) = 4x^3$ changes from negative to positive; not an inflection point since $f''(x) = 12x^2 \geq 0$ on both sides. $g(x) = x^3$: not an extremum ($g' = 3x^2 \geq 0$ does not change sign); it is an inflection point since $g''(x) = 6x$ changes sign at 0.
9. $f'(x) = \frac{1}{3}x^{-2/3}$ and $f''(x) = -\frac{2}{9}x^{-5/3}$, which does not exist at 0. For $x < 0$, $x^{-5/3} < 0$ so $f'' > 0$ (concave up); for $x > 0$, $f'' < 0$ (concave down). Since f is continuous at 0 and the concavity changes, $(0, 0)$ is an inflection point.

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10. Concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$ (read from where the graph changes from cap-shaped to cup-shaped). Inflection point $\left(2, \frac{2}{3}\right)$. Check: $f''(x) = 2x - 4$ changes sign at $x = 2$ and $f(2) = \frac{8}{3} - 8 + 6 = \frac{2}{3}$.
11. $P'(t) = \dfrac{4500e^{-0.5t}}{\left(1 + 9e^{-0.5t}\right)^2}$ and P'' changes sign from positive to negative when $9e^{-0.5t} = 1$, that is at $t = \ln 81 \approx 4.394$ hours. There $P = \frac{1000}{1+1} = 500$. Before this time P' is increasing (concave up) and afterward P' is decreasing (concave down), so the population grows fastest at the inflection point, when it is 500.
12. $Q'(h) = -3h^2 + 24h + 20$ and $Q''(h) = -6h + 24 = 6(4 - h)$. $Q'' > 0$ for $h < 4$ and $Q'' < 0$ for $h > 4$, so Q' increases then decreases: its maximum is at $h = 4$ hours, with $Q'(4) = -48 + 96 + 20 = 68$ units per hour.
13. $T'(t) = -18e^{-0.3t} < 0$ and $T''(t) = 5.4e^{-0.3t} > 0$ for all t, so T is decreasing and concave up. $T'(2) = -18e^{-0.6} \approx -9.879$ degrees per minute. Concave up means T' is increasing toward 0: the liquid cools more and more slowly.
14. D. $f''(x) = x(x-2)^2$ is negative for $x < 0$, positive for $0 < x < 2$ and positive for $x > 2$. The sign changes only at $x = 0$; the squared factor prevents a sign change at $x = 2$.
15. E. When $f''(3) = 0$ the Second Derivative Test is inconclusive. The function could have a minimum (x^4-type), a maximum, or neither, and it may or may not have an inflection point, so A to D are not forced.
16. C. $f''(x) = 3x^2 - 3 = 3(x-1)(x+1)$ is negative exactly on $(-1, 1)$.

SET C · CHALLENGE AND AP-STYLE

17. $a = -3$, $b = 9$. $f''(x) = 6ax + 2b$ and $f''(1) = 0$ gives $6a + 2b = 0$, so $b = -3a$. $f(1) = a + b = 6$ gives $-2a = 6$, $a = -3$, $b = 9$. Then $f''(x) = -18x + 18$, positive for $x < 1$ and negative for $x > 1$, so the concavity changes at $x = 1$.
18. (a) $f'(x) = (x-3)(x+1)$ is positive for $x < -1$ and $x > 3$ and negative on $(-1, 3)$. f is increasing on $(-\infty, -1)$ and $(3, \infty)$. Relative maximum at $x = -1$, relative minimum at $x = 3$.

    (b) f'' = slope of f': f' is decreasing for $x < 1$ and increasing for $x > 1$. Concave down on $(-\infty, 1)$, concave up on $(1, \infty)$.

    (c) Yes, at $x = 1$, because f' changes from decreasing to increasing there (equivalently, $f''(x) = 2x - 2$ changes sign at $x = 1$).
19. $f''(x) = 6ax + 2b$ is a linear function with nonzero slope $6a$, so it is zero at exactly one number, $x = -\dfrac{b}{3a}$, and it changes sign there (positive on one side, negative on the other). So there is exactly one inflection point, at $x = -\frac{b}{3a}$.
20. $f'(x) = 1 - \frac{k}{x} = \frac{x-k}{x}$, zero only at $x = k$. $f''(x) = \frac{k}{x^2} > 0$, so $f''(k) > 0$: relative minimum. $f' < 0$ on $(0, k)$ and $f' > 0$ on (k, ∞), so it is the absolute minimum, $f(k) = k - k\ln k = k(1 - \ln k)$. This is 0 when $\ln k = 1$, so $k = e$.

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21. $x=1$: $f'(1)=0$ and $f''(1)=-2<0$, so a relative maximum. $x=3$: $f'(3)=0$ and $f''(3)=4>0$, so a relative minimum. $x=2$: $f'(2)=-1\neq 0$, so it is not a critical number and there is no extremum there. $f''(2)=0$ alone does not show an inflection point; that needs a sign change of f'' at 2, which the table does not give.

3.5 Reading the Graphs of f, f', and f"

SET A · SKILLS

1. Relative minima at $x=0$ and $x=2\pi\approx 6.283$; relative maximum at $x=\pi\approx 3.142$. $f'=3\sin x$ changes from negative to positive at 0 and 2π, and from positive to negative at π.
2. $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$. f' has a relative maximum at $\frac{\pi}{2}$ and a relative minimum at $\frac{3\pi}{2}$, so $f''=3\cos x$ changes sign there (positive to negative, then negative to positive), and the concavity of f changes.
3. At $x=-0.5$: $f'<0$ (graph falling), $f''>0$ (cup-shaped). At $x=1$: $f'<0$, $f''<0$ (falling, cap-shaped). At $x=3$: $f'=0$ (valley), $f''>0$ (cup-shaped). (Check: $f'=\frac{x^2(x-3)}{2}$, $f''=\frac{3x(x-2)}{2}$.)
4. A. Increasing means $f'>0$; concave down means f' is decreasing, so $f''<0$.
5. (a) f' is positive at $x=1$ and negative at $x=2$, so by continuity (f is twice differentiable, so f' is continuous) it changes from positive to negative between 1 and 2: a relative maximum there. It is negative at $x=4$ and positive at $x=5$, so it changes from negative to positive between 4 and 5: a relative minimum there. (b) The Intermediate Value Theorem applied to the continuous function f' on $[1,2]$ gives a zero of f' between $f'(1)=1$ and $f'(2)=-2$.
6. $f'(x)=(x-1)(x-3)$, $f''(x)=2x-4$. Increasing on $(-\infty,1)$ and $(3,\infty)$; decreasing on $(1,3)$. Relative maximum at $x=1$, relative minimum at $x=3$. Concave down on $(-\infty,2)$, concave up on $(2,\infty)$, inflection point at $x=2$.
7. $f'(x)<0$ for $x<0$ and >0 for $x>0$: decreasing on $(-\infty,0)$, increasing on $(0,\infty)$, relative minimum at $x=0$. $f''(x)=(1-x)e^{-x}$ is positive for $x<1$ and negative for $x>1$: concave up on $(-\infty,1)$, concave down on $(1,\infty)$, inflection at $x=1$.
8. $f''>0$ for $x<-2$ and $x>1$; $f''<0$ for $-2<x<1$. Concave up on $(-\infty,-2)$ and $(1,\infty)$; concave down on $(-2,1)$; inflection points at $x=-2$ and $x=1$. No: f'' alone does not give f', so critical numbers and relative extrema cannot be located.
9. E. A line with positive slope means f' is increasing, so $f''>0$ (concave up everywhere). f' changes from negative to positive at $x=2$, so f has a relative minimum there.

SET B · APPLICATIONS AND MIXED

10. (a) f' crosses zero at $x=0.5$ (negative to positive), $x=3$ (positive to negative) and $x=4.5$ (negative to positive). Relative minima at $x=0.5$ and $x=4.5$; relative maximum at $x=3$.

    (b) The slope of f' is 4 on $(0,1)$, -1 on $(1,4)$, 2 on $(4,6)$, -1 on $(6,8)$. f'' changes sign at $x=1$, 4, 6: inflection points at $x=1$, $x=4$, $x=6$.

    (c) f is decreasing where $f'<0$: $(0,0.5)$ and $(3,4.5)$. It is concave down where f' is decreasing: $(1,4)$ and $(6,8)$. Both hold on $(3,4)$.
11. (a) $W'=R(t)=-(t-2)(t-8)$ is negative for $t<2$, positive for $2<t<8$, negative for $t>8$. W has a relative minimum at $t=2$ and a relative maximum at $t=8$.

    (b) W increases most rapidly where R is largest: $R'(t)=10-2t=0$ at $t=5$, $R(5)=9$ liters per minute, and R increases before and decreases after, so this is the maximum.

    (c) $W''=R'(t)=10-2t<0$ for $t>5$: concave down on $(5,10)$.
12. C. On $(-1,3)$, $f'>0$, so f is increasing. f' is decreasing for $x>1$ (the vertex of the parabola is at $x=1$), so f is concave down on $(1,\infty)$. Both hold on $(1,3)$.
13. D. Since $f''<0$ on $(0,5)$, f' is decreasing; with $f'(2)=0$ we get $f'>0$ on $(0,2)$ and $f'<0$ on $(2,5)$. So f increases up to $x=2$ and decreases after, and $f(2)$ is the greatest value on $(0,5)$. Choices A, B, C, E are false.
14. B. Point Q ($x=0.5$) is on the rising part with the cap shape. P is rising and cup-shaped ($f'>0$, $f''>0$); R is a valley ($f'=0$); S is a peak ($f'=0$); T is falling at the inflection point ($f''=0$).
15. $f'(x)=e^{-x/2}\left(1-\frac{x}{2}\right)$ is positive on $[0,2)$ and negative for $x>2$: absolute maximum $f(2)=2e^{-1}\approx 0.736$ at $x=2$. $f''(x)=\frac{x-4}{4}e^{-x/2}$: concave down on $(0,4)$, concave up on $(4,\infty)$; inflection point $(4,\,4e^{-2})\approx(4,0.541)$.
16. Average rates: 0.8, 0.6, 0.4, 0.2 meters per year. These are positive and decreasing, which suggests $H'>0$ and $H''<0$: the sapling keeps growing but more slowly each period (the graph is increasing and concave down).

SET C · CHALLENGE AND AP-STYLE

17. Relative maximum at $x=0$ and relative minimum at $x=4$; inflection points at $x=3-\sqrt{5}$ and $x=3+\sqrt{5}$. f' is positive for $x<0$, negative on $(0,4)$ and positive for $x>4$ (since $e^{-x}>0$). $f''(x)=e^{-x}\left(2x-4-x^2+4x\right)=e^{-x}(-x^2+6x-4)$, zero at $x=3\pm\sqrt{5}\approx 0.764,\ 5.236$, negative before 0.764, positive between, negative after 5.236: sign changes at both.
18. Let $x>1$. By the Mean Value Theorem, $f(x)-f(1)=f'(c)(x-1)$ for some c in $(1,x)$. Since $f''>0$, f' is increasing, so $f'(c)>f'(1)=2$, and $f(x)-3>2(x-1)$. Let $x<1$: MVT gives c in $(x,1)$ with $f(x)-f(1)=f'(c)(x-1)$; now $f'(c)<2$ and $x-1<0$, so $f'(c)(x-1)>2(x-1)$. Hence $f(x)>3+2(x-1)$ for all $x\neq 1$.
19. $a=-6$, $b=12$. $f''(x)=12x^2+6ax+2b$ must equal $12(x-1)(x-2)=12x^2-36x+24$, so $6a=-36$ and $2b=24$. Then $f'(x)=4x^3-18x^2+24x=2x(2x^2-9x+12)$; the quadratic has discriminant $81-96<0$, so the only critical number is $x=0$, with $f''(0)=24>0$: relative minimum (value 0). It is also the absolute minimum.

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20. Since $f'' > 0$ before 2 and $f'' < 0$ after 2, f' increases up to $x = 2$ and then decreases. With $f'(2) = 0$, f' has its maximum value 0 at $x = 2$, so $f'(x) \le 0$ everywhere near 2: f is decreasing on both sides, so no relative extremum. The concavity changes (up to down), so $(2, f(2))$ is an inflection point. Example: $f(x) = -(x-2)^3$, with $f'(x) = -3(x-2)^2$ and $f''(x) = -6(x-2)$.
21. Since f is concave down, f' is decreasing, so $f'(1) > f'(3)$. By the MVT, $\frac{f(3)-f(1)}{2} = f'(c)$ for some c in $(1,3)$, and $f'(3) < f'(c) < f'(1)$. So $f'(3) < \frac{f(3)-f(1)}{2} < f'(1)$. For $f = \ln x$ ($f'' = -\frac{1}{x^2} < 0$): $\frac{1}{3} < \frac{\ln 3}{2} < 1$, which is consistent ($\frac{\ln 3}{2} \approx 0.549$).

3.6 Curve Sketching

SET A · SKILLS

1. Domain: $x \neq -2, 3$. x-intercept $(1, 0)$; y-intercept $(0, \frac{1}{6})$. Since $x^2 - x - 6 = (x-3)(x+2)$ and the numerator is nonzero at $x = 3$ and $x = -2$: vertical asymptotes $x = 3$ and $x = -2$. Degree of numerator is less than degree of denominator, so horizontal asymptote $y = 0$.
2. $f'(x) = 3(x-3)(x+1)$. $f' > 0$ on $(-\infty, -1)$ and $(3, \infty)$; $f' < 0$ on $(-1, 3)$. Increasing on $(-\infty, -1)$ and $(3, \infty)$, decreasing on $(-1, 3)$. Local maximum $f(-1) = 7$ at $x = -1$; local minimum $f(3) = -25$ at $x = 3$.
3. $f''(x) = 12x^2 - 16 = 4(3x^2 - 4)$, zero at $x = \pm\frac{2\sqrt{3}}{3}$. $f'' > 0$ for $|x| > \frac{2\sqrt{3}}{3}$ (concave up) and $f'' < 0$ between (concave down). The sign changes at both zeros, so the inflection points are $\left(\pm\frac{2\sqrt{3}}{3}, -\frac{80}{9}\right)$.
4. Divide: $f(x) = x + 2 - \frac{1}{x+1}$. Vertical asymptote $x = -1$ (numerator is $-1 \neq 0$ there). Slant asymptote $y = x + 2$, since $-\frac{1}{x+1} \to 0$.
5. For large $|x|$, $\sqrt{4x^2+1} \approx 2|x|$. So the limit as $x \to \infty$ is $\frac{6x}{2x} = 3$, and as $x \to -\infty$ it is $\frac{6x}{2(-x)} = -3$. Horizontal asymptotes: $y = 3$ and $y = -3$.
6. Domain $x > 0$. $f'(x) = \ln x + 1 = 0$ at $x = \frac{1}{e}$; $f' < 0$ on $(0, \frac{1}{e})$ and $f' > 0$ on $(\frac{1}{e}, \infty)$. Local (and absolute) minimum $f(\frac{1}{e}) = -\frac{1}{e}$. $f''(x) = \frac{1}{x} > 0$ on the whole domain, so f is concave up everywhere.
7. $f'(x) = 3(x-2)(x+2)$, critical numbers ± 2. $f''(x) = 6x$. $f''(-2) = -12 < 0$: local maximum $f(-2) = 21$. $f''(2) = 12 > 0$: local minimum $f(2) = -11$.
8. (a) Local maximum at $x = -2$ (f' changes + to -); local minimum at $x = 5$ (f' changes - to +).
   (b) Inflection point at $x = 1$ (f'' changes - to +); f'' does not change sign at -2 or 5.
   (c) $(5, \infty)$, where both signs are +.
9. Factor: $f(x) = \dfrac{(x-3)(x+2)}{(x-3)(x+3)}$. For $x \neq 3$, $f(x) = \frac{x+2}{x+3}$. Vertical asymptote $x = -3$. The factor $x - 3$ cancels, so there is a hole at $(3, \frac{5}{6})$. Horizontal asymptote $y = 1$ (equal degrees, ratio of leading coefficients).

SET B · APPLICATIONS AND MIXED PROBLEMS

10. (a) $x = -3$ and $x = 1$.
    (b) At $x = -3$, f' changes from negative to positive: local minimum. At $x = 1$, $(x-1)^2 \ge 0$ so f' does not change sign: no extremum.
    (c) $f''(x) = 2(x-1)(x+3) + (x-1)^2 = (x-1)(3x+5)$. It changes sign at $x = -\frac{5}{3}$ and $x = 1$, so both are inflection points.
11. $f''(x) = 6x + 2a = 0$ at $x = 1$ gives $a = -3$. $f'(-1) = 3 - 2a + b = 0$ gives $b = -9$. Check: $f'(x) = 3(x-3)(x+1)$ changes from + to - at $x = -1$ (local max) and $f''(x) = 6x - 6$ changes sign at $x = 1$.
12. (a) $(-1, 3)$: the graph falls between the peak and the valley.
    (b) $(1, \infty)$: the graph bends upward to the right of the inflection point at $x = 1$.
    (c) $x = 1$. To the left of 1 the graph is concave down, so f' is decreasing; to the right it is concave up, so f' is increasing. Hence f' is smallest at $x = 1$.
13. $C'(x) = 3x^2 - 24x + 60 = 3(x-4)^2 + 12 > 0$, so C is increasing. $C''(x) = 6x - 24$ changes from - to + at $x = 4$, so C' has its minimum there: $C'(4) = 12$. Smallest marginal cost: 12 thousand dollars per hundred units, at $x = 4$ (400 units).
14. (a) $C'(t) = \frac{20(4-t^2)}{(t^2+4)^2}$; $C' > 0$ on $(0, 2)$ and $C' < 0$ for $t > 2$, so the maximum is $C(2) = 5$ mg/L at $t = 2$ hours.
    (b) $C(t) \to 0$; the t-axis is a horizontal asymptote.
    (c) $C'' < 0$ on $(0, 2\sqrt{3})$ and > 0 after, so C' is smallest at $t = 2\sqrt{3}$ hours, where $C'(2\sqrt{3}) = -\frac{5}{8}$ mg/L per hour.
15. Domain: $|x| > 1$. x-intercepts: $x^2 - 1 = 1$ gives $(\pm\sqrt{2}, 0)$; no y-intercept. Vertical asymptotes $x = \pm 1$ ($f \to -\infty$). $f'(x) = \frac{2x}{x^2-1}$: negative on $(-\infty, -1)$, positive on $(1, \infty)$; the only zero of f' is $x = 0$, outside the domain, so no extrema. $f''(x) = -\frac{2(x^2+1)}{(x^2-1)^2} < 0$: concave down on both pieces, no inflection points.
16. (a) $f'(x) = \frac{5(x-2)}{3x^{1/3}}$: zero at $x = 2$, undefined at $x = 0$ where f is defined. Critical numbers 0 and 2.
    (b) $f' > 0$ for $x < 0$, $f' < 0$ on $(0, 2)$, $f' > 0$ for $x > 2$. Local maximum $f(0) = 0$ (a cusp) and local minimum $f(2) = -3 \cdot 2^{2/3}$.
    (c) $x^{4/3} > 0$, so f'' changes sign only at $x = -1$: inflection point $(-1, -6)$. At $x = 0$ there is no sign change.

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17. $f'(x) = \frac{x(4-x)}{2}e^{-x/2}$: local minimum $(0,0)$, local maximum $(4, 16e^{-2}) \approx (4, 2.165)$. $f''(x) = \frac{1}{4}(x^2 - 8x + 8)e^{-x/2} = 0$ at $x = 4 \pm 2\sqrt{2}$: inflection points $(1.172, 0.764)$ and $(6.828, 1.534)$. As $x \to \infty$, $f \to 0$: $y = 0$.
18. (a) $P(0) = 100$; the limit is 1000.
    (b) $P'(t) = \frac{4500e^{-0.5t}}{(1+9e^{-0.5t})^2} > 0$.
    (c) The growth rate is largest where P'' changes from + to -, at $t = 2\ln 9 \approx 4.394$ hours. Then $P = 500$ and $P' = 125$ bacteria per hour.
19. (a) $f' = 3(x-1)(x-3)$: local maximum $f(1) = 2$, local minimum $f(3) = -2$.
    (b) f increases from $-\infty$ to 2 on $(-\infty, 1)$ (one zero), decreases from 2 to -2 on $(1, 3)$ (one zero), increases from -2 to ∞ on $(3, \infty)$ (one zero). By the IVT and monotonicity, exactly one zero in each interval.
    (c) $f(x) = (x-2)(x^2 - 4x + 1)$, zeros 2 and $2 \pm \sqrt{3}$. Smallest: 0.268.
20. (a) $f' = 0$ at $x = \frac{\pi}{3} \approx 1.047$ (f' changes + to -: local maximum) and $x = \frac{5\pi}{3} \approx 5.236$ (- to +: local minimum).
    (b) $f''(x) = -\sin x$ changes from - to + at $x = \pi \approx 3.142$.
    (c) $f(x) = 1 + \sin x - \frac{x}{2}$, so $f(\pi/3) = 1 + 0.866 - 0.524 \approx 1.342$.
21. (a) $f' = 1 - 2\sin x = 0$ at $x = \frac{\pi}{6} \approx 0.524$ (local max) and $x = \frac{5\pi}{6} \approx 2.618$ (local min).
    (b) $f'' = -2\cos x$ changes sign at $x = \frac{\pi}{2} \approx 1.571$ and $x = \frac{3\pi}{2} \approx 4.712$.
    (c) Compare $f(0) = 2$, $f(0.524) = 2.256$, $f(2.618) = 0.886$, $f(2\pi) = 8.283$. Absolute maximum 8.283 (at $x = 2\pi$), absolute minimum 0.886 (at $x = \frac{5\pi}{6}$).

3.7 Particle Motion Along a Line

SET A · SKILLS

1. $v(t) = 2t - 6$, $a(t) = 2$. At rest when $2t - 6 = 0$, so $t = 3$. Position $s(3) = 9 - 18 + 5 = -4$.
2. $v(t) = 3t^2 - 12 = 3(t-2)(t+2)$. For $t \geq 0$, $v < 0$ on $[0, 2)$ and $v > 0$ for $t > 2$. It moves left on $(0, 2)$ and right on $(2, \infty)$.
3. $a(t) = 2t - 4$. $t = 1$: $v = -3, a = -2$, same sign: speeding up. $t = 3$: $v = -3, a = 2$, opposite: slowing down. $t = 5$: $v = 5, a = 6$, same: speeding up.
4. $v = 6(t-2)(t-3)$, $a = 12t - 30$ (zero at $t = 2.5$). $(0, 2)$: $v > 0, a < 0$ slowing. $(2, 2.5)$: $v < 0, a < 0$ speeding up. $(2.5, 3)$: $v < 0, a > 0$ slowing. $(3, \infty)$: $v > 0, a > 0$ speeding up. Speeding up on $(2, \frac{5}{2})$ and $(3, \infty)$.
5. $v = 2t - 4$ changes sign at $t = 2$. $s(0) = 0$, $s(2) = -4$, $s(5) = 5$. Net displacement $= 5$. Total distance $= |-4 - 0| + |5 - (-4)| = 4 + 9 = 13$.
6. $s(t) = \frac{t^3}{3} - \frac{t^2}{2} - 2t + 1$. $v = (t-2)(t+1)$ changes sign at $t = 2$. $s(2) = -\frac{7}{3}$, $s(3) = -\frac{1}{2}$. Net displacement $= -\frac{1}{2} - 1 = -\frac{3}{2}$. Distance $= \left|-\frac{7}{3} - 1\right| + \left|-\frac{1}{2} + \frac{7}{3}\right| = \frac{10}{3} + \frac{11}{6} = \frac{31}{6}$.
7. $v = -32t + 64 = 0$ at $t = 2$, so the maximum height is $s(2) = 64$ ft. $s(t) = -16t(t-4) = 0$ at $t = 4$ s. $v(4) = -64$, so the landing speed is 64 ft/s.
8. $a(t) = 2t - 6 = 0$ at $t = 3$. a changes from negative to positive, so v has a minimum at $t = 3$: $v(3) = 9 - 18 + 5 = -4$.
9. (a) $\frac{-3-4}{5-0} = -\frac{7}{5}$.
    (b) $a(2) \approx \frac{v(3)-v(1)}{3-1} = \frac{-1-2}{2} = -\frac{3}{2}$.
    (c) $v(1) = 2 > 0$ and $v(3) = -1 < 0$ and v is continuous, so by the Intermediate Value Theorem $v(t) = 0$ somewhere in $(1, 3)$.

SET B · APPLICATIONS AND MIXED PROBLEMS

10. $v = (1-t)e^{-t}$, $a = (t-2)e^{-t}$.
    (a) At rest at $t = 1$; $x(1) = \frac{1}{e}$.
    (b) $v > 0$ on $(0, 1)$, $v < 0$ for $t > 1$; $a < 0$ on $(0, 2)$, $a > 0$ for $t > 2$. Speeding up when signs agree: only on $(1, 2)$.
    (c) At $t = 0$ the speed is $v(0) = 1$. For $0 < t < 1$ the speed decreases to 0. For $t > 1$ the speed $(t-1)e^{-t}$ is largest at $t = 2$, where it is only $e^{-2} \approx 0.135$. Greatest speed: 1, at $t = 0$.
11. (a) v crosses zero at $t = 1$ and $t = 5$, changing sign each time.
    (b) Speeding up when v and a (slope) share a sign: $(1, 2)$ ($v < 0$, slope -2) and $(5, 7)$ ($v > 0$, slope 2). On $(2, 4)$ and $(7, 8)$ the speed is constant.
    (c) Areas: 0 to 1: $\frac{1}{2}(1)(2) = 1$; 1 to 2: 1; 2 to 4: $2 \cdot 2 = 4$; 4 to 5: $\frac{1}{2}(1)(2) = 1$. Total distance $= 7$.
12. $v = 0$ when $\sin t = \frac{1}{2}$: $t = \frac{\pi}{6}, \frac{5\pi}{6}$, and v changes sign at both. $a = 2\cos t$. Signs: $(0, \frac{\pi}{6})$: $v < 0, a > 0$ slowing. $(\frac{\pi}{6}, \frac{\pi}{2})$: $v > 0, a > 0$ speeding up. $(\frac{\pi}{2}, \frac{5\pi}{6})$: $v > 0, a < 0$ slowing. $(\frac{5\pi}{6}, \frac{3\pi}{2})$: $v < 0, a < 0$ speeding up. $(\frac{3\pi}{2}, 2\pi)$: $v < 0, a > 0$ slowing. Speeding up on $(\frac{\pi}{6}, \frac{\pi}{2})$ and $(\frac{5\pi}{6}, \frac{3\pi}{2})$.
13. $v = -10t + 20 = 0$ at $t = 2$; $h(2) = 80$ m maximum. $h = -5(t-6)(t+2) = 0$ at $t = 6$ s. $v(6) = -40$ m/s (downward).
14. (a) $v = 6\cos 2t$, $a = -12\sin 2t$.
    (b) $-4(s-5) = -4 \cdot 3\sin 2t = -12\sin 2t = a$.
    (c) $v(\frac{\pi}{3}) = 6\cos\frac{2\pi}{3} = -3$ and $a(\frac{\pi}{3}) = -12\sin\frac{2\pi}{3} = -6\sqrt{3}$. Both negative: speeding up.
15. (a) $t^2 - 6t + 8 = 2t - 4$ gives $t^2 - 8t + 12 = (t-2)(t-6) = 0$: $t = 2$ and $t = 6$.
    (b) $v_A = 2t - 6$, $v_B = 2$. At $t = 2$: $v_A = -2$, $v_B = 2$. At $t = 6$: $v_A = 6$, $v_B = 2$.
    (c) At $t = 6$ both move right; A is faster ($6 > 2$).
16. $v = 3t^2 - 12t + 9 = 3(t-1)(t-3)$; $s = t^3 - 6t^2 + 9t + 2$. At rest at $t = 1$ and $t = 3$. $s(0) = 2$, $s(1) = 6$, $s(3) = 2$, $s(4) = 6$. Distance $= 4 + 4 + 4 = 12$.

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17. (a) $v = 3t^2 - 14t + 10 = 0$ at $t = \frac{7 \pm \sqrt{19}}{3}$, i.e. $t = 0.880$ and $t = 3.786$; $s(0.880) = 7.061$, $s(3.786) = -5.209$.
(b) $v(1) = -1$, $a(1) = -8$: same sign, speeding up.
(c) $s(0) = 3$, $s(5) = 3$. Distance $= |7.061 - 3| + |-5.209 - 7.061| + |3 - (-5.209)| = 4.061 + 12.269 + 8.209 = 24.539$.

18. (a) $v = 0$ for $t > 0$ when $\cos t = 0$: $t = \frac{\pi}{2} = 1.571$ and $t = \frac{3\pi}{2} = 4.712$ (sign changes both).
(b) $s(1.571) = 0.571$, $s(4.712) = -5.712$, $s(6) = -1.716$.
(c) $0.571 + |-5.712 - 0.571| + |-1.716 + 5.712| = 0.571 + 6.283 + 3.996 = 10.850$.

19. (a) $v = 0$ at $t = e \approx 2.718$; $v > 0$ before, $v < 0$ after, so it turns around there. $s(e) = \frac{1}{e} \approx 0.368$.
(b) $a = 0$ at $t = e^{3/2} \approx 4.482$; $a < 0$ before, $a > 0$ after. $v < 0, a < 0$ on $(2.718, 4.482)$: speeding up. On $(0, e)$: $v > 0, a < 0$, slowing; on $t > 4.482$: $v < 0, a > 0$, slowing. Speeding up only on $(2.718, 4.482)$.

20. (a) Product rule: $x' = -\frac{1}{2}e^{-t/2}\sin 3t + 3e^{-t/2}\cos 3t$, which equals the stated form.
(b) $6\cos 3t = \sin 3t$ means $\tan 3t = 6$: $3t = \arctan 6 = 1.406$, so $t = 0.469$; next $3t = 1.406 + \pi$, so $t = 1.516$. $x(0.469) = 0.780$ and $x(1.516) = -0.462$.
(c) $x'(1) = -1.844 < 0$ and $x''(1) = 1.052 > 0$: opposite signs, slowing down.

21. (a) $v(2) > 0 > v(4)$ and $v(6) < 0 < v(8)$; by the IVT $v = 0$ somewhere in $(2, 4)$ and in $(6, 8)$, with sign changes.
(b) $\frac{1-5}{8} = -0.500$ m/s^2.
(c) $2(5 + 3 - 2 - 4) = 4$ m.
(d) $2(5 + 3 + 2 + 4) = 28$ m.

3.8 Optimization I

SET A · SKILLS

1. $y = 40 - x$, $P = 40x - x^2$, $P' = 40 - 2x = 0$ at $x = 20$; $P'' = -2 < 0$. Numbers 20 and 20, product 400.

2. $S = 3x + \frac{12}{x}$, $S' = 3 - \frac{12}{x^2} = 0$ gives $x = 2$. $S'' = \frac{24}{x^3} > 0$ so a minimum. $y = 6$ and $S = 12$.

3. $y = 18 - x$, $A = 18x - x^2$, $A' = 18 - 2x = 0$ at $x = 9$, $A'' = -2 < 0$. A 9 by 9 square, area 81.

4. Let x be the two sides perpendicular to the river, $y = \frac{800}{x}$. $F = 2x + \frac{800}{x}$, $F' = 2 - \frac{800}{x^2} = 0$ at $x = 20$. $F'' > 0$: minimum. Dimensions 20 m by 40 m, fencing 80 m.

5. $h = \frac{32}{x^2}$, $S = x^2 + 4xh = x^2 + \frac{128}{x}$. $S' = 2x - \frac{128}{x^2} = 0$ gives $x = 4$; $S'' = 2 + \frac{256}{x^3} > 0$. Base 4 m, height 2 m, $S = 48$ m^2.

6. $V = x(12 - 2x)^2$, $V' = (12 - 2x)(12 - 6x)$, critical number $x = 2$ in $(0, 6)$. $V' > 0$ before and < 0 after, so a maximum. $V(2) = 2 \cdot 8^2 = 128$ cm^3.

7. $f = x(12 - x)^2$ on $[0, 12]$, $f' = (12 - x)(12 - 3x)$, critical number $x = 4$. $f(0) = f(12) = 0$, $f(4) = 4 \cdot 64 = 256$. Maximum 256 at $x = 4$, $y = 8$.

8. $D^2 = x^2 + (2x + 1)^2 = 5x^2 + 4x + 1$; derivative $10x + 4 = 0$ at $x = -\frac{2}{5}$ (second derivative $10 > 0$). Point $\left(-\frac{2}{5}, \frac{1}{5}\right)$, $D^2 = \frac{1}{5}$, distance $\frac{\sqrt{5}}{5}$.

9. $D^2 = (x - 4)^2 + x$; derivative $2(x - 4) + 1 = 0$ at $x = \frac{7}{2}$, second derivative $2 > 0$. Point $\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$, $D^2 = \frac{1}{4} + \frac{7}{2} = \frac{15}{4}$, $D = \frac{\sqrt{15}}{2}$.

SET B · APPLICATIONS AND MIXED PROBLEMS

10. Width x, height $y = \frac{600}{x}$. Printed area $(x - 4)(y - 6) = 624 - 6x - \frac{2400}{x}$. Derivative $-6 + \frac{2400}{x^2} = 0$ at $x = 20$ (second derivative $-\frac{4800}{x^3} < 0$). Poster 20 by 30 cm, printed area $16 \cdot 24 = 384$ cm^2.

11. Three fences of length x and two of length y: $3x + 2y = 600$, $y = 300 - \frac{3}{2}x$. $A = 300x - \frac{3}{2}x^2$, $A' = 300 - 3x = 0$ at $x = 100$, $A'' < 0$. Dimensions 100 m by 150 m, area 15000 m^2.

12. $h = \frac{27}{x^2}$. $C = 10x^2 + 5 \cdot 4xh = 10x^2 + \frac{540}{x}$. $C' = 20x - \frac{540}{x^2} = 0$ at $x = 3$; $C'' > 0$. Base 3 by 3 m, height 3 m, cost $90 + 180 = 270$ dollars.

13. $y = \frac{32}{x^2}$, $S = x + \frac{32}{x^2}$, $S' = 1 - \frac{64}{x^3} = 0$ at $x = 4$. $S' < 0$ on $(0, 4)$ and > 0 on $(4, \infty)$, so absolute minimum. $y = 2$, $S = 6$.

14. Vertices $(\pm x, 12 - x^2)$, $0 < x < \sqrt{12}$. $A = 2x(12 - x^2) = 24x - 2x^3$, $A' = 24 - 6x^2 = 0$ at $x = 2$; $A'' = -12x < 0$. Width 4, height 8, area 32.

15. Sides x and $\sqrt{100 - x^2}$. Maximize $A^2 = x^2(100 - x^2)$: derivative $200x - 4x^3 = 0$ at $x^2 = 50$; it is a maximum since A^2 is 0 at the ends of $(0, 10)$. Square of side $5\sqrt{2}$, area 50.

16. Radius r, rectangle height h: perimeter $2h + 2r + \pi r = 4 + \pi$, so $h = \frac{4 + \pi - 2r - \pi r}{2}$. $A = 2rh + \frac{\pi r^2}{2} = (4 + \pi)r - \left(2 + \frac{\pi}{2}\right)r^2$. $A' = (4 + \pi) - (4 + \pi)r = 0$ at $r = 1$; $A'' < 0$. $r = 1$ m, $h = 1$ m, $A = 2 + \frac{\pi}{2}$ m^2.

SET C · CHALLENGE AND AP-STYLE

17. $V = x(30 - 2x)(20 - 2x)$, $0 < x < 10$. $V' = 12x^2 - 200x + 600 = 0$ gives $x = \frac{25 \pm 5\sqrt{7}}{3}$: 3.924 or 12.743 (reject). $V' > 0$ before and < 0 after, so a maximum. Box $22.153 \times 12.153 \times 3.924$ cm, $V = 1056.306$ cm^3.

18. Profit $P(x) = xp(x) - C(x) = 70x - 0.02x^2 - 0.00004x^3 - 1000$. $P'(x) = 70 - 0.04x - 0.00012x^2 = 0$ gives $x = 615.069$ (the other root is negative). $P''(x) = -0.04 - 0.00024x < 0$ for $x \geq 0$, so this is the absolute maximum. Price $p = 67.699$ dollars, profit $P = 25181.165$ dollars.

19. $D^2 = (x - 3)^2 + (\sqrt{x} - 1)^2$; $\frac{d(D^2)}{dx} = 2(x - 3) + 1 - \frac{1}{\sqrt{x}} = 0$ gives $x = 2.799$ (numerical). The derivative is negative before and positive after. Point $(2.799, 1.673)$, distance 0.702.

20. Square piece length $x \in [0, 100]$: $A = \frac{x^2}{16} + \frac{(100 - x)^2}{4\pi}$. $A' = \frac{x}{8} - \frac{100 - x}{2\pi} = 0$ at $x = 56.010$. $A(56.010) = 350.062$, $A(0) = 795.775$, $A(100) = 625$.
(a) Minimum area 350.062 cm^2 with 56.010 cm for the square.
(b) Maximum area 795.775 cm^2 using all the wire for the circle.

21. Vertices $(\pm x, \pm y)$. Using $y = \frac{3}{4}\sqrt{16 - x^2}$, the area is $A = 4xy = 3x\sqrt{16 - x^2}$ for $0 < x < 4$. $A^2 = 9x^2(16 - x^2)$ is maximal at $x^2 = 8$. $x = 2.828$, $y = 2.121$. The rectangle is 5.657 by 4.243 with area 24.000.

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3.9 Optimization II

SET A · SKILLS

1. $h=\frac{54}{r^2}$, $S=2\pi r^2+\frac{108\pi}{r}$. $S'=4\pi r-\frac{108\pi}{r^2}=0$ gives $r=3$. $S''>0$ for $r>0$. $h=6$ cm, $S=18\pi+36\pi=54\pi$ cm^2.
2. $h=\frac{64}{r^2}$, $S=\pi r^2+2\pi rh=\pi r^2+\frac{128\pi}{r}$. $S'=2\pi r-\frac{128\pi}{r^2}=0$ gives $r=4$; $S''=2\pi+\frac{256\pi}{r^3}>0$. $r=4$, $h=4$, $S=16\pi+32\pi=48\pi$ m^2.
3. Cylinder height h, radius r: $r^2=9-\frac{h^2}{4}$. $V=\pi\left(9h-\frac{h^3}{4}\right)$, $0<h<6$. $V'=\pi\left(9-\frac{3}{4}h^2\right)=0$ at $h=2\sqrt{3}$; V' changes $+$ to $-$. Then $r^2=6$ and $V=\pi\cdot 6\cdot 2\sqrt{3}=12\sqrt{3}\,\pi$.
4. $r^2=100-h^2$, $V=\frac{\pi}{3}(100h-h^3)$, $0<h<10$. $V'=\frac{\pi}{3}(100-3h^2)=0$ at $h=\frac{10\sqrt{3}}{3}$; V' changes $+$ to $-$. $r^2=\frac{200}{3}$. $V=\frac{\pi}{3}\cdot\frac{200}{3}\cdot\frac{10\sqrt{3}}{3}=\frac{2000\sqrt{3}\,\pi}{27}$.
5. $D(x)=\sqrt{x^2+4}+\sqrt{(6-x)^2+16}$. $D'(x)=\frac{x}{\sqrt{x^2+4}}-\frac{6-x}{\sqrt{(6-x)^2+16}}$. At $x=2$: $\frac{2}{\sqrt{8}}-\frac{4}{\sqrt{32}}=0$. D' is increasing (the first term increases, the second decreases), so $x=2$ is the absolute minimum. $D(2)=2\sqrt{2}+4\sqrt{2}=6\sqrt{2}$.
6. $h=\frac{1000}{x^2}$, $S=2x^2+\frac{4000}{x}$. $S'=4x-\frac{4000}{x^2}=0$ gives $x=10$; $S''>0$. A cube, $10\times10\times10$, $S=600$ cm^2.
7. $h=\frac{32}{r^2}$. $C=2(2\pi r^2)+2\pi rh=4\pi r^2+\frac{64\pi}{r}$. $C'=8\pi r-\frac{64\pi}{r^2}=0$ gives $r=2$; $C''>0$. $h=8$ cm. $C=16\pi+32\pi=48\pi$ cents.
8. $A=2x+\frac{50}{x}$, $A'=2-\frac{50}{x^2}=0$ at $x=5$. $A''=\frac{100}{x^3}>0$ for $x>0$: absolute minimum $A(5)=20$. $C'(x)=4x$ so $C'(5)=20=A(5)$: at minimum average cost, marginal cost equals average cost.
9. $A=xe^{-x}$, $A'=(1-x)e^{-x}$, zero at $x=1$ with A' changing $+$ to $-$. Maximum $A(1)=\frac{1}{e}$.

SET B · APPLICATIONS AND MIXED PROBLEMS

10. $w^2+h^2=144$, so $h^2=144-w^2$ and the strength is $S=kw(144-w^2)$, $0<w<12$. $S'=k(144-3w^2)=0$ at $w=4\sqrt{3}$; $S'>0$ before and <0 after, so this is a maximum. $h^2=144-48=96$. $w=4\sqrt{3}$ dm and $h=4\sqrt{6}$ dm.
11. By similar triangles, $\frac{h}{12}=\frac{6-r}{6}$, so $h=12-2r$. $V=\pi r^2(12-2r)$, $0<r<6$. $V'=6\pi r(4-r)$, positive on $(0,4)$ and negative on $(4,6)$. $r=4$, $h=4$, $V=64\pi$.
12. $T(x)=\frac{\sqrt{x^2+4}}{3}+\frac{6-x}{5}$, $0\le x\le 6$. $T'=\frac{x}{3\sqrt{x^2+4}}-\frac{1}{5}=0$ gives $5x=3\sqrt{x^2+4}$, $16x^2=36$, $x=\frac{3}{2}$. $T\left(\frac{3}{2}\right)=\frac{5}{6}+\frac{9}{10}=\frac{26}{15}$ h. Endpoints: $T(0)=\frac{28}{15}$, $T(6)=\frac{2\sqrt{10}}{3}\approx 2.11$, both larger. Minimum time $\frac{26}{15}$ h at $x=\frac{3}{2}$.
13. $C(x)=5000\sqrt{9+x^2}+4000(8-x)$, $0\le x\le 8$. $C'=0$ gives $5x=4\sqrt{9+x^2}$, $9x^2=144$, $x=4$. $C(4)=25000+16000=41000$. Endpoints: $C(0)=47000$, $C(8)=5000\sqrt{73}\approx 42720$. Least cost 41000 dollars.
14. Vertices $(\pm x,0)$ and $(\pm x,\sqrt{25-x^2})$. $A=2x\sqrt{25-x^2}$. Maximize $A^2=4(25x^2-x^4)$: derivative $4(50x-4x^3)=0$ at $x^2=\frac{25}{2}$, with $A^2=0$ at the ends of $(0,5)$. Width $5\sqrt{2}$, height $\frac{5\sqrt{2}}{2}$, area 25.
15. $2x^2+4xh=96$, $h=\frac{24}{x}-\frac{x}{2}$. $V=x^2h=24x-\frac{x^3}{2}$, $0<x<\sqrt{48}$. $V'=24-\frac{3}{2}x^2=0$ at $x=4$; $V''=-3x<0$. $h=4$. Cube of side 4, $V=64$ cm^3.
16. $A(\theta)=\frac{1}{2}\cdot 10\cdot 10\sin\theta=50\sin\theta$. $A'=50\cos\theta=0$ at $\theta=\frac{\pi}{2}$; A' changes $+$ to $-$ (or $A''=-50\sin\theta<0$ on the domain). Maximum area 50 at a right angle.

SET C · CHALLENGE AND AP-STYLE

17. $h=\frac{1000}{\pi r^2}$, $S=2\pi r^2+\frac{2000}{r}$. $S'=4\pi r-\frac{2000}{r^2}=0$ gives $r=\sqrt[3]{\frac{500}{\pi}}=5.419$. $S''=4\pi+\frac{4000}{r^3}>0$. $h=10.839$ cm and $S=553.581$ cm^2.
18. $h=\frac{500}{\pi r^2}$. $C=0.04(2\pi r^2)+0.02(2\pi rh)=0.08\pi r^2+\frac{20}{r}$. $C'=0.16\pi r-\frac{20}{r^2}=0$ gives $r=3.414$; $C''>0$. $h=13.656$ cm, $C=8.788$ dollars.
19. $r^2=100-\frac{h^2}{4}$, $V=\pi\left(100h-\frac{h^3}{4}\right)$. $V'=\pi\left(100-\frac{3}{4}h^2\right)=0$ at $h=\frac{20}{\sqrt{3}}=11.547$; V' changes $+$ to $-$. $r=8.165$. $V=2418.399$ cm^3.
20. $T(x)=\frac{\sqrt{900+x^2}}{5}+\frac{\sqrt{2500+(100-x)^2}}{1.5}$, $0\le x\le 100$. $T'(x)=0$ gives $x=85.248$ (numerically). $T(85.248)=52.828$ s, while $T(0)=80.536$ and $T(100)=54.214$. Minimum time 52.828 s.
21. $C(x)=80\sqrt{1.44+x^2}+50(4-x)$, $0\le x\le 4$. $C'=0$ gives $8x=5\sqrt{1.44+x^2}$, $39x^2=36$, $x=0.961$. $C(0.961)=274.940$, $C(0)=296$, $C(4)=334.090$. Least cost 274.940 thousand dollars at $x=0.961$ km.

3.10 Linearization and Differentials

SET A · SKILLS

1. $f(1)=3$, $f'(x)=3x^2+2$, $f'(1)=5$. $L(x)=3+5(x-1)=5x-2$.
2. $g(16)=4$, $g'(x)=\frac{1}{2\sqrt{x}}$, $g'(16)=\frac{1}{8}$. $L(x)=4+\frac{1}{8}(x-16)$. $\sqrt{17}\approx 4+\frac{1}{8}=\frac{33}{8}=4.125$.
3. $h(2)=\frac{1}{2}$, $h'(2)=-\frac{1}{4}$. $L(x)=\frac{1}{2}-\frac{1}{4}(x-2)$. $L(2.1)=0.5-0.025=0.475$. $h''=\frac{2}{x^3}>0$: concave up, tangent below the graph, so 0.475 is an underestimate.
4. $f\left(\frac{\pi}{4}\right)=1$, $f'(x)=\sec^2 x$, $f'\left(\frac{\pi}{4}\right)=2$. $L(x)=1+2\left(x-\frac{\pi}{4}\right)$. Estimate $1+2(0.05)=1.1$. $f''=2\sec^2 x\tan x>0$ on $\left(0,\frac{\pi}{2}\right)$, so it is an underestimate.
5. $f(1)=0$, $f'(1)=1$, $L(x)=x-1$. $\ln(1.2)\approx 0.2$. $f''=-\frac{1}{x^2}<0$: concave down, tangent above the graph, so 0.2 is too large.
6. $f(8)=2$, $f'(x)=\frac{1}{3}x^{-2/3}$, $f'(8)=\frac{1}{12}$. $L(8.06)=2+\frac{0.06}{12}=2.005$.
7. $dy=(3x^2-4)dx=(8)(-0.1)=-0.8$. $\Delta y=f(1.9)-f(2)=(6.859-7.6)-0=-0.741$.
8. $L(x)=7+2(x-3)$. $f(3.2)\approx 7.4$, $f(2.9)\approx 6.8$. Since $f''<0$ on $(2,4)$, the tangent line is above the graph there, so both are overestimates.
9. $V=x^3$, $dV=3x^2dx=3(25)(0.02)=1.5$ cm^3. Relative error $\frac{dV}{V}=3\cdot\frac{0.02}{5}=0.012$, or 1.2%.

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SET B · APPLICATIONS AND MIXED PROBLEMS

10. $dV = 4\pi r^2 dr = 4\pi(36)(0.05) = 7.2\pi$ cm^3.
11. $\frac{dT}{dL} = \frac{\pi}{\sqrt{10L}}$. At $L = 2.5$: $\sqrt{25} = 5$, so $\frac{dT}{dL} = \frac{\pi}{5}$. $dT = \frac{\pi}{5}(0.1) = \frac{\pi}{50} \approx 0.063$ s.
12. $dI = -\frac{10}{R^2} dR = -\frac{10}{400}(0.5) = -0.0125$ A. $I = 0.5$ A, so relative error $\frac{dI}{I} = -0.025$, i.e. 2.5% (magnitude).
13. $N'(t) = -40e^{-0.05t}$, $N'(0) = -40$. $L(t) = 800 - 40t$, $N(2) \approx 720$. $N''(t) = 2e^{-0.05t} > 0$, concave up, tangent below the graph: underestimate (true value about 723.9).
14. $dA = 2\pi r\, dr = 2\pi(10)(0.2) = 4\pi$ cm^2. $A = 100\pi$, so $\frac{dA}{A} = 0.04$: 4%.
15. (a) $L(x) = 4 + 3(x - 1)$.
    (b) $L(1.2) = 4.6$. Since $g'' > 0$, the tangent line is below the graph, so 4.6 is an underestimate.
    (c) $4 + 3(x - 1) = 0$ gives $x = -\frac{1}{3}$.
16. (a) $f(4) = 2$. The tangent line passes through $(0, 1)$ and $(4, 2)$, so its slope is $\frac{1}{4}$.
    (b) $L(4.4) = 2 + \frac{1}{4}(0.4) = 2.1$.
    (c) The tangent line lies above the graph, as f is concave down, so 2.1 is too large (true value about 2.0976).

SET C · CHALLENGE AND AP-STYLE

17. (a) $dV = 4\pi r^2 dr = 4\pi(144)(0.1) = 180.956$ cm^3; $\frac{dV}{V} = 3 \cdot \frac{0.1}{12} = 2.5\%$.
    (b) $dS = 8\pi r\, dr = 8\pi(12)(0.1) = 30.159$ cm^2; $\frac{dS}{S} = 2 \cdot \frac{0.1}{12} = 1.667\%$.
18. $f(27) = 3$, $f'(27) = \frac{1}{27}$, $L(30) = 3 + \frac{3}{27} = 3.111$. The calculator gives $30^{1/3} = 3.107$. Since $f'' < 0$ the estimate is too large; the error is 0.004.
19. $f(1) = \frac{\pi}{4}$, $f'(x) = \frac{1}{1+x^2}$, $f'(1) = \frac{1}{2}$. $L(x) = \frac{\pi}{4} + \frac{1}{2}(x - 1)$; $L(1.1) = 0.785 + 0.05 = 0.835$. Calculator: 0.833. Since $f'' < 0$ for $x > 0$, the tangent line is above the graph, so the estimate is an overestimate by about 0.002.
20. $dV = \frac{\pi}{2} d^2\, dd = \frac{\pi}{2}(5.76)(0.1) = 0.905$ cm^3. $V = \frac{\pi}{6}(13.824) = 7.238$. Percent error $\frac{dV}{V} = 3 \cdot \frac{0.1}{2.4} = 12.5\%$.
21. (a) $L(t) = 60 - 1.5(t - 10)$: $T(12) \approx 57.000$, $T(9) \approx 61.500$.
    (b) $T'' > 0$ means the tangent line is below the graph: both are underestimates.
    (c) $60 - 1.5(t - 10) = 50$ gives $t = 16.667$. Since $T \geq L$ (concave up), $T(16.667) \geq 50$. Because T is decreasing, it must fall further to reach 50, so the actual time is later than 16.667 minutes.

3.11 Rates of Change in Context

SET A · SKILLS

1. At $t = 20$ minutes the area of the oil slick is increasing at a rate of 15 square meters per minute (about 15 more square meters over the next minute).
2. When 50 chairs are made, cost is increasing at 34 dollars per chair. Making the 51st chair costs about 34 dollars, so $C(51) - C(50) \approx 34$.
3. At a depth of 100 meters the temperature is decreasing at 0.04 degrees Celsius per meter of additional depth.
4. $A = \pi r^2$, $\frac{dA}{dr} = 2\pi r = 8\pi$ cm^2 per cm. At $r = 4$ the area increases by about 8π cm^2 per additional centimeter of radius; $2\pi r$ is the circumference.
5. $\frac{dV}{ds} = 3s^2 = 108$ cm^3 per cm. The surface area is $6s^2 = 216$, and $\frac{dV}{ds}$ is half of it (only three faces grow outward when the edge grows).
6. $R'(x) = 60 - 0.4x$, $R'(50) = 40$ dollars per item. Exact: $R(51) - R(50) = 2539.8 - 2500 = 39.8$ dollars.
7. $P'(x) = -4x + 80 = 0$ at $x = 20$; $P'' = -4 < 0$. Maximum profit $P(20) = -800 + 1600 - 350 = 450$ dollars.
8. $A = 0.5x + 8 + \frac{98}{x}$, $A' = 0.5 - \frac{98}{x^2} = 0$ gives $x = 14$; $A'' = \frac{196}{x^3} > 0$. $A(14) = 7 + 8 + 7 = 22$. $C'(x) = x + 8$, so $C'(14) = 22 = A(14)$.
9. (a) $Q'(15) \approx \frac{165-140}{20-10} = 2.5$ liters per minute.
    (b) $\frac{180-100}{30} = \frac{8}{3} \approx 2.67$ liters per minute.
    (c) Slopes $4,\ 2.5,\ 1.5$ decrease, which suggests $Q'' < 0$: the tank fills more and more slowly.

SET B · APPLICATIONS AND MIXED PROBLEMS

10. (a) $A'(t) = -8e^{-0.2t}$, $A'(0) = -8$: initially the amount is decreasing at 8 mg per hour.
    (b) $-8e^{-0.2t} = -0.2 \cdot 40e^{-0.2t}$. The rate of decrease is proportional to the amount present.
    (c) $e^{-0.2t} = \frac{1}{4}$, so $t = 5 \ln 4 = 10 \ln 2$ hours.
11. $H'(t) = 6 - 0.3t$.
    (a) $H'(10) = 3$: at $t = 10$ days the height is increasing at 3 centimeters per day.
    (b) $\frac{H(20)-H(0)}{20} = \frac{80-20}{20} = 3$ centimeters per day.
    (c) $6 - 0.3t = 1.5$ gives $t = 15$ days.
12. (a) $R = xp = 100x - 0.5x^2$, $R' = 100 - x$.
    (b) $R' = C'$: $100 - x = 20$, $x = 80$. $P'' = -1 < 0$. Price $p = 60$. $P = R - C = 4800 - 2100 = 2700$ dollars.
13. $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dr} = 4\pi r^2 = S$. At $r = 5$: 100π cm^3 per cm. $\Delta V \approx 100\pi(0.02) = 2\pi \approx 6.28$ cm^3.
14. (a) $P' = -\frac{24}{V^2}$, $P'(4) = -1.5$: at 4 liters the pressure decreases at 1.5 atmospheres per liter of added volume.
    (b) $\frac{6-12}{2} = -3$ atm per liter.
    (c) $-\frac{24}{V^2} = -3$ gives $V = 2\sqrt{2}$ liters. This is the Mean Value Theorem.

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15. (a) The graph falls between $t = 2$ and $t = 6$, so water leaves during $2 < t < 6$.
(b) The steepest downward slope is at the inflection point $t = 4$ (where the graph changes from concave down to concave up).
(c) At $t = 1$ the graph is concave down, so W' is decreasing.
16. $\frac{dA}{ds} = \frac{\sqrt{3}}{2}s = 3\sqrt{3}$ cm^2 per cm. When the side is 6 cm, the area increases by about $3\sqrt{3} \approx 5.2$ cm^2 for each extra centimeter of side.

SET C · CHALLENGE AND AP-STYLE

17. $T'(t) = -4.76e^{-0.07t}$.
(a) $T'(5) = -3.354$: at 5 minutes the temperature is decreasing at 3.354 degrees Celsius per minute.
(b) $\frac{T(10)-T(0)}{10} = -3.423$ degrees per minute.
(c) $e^{-0.07t} = \frac{2}{4.76}$ gives $t = 12.387$ minutes.
18. (a) $P'(5) \approx \frac{200-150}{7-3} = 12.500$ thousand per day. $P'(14) \approx \frac{270-248}{16-12} = 5.500$ thousand per day.
(b) $\frac{270-120}{16} = 9.375$ thousand per day.
(c) Between $t = 7$ and $t = 12$ the slope is $\frac{48}{5} = 9.6$. $200 + 9.6(t - 7) = 230$ gives $t = 10.125$ days.
19. $P = 50xe^{-0.02x} - 300 - 8x$. $P' = R' - C' = 50e^{-0.02x}(1 - 0.02x) - 8 = 0$ gives $x = 34.159$ (numerical). $P''(34.159) = -0.665 < 0$, a maximum. $P = 289.257$ dollars, $R = 862.527$ dollars.
20. (a) $\frac{dV}{dh} = \frac{\pi}{4}h^2 = 9\pi = 28.274$ cubic meters per meter of depth. It equals the area $\pi\left(\frac{h}{2}\right)^2$ of the water surface.
(b) $\Delta V \approx 28.274(0.05) = 1.414$ cubic meters.
21. Average: $\frac{C(4)-C(0)}{4} = \frac{2.506-6}{4} = -0.874$. $C'(t) = -1.5e^{-0.3t} = -0.8735$ gives $t = 1.802$ seconds. The Mean Value Theorem guarantees such a time (continuous on $[0, 4]$, differentiable on $(0, 4)$).

3.12 Multiple-Choice Practice on Applications of Derivatives

SET A · PART A [NO CALCULATOR]

1. B. $f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$, which is negative exactly on $(1, 3)$.
2. D. $f'(x) = 3x^2 - 12 = 0$ gives $x = \pm 2$. Compare $f(-3) = 9$, $f(-2) = 16$, $f(2) = -16$, $f(5) = 65$. The largest is 65.
3. A. The average rate is $\frac{3-1}{8} = \frac{1}{4}$. Solve $\frac{1}{2\sqrt{c+1}} = \frac{1}{4}$ to get $\sqrt{c+1} = 2$, so $c = 3$.
4. C. f'' changes sign at -1 and 0 (odd powers of the factors). The factor $(x - 2)^2$ does not change sign, so $x = 2$ is not an inflection point.
5. E. A local minimum occurs where f' changes from negative to positive: at $x = -2$ and $x = 3$. At $x = 1$ it changes from positive to negative (a local maximum).
6. E. $x^2 + y^2 = 289$ gives $x\,x' + y\,y' = 0$. With $x = 8$, $y = 15$, $x' = 3$: $y' = -\frac{8 \cdot 3}{15} = -\frac{8}{5}$, so the top slides down at $\frac{8}{5}$ ft/s.
7. C. $f(64) = 4$ and $f'(64) = \frac{1}{3} \cdot 64^{-2/3} = \frac{1}{48}$. So $L(65) = 4 + \frac{1}{48} = \frac{193}{48}$.
8. B. With base side x, height $32/x^2$: $S = x^2 + \frac{128}{x}$. $S' = 2x - \frac{128}{x^2} = 0$ gives $x = 4$, and $S(4) = 16 + 32 = 48$. It is a minimum since $S'' > 0$.
9. A. The form is $\frac{0}{0}$. Apply L'Hopital's Rule twice: $\frac{2e^{2x}-2}{2x} \to \frac{4e^{2x}}{2} = 2$ as $x \to 0$.
10. D. By the Second Derivative Test, $f'(1) = 0$ and $f''(1) > 0$ gives a local minimum; $f'(5) = 0$ and $f''(5) < 0$ gives a local maximum. At $x = 3$ the test is inconclusive.
11. A. $v(2) = 3(4) - 18(2) + 15 = -9 < 0$ and $a(2) = 6(2) - 18 = -6 < 0$. Velocity and acceleration have the same sign, so the particle moves left and speeds up.
12. D. $f'(x) = \frac{5(x-2)}{3x^{1/3}}$. It equals 0 at $x = 2$ and is undefined at $x = 0$, which is in the domain of f. That gives two critical points.
13. E. $x^{5/3}$ is continuous on $[-1, 1]$ with $f'(x) = \frac{5}{3}x^{2/3}$ defined everywhere. $|x|$ and $x^{2/3}$ have corners or cusps at 0, $\frac{1}{x}$ is undefined at 0, and $\ln(x + 1)$ is undefined at -1.
14. B. f' changes from positive to negative only at $x = -1$. At $x = 0$ the factor x^2 does not change the sign, and at $x = 3$ the sign changes from negative to positive (a local minimum).
15. C. $f'(x) = (1 - x)e^{-x}$ and $f''(x) = (x - 2)e^{-x}$. f'' changes from negative to positive at $x = 2$.
16. C. Minimize $D^2 = (x - 5)^2 + x$. $\frac{d}{dx}D^2 = 2(x - 5) + 1 = 0$ gives $x = \frac{9}{2}$. The parabola D^2 opens upward, so this is the minimum.
17. E. $V = \pi r^2 h$, so $V' = \pi(2rr'h + r^2h') = \pi(2 \cdot 4 \cdot 2 \cdot 10 + 16 \cdot (-3)) = 112\pi$, which is positive.
18. B. $L(3.2) = 2 + 4(0.2) = 2.8$. Since $f'' > 0$, the graph is concave up and lies above its tangent line, so the approximation underestimates.
19. D. By the Mean Value Theorem there is c with $f'(c) = \frac{11-3}{5-1} = 2$. The other statements need not hold, since f may wiggle or be nonmonotonic.
20. A. $f'(x) = e^{-x^2}(1 - 2x^2) = 0$ gives $x = \frac{1}{\sqrt{2}}$. Then $f = \frac{1}{\sqrt{2}}e^{-1/2} = \frac{1}{\sqrt{2e}}$. Since $f(0) = 0$ and $f \to 0$ as $x \to \infty$, this is the maximum.

SET B · PART B [CALCULATOR]

21. A. The graph is falling on $(-1, 1)$ and concave up for $x > 0$. Both hold only on $(0, 1)$.
22. B. $f'(x) = e^x - 6x = 0$ at $x \approx 0.204$ in the interval, where $f \approx 1.101$ (a local maximum). Endpoints: $f(0) = 1$, $f(2) \approx -4.611$. The minimum is -4.611.
23. C. The average rate is $\frac{\ln 4+3}{3}$. Solve $\frac{1}{c} + 1 = \frac{\ln 4+3}{3}$ to get $c = \frac{3}{\ln 4} \approx 2.164$, which lies in $(1, 4)$.
24. D. At depth h the radius is $h/2$, so $V = \frac{\pi}{12}h^3$ and $V' = \frac{\pi}{4}h^2h'$. Then $5 = \frac{9\pi}{4}h'$ gives $h' = \frac{20}{9\pi} \approx 0.707$.
25. E. $A = 2x \cdot 3\cos x = 6x\cos x$. $A' = 6(\cos x - x\sin x) = 0$ at $x \approx 0.860$, giving $A \approx 3.367$.
26. E. $f'' = 0$ when $e^x = 4x$, which has solutions $x \approx 0.357$ and $x \approx 2.153$ in $[0, 3]$. f'' changes sign at each: positive, then negative, then positive.

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27. D. $L(4.3) = \ln 4 + \frac{1}{4}(0.3) \approx 1.4613$ and $\ln 4.3 \approx 1.4586$. Since $\ln x$ is concave down, the tangent line lies above the graph, an overestimate of about 0.003.

28. C. $v'(t) = (2t - t^2)e^{-t} = 0$ at $t = 2$. $v(0) = 0$, $v(2) = 4e^{-2} \approx 0.541$, $v(5) \approx 0.168$. The greatest is 0.541.

29. B. Minimum area occurs when $h = 2r$, so $V = 2\pi r^3 = 500$ gives $r \approx 4.301$. Then $S = 6\pi r^2 \approx 348.734$.

30. A. $s^2 = x^2 + 9$, so $s\,s' = x\,x'$. With $x = 2$, $s = \sqrt{13}$: $s' = \frac{2 \cdot 400}{\sqrt{13}} \approx 221.880$.

3.13 Free-Response Practice on Applications of Derivatives

QUESTION 1 · QUESTION 1

1. (a) $v(t) = x'(t) = \left(1 - \frac{t}{2}\right)e^{-t/2}$ and $a(t) = v'(t) = \left(\frac{t}{4} - 1\right)e^{-t/2}$. So $v(3) \approx -0.112$ cm/s and $a(3) \approx -0.056$ cm/s^2. Since $v(3) < 0$ and $a(3) < 0$ have the same sign, the speed is increasing at $t = 3$.

(b) $v(t) = 0$ only when $1 - \frac{t}{2} = 0$, so $t = 2$. For $0 < t < 2$, $v(t) > 0$; for $t > 2$, $v(t) < 0$. Because v changes sign from positive to negative at $t = 2$, the particle changes direction at $t = 2$.

(c) Candidates: $x(0) = 0$, $x(2) = 2e^{-1} \approx 0.736$, $x(8) = 8e^{-4} \approx 0.147$. The greatest position is $\frac{2}{e} \approx 0.736$ cm, since it is the largest of the values at the critical point and endpoints (Candidates Test on a closed interval).

(d) Average velocity $= \frac{x(8) - x(0)}{8 - 0} = e^{-4} \approx 0.018$ cm/s. x is continuous on $[0, 8]$ and differentiable on $(0, 8)$, so by the Mean Value Theorem there is a c with $v(c) = e^{-4}$. Solving $\left(1 - \frac{c}{2}\right)e^{-c/2} = e^{-4}$ on $(0, 2)$ gives $c \approx 1.905$.

Scoring: 9 points. (a) 2 points: 1 for $v(3)$ and $a(3)$, 1 for the conclusion with reason. (b) 2 points: 1 for $t = 2$, 1 for the sign change of v. (c) 2 points: 1 for evaluating the candidates, 1 for the answer with justification. (d) 3 points: 1 for the average velocity, 1 for the hypotheses and MVT statement, 1 for c.

QUESTION 2 · QUESTION 2

2. (a) $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. At $t = 3$, $\frac{dV}{dt} = \frac{40}{4} = 10$, so $10 = 4\pi(25)\frac{dr}{dt}$ and $\frac{dr}{dt} = \frac{1}{10\pi} \approx 0.032$ cm/s.

(b) $\frac{dS}{dt} = 8\pi r \frac{dr}{dt} = 8\pi(5) \cdot \frac{1}{10\pi} = 4$ cm^2/s (surface area is increasing).

(c) From $\frac{dr}{dt} = \frac{V'}{4\pi r^2}$, differentiate with the quotient rule: $\frac{d^2r}{dt^2} = \frac{V''(4\pi r^2) - V'(8\pi r\, r')}{(4\pi r^2)^2}$. Here $V'' = -\frac{40}{(1+t)^2} = -2.5$, so the numerator is $-2.5(100\pi) - 10(4) = -250\pi - 40$ and $\frac{d^2r}{dt^2} = -\frac{25\pi + 4}{1000\pi^2} \approx -0.008$ cm/s^2. Because it is negative, $\frac{dr}{dt}$ is decreasing, so the radius is growing more slowly at $t = 3$ than a moment earlier (still growing, since $\frac{dr}{dt} > 0$).

(d) $r(3.2) \approx 5 + \frac{1}{10\pi}(0.2) \approx 5.006$ cm. If r is concave down, its graph lies below the tangent line, so the tangent line approximation is an overestimate.

Scoring: 9 points. (a) 2 points: 1 for the related-rates equation, 1 for the value with units. (b) 2 points: 1 for $\frac{dS}{dt}$ formula, 1 for the value with units. (c) 3 points: 1 for the derivative setup, 1 for the value, 1 for the conclusion with reason. (d) 2 points: 1 for the approximation, 1 for overestimate with reason.

QUESTION 3 · QUESTION 3

3. (a) The underwater length is $PQ = \sqrt{9 + x^2}$ and the shore length is $QB = 8 - x$. So $C(x) = 5\sqrt{9 + x^2} + 3(8 - x)$.

(b) $C'(x) = \frac{5x}{\sqrt{9 + x^2}} - 3 = 0$ gives $5x = 3\sqrt{9 + x^2}$, so $25x^2 = 81 + 9x^2$, $16x^2 = 81$, and $x = \frac{9}{4}$ (positive root, and $5x > 0$ checks).

(c) $C''(x) = \frac{5 \cdot 9}{(9 + x^2)^{3/2}} = \frac{45}{(9 + x^2)^{3/2}}$, which is positive for every x. In particular $C''\left(\frac{9}{4}\right) = \frac{45}{(15/4)^3} = \frac{64}{75} > 0$, so C has a local minimum at $x = \frac{9}{4}$.

(d) $C\left(\frac{9}{4}\right) = 5 \cdot \frac{15}{4} + 3 \cdot \frac{23}{4} = \frac{75}{4} + \frac{69}{4} = 36$. Endpoints: $C(0) = 15 + 24 = 39 > 36$ and $C(8) = 5\sqrt{73}$; since $25 \cdot 73 = 1825 > 1296 = 36^2$, we have $5\sqrt{73} > 36$. Only one critical point exists, so the absolute minimum cost is 36 thousand dollars.

Scoring: 9 points. (a) 2 points: 1 for each length. (b) 3 points: 1 for C', 1 for setting equal to zero and solving, 1 for $x = \frac{9}{4}$. (c) 2 points: 1 for C'', 1 for the sign and conclusion. (d) 2 points: 1 for $C\left(\frac{9}{4}\right) = 36$, 1 for the endpoint comparison.

QUESTION 4 · QUESTION 4

4. (a) The segments have equations $f'(x) = -2x - 6$ on $[-4, -2]$, $f'(x) = x$ on $[-2, 2]$, and $f'(x) = -2x + 6$ on $[2, 4]$. So $f' = 0$ at $x = -3$, 0, and 3. f' changes from positive to negative at $x = -3$ and at $x = 3$, so f has local maxima there. f' changes from negative to positive at $x = 0$, so f has a local minimum at $x = 0$.

(b) f'' is the slope of f': -2 on $(-4, -2)$, 1 on $(-2, 2)$, and -2 on $(2, 4)$. So f is concave up on $(-2, 2)$. f'' changes sign at $x = -2$ and at $x = 2$ (from negative to positive, then back), so f has points of inflection at $x = -2$ and $x = 2$.

(c) $f'(1) = 1$, so the tangent line is $y = 4 + 1(x - 1) = x + 3$. Then $f(1.5) \approx 4.5$. Since $f'' > 0$ near $x = 1$, f is concave up and lies above its tangent line, so 4.5 is an underestimate.

(d) $f'(x) > 0$ for $-4 < x < -3$ (the graph of f' is above the axis there), so f is increasing, and $f'' = -2 < 0$ there, so f is concave down. The interval is $(-4, -3)$.

Scoring: 9 points. (a) 3 points: 1 for the zeros of f', 1 for the sign changes, 1 for correct classification. (b) 3 points: 1 for f'' information, 1 for the concave up interval, 1 for the inflection points with justification. (c) 2 points: 1 for tangent line and approximation, 1 for under/over with reason. (d) 1 point: both conditions with reasons.

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Chapter 4: Integrals.

14 sections with explanations and worked examples, then 286 practice problems. Answers and full explanations are at the end of the chapter.

4.1 Antiderivatives and Indefinite Integrals

Learn to reverse differentiation: find all antiderivatives of a function, use the power rule backwards, and pin down the constant with an initial condition.

Differentiation takes a function and gives its rate of change. In this chapter we run the process backwards: given the rate of change, recover the function. Finding a function from its derivative is called **antidifferentiation**, and it is the foundation of integral calculus.

Definition: Antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for every x in I.
For example, $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$. So are $x^3 + 5$ and $x^3 - \pi$.

Because the derivative of a constant is 0, adding a constant never changes a derivative. In fact, that is the only freedom there is: if two functions have the same derivative on an interval, they differ by a constant. This is why a single $+C$ describes every antiderivative.

The Indefinite Integral

If $F'(x) = f(x)$ on an interval, then

$$\int f(x)\,dx = F(x) + C,$$

where C is an arbitrary constant called the **constant of integration**.
Read $\int f(x)\,dx$ as \"the indefinite integral of f with respect to x.\" It represents the whole family of antiderivatives, and dx tells you the variable.

Power Rule and Basic Properties

If $n \neq -1$, then

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Constant multiple: $\int k\,f(x)\,dx = k\int f(x)\,dx$. Sum and difference: $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$. Also $\int k\,dx = kx + C$.
In words: add 1 to the exponent, then divide by the new exponent. You can always check by differentiating your answer.

The power rule fails for $n = -1$ because it would divide by zero. The function $\frac{1}{x}$ has a different antiderivative, which you will meet in the next section. Before integrating, rewrite every radical and every quotient as a sum of terms of the form ax^n: $\sqrt[3]{x} = x^{1/3}$, $\frac{1}{x^4} = x^{-4}$, $\frac{1}{\sqrt{x}} = x^{-1/2}$.

EXAMPLE 1

Find $\int (4x^3 - 6x + 1)\,dx$.

SOLUTION

Integrate term by term with the power rule.
$\int 4x^3\,dx = x^4$, $\int -6x\,dx = -3x^2$, $\int 1\,dx = x$.
$\int (4x^3 - 6x + 1)\,dx = x^4 - 3x^2 + x + C$.
Check: $\frac{d}{dx}(x^4 - 3x^2 + x + C) = 4x^3 - 6x + 1$.

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EXAMPLE 2

Find $\int \left(3\sqrt{x} - \frac{2}{x^2}\right) dx$.

SOLUTION

Rewrite with exponents: $3x^{1/2} - 2x^{-2}$.

$\int 3x^{1/2}\,dx = 3 \cdot \frac{x^{3/2}}{3/2} = 2x^{3/2}$.

$\int -2x^{-2}\,dx = -2 \cdot \frac{x^{-1}}{-1} = 2x^{-1} = \frac{2}{x}$.

$$\int \left(3\sqrt{x} - \frac{2}{x^2}\right) dx = 2x^{3/2} + \frac{2}{x} + C.$$

EXAMPLE 3

Find $\int \frac{x^3 - 8}{x^2}\,dx$.

SOLUTION

There is no quotient rule for integrals. Divide each term by x^2 first: $\frac{x^3-8}{x^2} = x - 8x^{-2}$.

$\int x\,dx = \frac{x^2}{2}$ and $\int -8x^{-2}\,dx = -8 \cdot \frac{x^{-1}}{-1} = \frac{8}{x}$.

$$\int \frac{x^3 - 8}{x^2}\,dx = \frac{x^2}{2} + \frac{8}{x} + C.$$

Common mistakes

(1) Forgetting $+C$ on an indefinite integral. (2) Trying to integrate a product or quotient factor by factor: $\int x^2 \cdot x^3\,dx \neq \frac{x^3}{3} \cdot \frac{x^4}{4}$. Multiply out first to get $\int x^5\,dx = \frac{x^6}{6} + C$. (3) Applying the power rule to $\frac{1}{x}$, which has $n = -1$.

A single antiderivative is picked out of the family by an **initial condition**, a known value of the function. A problem that asks for a function from its derivative and one such value is an **initial value problem**.

1. **Integrate.** Find the general antiderivative, including $+C$.
2. **Substitute.** Put the given point into the general antiderivative.
3. **Solve for C.** Isolate the constant.
4. **Write the answer.** Put the value of C back and state the specific function.

EXAMPLE 4

Find f if $f'(x) = 6x^2 - 4x$ and $f(2) = 9$.

SOLUTION

$f(x) = 2x^3 - 2x^2 + C$.

$f(2) = 16 - 8 + C = 8 + C = 9$, so $C = 1$.

$f(x) = 2x^3 - 2x^2 + 1$.

EXAMPLE 5

A particle moves along a line with acceleration $a(t) = 6t - 4$ (in m/s²). At $t = 0$ its velocity is 2 m/s and its position is 1 m. Find the position $s(t)$.

SOLUTION

Velocity is the antiderivative of acceleration: $v(t) = 3t^2 - 4t + C_1$. Since $v(0) = 2$, $C_1 = 2$, so $v(t) = 3t^2 - 4t + 2$.

Position is the antiderivative of velocity: $s(t) = t^3 - 2t^2 + 2t + C_2$. Since $s(0) = 1$, $C_2 = 1$.

$s(t) = t^3 - 2t^2 + 2t + 1$.

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Practice

SET A Skills

Find each indefinite integral. [No calculator]

1. $\int (6x^2 - 4x + 7)\,dx$
2. $\int (10x^4 + 9x^2 - 1)\,dx$
3. $\int \frac{4}{x^3}\,dx$
4. $\int 6\sqrt{x}\,dx$
5. $\int 12\sqrt[3]{x}\,dx$
6. $\int (x^2 + 1)^2\,dx$
7. $\int \frac{x^3 - 5x^2 + 4}{x^2}\,dx$
8. $\int (2x + 3)(x - 1)\,dx$
9. $\int \left(\frac{5}{\sqrt{x}} + 2x\sqrt{x}\right)\,dx$

SET B Applications and Mixed

Solve each problem. Give exact answers unless a decimal is requested. [Calculator]

10. Find f if $f'(x) = 9x^2 - 2x$ and $f(1) = 5$.
11. Find f if $f'(x) = 3\sqrt{x}$ and $f(4) = 20$.
12. Find f if $f''(x) = 12x - 6$, $f'(1) = 0$, and $f(0) = 2$.
13. A ball is thrown straight up from a height of 2 m with an initial velocity of 20 m/s. Its acceleration is -9.8 m/s^2. (a) Find the height $h(t)$. (b) Find the height at $t = 1.5$ s. (c) Find the maximum height. Round to three decimal places.
14. A factory's marginal cost is $C'(x) = 0.06x + 4$ dollars per unit, where x is the number of units produced. The fixed cost (cost at $x = 0$) is 500 dollars. Find $C(x)$ and the cost of producing 100 units.
15. A particle moves on a line with velocity $v(t) = 3t^2 - 12t + 9$ m/s and position $s(0) = 4$ m. Find $s(t)$ and the position at $t = 2$.
16. A curve passes through $(1, 4)$ and has slope $3x^2 - 2x$ at each point (x, y). Find its equation and the value of y when $x = 2$.

SET C Challenge and AP-Style

Solve each problem and explain your reasoning. [No calculator]

17. Find f if $f'''(x) = 24x$, $f''(0) = 2$, $f'(0) = -1$, and $f(0) = 5$.
18. A student claims that $\int x^2 \cdot x^4\,dx = \frac{x^3}{3} \cdot \frac{x^5}{5} + C$. Is this correct? Explain and give the correct answer.
19. Find the function f with $f'(x) = |x|$ and $f(0) = 1$.
20. A particle has velocity $v(t) = t^2 - 4t + 3$ for $0 \le t \le 4$ and starts at $s(0) = 0$. Find the total distance traveled on $[0, 4]$.
21. Find constants a and b so that $F(x) = (ax + b)\sqrt{x}$ is an antiderivative of $f(x) = \frac{6x + 5}{2\sqrt{x}}$ for $x > 0$.

4.2 Derivative Rules Backwards

Turn the derivative formulas for trigonometric, exponential, and logarithmic functions into antiderivative formulas, including linear inner functions.

Every derivative formula you know becomes an antiderivative formula when you read it backwards. Since $\frac{d}{dx}\sin x = \cos x$, we have $\int \cos x\,dx = \sin x + C$. The table below collects the ones you must know by heart, along with the rule for handling a linear inner function like $3x - 1$.

Basic Antiderivatives

$\int \cos x\,dx = \sin x + C$ and $\int \sin x\,dx = -\cos x + C$.
$\int \sec^2 x\,dx = \tan x + C$ and $\int \csc^2 x\,dx = -\cot x + C$.
$\int \sec x \tan x\,dx = \sec x + C$ and $\int \csc x \cot x\,dx = -\csc x + C$.
$\int e^x\,dx = e^x + C$ and $\int \frac{1}{x}\,dx = \ln|x| + C$ (on any interval not containing 0).
$\int \frac{1}{1+x^2}\,dx = \arctan x + C$ and $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$.

Notice the signs. The derivative of $\cos x$ is $-\sin x$, so $\int \sin x\,dx$ picks up a minus sign. Every antiderivative with \"co\" in its name (cosine, cotangent, cosecant) carries a minus sign.

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Linear Inner Function Rule

If $\int f(x)\,dx = F(x) + C$ and $a \neq 0$, then

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C.$$

The chain rule multiplies by a when you differentiate, so you divide by a to undo it. This works only when the inside is linear.

EXAMPLE 1

Find $\int (3\sin x - 2\cos x + e^x)\,dx$.

SOLUTION

Integrate each term using the table.

$\int 3\sin x\,dx = -3\cos x$, $\int -2\cos x\,dx = -2\sin x$, $\int e^x\,dx = e^x$.

$$\int (3\sin x - 2\cos x + e^x)\,dx = -3\cos x - 2\sin x + e^x + C.$$

EXAMPLE 2

Find $\int \left(\sec^2 x + \frac{4}{x}\right)dx$ for $x > 0$.

SOLUTION

$\int \sec^2 x\,dx = \tan x$.

$\int \frac{4}{x}\,dx = 4\ln|x|$, and since $x > 0$ this is $4\ln x$.

$$\int \left(\sec^2 x + \frac{4}{x}\right)dx = \tan x + 4\ln x + C.$$

EXAMPLE 3

Find (a) $\int e^{3x-1}\,dx$, (b) $\int \cos(5x)\,dx$, (c) $\int \frac{1}{2x+3}\,dx$.

SOLUTION

(a) Here $a = 3$: $\int e^{3x-1}\,dx = \frac{1}{3}e^{3x-1} + C$.

(b) Here $a = 5$: $\int \cos(5x)\,dx = \frac{1}{5}\sin(5x) + C$.

(c) Here $a = 2$ and the outer antiderivative is $\ln|u|$: $\int \frac{1}{2x+3}\,dx = \frac{1}{2}\ln|2x+3| + C$.

Check (a): $\frac{d}{dx}\left[\frac{1}{3}e^{3x-1}\right] = \frac{1}{3}\cdot 3e^{3x-1} = e^{3x-1}$.

Common mistakes

(1) Writing $\int \sin x\,dx = \cos x + C$ (the sign is wrong). (2) Using the power rule on e^x: $\int e^x\,dx$ is not $\frac{e^{x+1}}{x+1}$. (3) Forgetting to divide by the coefficient: $\int \cos(5x)\,dx$ is $\frac{1}{5}\sin 5x$, not $\sin 5x$. (4) Writing $\ln x$ instead of $\ln|x|$ when x may be negative.

EXAMPLE 4

Find $\int \tan^2 x\,dx$.

SOLUTION

No table entry has $\tan^2 x$, but the identity $\tan^2 x = \sec^2 x - 1$ rewrites it.

$\int (\sec^2 x - 1)\,dx = \tan x - x$.

$$\int \tan^2 x\,dx = \tan x - x + C.$$

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EXAMPLE 5

Find f if $f'(x) = 2\sin x + 3e^x$ and $f(0) = 5$.

SOLUTION

$f(x) = -2\cos x + 3e^x + C$.

$f(0) = -2 + 3 + C = 1 + C$. Setting $1 + C = 5$ gives $C = 4$.

$f(x) = -2\cos x + 3e^x + 4$.

Practice

SET A Skills

Find each indefinite integral. [No calculator]

1. $\int (5\cos x - 3\sin x)\,dx$
2. $\int (e^x + 2x)\,dx$
3. $\int \left(\frac{7}{x} + 3x^2\right)\,dx$
4. $\int 6\sec^2 x\,dx$
5. $\int (\sec x \tan x - \csc^2 x)\,dx$
6. $\int \cos(4x)\,dx$
7. $\int 6e^{3x}\,dx$
8. $\int \sin\left(\frac{x}{2}\right)\,dx$
9. $\int \frac{1}{3x - 5}\,dx$

SET B Applications and Mixed

Solve each problem. Give exact answers unless a decimal is requested. [Calculator]

10. Find f if $f'(x) = 4\cos x + 2e^x$ and $f(0) = 7$.
11. Find f if $f'(x) = \sec^2 x - 2$ and $f\left(\frac{\pi}{4}\right) = 3$. Give an exact answer.
12. A particle moves on a line with velocity $v(t) = 4\cos(2t)$ and starts at $s(0) = 1$. Find $s(t)$ and $s\left(\frac{\pi}{4}\right)$.
13. A curve has slope $\frac{dy}{dx} = e^{x/2}$ at each point and passes through $(0, 5)$. Find y when $x = 2\ln 3$.
14. Find $f(e)$ if $f'(x) = \frac{5}{x} - 2$ for $x > 0$ and $f(1) = 4$.
15. A bacterial culture grows at the rate $P'(t) = 120e^{0.2t}$ bacteria per hour and starts with $P(0) = 500$. How many bacteria are present at $t = 5$ hours? Round to three decimal places.
16. The current in a wire is $I(t) = 4\sin(2t)$ amperes, which equals $Q'(t)$, where Q is the charge in coulombs that has passed a point. If $Q(0) = 0.5$, find $Q(1)$. Round to three decimal places.

SET C Challenge and AP-Style

Solve each problem and explain your reasoning. [No calculator]

17. Find $\int (\sin x + \cos x)^2\,dx$.
18. Find $\int \frac{e^{2x} + 1}{e^x}\,dx$.
19. Find $\int \frac{x^3 + 2x^2 - 5}{x}\,dx$ for $x > 0$.
20. Find f if $f''(x) = -9\cos(3x)$, $f'(0) = 3$, and $f(0) = 1$.
21. A student writes $\int \cos(3x)\,dx = \sin(3x) + C$. Show by differentiation that this is wrong and give the correct antiderivative.

4.3 The Method of Substitution

Learn to spot an inside function and its derivative so that a complicated integral becomes a basic one. Substitution is the chain rule run backwards.

The chain rule says $\frac{d}{dx}[F(g(x))] = F'(g(x))\,g'(x)$. Reading this backwards gives a powerful way to integrate: if an integrand looks like \"a function of an inside function, times the derivative of that inside function,\" then substitution turns it into a simpler integral.

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The Method of Substitution (u-Substitution)

Suppose $u = g(x)$ is differentiable, and f is continuous on the range of g. Then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du.$$

Here $du = g'(x)\, dx$. After integrating in u, always substitute back to express the answer in x.

Check: differentiate your final answer in x; you must recover the original integrand.

1. **Choose u.** Pick the inside function: the expression inside parentheses, under a root, in an exponent, or in a denominator, whose derivative also appears (up to a constant) in the integrand.
2. **Compute du.** Differentiate: $du = g'(x)\, dx$.
3. **Adjust constants.** If du differs from what the integrand supplies by a constant factor, solve for the piece you need. For example, when $u = x^2 + 1$ we have $du = 2x\, dx$, so $x\, dx = \frac{1}{2}\, du$.
4. **Rewrite everything in u.** No x may remain inside the new integral.
5. **Integrate and back-substitute.** Find the antiderivative in u, add $+C$, and replace u by $g(x)$.

Two Special Forms

Power form: $\int [g(x)]^n\, g'(x)\, dx = \frac{[g(x)]^{n+1}}{n+1} + C$ for $n \neq -1$.

Logarithm form: $\int \frac{g'(x)}{g(x)}\, dx = \ln|g(x)| + C$, valid on any interval where $g(x) \neq 0$.

- Clue: a power of an expression, such as $(x^2 + 3)^4$, suggests $u = x^2 + 3$.
- Clue: a root, such as $\sqrt{x^3 + 5}$, suggests $u = x^3 + 5$.
- Clue: $e^{\text{something}}$ suggests $u =$ that exponent.
- Clue: a fraction whose numerator is (a multiple of) the derivative of the denominator suggests $u =$ the denominator, giving $\ln|u|$.

EXAMPLE 1

Find $\int 2x(x^2 + 3)^4\, dx$.

SOLUTION

Let $u = x^2 + 3$. Then $du = 2x\, dx$, which is exactly what appears.

$\int u^4\, du = \frac{u^5}{5} + C$.

Back-substitute: $\int 2x(x^2 + 3)^4\, dx = \dfrac{(x^2 + 3)^5}{5} + C$.

EXAMPLE 2

Find $\int x^2 \sqrt{x^3 + 5}\, dx$.

SOLUTION

Let $u = x^3 + 5$. Then $du = 3x^2\, dx$, so $x^2\, dx = \frac{1}{3}\, du$.

$\int \sqrt{u} \cdot \frac{1}{3}\, du = \frac{1}{3} \cdot \frac{u^{3/2}}{3/2} = \frac{2}{9} u^{3/2}$.

$\int x^2 \sqrt{x^3 + 5}\, dx = \dfrac{2}{9}(x^3 + 5)^{3/2} + C$.

Common mistakes

(1) Forgetting the constant factor: with $u = x^2 + 1$, $x\, dx = \frac{1}{2}\, du$, not du. So $\int x(x^2 + 1)^2\, dx = \frac{(x^2+1)^3}{6} + C$, not $\frac{(x^2+1)^3}{3} + C$. (2) Leaving some x's in the integral after substituting. (3) Forgetting to switch back from u to x at the end. (4) Substituting when the derivative of the inside is not present: $\int \sin(x^2)\, dx$ cannot be done with $u = x^2$ because there is no x factor to supply $du = 2x\, dx$.

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EXAMPLE 3

Find $\int \cos x\, e^{\sin x}\, dx$.

SOLUTION

Let $u = \sin x$, so $du = \cos x\, dx$.

$\int e^u\, du = e^u + C$.

$\int \cos x\, e^{\sin x}\, dx = e^{\sin x} + C$.

EXAMPLE 4

Find $\int \frac{x}{x^2+1}\, dx$.

SOLUTION

Let $u = x^2 + 1$, so $du = 2x\, dx$ and $x\, dx = \frac{1}{2}\, du$.

$\int \frac{1}{u} \cdot \frac{1}{2}\, du = \frac{1}{2} \ln|u| + C$.

Since $x^2 + 1 > 0$, the absolute value is not needed: $\int \frac{x}{x^2+1}\, dx = \frac{1}{2}\ln(x^2+1) + C$.

EXAMPLE 5

Find $\int x\sqrt{x+1}\, dx$.

SOLUTION

The derivative of $x + 1$ is 1, so there is a leftover x. Let $u = x + 1$; then $du = dx$ and $x = u - 1$.

$\int (u-1)u^{1/2}\, du = \int (u^{3/2} - u^{1/2})\, du = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}$.

$\int x\sqrt{x+1}\, dx = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$.

Practice

SET A Skills

Find each indefinite integral using substitution. [No calculator]

1. $\int 2x(x^2+1)^5\, dx$
2. $\int x^2(x^3+4)^4\, dx$
3. $\int x\sqrt{x^2+9}\, dx$
4. $\int \cos x\, \sin^4 x\, dx$
5. $\int x^2 e^{x^3}\, dx$
6. $\int \frac{x}{x^2+4}\, dx$
7. $\int \tan x\, \sec^2 x\, dx$
8. $\int \frac{3x^2+1}{x^3+x}\, dx$
9. $\int \frac{(\ln x)^2}{x}\, dx$ for $x > 0$

SET B Applications and Mixed

Solve each problem. Give exact answers unless a decimal is requested. [Calculator]

10. $\int (4x+2)(x^2+x)^3\, dx$
11. $\int x^2 \sin(x^3)\, dx$
12. $\int \frac{e^{2x}}{1+e^{2x}}\, dx$
13. Find $f(1)$ if $f'(x) = 6x(x^2+1)^2$ and $f(0) = 1$.
14. A particle moves on a line with velocity $v(t) = t\cos(t^2)$ m/s and position $s(0) = 2$ m. Find $s\left(\sqrt{\pi/2}\right)$.
15. The concentration $c(t)$ of a drug in the blood (in mg per liter) changes at the rate $c'(t) = 12te^{-t^2}$ for $t \geq 0$ in hours, with $c(0) = 0$. Find $c(2)$ to three decimal places.

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16. A curve has slope $\frac{dy}{dx} = \sin x \cos^2 x$ and passes through $(0, 1)$. Find $y\left(\frac{\pi}{2}\right)$.

SET C Challenge and AP-Style

Solve each problem and explain your reasoning. [No calculator]

17. $\int x\sqrt{x-3}\,dx$
18. $\int e^x\sqrt{1+e^x}\,dx$
19. $\int \frac{\sin x}{\cos^3 x}\,dx$
20. Use $u = \sin x$ to show $\int 2\sin x\cos x\,dx = \sin^2 x + C$, and use $u = \cos x$ to get $-\cos^2 x + C$. These look different. Are both correct?
21. Find $f(0)$ if $f'(x) = \frac{x}{\sqrt{x^2+16}}$ and $f(3) = 7$.

4.4 Using Geometry for Definite Integrals

You will learn to evaluate definite integrals as signed areas of triangles, rectangles and circles, and to use symmetry. This is the first way to see what an integral means.

The definite integral $\int_a^b f(x)\,dx$ is defined as a limit of sums, but for many functions you can find its value without any limit at all. When the graph of f is made of lines and circular arcs, the integral is just a combination of areas of triangles, rectangles, trapezoids and pieces of circles. The one new idea is that area is counted with a sign.

The Definite Integral as Signed Area

Let f be continuous on $[a, b]$. Then $\int_a^b f(x)\,dx$ equals the area of the regions between the graph and the x-axis that lie **above** the axis, **minus** the area of the regions that lie **below** the axis.

Regions above the x-axis count positive, regions below count negative. The result can be positive, negative or zero, but a geometric area is never negative.

Area Formulas You Need

Rectangle: bh. Triangle: $\frac{1}{2}bh$. Trapezoid: $\frac{1}{2}(b_1 + b_2)h$ (the parallel sides are b_1, b_2, and h is the distance between them).

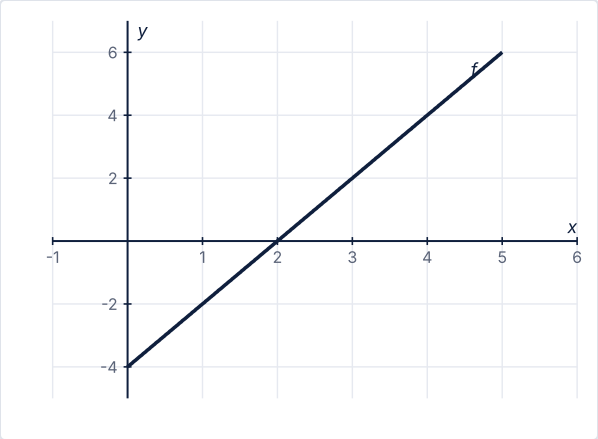
Circle of radius r: πr^2. Semicircle: $\frac{1}{2}\pi r^2$. Quarter circle: $\frac{1}{4}\pi r^2$. The graph of $y = \sqrt{r^2 - (x-h)^2}$ is the **upper** half of a circle of radius r centered at $(h, 0)$; $y = -\sqrt{r^2 - (x-h)^2}$ is the lower half.

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EXAMPLE 1

Use the graph of $f(x) = 2x - 4$ on $[0, 5]$ to evaluate $\int_0^5 (2x - 4)\,dx$.



Graph of $f(x) = 2x - 4$ on $[0, 5]$.

SOLUTION

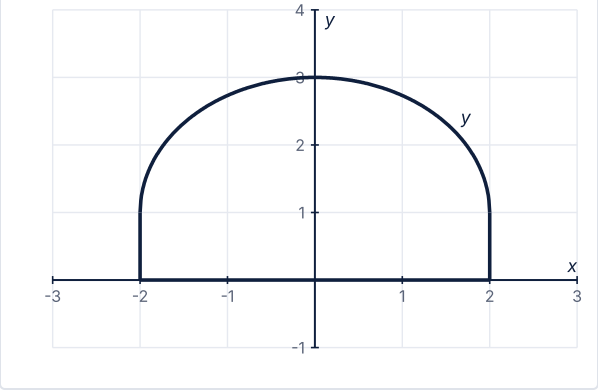
The line crosses the x-axis at $x = 2$. On $[0, 2]$ the region is a triangle below the axis with base 2 and height 4: area $\frac{1}{2}(2)(4) = 4$, counted as -4.

On $[2, 5]$ the region is a triangle above the axis with base 3 and height 6: area $\frac{1}{2}(3)(6) = 9$.

$\int_0^5 (2x - 4)\,dx = 9 - 4 = 5$.

EXAMPLE 2

Evaluate $\int_{-2}^{2} \left(1 + \sqrt{4 - x^2}\right) dx$.



Graph of $y = 1 + \sqrt{4 - x^2}$ on $[-2, 2]$.

SOLUTION

The graph of $y = 1 + \sqrt{4 - x^2}$ is a semicircle of radius 2 sitting on top of the horizontal line $y = 1$. The region under it is a rectangle plus a semicircle.

Rectangle: width 4, height 1, area 4. Semicircle: $\frac{1}{2}\pi(2)^2 = 2\pi$.

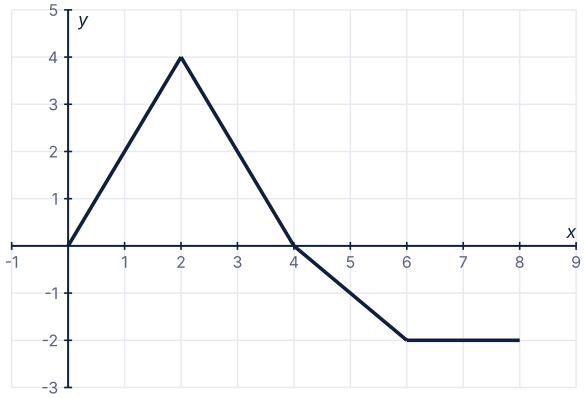
$\int_{-2}^{2} \left(1 + \sqrt{4 - x^2}\right) dx = 4 + 2\pi$.

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EXAMPLE 3

The graph of f consists of line segments. Find $\int_0^8 f(x)\,dx$ and $\int_0^8 |f(x)|\,dx$.



Graph of f for $0 \le x \le 8$ made of line segments.

SOLUTION

$[0, 4]$: triangle above the axis, base 4, height 4: area 8.

$[4, 6]$: triangle below the axis, base 2, height 2: area 2, counted as -2.

$[6, 8]$: rectangle below the axis, width 2, height 2: area 4, counted as -4.

$\int_0^8 f = 8 - 2 - 4 = 2$. For $|f|$ every region counts positive: $\int_0^8 |f| = 8 + 2 + 4 = 14$.

EXAMPLE 4

Evaluate $\int_{-3}^{3} \left(x^3 + 2\sqrt{9 - x^2}\right) dx$.

SOLUTION

Split the integral: $\int_{-3}^{3} x^3\,dx + 2\int_{-3}^{3} \sqrt{9 - x^2}\,dx$.

The function x^3 is odd, so the area above the axis on $[0, 3]$ cancels the area below the axis on $[-3, 0]$: $\int_{-3}^{3} x^3\,dx = 0$.

$\int_{-3}^{3} \sqrt{9 - x^2}\,dx$ is a semicircle of radius 3: $\frac{1}{2}\pi(9) = \frac{9\pi}{2}$.

Total: $0 + 2 \cdot \frac{9\pi}{2} = 9\pi$.

Symmetry and Basic Properties

If f is **odd** ($f(-x) = -f(x)$) and continuous on $[-a, a]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

If f is **even** ($f(-x) = f(x)$) and continuous on $[-a, a]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

Also: $\int_a^a f\,dx = 0$, $\int_b^a f\,dx = -\int_a^b f\,dx$, $\int_a^b kf\,dx = k\int_a^b f\,dx$, $\int_a^b (f \pm g)\,dx = \int_a^b f\,dx \pm \int_a^b g\,dx$, and $\int_a^b k\,dx = k(b - a)$.

Common mistake

Do not treat $\int_a^b f(x)\,dx$ as the total area when part of the graph is below the x-axis. Signed area subtracts the part below. If a problem asks for the **area** of the region, or the total distance traveled, use $\int_a^b |f(x)|\,dx$. Also watch the shape: $\sqrt{r^2 - x^2}$ over $[-r, r]$ is a semicircle, not a whole circle.

Practice

SET A ### Skills

Evaluate each integral using geometry. Give exact answers. [No calculator]

1. $\int_1^5 3\,dx$
2. $\int_0^6 \frac{x}{3}\,dx$
3. $\int_{-4}^{2} (x + 1)\,dx$
4. $\int_0^5 |x - 2|\,dx$

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5. $\displaystyle\int_{-5}^{5} \sqrt{25 - x^2}\, dx$

6. $\displaystyle\int_{0}^{3} \sqrt{9 - x^2}\, dx$

7. $\displaystyle\int_{-1}^{3} \left(-\sqrt{4 - (x-1)^2}\right) dx$

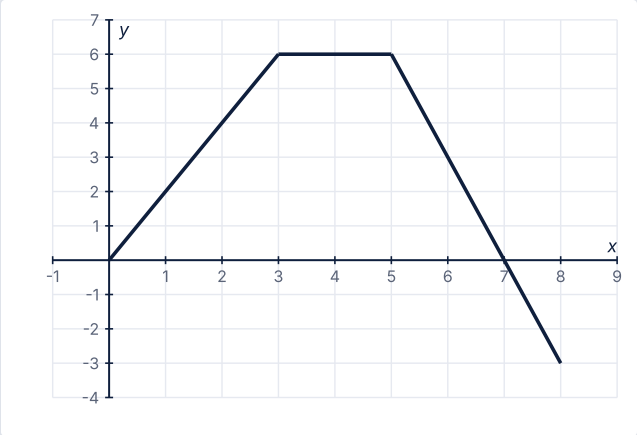
8. $\displaystyle\int_{0}^{4} \left(2 - \sqrt{4 - (x-2)^2}\right) dx$

9. The function f is odd and continuous, and $\int_0^3 f(x)\, dx = 4$. Find $\int_{-3}^{3} (f(x) + 2)\, dx$.

SET B Applications and Mixed

Answer each question. Give exact answers. [No calculator]

10. A particle moves along a line with velocity $v(t)$ in meters per second, shown in the graph for $0 \le t \le 8$ (line segments). (a) Find the displacement from $t = 0$ to $t = 8$. (b) Find the total distance traveled. (c) If the particle is at position $s = 4$ when $t = 0$, find its position at $t = 8$.



Velocity $v(t)$ in m/s, $0 \le t \le 8$.

11. Water flows into a tank at a rate $r(t) = 12 - 3t$ liters per minute for $0 \le t \le 4$. The tank holds 15 liters at $t = 0$. Use a geometric argument to find the amount of water in the tank at $t = 4$.

12. The current in a circuit is $i(t) = \sqrt{4 - t^2}$ amperes for $0 \le t \le 2$ seconds. The charge that passes is $\int_0^2 i(t)\, dt$ coulombs. Find it exactly.

13. A spring exerts a force $F(x) = 20x$ newtons when stretched x meters. The work done stretching it from $x = 0$ to $x = 0.3$ is $\int_0^{0.3} F(x)\, dx$ joules. Find the work.

14. Suppose $\int_1^4 f(x)\, dx = 7$, $\int_4^9 f(x)\, dx = -2$. Find (a) $\int_1^9 f(x)\, dx$ and (b) $\int_9^1 (2f(x) + 1)\, dx$.

15. What is the value of $\displaystyle\int_{-4}^{4} \left(x^3 + |x|\right) dx$?

(A) 0 **(B)** 8 **(C)** 16 **(D)** 32 **(E)** 64

16. For $k > 0$, $\int_0^k 2x\, dx = 25$. Find k.

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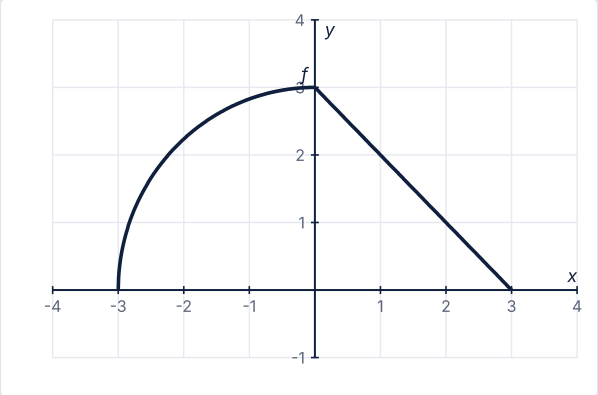
SET C Challenge and AP-Style

Round decimal answers to three places. [Calculator]

17. Use geometry to find $\int_{-3.5}^{3.5} \sqrt{12.25 - x^2}\,dx$. Give a decimal to three places.

18. Find $\int_{-4}^{4} \left(\sqrt{16 - x^2} - 3\right) dx$ to three decimal places.

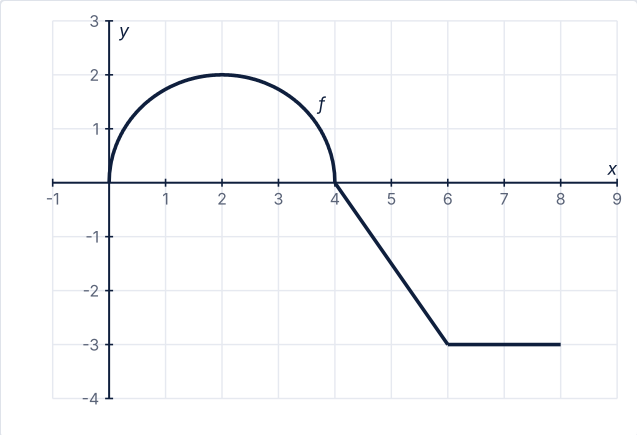
19. Let $f(x) = \begin{cases} \sqrt{9 - x^2}, & -3 \le x \le 0 \\ 3 - x, & 0 < x \le 3. \end{cases}$ Use the graph to find $\int_{-3}^{3} f(x)\,dx$ to three decimal places.



Graph of f on $[-3, 3]$.

20. For $a > 0$, $\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = 50$. Find a to three decimal places.

21. The graph of f is a semicircle on $[0, 4]$ followed by line segments, as shown. Find to three decimal places: (a) $\int_0^4 f(x)\,dx$; (b) $\int_0^8 f(x)\,dx$; (c) $\int_0^8 |f(x)|\,dx$.



Graph of f: a semicircle, then two line segments.

4.5 Riemann Sums

You will learn to build left, right and midpoint Riemann sums from functions and tables, judge over- and underestimates, and write a limit of sums as an integral.

Many quantities, such as distance from a velocity, total flow from a rate, or area under a curve, are found by adding up many small pieces. A **Riemann sum** does this: it slices an interval into subintervals, builds a rectangle on each one, and adds the areas. Using more and more rectangles gives the definite integral.

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Riemann Sums with Equal Subintervals

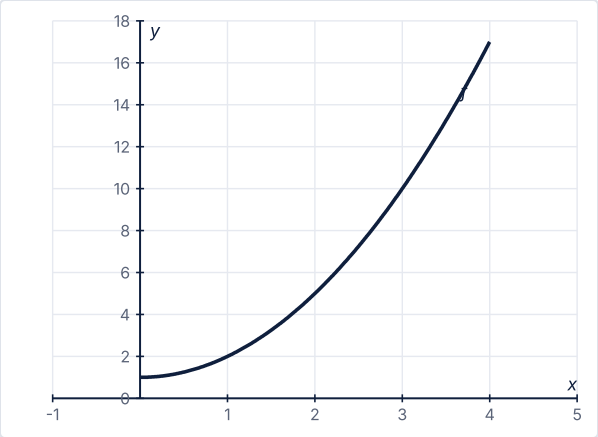
Divide $[a, b]$ into n subintervals of equal width $\Delta x = \dfrac{b-a}{n}$, with endpoints $x_i = a + i\,\Delta x$ for $i = 0, 1, \ldots, n$. For a function f defined on $[a, b]$:

Left sum: $L_n = \sum_{i=0}^{n-1} f(x_i)\,\Delta x$. Right sum: $R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$. Midpoint sum: $M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$.

Each term is (height) times (width). The left, right or midpoint sum uses the height of f at the left end, right end or midpoint of each subinterval.

EXAMPLE 1

Let $f(x) = x^2 + 1$. Find L_4, R_4 and M_4 for $\int_0^4 f(x)\,dx$.



Graph of $f(x) = x^2 + 1$ on $[0, 4]$.

SOLUTION

$\Delta x = \frac{4-0}{4} = 1$, so the endpoints are $0, 1, 2, 3, 4$ and the midpoints are $0.5, 1.5, 2.5, 3.5$.

$L_4 = 1\,[f(0) + f(1) + f(2) + f(3)] = 1 + 2 + 5 + 10 = 18$.

$R_4 = 1\,[f(1) + f(2) + f(3) + f(4)] = 2 + 5 + 10 + 17 = 34$.

$M_4 = 1\,[f(0.5) + f(1.5) + f(2.5) + f(3.5)] = 1.25 + 3.25 + 7.25 + 13.25 = 25$.

The exact value is $\int_0^4 (x^2+1)\,dx = \frac{64}{3} + 4 \approx 25.33$, so L_4 is too low, R_4 is too high, and M_4 is close.

Over- and Underestimates

If f is **increasing** on $[a, b]$, then $L_n \le \int_a^b f \le R_n$ (left underestimates, right overestimates). If f is **decreasing**, the inequalities reverse: $R_n \le \int_a^b f \le L_n$.

If f is **concave up**, the midpoint sum underestimates the integral. If f is **concave down**, the midpoint sum overestimates it.

These statements need f to be monotonic (or concave) on the whole interval. If f goes up and then down, you cannot say which sum is larger without more work.

EXAMPLE 2

Water flows into a tank at a rate $r(t)$ liters per minute, measured at unequal times: $r(0) = 3$, $r(2) = 5$, $r(5) = 4$, $r(9) = 6$, $r(10) = 8$. Use left and right sums with the four subintervals $[0, 2], [2, 5], [5, 9], [9, 10]$ to estimate the total water entering in 10 minutes.

SOLUTION

With unequal subintervals each rectangle has its own width: $\Delta t = 2, 3, 4, 1$.

Left: $3(2) + 5(3) + 4(4) + 6(1) = 6 + 15 + 16 + 6 = 43$.

Right: $5(2) + 4(3) + 6(4) + 8(1) = 10 + 12 + 24 + 8 = 54$.

The total is estimated between 43 and 54 liters. (The rate is not monotonic, so we cannot claim which one is an over- or underestimate.)

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EXAMPLE 3

Write $\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+\frac{3i}{n}}\cdot\frac{3}{n}$ as a definite integral and evaluate it.

SOLUTION

Match to $\sum f(x_i)\Delta x$: the width is $\Delta x=\frac{3}{n}=\frac{b-a}{n}$, so $b-a=3$. The sample point is $x_i=1+\frac{3i}{n}=a+i\,\Delta x$, so $a=1$ and $b=4$.

Then $f(x)=\sqrt{x}$, and the limit is $\int_1^4\sqrt{x}\,dx$.

$\int_1^4\sqrt{x}\,dx=\frac{2}{3}x^{3/2}\Big|_1^4=\frac{2}{3}(8-1)=\frac{14}{3}$.

The Definite Integral as a Limit

If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\,\Delta x,\qquad \Delta x=\frac{b-a}{n},\ x_i=a+i\,\Delta x.$$

The same limit results from left sums or midpoint sums. Useful sums: $\sum_{i=1}^{n}i=\frac{n(n+1)}{2}$ and $\sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6}$.

EXAMPLE 4

Find a formula for R_n for $f(x)=x^2$ on $[0,3]$ and take the limit as $n\to\infty$.

SOLUTION

$\Delta x=\frac{3}{n}$ and $x_i=\frac{3i}{n}$. So $R_n=\sum_{i=1}^{n}\left(\frac{3i}{n}\right)^2\frac{3}{n}=\frac{27}{n^3}\sum_{i=1}^{n}i^2$.

Using the sum formula, $R_n=\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}=\frac{9(n+1)(2n+1)}{2n^2}=9+\frac{27}{2n}+\frac{9}{2n^2}$.

As $n\to\infty$ the last two terms go to 0, so $\int_0^3 x^2\,dx=9$. Note $R_n>9$ for every n, as expected for an increasing function.

Common mistake

With a table, the rectangle widths are the **differences of the x-values**, which may not be equal. Do not multiply every value by the same Δx unless the spacing really is constant. Also remember that a left sum drops the last table value and a right sum drops the first.

Practice

SET A Skills

Answer each question. [No calculator]

1. Let $f(x)=2x+1$. For $\int_0^6 f(x)\,dx$ find (a) L_3, (b) R_3, (c) M_3.
2. Find L_4 and R_4 for $\int_0^2 x^3\,dx$.
3. The table gives values of a continuous function h. Find (a) the left sum and (b) the right sum with four subintervals for $\int_0^8 h(x)\,dx$, and (c) the midpoint sum with two subintervals.

x	0	2	4	6	8
$h(x)$	5	8	12	9	3

4. Evaluate $\sum_{k=1}^{5}(2k-1)$.
5. Write the limit $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^3\frac{2}{n}$ as a definite integral and evaluate it.
6. Write $\lim_{n\to\infty}\sum_{i=1}^{n}e^{2+4i/n}\cdot\frac{4}{n}$ as a definite integral.
7. For $f(x)=\frac{1}{x}$ on $[1,5]$, explain whether L_n and R_n overestimate or underestimate $\int_1^5 f(x)\,dx$.

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8. Find M_2 for $\int_1^5 x^2\,dx$. Is it an overestimate or an underestimate? Justify.

9. Which expression equals $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{3i}{n}\right)^2\frac{3}{n}$?

(A) $\int_0^3 x^2\,dx$ (B) $\int_1^4 x^2\,dx$ (C) $\int_1^3 x^2\,dx$
(D) $\int_0^1 (1+3x)^2\,dx$ (E) $\int_1^4 3x^2\,dx$

SET B Applications and Mixed

Answer each question. [No calculator]

10. The rate $r(t)$ at which water enters a tank is measured at the times shown. Use a left sum and a right sum with the three subintervals to estimate the total water that entered from $t=0$ to $t=12$ minutes.

t (min)	0	3	7	12
$r(t)$ (L/min)	4	6	5	9

11. A car's velocity $v(t)$ in meters per second is recorded every 2 seconds, and v is increasing. (a) Find the left and right sums with four subintervals for the distance traveled in 8 seconds. (b) Which sum is an upper bound? Explain.

t (s)	0	2	4	6	8
$v(t)$	0	6	10	12	13

12. For $f(x)=x^2$ on $[0,2]$: (a) find a formula for R_n in terms of n; (b) find $\lim_{n\to\infty} R_n$.

13. Let f be continuous, increasing and concave down on $[a,b]$. Which statement about the sums with the same n is true? (Let $I=\int_a^b f(x)\,dx$.)

(A) $L_n < M_n < I < R_n$ (B) $L_n < I < M_n < R_n$ (C) $I < L_n < M_n < R_n$
(D) $L_n < I < R_n < M_n$ (E) $M_n < L_n < I < R_n$

14. Find M_3 for $\int_0^6 3x^2\,dx$, then find the error $\left|M_3-\int_0^6 3x^2\,dx\right|$.

15. Evaluate $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{2n}\cos\left(\frac{\pi i}{2n}\right)$ by recognizing it as an integral.

16. Find M_4 for $\int_1^9 \frac{1}{x}\,dx$ as an exact fraction.

SET C Challenge and AP-Style

Round decimal answers to three places. [Calculator]

17. For $f(x)=\sqrt{1+x^3}$ on $[0,2]$ find L_4, R_4 and M_4 for $\int_0^2 f(x)\,dx$. Round to three decimal places.

18. Use a midpoint sum with $n=3$ to approximate $\int_0^{1.5}\sin(x^2)\,dx$. Also find L_3 and R_3. Round to three decimal places.

19. A drug is infused at the rate $r(t)=6te^{-t/2}$ mg per minute for $0\le t\le 8$. Find the right sum R_4 and the midpoint sum M_4 for the total amount infused. Round to three decimal places. Then find the exact total with your calculator's integration and compare.

20. Values of a continuous function f are given at unequal spacings. Find the left sum and the right sum for $\int_0^8 f(t)\,dt$ using the four subintervals determined by the table. Give three decimal places.

t	0	1.5	4	5	8
$f(t)$	2.4	3.1	3.8	2.9	1.7

21. Write $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\sqrt{4+\frac{2i}{n}}$ as a definite integral and evaluate it to three decimal places.

4.6 Trapezoidal Rule and Simpson's Rule

You will learn to approximate integrals with trapezoids, including unequal subintervals, to compare estimates using concavity, and to apply Simpson's rule.

Rectangles are a crude way to approximate area. The **trapezoidal rule** replaces each rectangle with a trapezoid whose top edge joins two points on the graph, which usually follows the curve far better. **Simpson's rule** goes further and fits parabolas. Both work from a formula or from a table of values.

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The Trapezoidal Rule

For f on $[a,b]$ with subintervals of **equal** width $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\,\Delta x$:

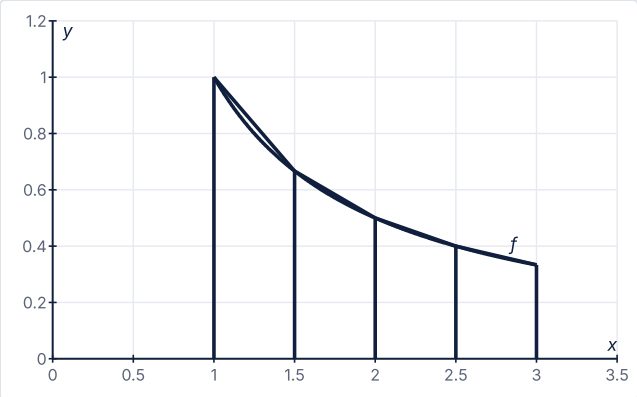
$$T_n = \frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big].$$

For **unequal** subintervals add the individual trapezoid areas: $\sum_{i=1}^{n} \frac{1}{2}\big[f(x_{i-1}) + f(x_i)\big](x_i - x_{i-1})$.

The trapezoid sum is the average of the left and right sums: $T_n = \frac{1}{2}(L_n + R_n)$. Interior values are doubled because each is used by two trapezoids.

EXAMPLE 1

Approximate $\int_1^3 \frac{1}{x}\,dx$ with the trapezoidal rule using $n = 4$. Is the result an overestimate or an underestimate?



Four trapezoids under $f(x) = 1/x$ on $[1,3]$: the chords lie above the curve.

SOLUTION

$\Delta x = \frac{3-1}{4} = 0.5$. The values are $f(1) = 1$, $f(1.5) = \frac{2}{3}$, $f(2) = \frac{1}{2}$, $f(2.5) = \frac{2}{5}$, $f(3) = \frac{1}{3}$.

$T_4 = \frac{0.5}{2}\left[1 + 2\cdot\frac{2}{3} + 2\cdot\frac{1}{2} + 2\cdot\frac{2}{5} + \frac{1}{3}\right] = 0.25\left[\frac{67}{15}\right] = \frac{67}{60} \approx 1.117$.

Because $f''(x) = \frac{2}{x^3} > 0$ on $[1,3]$, the graph is concave up, the chords lie above the curve, and T_4 is an **overestimate**. (The exact value is $\ln 3 \approx 1.099$.)

Trapezoid Error and Concavity

If $f'' > 0$ on $[a,b]$ (concave up), then T_n **overestimates** $\int_a^b f$. If $f'' < 0$ (concave down), then T_n **underestimates** it.

The midpoint sum errs in the opposite direction: concave up gives $M_n < \int_a^b f < T_n$, and concave down gives $T_n < \int_a^b f < M_n$.

For a justification, name the concavity (cite the sign of f'' or the shape of the graph) and then state the conclusion.

EXAMPLE 2

A river's depth d (meters) is measured at distances x meters from one bank: $d(0) = 0$, $d(2) = 3$, $d(5) = 4$, $d(9) = 0$. Estimate the cross-sectional area $\int_0^9 d(x)\,dx$ with trapezoids.

SOLUTION

The spacing is unequal: widths $2, 3, 4$.

$\frac{1}{2}(0+3)(2) = 3$, $\frac{1}{2}(3+4)(3) = 10.5$, $\frac{1}{2}(4+0)(4) = 8$.

Area $\approx 3 + 10.5 + 8 = 21.5\ \text{m}^2$.

Simpson's Rule (extension beyond the exam)

For an **even** number n of equal subintervals,

$$S_n = \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big].$$

The coefficients run $1, 4, 2, 4, 2, \ldots, 4, 1$. Each pair of subintervals is fitted with a parabola through three points. Simpson's rule is exact for every polynomial of degree 3 or less.

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EXAMPLE 3

Use Simpson's rule with $n = 4$ for $\int_1^3 \frac{1}{x}\,dx$ and compare with Example 1.

SOLUTION

$\Delta x = 0.5$, so $\frac{\Delta x}{3} = \frac{1}{6}$. Coefficients $1, 4, 2, 4, 1$.

$S_4 = \frac{1}{6}\left[1 + 4\cdot\frac{2}{3} + 2\cdot\frac{1}{2} + 4\cdot\frac{2}{5} + \frac{1}{3}\right] = \frac{1}{6}(6.6) = \frac{11}{10} = 1.1.$

The exact value is $\ln 3 \approx 1.0986$. The Simpson error (0.0014) is much smaller than the trapezoid error (0.0181).

EXAMPLE 4

Find T_2 and S_2 for $\int_0^2 x^3\,dx$ and compare with the exact value.

SOLUTION

$\Delta x = 1$ and the values are $0, 1, 8$.

$T_2 = \frac{1}{2}[0 + 2(1) + 8] = 5$. $S_2 = \frac{1}{3}[0 + 4(1) + 8] = 4$.

The exact value is $\frac{2^4}{4} = 4$. Simpson's rule is exact for cubics, while T_2 overestimates because x^3 is concave up on $[0, 2]$.

Common mistake

Do not double the first and last values in the trapezoidal rule, and do not forget to double all the interior ones. The factor in front is $\frac{\Delta x}{2}$ for trapezoids but $\frac{\Delta x}{3}$ for Simpson's rule. Simpson's rule requires an even number of subintervals. With unequal widths, use each trapezoid's own width.

Practice

SET A Skills

Answer each question. Give exact answers. [No calculator]

1. Find T_4 for $\int_0^2 x^2\,dx$.
2. Find T_2 for $\int_0^4 (16 - x^2)\,dx$. Is it an overestimate or an underestimate? Justify.
3. Use the trapezoidal rule with four subintervals to approximate $\int_0^4 f(x)\,dx$.

x	0	1	2	3	4
$f(x)$	2	5	6	4	1

4. Use trapezoids on the subintervals given by the table to approximate $\int_0^6 g(x)\,dx$.

x	0	1	4	6
$g(x)$	3	5	2	4

5. Find Simpson's rule S_4 for $\int_0^4 x^3\,dx$ and compare with the exact value.
6. Use Simpson's rule with $n = 4$ to approximate $\int_0^8 h(x)\,dx$.

x	0	2	4	6	8
$h(x)$	1	4	6	5	2

7. For a decreasing function f on $[a, b]$, $L_5 = 34$ and $R_5 = 26$. Find T_5.
8. Explain whether the trapezoidal rule overestimates or underestimates $\int_0^1 e^x\,dx$.
9. What is T_2 for $\int_0^2 x^3\,dx$?

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

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SET B Applications and Mixed

Answer each question. Exact values or simple decimals are fine. [No calculator]

10. A stream's depth d (meters) is measured at distances x (meters) from the bank. Estimate the cross-sectional area $\int_0^{12} d(x)\,dx$ using trapezoids.

x	0	3	5	9	12
$d(x)$	0	1.5	2.5	2	0

11. Fuel flows to an engine at a rate $f(t)$ liters per hour, measured each hour for 6 hours. (a) Estimate the total fuel used with the trapezoidal rule. (b) Estimate it again with Simpson's rule.

t	0	1	2	3	4	5	6
$f(t)$	8	9	11	10	8	4	2

12. A student uses $T_4 = 7.4$ to approximate $\int_0^4 g(x)\,dx$, where $g''(x) < 0$ for all x in $[0, 4]$. Is 7.4 greater than, less than, or equal to the integral? Justify.

13. For a continuous function f, T_4 for $\int_0^8 f(x)\,dx$ equals 38, where $f(0) = 3$, $f(2) = a$, $f(4) = 7$, $f(6) = 5$, $f(8) = 1$. Find a.

14. For $f(x) = x^2$ on $[0, 4]$ find T_2, M_2 and the exact integral. Then compute $\frac{2M_2 + T_2}{3}$ and compare it with Simpson's S_2.

15. The horizontal cross-section of a tank at height h meters has area $A(h)$ square meters. Estimate the volume $\int_0^4 A(h)\,dh$ using (a) the trapezoidal rule and (b) Simpson's rule.

h	0	1	2	3	4
$A(h)$	12	10	7	5	2

16. For a continuous function f on $[a, b]$, the trapezoidal rule T_n is guaranteed to overestimate $\int_a^b f(x)\,dx$ when f is

(A) increasing on $[a, b]$ **(B)** decreasing on $[a, b]$ **(C)** concave up on $[a, b]$
(D) concave down on $[a, b]$ **(E)** positive on $[a, b]$

SET C Challenge and AP-Style

Round decimal answers to three places. [Calculator]

17. Use $n = 4$ to find T_4 and S_4 for $\int_0^2 e^{-x^2}\,dx$, and compare both with the value from your calculator's numerical integration. Round to three decimal places.

18. Find T_5 for $\int_1^{2.5} \ln x\,dx$ to three decimal places, and state with a reason whether it over- or underestimates the integral.

19. Find T_4 for $\int_1^9 \sqrt{x}\,dx$ and the exact value. Round to three decimals and justify whether T_4 is an over- or underestimate.

20. The speed of a water plume $c(t)$ is recorded at irregular times. Use trapezoids to approximate $\int_0^6 c(t)\,dt$ to three decimal places.

t	0	1.2	2.5	4.1	6
$c(t)$	0	3.4	5.1	4.2	1.5

21. Let $g(x) = \dfrac{1}{1 + x^2}$. Find T_6 and S_6 for $\int_0^3 g(x)\,dx$ and compare them with the exact value $\arctan 3$. Round to three decimal places.

4.7 The Fundamental Theorem of Calculus

You will learn how to differentiate a function defined by an integral, including the chain rule, and how to evaluate definite integrals with antiderivatives.

Differentiation and integration look unrelated at first: one is a slope, the other an area. The **Fundamental Theorem of Calculus** shows that they undo each other. It gives you two powerful tools: a way to differentiate a function defined as an integral, and a way to evaluate a definite integral without any Riemann sums.

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The Fundamental Theorem of Calculus, Part 1

Let f be continuous on an interval I and let a be in I. Define $F(x) = \int_a^x f(t)\,dt$. Then F is differentiable on I and

$$F'(x) = f(x).$$

In words: the derivative of an accumulation function is the integrand evaluated at the upper limit. The lower limit does not matter.

Part 1 with the Chain Rule

If f is continuous and $u(x)$, $v(x)$ are differentiable, then

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))\,u'(x)$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = f(u(x))\,u'(x) - f(v(x))\,v'(x).$$

A variable **lower** limit gives a minus sign: $\frac{d}{dx}\int_x^b f(t)\,dt = -f(x)$.

EXAMPLE 1

Find (a) $\frac{d}{dx}\int_1^x \sqrt{t^3+1}\,dt$ and (b) $\frac{d}{dx}\int_x^4 \sqrt{t^3+1}\,dt$ for $x > 0$.

SOLUTION

The integrand $\sqrt{t^3+1}$ is continuous for $t \geq 0$, so Part 1 applies.

(a) Replace t by the upper limit: $\sqrt{x^3+1}$.

(b) Reverse the limits first: $\int_x^4 = -\int_4^x$, so the derivative is $-\sqrt{x^3+1}$.

EXAMPLE 2

Find $\frac{d}{dx}\int_1^{x^2} \frac{dt}{t^2+3}$ and $\frac{d}{dx}\int_x^{2x} \cos(t^2)\,dt$.

SOLUTION

First: $f(t) = \frac{1}{t^2+3}$ is continuous everywhere. Upper limit $u = x^2$, $u' = 2x$.

$$\frac{d}{dx}\int_1^{x^2} \frac{dt}{t^2+3} = \frac{1}{(x^2)^2+3}\cdot 2x = \frac{2x}{x^4+3}.$$

Second: both limits vary. Upper limit $2x$ (derivative 2), lower limit x (derivative 1).

$$\frac{d}{dx}\int_x^{2x} \cos(t^2)\,dt = \cos\left((2x)^2\right)\cdot 2 - \cos(x^2)\cdot 1 = 2\cos(4x^2) - \cos(x^2).$$

The Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a,b]$ and F is **any** antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Notation: $F(x)\Big|_a^b = F(b) - F(a)$. The constant of integration cancels, so you may leave it out.

Equivalent form (net change): $\int_a^b F'(x)\,dx = F(b) - F(a)$. The integral of a rate of change is the net change in the quantity.

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EXAMPLE 3

Evaluate $\displaystyle\int_1^4 \left(3\sqrt{x} - \frac{2}{x^2}\right) dx$.

SOLUTION

Rewrite: $3x^{1/2} - 2x^{-2}$. An antiderivative is $F(x) = 2x^{3/2} + \frac{2}{x}$.

$F(4) = 2(8) + \frac{2}{4} = \frac{33}{2}$ and $F(1) = 2 + 2 = 4$.

$\int_1^4 = F(4) - F(1) = \frac{33}{2} - 4 = \frac{25}{2}$.

EXAMPLE 4

Evaluate $\displaystyle\int_0^{\pi/3} (4\sin x + \sec^2 x)\, dx$, and explain why the theorem cannot be used for $\displaystyle\int_{-1}^{3} \frac{1}{x^2}\, dx$.

SOLUTION

An antiderivative is $-4\cos x + \tan x$. At $\frac{\pi}{3}$: $-4 \cdot \frac{1}{2} + \sqrt{3} = \sqrt{3} - 2$. At 0: $-4 + 0 = -4$.

$\int_0^{\pi/3} = (\sqrt{3} - 2) - (-4) = 2 + \sqrt{3}$.

For $\frac{1}{x^2}$, the function is not defined at $x = 0$, which lies in $[-1, 3]$, so f is **not continuous** on the interval and Part 2 does not apply. Computing $-\frac{1}{x}\Big|_{-1}^{3} = -\frac{4}{3}$ would be wrong: the integrand is positive, so no valid value can be negative. (The integral actually diverges.)

EXAMPLE 5

A tank's volume changes at the rate $V'(t) = 3\sqrt{t}$ liters per minute. If $V(0) = 10$, find $V(9)$.

SOLUTION

By net change, $V(9) - V(0) = \int_0^9 3\sqrt{t}\, dt = 2t^{3/2}\Big|_0^9 = 2(27) = 54$.

$V(9) = 10 + 54 = 64$ liters.

Common mistakes

(1) Part 1 says $\frac{d}{dx}\int_a^{u(x)} f(t)\, dt = f(u(x))\, u'(x)$. Do not forget the factor $u'(x)$. (2) Do not integrate first and then substitute the wrong limit order: always compute $F(\text{upper}) - F(\text{lower})$. (3) Check continuity on the whole interval before using Part 2. (4) The answer to a derivative of an accumulation function is a **function of** x, with no t left.

Practice

SET A ### Skills

Answer each question. [No calculator]

1. Find $\displaystyle\frac{d}{dx}\int_0^x \cos(t^2)\, dt$.
2. Find $\displaystyle\frac{d}{dx}\int_x^5 \sqrt{t^2+1}\, dt$.
3. Find $\displaystyle\frac{d}{dx}\int_1^{x^3} \ln t\, dt$ for $x > 0$.
4. Find $\displaystyle\frac{d}{dx}\int_{2x}^7 e^{t^2}\, dt$.
5. Find $\displaystyle\frac{d}{dx}\int_x^{x^2} \sin t\, dt$.
6. Evaluate $\displaystyle\int_0^2 (3x^2 - 4x + 1)\, dx$.
7. Evaluate $\displaystyle\int_1^8 x^{1/3}\, dx$.
8. Evaluate $\displaystyle\int_1^{e^2} \frac{1}{x}\, dx + \int_0^{\pi/4} \sec^2 x\, dx$.
9. Evaluate $\displaystyle\int_0^1 \left(e^x + \frac{1}{1+x^2}\right) dx$.

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SET B Applications and Mixed

Answer each question. [No calculator]

10. Let $g(x) = \int_0^x (t^2 - 4t + 3)\,dt$. Find all critical points of g on $(-\infty, \infty)$ and classify each as a local maximum or minimum, with justification.
11. A particle moves on a line with velocity $v(t) = t^2 - 6t + 8$ meters per second for $0 \le t \le 5$. Find (a) the displacement and (b) the total distance traveled.
12. A sapling's height $h(t)$ in centimeters changes at the rate $h'(t) = \dfrac{6}{2t+1}$ centimeters per week, and $h(0) = 40$. Find $h(4)$.
13. Let $F(x) = \int_2^x \sqrt{t^2 + 5}\,dt$. Write an equation of the tangent line to the graph of F at $x = 2$, then use it to approximate $F(2.1)$.
14. A student evaluates $\int_0^{\pi} \sec^2 x\,dx = \tan x\Big|_0^{\pi} = 0 - 0 = 0$. Explain why this cannot be correct.
15. Find $f(x)$ if $\int_0^x f(t)\,dt = x^2 e^x$ for all x.
16. What is $\dfrac{d}{dx}\int_0^{x^2} \dfrac{1}{1+t^2}\,dt$?

(A) $\dfrac{1}{1+x^2}$ (B) $\dfrac{1}{1+x^4}$ (C) $\dfrac{2x}{1+x^2}$ (D) $\dfrac{2x}{1+x^4}$ (E) $2x\arctan(x^2)$

SET C Challenge and AP-Style

Round decimal answers to three places. [Calculator]

17. Let $F(x) = \int_0^x e^{-t^2}\,dt$. Find $F'(1.2)$ and, using your calculator's numerical integration, $F(1.2)$. Round to three decimal places.
18. Let $H(x) = \int_1^{x^2} \sqrt{t+3}\,dt$. Find $H'(1.5)$ to three decimal places.
19. Let $G(x) = \int_0^x \cos(t^2)\,dt$. Find the tangent line to G at $x = 1$, use it to approximate $G(1.2)$, and compare with the value from numerical integration. Round to three decimals.
20. Find the value of $k > 0$ for which $\int_0^k 4e^{-t/2}\,dt = 5$. Round to three decimal places.
21. A particle's velocity is $v(t) = t\cos t$ for $0 \le t \le 5$. Find (a) its displacement and (b) the total distance traveled. Round to three decimal places.

4.8 Accumulation Functions and the FTC Graphically

You will learn to read values, extrema, and concavity of an accumulation function from the graph of its integrand and to write AP-style justifications.

An **accumulation function** $g(x) = \int_a^x f(t)\,dt$ keeps a running total of the signed area under f. The Fundamental Theorem tells us how g behaves in terms of the graph of f, so from a picture of f alone you can find values of g and describe where g rises, falls, bends and peaks. This is a favorite kind of free-response question.

Reading an Accumulation Function from the Graph of f

Let f be continuous and $g(x) = c + \int_a^x f(t)\,dt$. Then $g(a) = c$, $g'(x) = f(x)$, and $g''(x) = f'(x)$ wherever f is differentiable.

Values: $g(b) - g(a) = \int_a^b f(t)\,dt$ is the signed area between a and b. To get $g(x)$ at a new point, add the signed area to the last known value.

Heights of f are **slopes** of g. Areas under f are **changes in the values** of g.

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Dictionary: What the Graph of f Says About g

g is **increasing** where $f > 0$ and **decreasing** where $f < 0$.

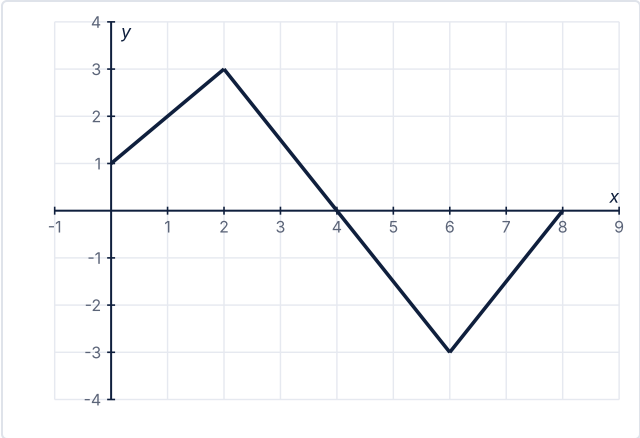
g has a **local maximum** where f changes from positive to negative, and a **local minimum** where f changes from negative to positive. A zero of f with no sign change gives no extremum.

g is **concave up** where f is increasing ($f' > 0$) and **concave down** where f is decreasing ($f' < 0$). g has an **inflection point** where f has a local maximum or minimum (where f' changes sign).

For absolute extrema on $[a, b]$, compare the values of g at the critical points and at the endpoints.

EXAMPLE 1

The graph of f is shown. Let $g(x) = \int_0^x f(t)\,dt$. Find $g(2), g(4), g(6), g(8)$; where g has a local extremum; and the intervals where g is concave down. Find the absolute maximum and minimum of g on $[0, 8]$.



Graph of f on $[0, 8]$, made of line segments.

SOLUTION

$g(0) = 0$. $[0, 2]$ is a trapezoid with parallel sides 1 and 3 and width 2: area 4, so $g(2) = 4$.

$[2, 4]$: triangle above, area $\frac{1}{2}(2)(3) = 3$, so $g(4) = 7$. $[4, 6]$: triangle below, area 3, so $g(6) = 4$. $[6, 8]$: triangle below, area 3, so $g(8) = 1$.

Because $g' = f$ changes from positive to negative at $x = 4$, g has a local maximum there, $g(4) = 7$.

f is decreasing on $(2, 6)$, so $g'' = f' < 0$ and g is concave down on $(2, 6)$. It is concave up on $(0, 2)$ and $(6, 8)$, so $x = 2$ and $x = 6$ are inflection points.

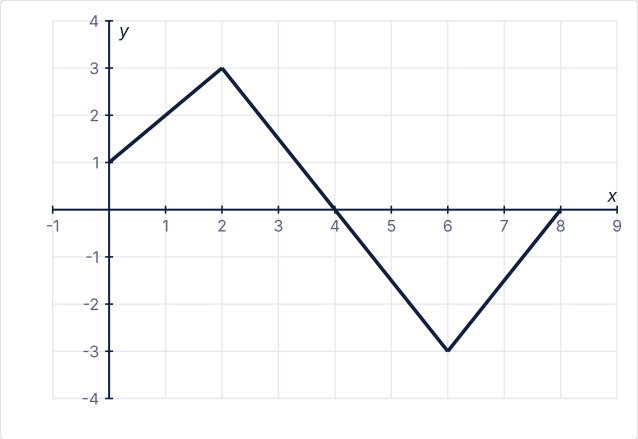
Candidates: $g(0) = 0$, $g(4) = 7$, $g(8) = 1$. Absolute maximum 7 at $x = 4$; absolute minimum 0 at $x = 0$.

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EXAMPLE 2

With the same graph of f, let $h(x) = 5 + \int_2^x f(t)\,dt$. Find $h(2)$, $h(4)$, $h(8)$, $h'(6)$ and $h''(5)$.



Graph of f on $[0, 8]$, made of line segments.

SOLUTION

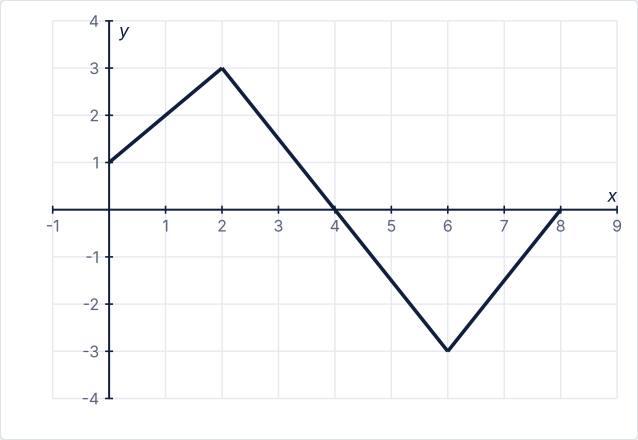
$h(2) = 5 + \int_2^2 f = 5$. $h(4) = 5 + 3 = 8$. $h(8) = 5 + \int_2^8 f = 5 + (3 - 3 - 3) = 2$.

$h'(6) = f(6) = -3$.

$h''(x) = f'(x)$. On $(4, 6)$ the graph of f is a line from $(4, 0)$ to $(6, -3)$ with slope $-\frac{3}{2}$, so $h''(5) = -\frac{3}{2}$.

EXAMPLE 3

With the same f, let $k(x) = \displaystyle\int_0^{2x} f(t)\,dt$ for $0 \le x \le 4$. Find $k'(1)$ and $k'(3)$.



Graph of f on $[0, 8]$, made of line segments.

SOLUTION

By the chain rule form of Part 1, $k'(x) = f(2x) \cdot 2$.

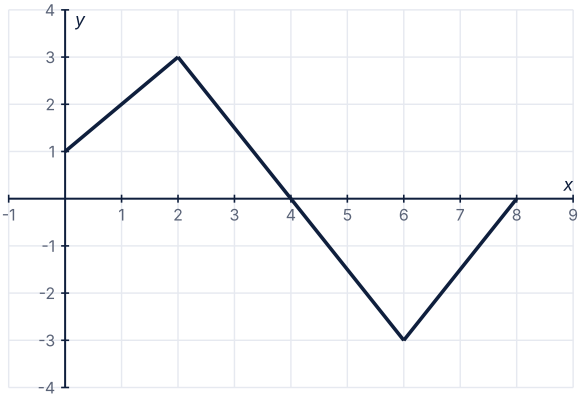
$k'(1) = 2f(2) = 2(3) = 6$ and $k'(3) = 2f(6) = 2(-3) = -6$.

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EXAMPLE 4

Write an AP-style justification that g in Example 1 has a local maximum at $x = 4$, and explain why g has no local extremum at $x = 2$, even though f has its highest point there.



Graph of f on $[0, 8]$, made of line segments.

SOLUTION

Local maximum: "Since $g'(x) = f(x)$, and $f(x) > 0$ for $0 < x < 4$ and $f(x) < 0$ for $4 < x < 8$, g' changes sign from positive to negative at $x = 4$. Therefore g has a local maximum at $x = 4$."

At $x = 2$: $f(x) > 0$ on both sides of 2, so $g' = f$ does not change sign and g keeps increasing through $x = 2$. What changes at $x = 2$ is that f switches from increasing to decreasing, so $g'' = f'$ changes sign and g has an inflection point there.

Common mistakes

(1) Do not look for extrema of g at the highest or lowest points of the graph of f; those are where g has **inflection points**. Extrema of g occur where f crosses the x-axis. (2) Concavity of g comes from the slope of f, not the sign of f. (3) A value like $g(6)$ must be built from the starting value of the accumulation, so check the lower limit and any constant added. (4) To justify, always name the function whose sign you use: "$g' = f$" or "$g'' = f'$".

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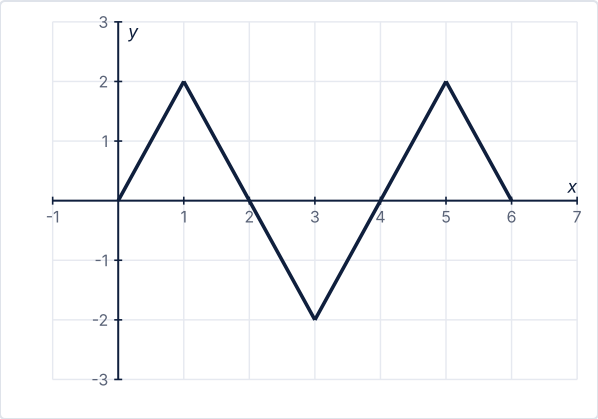
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Practice

SET A Skills

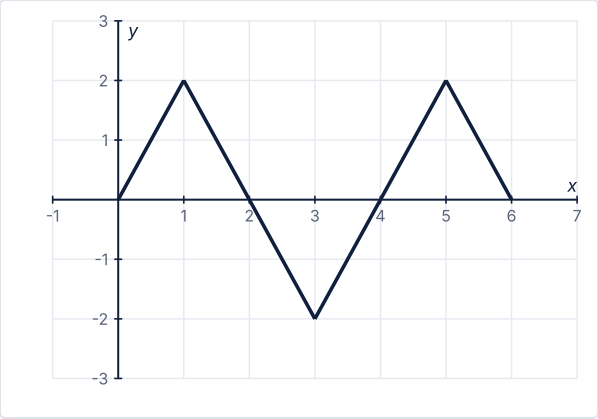
Answer each question. [No calculator]

1. The graph of f is shown, and $g(x) = \int_0^x f(t)\,dt$. Find $g(1)$, $g(2)$, $g(3)$, $g(4)$, $g(5)$ and $g(6)$.



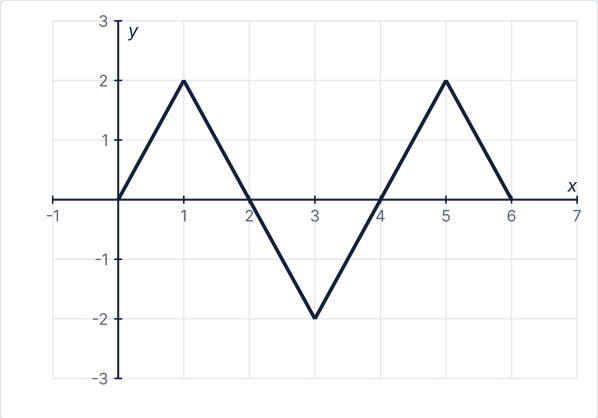
Graph of f on $[0, 6]$, made of line segments.

2. For g as in the graph of f shown (with $g(x) = \int_0^x f(t)\,dt$), find the x-values in $(0, 6)$ where g has a local maximum or minimum, and justify.



Graph of f on $[0, 6]$, made of line segments.

3. For $g(x) = \int_0^x f(t)\,dt$ with f shown, state the open intervals on which g is increasing and on which it is decreasing.

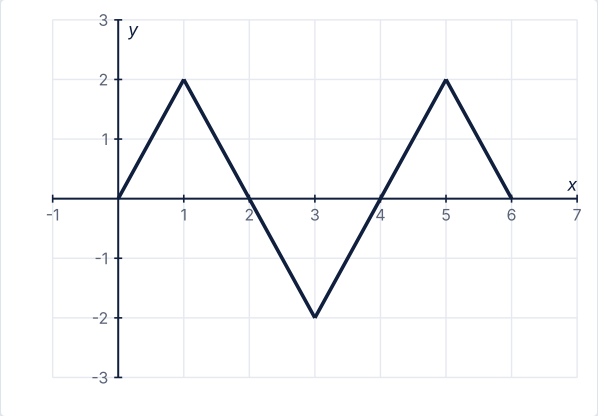


Graph of f on $[0, 6]$, made of line segments.

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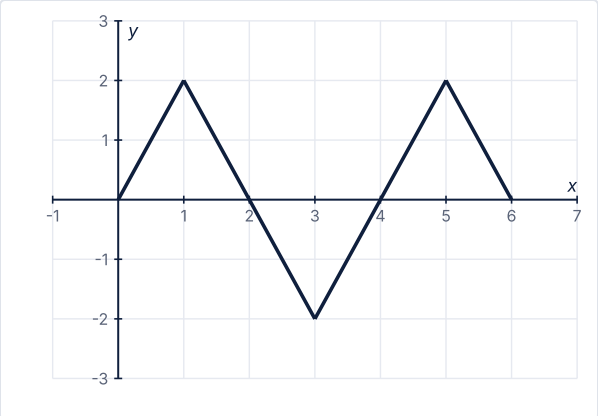
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4. For $g(x) = \int_0^x f(t)\,dt$ with f shown, find the open intervals on which g is concave up and concave down, and the x-coordinates of the points of inflection of g.



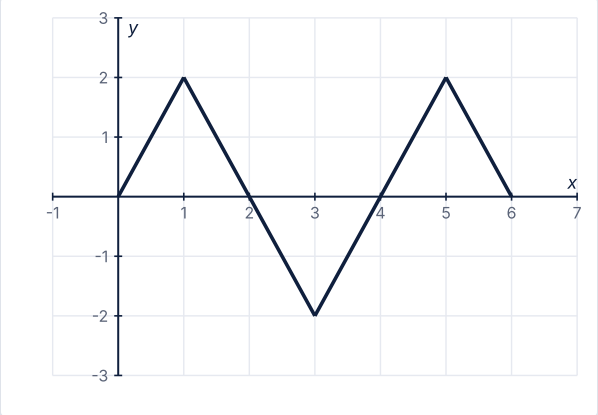
Graph of f on $[0, 6]$, made of line segments.

5. For $g(x) = \int_0^x f(t)\,dt$ with f shown, find the absolute maximum and absolute minimum values of g on $[0, 6]$.



Graph of f on $[0, 6]$, made of line segments.

6. For $g(x) = \int_0^x f(t)\,dt$ with f shown, find $g'(4.5)$ and $g''(4.5)$.



Graph of f on $[0, 6]$, made of line segments.

7. Let f be continuous with $\int_0^3 f(x)\,dx = 5$ and $\int_3^7 f(x)\,dx = -2$. If $g(x) = 2 + \int_0^x f(t)\,dt$, find $g(3)$ and $g(7)$.

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8. The table gives values of a continuous function f. Let $g(x) = 7 + \int_3^x f(t)\,dt$. Find $g(3)$ and decide whether g is increasing or decreasing at $x = 2$, $x = 3$ and $x = 5$. Justify.

x	1	2	3	4	5
$f(x)$	3	1	-2	-1	4

9. The function f is continuous with $f(x) > 0$ for $x < 4$ and $f(x) < 0$ for $x > 4$. Let $g(x) = \int_0^x f(t)\,dt$. Which statement is true?

(A) g has a local minimum at $x = 4$

(B) g has a local maximum at $x = 4$

(C) g has an inflection point at $x = 4$

(D) $g(4) = 0$

(E) $g'(4) > 0$

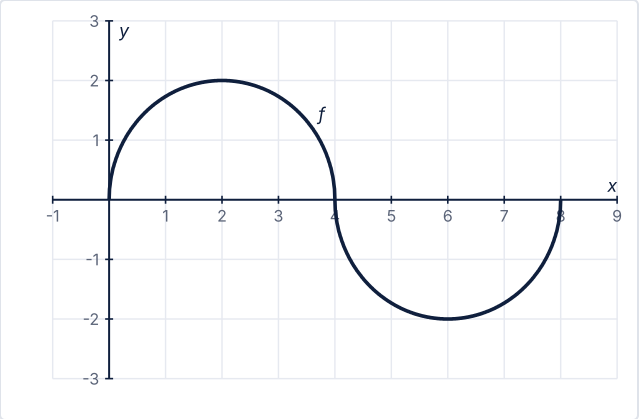
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SET B Applications and Mixed

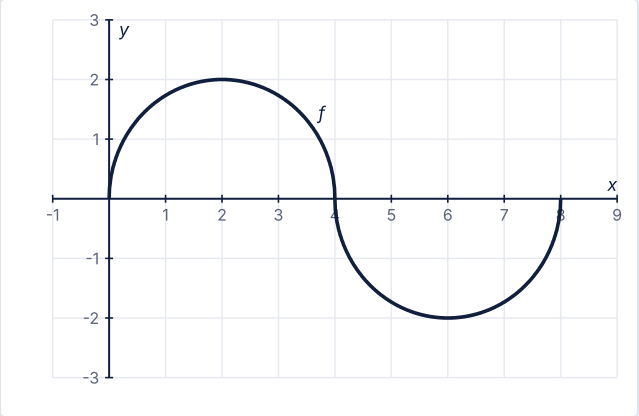
Answer each question. [No calculator]

10. Let $g(x) = \int_0^x f(t)\,dt$, where the graph of f is two semicircles of radius 2. Find $g(0)$, $g(2)$, $g(4)$, $g(6)$ and $g(8)$ exactly.



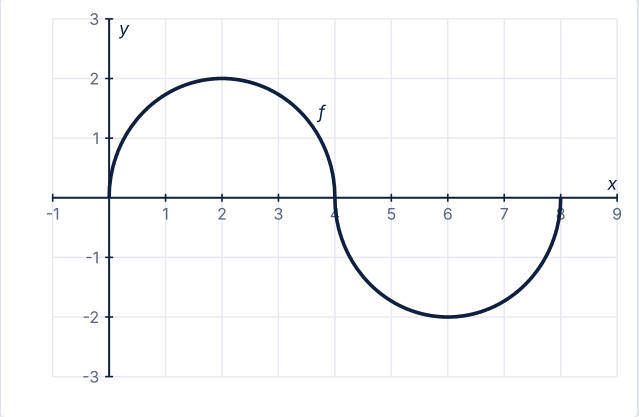
Graph of f on $[0, 8]$: two semicircles of radius 2.

11. For g as above (f two semicircles), find the local extrema of g on $(0, 8)$ and the absolute maximum and minimum values of g on $[0, 8]$. Justify.



Graph of f on $[0, 8]$: two semicircles of radius 2.

12. For g as above (f two semicircles), find the open intervals where g is concave down and the x-coordinates of the inflection points of g. Justify.

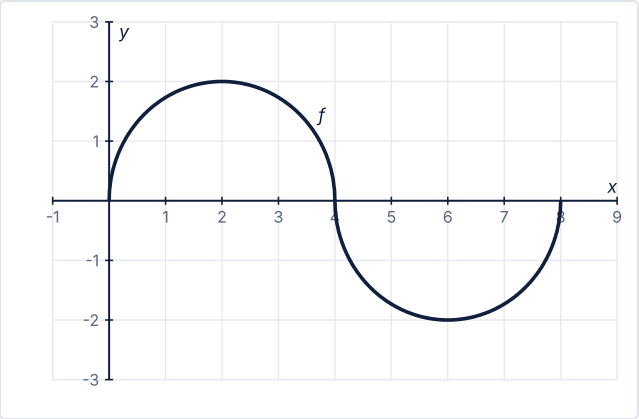


Graph of f on $[0, 8]$: two semicircles of radius 2.

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13. Let $k(x) = \int_0^{2x} f(t)\,dt$ for $0 \le x \le 4$, where f is as shown. Find $k(2)$, $k'(1)$ and $k'(3)$ exactly.



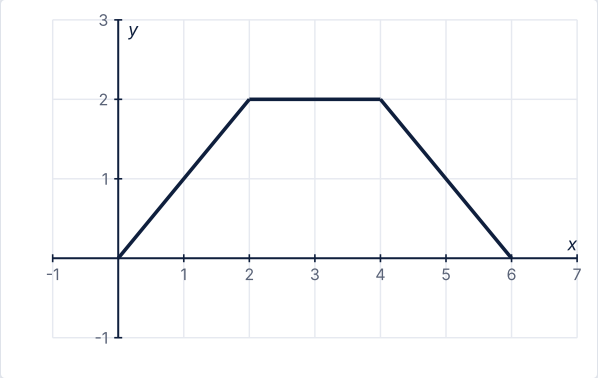
Graph of f on $[0, 8]$: two semicircles of radius 2.

14. A continuous function f is linear between consecutive table points. Let $g(x) = \int_0^x f(t)\,dt$. Find $g(6)$, and find the maximum value of g on $[0, 6]$ and where it occurs.

x	0	1	2	3	4	5	6
$f(x)$	0	2	3	1	-1	-2	0

15. Let $g(x) = \displaystyle\int_1^x \left(t^2 - 3t + 2\right) dt$. Find the critical points of g and classify them, find $g(2)$, and find the x-coordinate of the inflection point of g.

16. The graph of f is shown, and $g(x) = \int_0^x f(t)\,dt$. What is $g(6)$?



Graph of f on $[0, 6]$.

(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

SET C Challenge and AP-Style

Round decimal answers to three places. [Calculator]

17. Let $g(x) = \displaystyle\int_0^x t\cos t\,dt$ on $[0, 6]$. Find the x-values of the local extrema of g, and the absolute maximum and minimum values of g on $[0, 6]$. Round to three decimal places.

18. For g in the previous problem, $g''(x) = \cos x - x\sin x$. Find the x-coordinates (to three decimals) of all inflection points of g in $(0, 6)$ and justify.

19. Sand enters a pile at the rate $F(t) = 3 + 2\sin\left(\dfrac{t^2}{10}\right)$ cubic feet per hour for $0 \le t \le 4$. The pile holds 5 cubic feet at $t = 0$. (a) Find the amount at $t = 4$. (b) Is the amount increasing at $t = 4$? Is the rate of increase itself increasing or decreasing? Justify. Round to three decimals.

20. Find the positive value of x for which $\displaystyle\int_0^x te^{-t}\,dt = 0.5$. Round to three decimal places.

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21. Let $A(x) = \int_0^{x^2} \sqrt{1+t^2}\,dt$. Find $A'(1.3)$ and $A''(1.3)$ to three decimal places.

4.9 Definite Integrals with Substitution and Properties

You will learn to change limits in u-substitution, use additivity, symmetry and reversed limits, and integrate piecewise and absolute value functions.

Definite integrals follow the same rules as the antiderivatives you already know, with one extra job: the limits of integration. This section covers substitution in a definite integral, the algebra of integral properties (adding, splitting, reversing), symmetry, and integrals of piecewise and absolute value functions.

Substitution in a Definite Integral

Let g' be continuous on $[a,b]$ and f continuous on the range of g. Then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du, \qquad u = g(x).$$

When you substitute, **change the limits**: they become the values of u at the old limits. Then finish in u and do not go back to x. The upper u-limit may be smaller than the lower one; that is fine.

1. **Choose u.** Pick an inside function whose derivative (up to a constant) is also in the integrand.
2. **Compute du and rewrite.** Replace $g'(x)\,dx$ with du, adjusting by a constant factor if needed.
3. **Change the limits.** Evaluate $u = g(x)$ at $x = a$ and $x = b$.
4. **Integrate and evaluate.** Use the new limits in the antiderivative in u.

EXAMPLE 1

Evaluate $\int_0^2 x\,(x^2+1)^3\,dx$.

SOLUTION

Let $u = x^2+1$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}du$.

Limits: $x=0 \Rightarrow u=1$ and $x=2 \Rightarrow u=5$.

$\frac{1}{2}\int_1^5 u^3\,du = \frac{1}{2}\cdot\frac{u^4}{4}\Big|_1^5 = \frac{1}{8}(625-1) = 78.$

EXAMPLE 2

Evaluate $\int_0^{\pi/2} \frac{\sin x}{(1+\cos x)^2}\,dx$ and $\int_0^{\ln 3} \frac{e^x}{1+e^x}\,dx$.

SOLUTION

First: $u = 1+\cos x$, $du = -\sin x\,dx$. Limits: $x=0 \Rightarrow u=2$, $x=\frac{\pi}{2} \Rightarrow u=1$.

$\int_2^1 -u^{-2}\,du = \int_1^2 u^{-2}\,du = -\frac{1}{u}\Big|_1^2 = -\frac{1}{2}+1 = \frac{1}{2}$. (Reversing the limits absorbed the minus sign.)

Second: $u = 1+e^x$, $du = e^x\,dx$. Limits: $x=0 \Rightarrow u=2$, $x=\ln 3 \Rightarrow u=4$.

$\int_2^4 \frac{du}{u} = \ln 4 - \ln 2 = \ln 2.$

Properties of Definite Integrals

Assume f and g are continuous on an interval containing a, b, c.

Additivity: $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$ (for any c, even outside $[a,b]$ as long as f is continuous there). Reversing limits: $\int_b^a f\,dx = -\int_a^b f\,dx$. Same limits: $\int_a^a f\,dx = 0$.

Linearity: $\int_a^b [kf+mg]\,dx = k\int_a^b f\,dx + m\int_a^b g\,dx$. Symmetry: if f is even, $\int_{-a}^a f = 2\int_0^a f$; if f is odd, $\int_{-a}^a f = 0$.

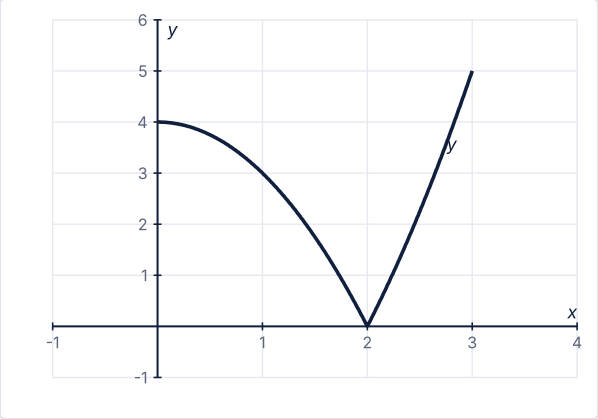
Absolute value and piecewise functions: split the interval at the places where the formula changes (for $|h(x)|$, at the zeros of h where h changes sign), then integrate each piece.

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EXAMPLE 3

Evaluate $\int_0^3 |x^2 - 4|\,dx$.



Graph of $y = |x^2 - 4|$ on $[0, 3]$.

SOLUTION

$x^2 - 4$ changes sign at $x = 2$: it is negative on $[0, 2)$ and positive on $(2, 3]$. So $|x^2 - 4| = 4 - x^2$ on $[0, 2]$ and $x^2 - 4$ on $[2, 3]$.

$\int_0^2 (4 - x^2)\,dx = 8 - \frac{8}{3} = \frac{16}{3}$ and $\int_2^3 (x^2 - 4)\,dx = (9 - 12) - \left(\frac{8}{3} - 8\right) = \frac{7}{3}$.

Total: $\frac{16}{3} + \frac{7}{3} = \frac{23}{3}$.

EXAMPLE 4

Evaluate $\int_{-2}^{2} \left(x^3 e^{x^2} + 3x^2\right) dx$ using symmetry.

SOLUTION

$x^3 e^{x^2}$ is odd (odd times even), and it is continuous, so its integral over $[-2, 2]$ is 0.

$3x^2$ is even: $\int_{-2}^{2} 3x^2\,dx = 2\int_0^2 3x^2\,dx = 2\left[x^3\right]_0^2 = 16$.

The total is $0 + 16 = 16$. No substitution was needed.

EXAMPLE 5

Evaluate $\int_0^3 x\sqrt{x+1}\,dx$.

SOLUTION

Let $u = x + 1$, so $x = u - 1$ and $dx = du$. Limits: $u = 1$ to $u = 4$.

$\int_1^4 (u-1)u^{1/2}\,du = \int_1^4 \left(u^{3/2} - u^{1/2}\right) du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^4$.

At 4: $\frac{2}{5}(32) - \frac{2}{3}(8) = \frac{64}{5} - \frac{16}{3} = \frac{112}{15}$. At 1: $\frac{2}{5} - \frac{2}{3} = -\frac{4}{15}$.

Result: $\frac{112}{15} + \frac{4}{15} = \frac{116}{15}$.

Common mistakes

(1) Changing to u but keeping the x-limits. If you convert limits, the final answer comes from the u-antiderivative with the u-**limits**. (2) Forgetting the constant factor, such as the $\frac{1}{2}$ from $du = 2x\,dx$. (3) Integrating $|h(x)|$ as if it were $h(x)$: you must split at sign changes. (4) Using symmetry without checking that the interval is symmetric about 0 and that the function is truly even or odd.

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Practice

SET A Skills

Evaluate or answer. Give exact answers. [No calculator]

1. Evaluate $\int_0^1 2x\,(x^2+3)^4\,dx$.
2. Evaluate $\int_1^2 \dfrac{3x^2}{x^3+1}\,dx$.
3. Evaluate $\int_0^{\pi/2} \cos x\,\sin^3 x\,dx$.
4. Evaluate $\int_0^1 xe^{x^2}\,dx$.
5. Evaluate $\int_1^5 \dfrac{dx}{\sqrt{2x-1}}$.
6. Evaluate $\int_0^3 \dfrac{x}{\sqrt{x^2+16}}\,dx$.
7. The function f is continuous with $\int_2^5 f(x)\,dx = 6$ and $\int_2^8 f(x)\,dx = 1$. Find $\int_5^8 f(x)\,dx$ and $\int_8^5 f(x)\,dx$.
8. Evaluate $\int_{-2}^{2} \left(x^3 + x^2 + x\right)\,dx$ using symmetry.
9. Evaluate $\int_0^4 |x-3|\,dx$.

SET B Applications and Mixed

Answer each question. Give exact answers. [No calculator]

10. A particle has velocity $v(t) = \dfrac{6t}{1+t^2}$ meters per second. Find its displacement from $t=0$ to $t=3$.
11. A particle has velocity $v(t) = t^2 - 3t$ meters per second for $0 \le t \le 4$. Find (a) the displacement and (b) the total distance traveled.
12. Let $f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ 3 - 2x, & 1 < x \le 2. \end{cases}$ Verify that f is continuous at $x=1$ and find $\int_0^2 f(x)\,dx$.
13. The function f is even and g is odd, both continuous on $[-2,2]$, with $\int_0^2 f(x)\,dx = 5$ and $\int_0^2 g(x)\,dx = 3$. Find $\int_{-2}^{2} (3f(x) - 2g(x) + 1)\,dx$.
14. Evaluate $\int_0^{\pi/4} \tan x\,dx$.
15. Suppose f is continuous and $\int_0^4 f(u)\,du = 10$. Find (a) $\int_0^2 xf(x^2)\,dx$ and (b) $\int_0^1 f(4x)\,dx$.
16. Find the positive number a for which $\int_0^a \dfrac{x}{x^2+1}\,dx = \ln 5$.

SET C Challenge and AP-Style

Round decimal answers to three places. [Calculator]

17. Evaluate $\int_0^2 x\sqrt{x^2+5}\,dx$. Give the exact value and a decimal to three places.
18. Evaluate $\int_0^{\pi} \sin x\,e^{\cos x}\,dx$. Give the exact value and a decimal to three places.
19. Evaluate $\int_0^3 \left|x^3 - 3x\right|\,dx$.
20. Evaluate $\int_{-1}^{1} \left(x^5\cos x + \sqrt{1-x^2}\right)\,dx$ using symmetry and geometry. Give a decimal to three places.
21. Evaluate $\int_0^1 x^3\sqrt{1+x^2}\,dx$. Give a decimal to three places.

4.10 Net Change, Tables, and Data

Learn to turn a rate of change into a total change using integrals, and to estimate that total from a table of data. This is the heart of many AP free-response questions.

Many quantities in science are described by their **rate of change**: liters per minute, meters per second, dollars per unit. The Fundamental Theorem of Calculus tells you that adding up a rate over an interval gives the total change in the quantity. This section turns that idea into a method you can use with formulas, graphs, and tables of data.

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The Net Change Theorem

If F' is continuous on $[a, b]$, then

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

In words: the integral of a rate of change gives the **net change** in the quantity. Equivalently, $F(b) = F(a) + \int_a^b F'(x)\,dx$, so final amount = initial amount + net change.

The units of the answer come from multiplying the units of the rate by the units of the variable of integration. If $R(t)$ is in liters per minute and t is in minutes, then $\int_a^b R(t)\,dt$ is in (liters per minute)(minutes) = liters.

EXAMPLE 1 (A FORMULA FOR THE RATE)

A culture contains 500 cells at time $t = 4$ hours. Cells are added at the rate $P'(t) = 6\sqrt{t}$ cells per hour. How many cells are in the culture at $t = 9$?

SOLUTION

By the Net Change Theorem, $P(9) = P(4) + \int_4^9 6\sqrt{t}\,dt$.

$\int_4^9 6t^{1/2}\,dt = \left[4t^{3/2}\right]_4^9 = 4(27) - 4(8) = 76$.

$P(9) = 500 + 76 = 576$ cells.

When a rate is known only from a **table**, you cannot integrate exactly. You approximate the integral with a Riemann sum or the trapezoidal sum, using the subintervals the table gives you. The subintervals do not have to be equal.

Estimating from a Table

For data points $t_0 < t_1 < \cdots < t_n$ with rate values $R(t_i)$:

Left sum: $\sum_{i=0}^{n-1} R(t_i)\,(t_{i+1} - t_i)$. Right sum: $\sum_{i=1}^{n} R(t_i)\,(t_i - t_{i-1})$.

Trapezoidal sum: $\sum_{i=0}^{n-1} \dfrac{R(t_i) + R(t_{i+1})}{2}\,(t_{i+1} - t_i)$.

If R is increasing, the left sum underestimates and the right sum overestimates; if R is decreasing, the reverse holds.

t (minutes)	0	2	5	9	12
$R(t)$ (L/min)	4	7	9	8	5

EXAMPLE 2 (TRAPEZOIDS ON A TABLE)

Water flows into a tank at a rate $R(t)$ liters per minute. Selected values are shown. Use a trapezoidal sum with the four subintervals to approximate $\int_0^{12} R(t)\,dt$.

SOLUTION

Subinterval widths are 2, 3, 4, 3.

$\frac{4+7}{2}(2) + \frac{7+9}{2}(3) + \frac{9+8}{2}(4) + \frac{8+5}{2}(3) = 11 + 24 + 34 + 19.5$.

$\int_0^{12} R(t)\,dt \approx 88.5$ liters.

EXAMPLE 3 (INTERPRETATION SENTENCE)

In Example 2, the tank held 30 liters at $t = 0$ and no water left the tank. Interpret $\int_0^{12} R(t)\,dt$ and estimate the amount in the tank at $t = 12$.

SOLUTION

Interpretation: $\int_0^{12} R(t)\,dt$ is the total amount of water, in liters, that flowed into the tank during the 12 minutes from $t = 0$ to $t = 12$.

Using the estimate 88.5 liters, the amount at $t = 12$ is about $30 + 88.5 = 118.5$ liters.

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Net change versus total distance

The integral $\int_a^b v(t)\,dt$ gives **displacement** (net change in position), because motion in the negative direction subtracts. The **total distance** traveled is $\int_a^b |v(t)|\,dt$. To compute it, find where $v(t) = 0$, split the interval there, and add the absolute values of the pieces.

EXAMPLE 4 (DISPLACEMENT AND DISTANCE)

A particle moves along a line with velocity $v(t) = t^2 - 4t + 3$ meters per second for $0 \le t \le 4$. Find the displacement and the total distance traveled.

SOLUTION

Displacement: $\int_0^4 (t^2 - 4t + 3)\,dt = \left[\frac{t^3}{3} - 2t^2 + 3t\right]_0^4 = \frac{64}{3} - 32 + 12 = \frac{4}{3}$ meters.

Since $v(t) = (t-1)(t-3)$, $v > 0$ on $[0, 1)$ and $(3, 4]$, and $v < 0$ on $(1, 3)$.

$\int_0^1 v\,dt = \frac{4}{3}$, $\int_1^3 v\,dt = -\frac{4}{3}$, $\int_3^4 v\,dt = \frac{4}{3}$.

Total distance $= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$ meters.

Practice

SET A Skills

Solve each problem. [No calculator]

1. The function f satisfies $f'(x) = 3x^2 - 4x$ and $f(1) = 5$. Find $f(3)$.
2. A particle moves with velocity $v(t) = 4 - 2t$ meters per second. Find its displacement from $t = 0$ to $t = 5$.
3. For the particle in the previous problem, find the total distance traveled from $t = 0$ to $t = 5$.
4. Liquid leaks from a container at the rate $r(t) = \dfrac{4}{t+1}$ liters per hour. Find the total amount that leaks out from $t = 0$ to $t = 3$. Give an exact answer.
5. The table gives values of $g'(t)$, the rate of change of a quantity g. If $g(0) = 10$, use a trapezoidal sum with the three subintervals to estimate $g(6)$.

t	0	1	3	6
$g'(t)$	2	4	3	1

6. Use the table from the previous problem. Find the left Riemann sum and the right Riemann sum, each with three subintervals, for $\int_0^6 g'(t)\,dt$.

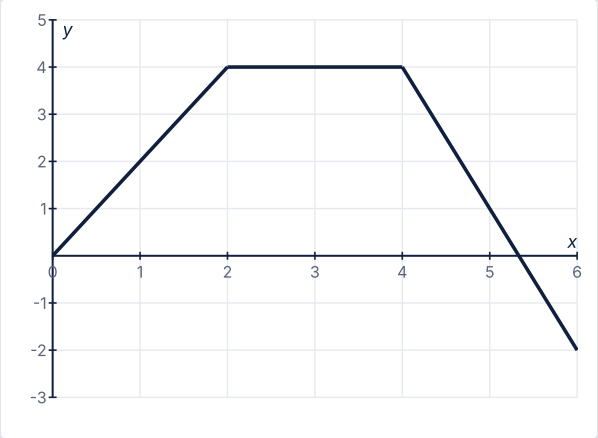
t	0	1	3	6
$g'(t)$	2	4	3	1

7. People enter a museum at the rate $R(t)$ people per minute, where t is in minutes after opening. Given that $\int_0^{30} R(t)\,dt = 420$, write a sentence interpreting this value, including units.
8. The function $h'(t)$ gives the rate of change of a plant's height in centimeters per day, where t is in days. State the units of $\int_2^9 h'(t)\,dt$ and of $\frac{1}{7}\int_2^9 h'(t)\,dt$, and say what each represents.

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9. The graph of f' consists of line segments, as shown. Given $f(0) = 3$, find $f(6)$.



Graph of f'.

SET B Applications

Solve each problem. Give decimal answers to three decimal places. [Calculator]

10. Water flows into a tank at rate $R(t)$ liters per minute. The tank holds 40 liters at $t = 0$. Use a trapezoidal sum with four subintervals to estimate the volume at $t = 15$, and write a sentence explaining what your sum represents.

t (min)	0	3	6	10	15
$R(t)$	5	8	10	7	4

11. A particle moves along a line with velocity $v(t) = 5\sin t - 2$ meters per second for $0 \le t \le 4$. Find the displacement of the particle.

12. For the particle in the previous problem, find the total distance traveled from $t = 0$ to $t = 4$.

13. Oil leaks from a damaged tank at the rate $r(t) = 50e^{-0.2t}$ liters per hour. The tank held 1000 liters at $t = 0$. How many liters remain at $t = 8$ hours?

14. Cars enter a parking garage at the rate $E(t)$ cars per hour, where t is hours after 6:00 a.m. Use the left sum and the right sum with four subintervals to estimate $\int_0^8 E(t)\,dt$, then give the average of the two and interpret it.

t	0	2	4	6	8
$E(t)$	12	30	44	38	20

15. A tank contains 100 gallons at $t = 0$ minutes. Water enters at the rate $10 + 3\sin(0.4t)$ gallons per minute and leaves at the rate $7 + 0.5t$ gallons per minute. (a) How many gallons are in the tank at $t = 6$? (b) At what time t in $0 < t < 10$ is the amount of water greatest? Justify your answer.

16. The mass of a bacterial colony changes at the rate $M'(t) = 0.6\,t\,e^{-0.3t}$ grams per hour. If $M(0) = 2$ grams, find the mass at $t = 10$ hours.

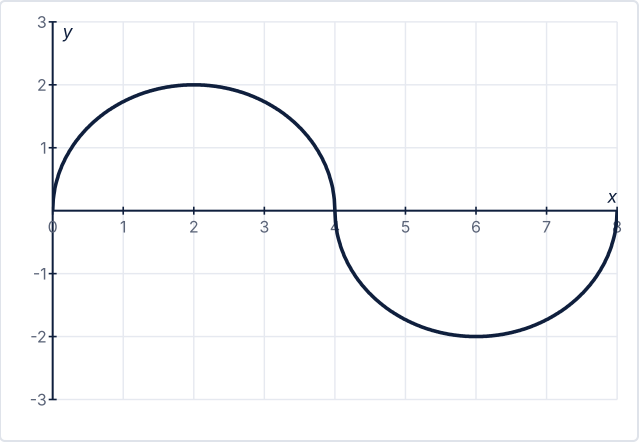
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SET C **Challenge**

Solve each problem. Justify your answers. [No calculator]

17. The graph shows the velocity $v(t)$ in meters per second, made of two semicircles, for $0 \le t \le 8$. The particle is at position $s(0) = -3$ meters. Find (a) the position at $t = 8$, and (b) the total distance traveled.



Graph of v.

18. The amount of water in a tank changes at the rate $A'(t) = 12 - 2t$ liters per minute, with $A(0) = 20$ liters. (a) At what time is the amount greatest, and what is the maximum amount? (b) At what time does the tank become empty? Give exact answers.
19. The function g is differentiable with $g(1) = 4$ and $-3 \le g'(x) \le 2$ for all x in $[1, 5]$. Find the smallest and largest possible values of $g(5)$, and justify your answer using the Net Change Theorem.
20. Let f be differentiable with $f'(x) = x^3 \cos x$. Without finding f, determine $f(2) - f(-2)$ and justify.
21. A particle has velocity $v(t) = t^2 - 2t$ for $0 \le t \le 4$. Find the displacement and the total distance traveled. Give exact values.

4.11 Average Value and the Mean Value Theorem for Integrals

Learn to find the average value of a function on an interval and to use the Mean Value Theorem for Integrals. Averages of rates appear often in AP free-response questions.

The average of a list of numbers is their sum divided by how many there are. A continuous function takes infinitely many values on an interval, and the integral plays the role of the sum. Dividing by the length of the interval gives the **average value** of the function.

Average Value of a Function

If f is continuous on $[a, b]$, the average value of f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Equivalently, $\int_a^b f(x)\,dx = f_{\text{avg}}\,(b-a)$: the area under the curve equals the area of a rectangle with height f_{avg} and the same width.

EXAMPLE 1 (A FORMULA)

Find the average value of $f(x) = 3x^2 - 2x$ on $[1, 3]$.

SOLUTION

$\int_1^3 (3x^2 - 2x)\,dx = \left[x^3 - x^2\right]_1^3 = (27 - 9) - (1 - 1) = 18.$

$f_{\text{avg}} = \frac{1}{3-1} \cdot 18 = 9.$

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Mean Value Theorem for Integrals

If f is **continuous** on $[a, b]$, then there is at least one number c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

In words: a continuous function must actually reach its average value somewhere. The theorem guarantees at least one such c; there may be more than one. Continuity is required.

EXAMPLE 2 (FINDING C)

Let $f(x) = x^2$ on $[0, 3]$. Find the average value of f and the value of c guaranteed by the Mean Value Theorem for Integrals.

SOLUTION

$\int_0^3 x^2\,dx = \frac{27}{3} = 9$, so $f_{\text{avg}} = \frac{9}{3} = 3$.

Solve $f(c) = 3$: $c^2 = 3$, so $c = \sqrt{3}$ (the value $-\sqrt{3}$ is not in $[0, 3]$).

Justification: f is continuous on $[0, 3]$, so by the theorem some c in $[0, 3]$ has $f(c) = 3$; indeed $c = \sqrt{3} \approx 1.732$.

t (hours)	0	2	3	7	10
$C(t)$ (mg/L)	0	12	15	6	2

EXAMPLE 3 (A TABLE)

A drug's concentration $C(t)$ in milligrams per liter is measured at the times shown. Use a trapezoidal sum with four subintervals to estimate the average concentration over $0 \le t \le 10$ hours.

SOLUTION

Integral estimate: $\frac{0+12}{2}(2) + \frac{12+15}{2}(1) + \frac{15+6}{2}(4) + \frac{6+2}{2}(3) = 12 + 13.5 + 42 + 12 = 79.5$.

Average $\approx \frac{79.5}{10} = 7.95$ milligrams per liter.

The unit of an average value is the unit of the function itself.

Average value is not the same as the average of the endpoint values

The average value of f uses all the values of f on the interval, not just $f(a)$ and $f(b)$. Also be careful about two different averages: the **average value of** f' on $[a, b]$ equals the **average rate of change** of f, namely $\frac{f(b)-f(a)}{b-a}$, by the Net Change Theorem.

EXAMPLE 4 (AVERAGE VELOCITY AND AVERAGE SPEED)

A particle has velocity $v(t) = t^2 - 4t + 3$ meters per second for $0 \le t \le 4$. Find its average velocity and its average speed.

SOLUTION

$\int_0^4 v\,dt = \frac{4}{3}$, so the average velocity is $\frac{1}{4} \cdot \frac{4}{3} = \frac{1}{3}$ meter per second.

Speed is $|v|$. Since $v < 0$ only on $(1, 3)$, $\int_0^4 |v|\,dt = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$.

Average speed $= \frac{1}{4} \cdot 4 = 1$ meter per second.

EXAMPLE 5 (MORE THAN ONE VALUE OF C)

Let $f(x) = \sin x$ on $[0, \pi]$. Find f_{avg} and every c in $[0, \pi]$ with $f(c) = f_{\text{avg}}$.

SOLUTION

$\int_0^\pi \sin x\,dx = 2$, so $f_{\text{avg}} = \frac{2}{\pi}$.

Solve $\sin c = \frac{2}{\pi} \approx 0.6366$ on $[0, \pi]$: $c \approx 0.690$ or $c = \pi - 0.690 \approx 2.451$.

The theorem promises at least one c; here there are two.

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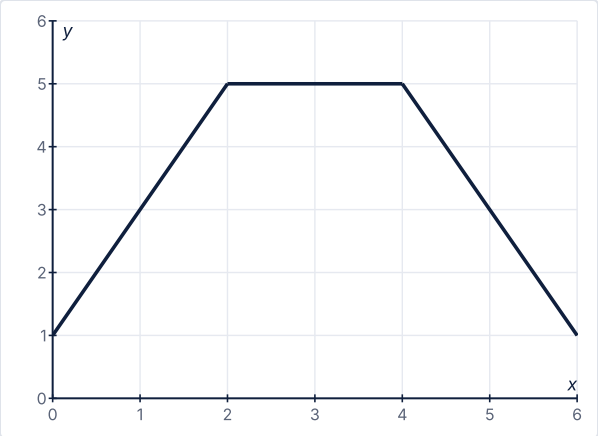
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Practice

SET A Skills

Solve each problem. [No calculator]

1. Find the average value of $f(x) = x^2 + 4x$ on $[0, 3]$.
2. Find the average value of $f(x) = \sqrt{x}$ on $[1, 9]$.
3. Find the average value of $f(x) = \dfrac{1}{x}$ on $[1, e^2]$.
4. Find the average value of $f(x) = \cos x$ on $\left[0, \frac{\pi}{2}\right]$.
5. Find the average value of $f(x) = e^{2x}$ on $[0, \ln 3]$.
6. Find the value of c in $(1, 3)$ guaranteed by the Mean Value Theorem for Integrals for $f(x) = 3x^2$ on $[1, 3]$.
7. The graph of f is made of line segments on $[0, 6]$. Find the average value of f on $[0, 6]$.



Graph of f.

8. The table gives values of a continuous function f. Use a trapezoidal sum with three subintervals to estimate the average value of f on $[0, 10]$.

x	0	2	6	10
$f(x)$	3	5	2	4

9. Let $f(x) = x^3 - 3x$. Find the average value of $f'(x)$ on $[0, 2]$, and explain how it is related to $f(2)$ and $f(0)$.

SET B Applications

Solve each problem. Give decimal answers to three decimal places. [Calculator]

10. The temperature in a greenhouse is $T(t) = 20 + 6\sin\left(\frac{\pi t}{12}\right)$ degrees Celsius, where t is hours after midnight. Find the average temperature from $t = 0$ to $t = 12$.
11. In a chemical reaction the rate of change of a product's concentration is $r(t) = 8e^{-0.3t}$ moles per liter per minute. Find the average rate over the first 5 minutes.
12. Gas flows through a pipe at a rate $g(t)$ cubic meters per hour, measured at the times shown. Use a trapezoidal sum with four subintervals to estimate the average flow rate over $0 \le t \le 4$, and interpret the result in a sentence.

t (hours)	0	1	2	3	4
$g(t)$	1.2	1.6	2.1	1.9	1.4

13. Let $f(x) = xe^{-x}$ on $[0, 2]$. Find the average value of f and all values of c in $[0, 2]$ for which $f(c)$ equals it.
14. A particle has velocity $v(t) = 3\cos t - 1$ meters per second for $0 \le t \le 4$. Find its average velocity and its average speed on $[0, 4]$.
15. People enter a stadium gate at the rate $R(t) = 40 + 25\sin(0.5t)$ people per minute for $0 \le t \le 10$. Find the average rate of entry over these 10 minutes.

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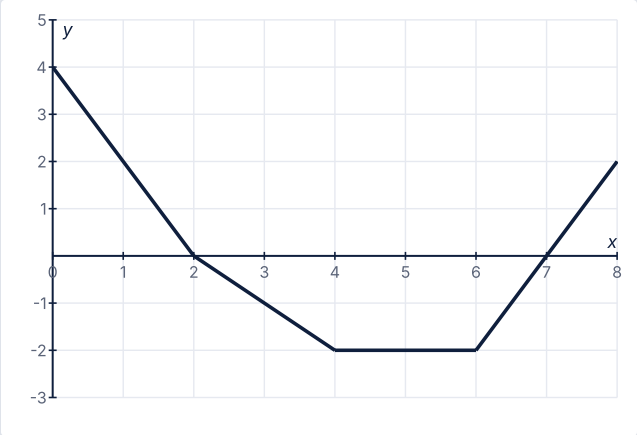
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16. Let $f(x) = \sqrt{25 - x^2}$ on $[-5, 5]$. Use the area of a semicircle to find the average value of f, then find the values of c in $(-5, 5)$ with $f(c)$ equal to it.

SET C Challenge

Solve each problem. Justify your answers. [No calculator]

17. The function f is continuous on $[1, 5]$ and $\int_1^5 f(x)\,dx = 12$. Show that $f(c) = 3$ for some c in $[1, 5]$, naming the theorem and checking its hypothesis.
18. Find the constant k so that the average value of $f(x) = kx^2$ on $[0, 3]$ equals 12.
19. Find the value of $b > 1$ such that the average value of $f(x) = x^2$ on $[1, b]$ is 7.
20. The graph of f consists of line segments on $[0, 8]$.
    (a) Find the average value of f on $[0, 8]$.
    (b) Explain why there must be a number c in $[0, 8]$ with $f(c)$ equal to that average.



Graph of f.

21. The function f has a continuous derivative with $f(2) = 7$ and $f(8) = 25$.
    (a) Find the average value of f' on $[2, 8]$.
    (b) Explain why $f'(c) = 3$ for some c in $[2, 8]$, and name the theorem you use.

4.12 Integrals Involving Logarithms, Exponentials, and Inverse Trig

Learn the integrals that produce natural logarithms, exponentials, arcsine, and arctangent, and how to rewrite an integrand until it fits one of these forms.

Three families of antiderivatives complete your toolkit for this chapter: the **natural logarithm**, the **exponential functions**, and the **inverse trigonometric functions**. In most problems you must recognize the pattern, often after a substitution or after rewriting the integrand.

Logarithm and Exponential Integrals

$$\int \frac{1}{u}\,du = \ln|u| + C \qquad \int e^u\,du = e^u + C \qquad \int a^u\,du = \frac{a^u}{\ln a} + C \;\; (a > 0,\ a \neq 1).$$

For a composite inside, let $u = g(x)$ and use $du = g'(x)\,dx$. In particular, $\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C$. The absolute value matters: $\ln|u|$ is defined for negative u as well.

EXAMPLE 1 (LOGARITHM FROM A SUBSTITUTION)

Evaluate $\int_0^2 \frac{x}{x^2 + 1}\,dx$.

SOLUTION

Let $u = x^2 + 1$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}du$. When $x = 0$, $u = 1$; when $x = 2$, $u = 5$.

$$\frac{1}{2}\int_1^5 \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_1^5 = \frac{1}{2}\ln 5.$$

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EXAMPLE 2 (EXPONENTIALS)

Find $\int xe^{-x^2}\,dx$ and $\int_0^2 2^x\,dx$.

SOLUTION

First: let $u=-x^2$, $du=-2x\,dx$, so $x\,dx=-\frac{1}{2}du$. Then $\int xe^{-x^2}dx=-\frac{1}{2}\int e^u\,du=-\frac{1}{2}e^{-x^2}+C$.

Second: $\int_0^2 2^x\,dx=\left.\frac{2^x}{\ln 2}\right|_0^2=\frac{4-1}{\ln 2}=\frac{3}{\ln 2}$.

Inverse Trigonometric Integrals

For a constant $a>0$:

$$\int\frac{du}{\sqrt{a^2-u^2}}=\arcsin\frac{u}{a}+C \qquad \int\frac{du}{a^2+u^2}=\frac{1}{a}\arctan\frac{u}{a}+C.$$

Reduce a given integral to one of these forms by factoring out a^2 or by a substitution $u=kx$. Do not forget the factor $\frac{1}{a}$ in the arctangent form. The arcsine form requires $|u|<a$ on the interval.

EXAMPLE 3 (ARCSINE AND ARCTANGENT)

Evaluate $\int_0^1\frac{dx}{\sqrt{4-x^2}}$ and find $\int\frac{3}{1+4x^2}\,dx$.

SOLUTION

First: $a=2$, $u=x$, so the integral is $\left.\arcsin\frac{x}{2}\right|_0^1=\arcsin\frac{1}{2}-0=\frac{\pi}{6}$.

Second: let $u=2x$, $du=2\,dx$. Then $\int\frac{3}{1+u^2}\cdot\frac{du}{2}=\frac{3}{2}\arctan u+C=\frac{3}{2}\arctan(2x)+C$.

Often the integrand is not in a recognizable form. **Rewrite first**: divide when the degree of the numerator is at least the degree of the denominator, split a fraction into separate terms, or complete the square in a quadratic.

EXAMPLE 4 (DIVIDE FIRST)

Evaluate $\int_0^2\frac{x^2}{x+1}\,dx$.

SOLUTION

Divide: $\frac{x^2}{x+1}=x-1+\frac{1}{x+1}$.

$$\int_0^2\left(x-1+\frac{1}{x+1}\right)dx=\left[\frac{x^2}{2}-x+\ln|x+1|\right]_0^2=(2-2+\ln 3)-0=\ln 3.$$

EXAMPLE 5 (SPLIT AND COMPLETE THE SQUARE)

Find $\int\frac{4x+1}{x^2+4}\,dx$ and $\int\frac{dx}{x^2+2x+5}$.

SOLUTION

Split the first: $\int\frac{4x}{x^2+4}\,dx+\int\frac{1}{x^2+4}\,dx=2\ln(x^2+4)+\frac{1}{2}\arctan\frac{x}{2}+C$ (the first term uses $u=x^2+4$; the second has $a=2$).

Complete the square in the second: $x^2+2x+5=(x+1)^2+4$. With $u=x+1$, $a=2$: $\frac{1}{2}\arctan\frac{x+1}{2}+C$.

Common mistakes

(1) $\int\frac{1}{x^2+4}\,dx$ is **not** $\ln(x^2+4)$; the numerator is not the derivative of the denominator, and the answer is an arctangent. (2) Do not write $\int\frac{1}{u^2}\,du$ as a logarithm; it is $-\frac{1}{u}+C$. (3) Forget the $\frac{1}{a}$ in the arctan formula and your answer is off by a constant factor. (4) $\int e^{x^2}\,dx$ cannot be found with these tools; e^{x^2} needs the extra factor $2x$. (5) Check the limits when you substitute in a definite integral.

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Practice

SET A Skills

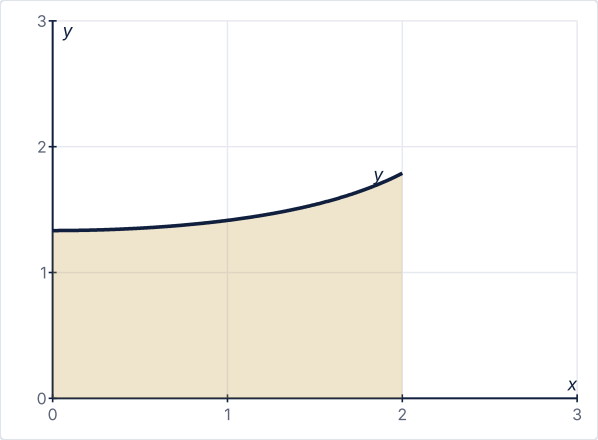
Evaluate or find each integral. [No calculator]

1. Find $\int \frac{5}{x}\,dx$.
2. Find $\int \frac{4x^3}{x^4+1}\,dx$.
3. Find $\int e^{5x}\,dx$.
4. Find $\int x^2 e^{x^3}\,dx$.
5. Evaluate $\int_1^e \frac{\ln x}{x}\,dx$.
6. Evaluate $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$.
7. Evaluate $\int_0^2 \frac{dx}{4+x^2}$.
8. Evaluate $\int_0^{\pi/4} \tan x\,dx$.
9. Evaluate $\int_1^e \frac{x^2+2}{x}\,dx$.

SET B Applications

Solve each problem. Give decimal answers to three decimal places unless told otherwise. [Calculator]

10. A radioactive sample decays at the rate $r(t) = 40e^{-0.05t}$ grams per day. How many grams decay during the first 30 days?
11. The marginal revenue from selling the qth unit is $R'(q) = \frac{200}{q+5}$ dollars per unit. Find the revenue gained when sales increase from $q = 0$ to $q = 45$ units.
12. Find the area of the shaded region under $y = \frac{4}{\sqrt{9-x^2}}$ for $0 \le x \le 2$.



Region under y = 4/sqrt(9 - x^2) for 0 <= x <= 2.

13. Find the average value of $f(x) = \frac{\ln x}{x}$ on $[1, 5]$.
14. A population grows at the rate $P'(t) = 300e^{0.2t}$ individuals per day. If $P(0) = 1000$, find $P(10)$ to the nearest whole number.
15. Evaluate $\int_0^4 \frac{3x+1}{x+1}\,dx$.
16. Evaluate $\int_0^3 \frac{dx}{x^2+6x+13}$.

SET C Challenge

Solve each problem. Show your reasoning. [No calculator]

17. Evaluate $\int_e^{e^2} \frac{dx}{x\ln x}$.
18. Evaluate $\int_0^{\ln 2} \frac{dx}{1+e^x}$. (Hint: multiply the numerator and denominator by e^{-x}.)
19. Evaluate $\int_0^1 \frac{3x+2}{x^2+1}\,dx$.
20. Evaluate $\int_1^2 \frac{dx}{\sqrt{3+2x-x^2}}$.

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21. The function f satisfies $f'(x) = \dfrac{4}{1+x^2}$ and $f(1) = 2\pi$. Find $f(\sqrt{3})$.

4.13 Multiple-Choice Practice on Integrals

Thirty multiple-choice problems on Chapter 4: antiderivatives, Riemann sums, the Fundamental Theorem, substitution, net change, and average value.

Work Part A without a calculator and Part B with a calculator. Give each problem a fair try before you read its answer, and check your reasoning, not just the letter.

SET A Part A [No calculator]

Choose the best answer. [No calculator]

1. Which of the following is $\int (6x^2 - 4x + 1)\,dx$?

(A) $12x - 4 + C$ (B) $2x^3 - 2x^2 + C$ (C) $2x^3 - 2x^2 + x + C$
(D) $6x^3 - 4x^2 + x + C$ (E) $2x^3 - 4x^2 + x + C$

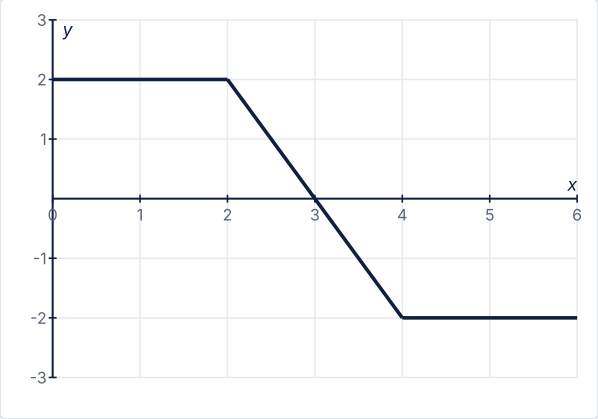
2. $\displaystyle\int_1^3 (2x+1)\,dx =$

(A) 10 (B) 12 (C) 8 (D) 7 (E) 14

3. $\dfrac{d}{dx}\displaystyle\int_0^{x^2} (t^3+1)\,dt =$

(A) $x^6 + 1$ (B) $\dfrac{x^8}{4} + x^2$ (C) $2x^6 + 2$ (D) $x^7 + x$ (E) $2x^7 + 2x$

4. The graph of f is shown, and $g(x) = \displaystyle\int_0^x f(t)\,dt$ for $0 \le x \le 6$. At which value of x does g attain its maximum on $[0,6]$?



Graph of f.

(A) $x = 0$ (B) $x = 3$ (C) $x = 2$ (D) $x = 4$ (E) $x = 6$

5. $\displaystyle\int x\sqrt{x^2+1}\,dx =$

(A) $\frac{2}{3}(x^2+1)^{3/2} + C$ (B) $\frac{1}{3}x^2(x^2+1)^{3/2} + C$ (C) $\dfrac{1}{2\sqrt{x^2+1}} + C$
(D) $\frac{1}{3}(x^2+1)^{3/2} + C$ (E) $(x^2+1)^{3/2} + C$

6. $\displaystyle\int_0^{\pi/2} \sin^2 x \cos x\,dx =$

(A) 1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) $\frac{\pi}{6}$

7. The left Riemann sum for $\displaystyle\int_0^4 x^2\,dx$ with four subintervals of equal width is

(A) 30 (B) 22 (C) $\frac{64}{3}$ (D) 14 (E) 16

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8. The table gives values of a function f. The right Riemann sum with three subintervals for $\int_0^7 f(x)\,dx$ is

x	0	1	3	7
$f(x)$	6	5	2	1

(A) 13 **(B)** 14 **(C)** 18.5 **(D)** 24 **(E)** 8

9. If $\int_1^5 f(x)\,dx = 7$ and $\int_1^3 f(x)\,dx = 2$, then $\int_3^5 (2f(x)+1)\,dx =$

(A) 11 **(B)** 10 **(C)** 12 **(D)** 15 **(E)** 16

10. The average value of $f(x) = 3x^2$ on $[0, 2]$ is

(A) 8 **(B)** 2 **(C)** 6 **(D)** 12 **(E)** 4

11. $\int_{-2}^{2} (x^3 + x^2)\,dx =$

(A) 0 **(B)** $\frac{8}{3}$ **(C)** 16 **(D)** $\frac{16}{3}$ **(E)** $\frac{32}{3}$

12. $\int_0^1 xe^{x^2}\,dx =$

(A) $e - 1$ **(B)** $\frac{e-1}{2}$ **(C)** $\frac{e+1}{2}$ **(D)** $\frac{e}{2}$ **(E)** $2(e-1)$

13. $\int_0^3 \frac{dx}{9+x^2} =$

(A) $\frac{\pi}{4}$ **(B)** $\frac{\pi}{6}$ **(C)** $\frac{\pi}{12}$ **(D)** $\frac{1}{2}\ln 2$ **(E)** $\frac{1}{3}$

14. $\int_1^e \left(\frac{3}{x} + 2x\right) dx =$

(A) $e^2 + 2$ **(B)** $e^2 + 3$ **(C)** $e^2 - 1$ **(D)** e^2 **(E)** $3e + e^2 - 1$

15. A particle moves on a line with velocity $v(t) = 3t^2 - 6t$ for $0 \le t \le 3$. The total distance it travels is

(A) 0 **(B)** 4 **(C)** 12 **(D)** 16 **(E)** 8

16. Let $G(x) = \int_2^x (t^2 - 3t)\,dt$. On which interval is G decreasing?

(A) $(0, 3)$ **(B)** $(-\infty, 0)$ **(C)** $(3, \infty)$ **(D)** $(-\infty, 3)$ **(E)** $(2, \infty)$

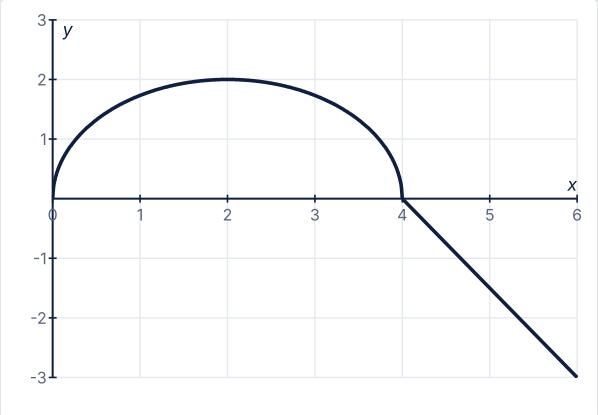
17. If $f'(x) = 6x^2 - 2$ and $f(1) = 5$, then $f(2) =$

(A) 11 **(B)** 16 **(C)** 12 **(D)** 21 **(E)** 17

18. $\int_0^1 \frac{2x+3}{x^2+3x+2}\,dx =$

(A) $\ln 6$ **(B)** $\ln 3$ **(C)** $\ln 2$ **(D)** $\ln 12$ **(E)** $\frac{5}{2}$

19. The graph of f consists of a semicircle and a line segment. The value of $\int_0^6 f(x)\,dx$ is



Graph of f: a semicircle and a line segment.

(A) $2\pi + 3$ **(B)** $4\pi - 3$ **(C)** $\pi - 3$ **(D)** $2\pi - 3$ **(E)** 2π

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20. On $[a, b]$ the function f is positive, increasing, and concave up. Which statement is true?

(A) The left sum overestimates $\int_a^b f(x)\,dx$.
(B) The right sum underestimates $\int_a^b f(x)\,dx$.
(C) The trapezoidal sum overestimates $\int_a^b f(x)\,dx$.
(D) The midpoint sum overestimates $\int_a^b f(x)\,dx$.
(E) The trapezoidal sum equals the midpoint sum.

SET B Part B [Calculator]

Choose the best answer. Decimal answers are given to three places. [Calculator]

21. $\displaystyle\int_0^2 e^{-x^2}\,dx \approx$

(A) 0.882 **(B)** 0.982 **(C)** 0.865 **(D)** 0.760 **(E)** 1.264

22. The average value of $f(x) = \ln(x^2 + 1)$ on $[0, 3]$ is approximately

(A) 1.179 **(B)** 3.406 **(C)** 2.303 **(D)** 1.135 **(E)** 0.693

23. A tank holds 30 liters at $t = 0$. Water enters at rate $R(t)$ liters per minute, with values in the table. Using a trapezoidal sum with three subintervals, the volume at $t = 9$ is approximately

t (min)	0	2	5	9
$R(t)$ (L/min)	5	9	6	2

(A) 52.5 liters **(B)** 82.5 liters **(C)** 74 liters
(D) 91 liters **(E)** 75 liters

24. A particle has velocity $v(t) = t \sin t$ for $0 \le t \le 4$. The total distance traveled is approximately

(A) 1.858 **(B)** 3.142 **(C)** 4.000 **(D)** 6.283 **(E)** 4.425

25. The Mean Value Theorem for Integrals guarantees a number c in $[1, 5]$ with $\sqrt{c}$ equal to the average value of $\sqrt{x}$ on $[1, 5]$. The value of c is approximately

(A) 1.697 **(B)** 3.000 **(C)** 2.879 **(D)** 2.236 **(E)** 2.449

26. A tank holds 50 liters at $t = 0$ minutes and leaks at the rate $8e^{-0.3t}$ liters per minute with no inflow. The volume at $t = 6$ is approximately

(A) 22.258 liters **(B)** 72.258 liters **(C)** 29.600 liters
(D) 2.000 liters **(E)** 27.741 liters

27. If $F(x) = \displaystyle\int_0^x \cos(t^2)\,dt$, then $F'(2) =$

(A) -0.757 **(B)** -0.654 **(C)** 0.654 **(D)** -2.615 **(E)** 0.757

28. What is the value of $k > 0$ for which $\displaystyle\int_0^k \frac{dx}{1 + x^2} = 1$?

(A) 0.785 **(B)** 1.000 **(C)** 1.557 **(D)** 2.718 **(E)** 1.718

29. The trapezoidal sum T_4 with four equal subintervals for $\displaystyle\int_0^2 \sqrt{1 + x^3}\,dx$ is approximately

(A) 3.283 **(B)** 2.783 **(C)** 3.783 **(D)** 3.241 **(E)** 3.000

30. A tank holds 100 gallons at $t = 0$ minutes. Water enters at the rate $15 + 4 \sin t$ gallons per minute and leaves at the constant rate 12 gallons per minute. The volume at $t = 5$ is approximately

(A) 17.865 gallons **(B)** 115.000 gallons **(C)** 132.000 gallons
(D) 117.865 gallons **(E)** 112.135 gallons

4.14 Free-Response Practice on Integrals

Four free-response questions on Chapter 4. Write complete solutions with units and justifications, then compare with the scoring guides.

Each question is worth 9 points. Show setups, state theorems by name, and label units. A justification must cite the reason, not just the result.

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QUESTION 1 Rate in context (table)

A calculator is allowed. Show your work. [Calculator]

1. A storage tank is filled by a pipeline. Fluid enters the tank at a rate $r(t)$ liters per minute, where r is a differentiable function of t, the time in minutes. Values of $r(t)$ are given in the table. Fluid also leaves the tank at the rate $L(t) = 3 + 0.4t$ liters per minute. At $t = 0$ the tank holds 50 liters.

   (a) Use a trapezoidal sum with the four subintervals given by the table to approximate $\int_0^{20} r(t)\,dt$. Using correct units, explain the meaning of $\int_0^{20} r(t)\,dt$ in the context of the problem.

   (b) Use your answer to (a) to approximate the amount of fluid in the tank at $t = 20$.

   (c) Approximate the average value of $r(t)$ on $0 \le t \le 20$ using your answer to (a).

   (d) Explain why there must be a time t with $4 < t < 9$ at which $r(t) = 20$.

t (minutes)	0	4	9	13	20
$r(t)$ (L/min)	12	18	22	16	10

QUESTION 2 Graph and accumulation

No calculator. Show your work and justify your answers. [No calculator]

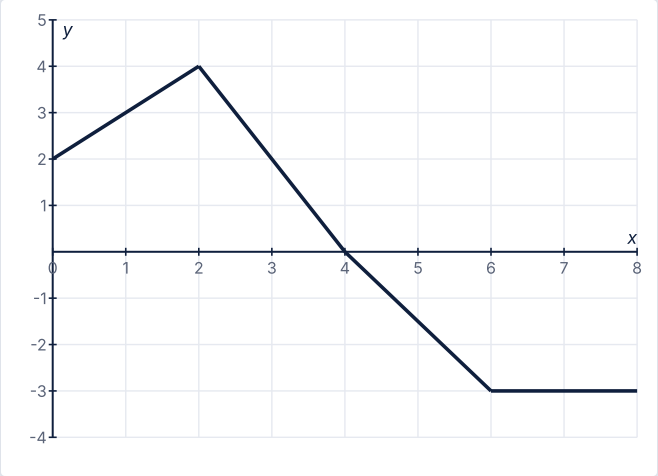
2. The graph of the continuous function f consists of four line segments, as shown. Let $g(x) = \int_0^x f(t)\,dt$ for $0 \le x \le 8$.

   (a) Find $g(4)$, $g(6)$, and $g(8)$.

   (b) Find $g'(3)$ and $g''(3)$.

   (c) On what open interval is the graph of g concave down? Give a reason for your answer.

   (d) Find the absolute maximum value of g on $[0, 8]$, and justify your answer.



Graph of f.

QUESTION 3 Particle motion

A calculator is allowed. Give decimal answers to three places. Show your work. [Calculator]

3. A particle moves along the x-axis with velocity $v(t) = 4\sin t - t$ meters per second for $0 \le t \le 5$, where t is in seconds. The particle is at $x = 2$ at $t = 0$.

   (a) Find the acceleration of the particle at $t = 2$. Include units.

   (b) Find the position of the particle at $t = 5$.

   (c) Find the total distance traveled by the particle from $t = 0$ to $t = 5$.

   (d) Is the speed of the particle increasing or decreasing at $t = 2$? Give a reason for your answer.

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QUESTION 4 **Rates and averages**

No calculator. Show your work and justify your answers. [No calculator]

4. The concentration $C(t)$ of a chemical in a reactor, in milligrams per liter, changes at the rate $C'(t) = \dfrac{4t}{t^2+4}$ milligrams per liter per hour for $0 \le t \le 2$, where t is in hours. At $t = 0$, $C(0) = 3$. A reagent is also added at the rate $A(t) = \dfrac{6}{t^2+4}$ grams per hour.

(a) Find $C(2)$. Show the work that leads to your answer.

(b) Find the average rate of change of C on $[0, 2]$.

(c) Explain why there is a time c with $0 < c < 2$ at which $C'(c)$ equals your answer in (b). Name the theorem you use and verify its hypotheses.

(d) Find the total amount of reagent added from $t = 0$ to $t = 2$.

Answers Chapter 4: Answers and Explanations

Answers for every practice problem, numbered to match. Read the explanation even when your answer is correct.

4.1 Antiderivatives and Indefinite Integrals

SET A · SKILLS

1. $2x^3 - 2x^2 + 7x + C$. Apply the power rule to each term.
2. $2x^5 + 3x^3 - x + C$.
3. $-\dfrac{2}{x^2} + C$. Rewrite as $4x^{-3}$; then $4 \cdot \frac{x^{-2}}{-2} = -2x^{-2}$.
4. $4x^{3/2} + C$. $6 \cdot \frac{x^{3/2}}{3/2} = 4x^{3/2}$.
5. $9x^{4/3} + C$. $12 \cdot \frac{x^{4/3}}{4/3} = 9x^{4/3}$.
6. $\dfrac{x^5}{5} + \dfrac{2x^3}{3} + x + C$. Expand first: $x^4 + 2x^2 + 1$.
7. $\dfrac{x^2}{2} - 5x - \dfrac{4}{x} + C$. Divide: $x - 5 + 4x^{-2}$.
8. $\dfrac{2x^3}{3} + \dfrac{x^2}{2} - 3x + C$. Expand: $2x^2 + x - 3$.
9. $10\sqrt{x} + \dfrac{4}{5}x^{5/2} + C$. Rewrite as $5x^{-1/2} + 2x^{3/2}$.

SET B · APPLICATIONS AND MIXED

10. $f(x) = 3x^3 - x^2 + 3$. General: $3x^3 - x^2 + C$; $f(1) = 2 + C = 5$ gives $C = 3$.
11. $f(x) = 2x^{3/2} + 4$. $f(4) = 2 \cdot 8 + C = 20$ gives $C = 4$.
12. $f(x) = 2x^3 - 3x^2 + 2$. $f'(x) = 6x^2 - 6x + C_1$ and $f'(1) = 0$ give $C_1 = 0$; then $f(0) = 2$ gives the constant 2.
13. (a) $h(t) = -4.9t^2 + 20t + 2$.
    (b) $h(1.5) = 20.975$ m.
    (c) $h'(t) = -9.8t + 20 = 0$ at $t = \frac{20}{9.8} \approx 2.041$, so the maximum height is $h(2.041) \approx 22.408$ m.
14. $C(x) = 0.03x^2 + 4x + 500$. $C(100) = 300 + 400 + 500 = 1200$ dollars.
15. $s(t) = t^3 - 6t^2 + 9t + 4$. $s(2) = 8 - 24 + 18 + 4 = 6$ m.
16. $y = x^3 - x^2 + C$; $4 = 1 - 1 + C$ gives $C = 4$. So $y = x^3 - x^2 + 4$ and $y(2) = 8 - 4 + 4 = 8$.

SET C · CHALLENGE AND AP-STYLE

17. $f''(x) = 12x^2 + 2$, $f'(x) = 4x^3 + 2x - 1$, $f(x) = x^4 + x^2 - x + 5$. Each constant comes from the given value at 0.
18. Incorrect. There is no rule that integrates a product factor by factor. Combine first: $x^2 \cdot x^4 = x^6$, so the integral is $\dfrac{x^7}{7} + C$.
19. $f(x) = \dfrac{x|x|}{2} + 1$, that is, $\frac{x^2}{2} + 1$ for $x \ge 0$ and $-\frac{x^2}{2} + 1$ for $x < 0$. For $x > 0$, $f' = x$; for $x < 0$, $f' = -x = |x|$; at 0 both pieces have derivative 0, so f is differentiable. $f(0) = 1$ fixes the constant.
20. $s(t) = \frac{t^3}{3} - 2t^2 + 3t$. $v = (t-1)(t-3)$ changes sign at $t = 1$ and $t = 3$. $s(1) = \frac{4}{3}$, $s(3) = 0$, $s(4) = \frac{4}{3}$. Distance $= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$.
21. $F'(x) = a\sqrt{x} + \frac{ax+b}{2\sqrt{x}} = \frac{3ax+b}{2\sqrt{x}}$. Match with $\frac{6x+5}{2\sqrt{x}}$: $3a = 6$, $b = 5$. So $a = 2$, $b = 5$.

4.2 Derivative Rules Backwards

SET A · SKILLS

1. $5\sin x + 3\cos x + C$.
2. $e^x + x^2 + C$.
3. $7\ln|x| + x^3 + C$.
4. $6\tan x + C$.
5. $\sec x + \cot x + C$. Note $\int -\csc^2 x\,dx = +\cot x$.
6. $\frac{1}{4}\sin(4x) + C$.
7. $2e^{3x} + C$. Divide 6 by the inner coefficient 3.
8. $-2\cos\left(\frac{x}{2}\right) + C$. Dividing by $\frac{1}{2}$ multiplies by 2.
9. $\frac{1}{3}\ln|3x - 5| + C$.

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SET B · APPLICATIONS AND MIXED

10. $f(x) = 4\sin x + 2e^x + 5$. General: $4\sin x + 2e^x + C$; $f(0) = 2 + C = 7$.
11. $f(x) = \tan x - 2x + C$. $f(\frac{\pi}{4}) = 1 - \frac{\pi}{2} + C = 3$ gives $C = 2 + \frac{\pi}{2}$. So $f(x) = \tan x - 2x + 2 + \frac{\pi}{2}$.
12. $s(t) = 2\sin(2t) + 1$. $s(\frac{\pi}{4}) = 2\sin\frac{\pi}{2} + 1 = 3$.
13. $y = 2e^{x/2} + C$; $5 = 2 + C$ gives $C = 3$. At $x = 2\ln 3$, $e^{x/2} = 3$, so $y = 2(3) + 3 = 9$.
14. $f(x) = 5\ln x - 2x + C$; $f(1) = -2 + C = 4$ gives $C = 6$. $f(e) = 5 - 2e + 6 = 11 - 2e$.
15. $P(t) = 600e^{0.2t} + C$; $P(0) = 600 + C = 500$ gives $C = -100$. $P(5) = 600e - 100 \approx 1530.969$.
16. $Q(t) = -2\cos(2t) + C$; $Q(0) = -2 + C = 0.5$ gives $C = 2.5$. $Q(1) = -2\cos 2 + 2.5 \approx 3.332$ coulombs.

SET C · CHALLENGE AND AP-STYLE

17. Expand: $\sin^2 x + \cos^2 x + 2\sin x\cos x = 1 + \sin 2x$. So the integral is $x - \frac{1}{2}\cos 2x + C$.
18. Divide: $\frac{e^{2x}+1}{e^x} = e^x + e^{-x}$. The integral is $e^x - e^{-x} + C$.
19. Divide: $x^2 + 2x - \frac{5}{x}$. The integral is $\frac{x^3}{3} + x^2 - 5\ln x + C$.
20. $f'(x) = -3\sin(3x) + C_1$; $f'(0) = C_1 = 3$. $f(x) = \cos(3x) + 3x + C_2$; $f(0) = 1 + C_2 = 1$ gives $C_2 = 0$. $f(x) = \cos(3x) + 3x$.
21. $\frac{d}{dx}\sin(3x) = 3\cos(3x) \neq \cos(3x)$. The correct result is $\frac{1}{3}\sin(3x) + C$, whose derivative is $\cos(3x)$.

4.3 The Method of Substitution

SET A · SKILLS

1. $\frac{(x^2+1)^6}{6} + C$. Let $u = x^2 + 1$, $du = 2x\,dx$.
2. $\frac{(x^3+4)^5}{15} + C$. Let $u = x^3 + 4$, $x^2\,dx = \frac{1}{3}du$.
3. $\frac{1}{3}(x^2+9)^{3/2} + C$. Let $u = x^2 + 9$, $x\,dx = \frac{1}{2}du$: $\frac{1}{2}\cdot\frac{u^{3/2}}{3/2}$.
4. $\frac{\sin^5 x}{5} + C$. Let $u = \sin x$.
5. $\frac{1}{3}e^{x^3} + C$. Let $u = x^3$.
6. $\frac{1}{2}\ln(x^2+4) + C$. Let $u = x^2 + 4$.
7. $\frac{\tan^2 x}{2} + C$. Let $u = \tan x$, $du = \sec^2 x\,dx$.
8. $\ln|x^3 + x| + C$. The numerator is the derivative of the denominator.
9. $\frac{(\ln x)^3}{3} + C$. Let $u = \ln x$, $du = \frac{1}{x}\,dx$.

SET B · APPLICATIONS AND MIXED

10. $\frac{(x^2+x)^4}{2} + C$. Let $u = x^2 + x$, $du = (2x+1)\,dx$, and $4x + 2 = 2(2x+1)$, so the integral is $2\int u^3\,du$.
11. $-\frac{1}{3}\cos(x^3) + C$. Let $u = x^3$.
12. $\frac{1}{2}\ln(1 + e^{2x}) + C$. Let $u = 1 + e^{2x}$, $du = 2e^{2x}\,dx$.
13. $f(x) = (x^2+1)^3 + C$; $f(0) = 1 + C = 1$ gives $C = 0$. $f(1) = 2^3 = 8$.
14. $s(t) = \frac{1}{2}\sin(t^2) + C$; $s(0) = C = 2$. $s\left(\sqrt{\pi/2}\right) = \frac{1}{2}\sin\frac{\pi}{2} + 2 = 2.5$ m.
15. $c(t) = -6e^{-t^2} + C$; $c(0) = 0$ gives $C = 6$. $c(2) = 6 - 6e^{-4} \approx 5.890$ mg/L.
16. Let $u = \cos x$: $y = -\frac{\cos^3 x}{3} + C$. $y(0) = -\frac{1}{3} + C = 1$ gives $C = \frac{4}{3}$. $y(\frac{\pi}{2}) = 0 + \frac{4}{3} = \frac{4}{3}$.

SET C · CHALLENGE AND AP-STYLE

17. Let $u = x - 3$, $x = u + 3$: $\int (u+3)u^{1/2}\,du = \frac{2}{5}u^{5/2} + 2u^{3/2}$. Answer: $\frac{2}{5}(x-3)^{5/2} + 2(x-3)^{3/2} + C$.
18. Let $u = 1 + e^x$, $du = e^x\,dx$: $\int u^{1/2}\,du = \frac{2}{3}(1+e^x)^{3/2} + C$.
19. Let $u = \cos x$, $du = -\sin x\,dx$: $-\int u^{-3}\,du = \frac{1}{2u^2}$. Answer: $\frac{1}{2\cos^2 x} + C = \frac{1}{2}\sec^2 x + C$.
20. Both are correct. With $u = \sin x$: $\int 2u\,du = u^2$. With $u = \cos x$: $-\int 2u\,du = -u^2$. They differ by $\sin^2 x - (-\cos^2 x) = 1$, a constant, which is absorbed into C.
21. Let $u = x^2 + 16$: $f(x) = \sqrt{x^2+16} + C$. $f(3) = 5 + C = 7$ gives $C = 2$. $f(0) = 4 + 2 = 6$.

4.4 Using Geometry for Definite Integrals

SET A · SKILLS

1. 12. Rectangle of width 4 and height 3.
2. 6. Triangle with base 6 and height $f(6) = 2$: $\frac{1}{2}(6)(2) = 6$.
3. 0. Triangle below the axis on $[-4,-1]$ has area $\frac{1}{2}(3)(3) = 4.5$; the triangle above on $[-1,2]$ has area 4.5. They cancel.
4. $\frac{13}{2}$. Two triangles: $\frac{1}{2}(2)(2) = 2$ on $[0,2]$ and $\frac{1}{2}(3)(3) = \frac{9}{2}$ on $[2,5]$; total $\frac{13}{2}$.
5. $\frac{25\pi}{2}$. The region is a semicircle of radius 5.
6. $\frac{9\pi}{4}$. The region is a quarter circle of radius 3.
7. -2π. The integrand is the lower half of a circle of radius 2, so the signed area is $-\frac{1}{2}\pi(2)^2$.
8. $8 - 2\pi$. Rectangle $4\cdot 2 = 8$ minus the upper semicircle of radius 2 (area 2π).
9. 12. The odd function contributes 0 over $[-3,3]$ and $\int_{-3}^{3} 2\,dx = 2(6) = 12$.

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SET B · APPLICATIONS AND MIXED

10. (a) 25.5 m. Areas: $9+12+6$ above the axis, -1.5 below (zero crossing at $t=7$). (b) 28.5 m: $9+12+6+1.5$. (c) $4+25.5=29.5$ m.
11. 39 liters. $r(t)$ falls from 12 to 0 over 4 minutes: triangle area $\frac{1}{2}(4)(12)=24$. Total $15+24=39$.
12. π coulombs. The graph is a quarter circle of radius 2: $\frac{1}{4}\pi(2)^2=\pi$.
13. 0.9 J. Triangle with base 0.3 and height $F(0.3)=6$: $\frac{1}{2}(0.3)(6)=0.9$.
14. (a) 5, by additivity: $7+(-2)$. (b) -18. Reversing limits: $\int_9^1 (2f+1)\,dx=-(2\cdot 5+(9-1))=-18$.
15. C. x^3 is odd so it contributes 0; $\int_{-4}^{4}|x|\,dx=2\cdot\frac{1}{2}(4)(4)=16$.
16. $k=5$. The region is a triangle with base k and height $2k$, area k^2. Then $k^2=25$ and $k>0$ gives $k=5$.

SET C · CHALLENGE AND AP-STYLE

17. 19.242. A semicircle of radius 3.5: $\frac{1}{2}\pi(12.25)\approx 19.242$.
18. 1.133. The semicircle has area $\frac{1}{2}\pi(16)=8\pi$; the constant -3 contributes $-3(8)=-24$. $8\pi-24\approx 1.133$.
19. 11.569. Quarter circle of radius 3: $\frac{9\pi}{4}\approx 7.069$; triangle with base 3 and height 3: 4.5. Sum ≈ 11.569.
20. $a\approx 5.642$. The integral is a semicircle: $\frac{1}{2}\pi a^2=50$, so $a=\sqrt{100/\pi}\approx 5.642$.
21. (a) 6.283: semicircle of radius 2, area 2π. (b) -2.717: add the triangle below ($\frac{1}{2}(2)(3)=3$) and the rectangle below ($2\cdot 3=6$): $2\pi-9$. (c) 15.283: $2\pi+3+6$.

4.5 Riemann Sums

SET A · SKILLS

1. (a) 30. (b) 54. (c) 42. With $\Delta x=2$: $L_3=2(1+5+9)$, $R_3=2(5+9+13)$, $M_3=2(3+7+11)$.
2. $L_4=\frac{9}{4}$ and $R_4=\frac{25}{4}$. With $\Delta x=\frac{1}{2}$: $L_4=\frac{1}{2}\left(0+\frac{1}{8}+1+\frac{27}{8}\right)=\frac{9}{4}$ and $R_4=\frac{1}{2}\left(\frac{1}{8}+1+\frac{27}{8}+8\right)=\frac{25}{4}$.
3. (a) 68: $2(5+8+12+9)$. (b) 64: $2(8+12+9+3)$. (c) 68: $\Delta x=4$, midpoints $x=2,6$: $4(8+9)$.
4. 25. The terms are $1,3,5,7,9$ (the first five odd numbers).
5. $\int_0^2 x^3\,dx=4$. Here $\Delta x=\frac{2}{n}$ and $x_i=\frac{2i}{n}$, so $a=0$, $b=2$, $f(x)=x^3$. Then $\left.\frac{x^4}{4}\right|_0^2=4$.
6. $\int_2^6 e^x\,dx$. Width $\frac{4}{n}$ means $b-a=4$; $x_i=2+\frac{4i}{n}$ gives $a=2$, $b=6$, $f(x)=e^x$.
7. f is decreasing, so L_n overestimates and R_n underestimates. On each subinterval the left height is the largest value and the right height is the smallest value.
8. $M_2=40$, an underestimate. $\Delta x=2$, midpoints 2 and 4: $2(4+16)=40$. Since x^2 is concave up, the midpoint sum underestimates. (The exact value is $\frac{124}{3}\approx 41.33$.)
9. B. $\Delta x=\frac{3}{n}$ and $x_i=1+\frac{3i}{n}$, so $a=1$, $b=4$ and $f(x)=x^2$.

SET B · APPLICATIONS AND MIXED

10. Left: $4(3)+6(4)+5(5)=61$. Right: $6(3)+5(4)+9(5)=83$. The widths are $3,4,5$. The total is about 61 to 83 liters.
11. (a) Left: $2(0+6+10+12)=56$ m. Right: $2(6+10+12+13)=82$ m. (b) The right sum, 82, is an upper bound because v is increasing, so each right height is the largest on its subinterval.
12. (a) $\Delta x=\frac{2}{n}$, $x_i=\frac{2i}{n}$: $R_n=\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}=\frac{4(n+1)(2n+1)}{3n^2}=\frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}$. (b) The limit is $\frac{8}{3}$.
13. B. Increasing gives $L_n<I<R_n$. Concave down means the midpoint sum overestimates, so $I<M_n$. And $M_n<R_n$ since the right sum is larger for an increasing function.
14. $M_3=210$ and the error is 6. $\Delta x=2$, midpoints $1,3,5$: $2\cdot 3(1+9+25)=210$. The exact value is $\left.x^3\right|_0^6=216$.
15. 1. The sum is a right sum for $\cos x$ with $\Delta x=\frac{\pi}{2n}$ on $[0,\frac{\pi}{2}]$. $\int_0^{\pi/2}\cos x\,dx=\left.\sin x\right|_0^{\pi/2}=1$.
16. $\frac{25}{12}$. $\Delta x=2$, midpoints $2,4,6,8$: $2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}\right)=2\cdot\frac{25}{24}$. (The exact value is $\ln 9\approx 2.197$.)

SET C · CHALLENGE AND AP-STYLE

17. $L_4\approx 2.783$, $R_4\approx 3.783$, $M_4\approx 3.220$. With $\Delta x=0.5$. (A numerical integral gives 3.241.) Left underestimates and right overestimates because f is increasing.
18. $L_3\approx 0.544$, $R_3\approx 0.933$, $M_3\approx 0.798$. With $\Delta x=0.5$, midpoints $0.25,0.75,1.25$. (The integral is about 0.778.)
19. $R_4\approx 20.668$ mg, $M_4\approx 22.773$ mg. With $\Delta t=2$. The exact total is $24-120e^{-4}\approx 21.802$ mg. The rate rises until $t=2$ then falls, so the sums are not guaranteed to be bounds.
20. Left: $2.4(1.5)+3.1(2.5)+3.8(1)+2.9(3)=23.850$. Right: $3.1(1.5)+3.8(2.5)+2.9(1)+1.7(3)=22.150$. The widths are $1.5,2.5,1,3$.
21. $\int_4^6\sqrt{x}\,dx\approx 4.465$. $\Delta x=\frac{2}{n}$ so $b-a=2$; $x_i=4+\frac{2i}{n}$ gives $a=4$, $b=6$. Value: $\frac{2}{3}\left(6^{3/2}-8\right)\approx 4.465$.

4.6 Trapezoidal Rule and Simpson's Rule

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1. $\frac{11}{4}$. $\Delta x = 0.5$: $\frac{0.5}{2}\,[0 + 2(0.25) + 2(1) + 2(2.25) + 4] = 0.25(11) = 2.75$.
2. 40, an underestimate. $\Delta x = 2$: $\frac{2}{2}[16 + 2(12) + 0] = 40$. Since $f''(x) = -2 < 0$, the graph is concave down, so the chords lie below the curve. (Exact value is $\frac{128}{3}$.)
3. 16.5. $\Delta x = 1$: $\frac{1}{2}[2 + 2(5) + 2(6) + 2(4) + 1] = \frac{1}{2}(33)$.
4. 20.5. $\frac{1}{2}(3 + 5)(1) + \frac{1}{2}(5 + 2)(3) + \frac{1}{2}(2 + 4)(2) = 4 + 10.5 + 6$.
5. $S_4 = 64$, exactly equal to $\int_0^4 x^3\,dx = 64$. $\frac{1}{3}[0 + 4(1) + 2(8) + 4(27) + 64] = \frac{192}{3}$.
6. 34. $\frac{2}{3}[1 + 4(4) + 2(6) + 4(5) + 2] = \frac{2}{3}(51)$.
7. 30. $T_5 = \frac{1}{2}(L_5 + R_5) = \frac{1}{2}(34 + 26)$.
8. Overestimates. $f''(x) = e^x > 0$, so the graph is concave up and the chords lie above the curve.
9. C. $\Delta x = 1$: $\frac{1}{2}[0 + 2(1) + 8] = 5$. Choice B is the exact value of the integral, not T_2.

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10. 18.25 m^2. $\frac{1}{2}(0 + 1.5)(3) + \frac{1}{2}(1.5 + 2.5)(2) + \frac{1}{2}(2.5 + 2)(4) + \frac{1}{2}(2 + 0)(3) = 2.25 + 4 + 9 + 3$.
11. (a) 47 liters. $\frac{1}{2}[8 + 2(9 + 11 + 10 + 8 + 4) + 2] = \frac{1}{2}(94)$. (b) $\frac{140}{3} \approx 46.67$ liters. $\frac{1}{3}[8 + 4(9) + 2(11) + 4(10) + 2(8) + 4(4) + 2] = \frac{140}{3}$.
12. Less than the integral. Because $g'' < 0$ on $[0, 4]$, the graph is concave down, so the chords lie below the curve and the trapezoidal rule underestimates. Hence $7.4 < \int_0^4 g(x)\,dx$.
13. $a = 5$. $\Delta x = 2$: $T_4 = 1 \cdot [3 + 2a + 2(7) + 2(5) + 1] = 28 + 2a = 38$, so $a = 5$.
14. $T_2 = 24$, $M_2 = 20$, exact $= \frac{64}{3} \approx 21.33$. Then $\frac{2(20)+24}{3} = \frac{64}{3}$, which equals $S_2 = \frac{2}{3}[0 + 4(4) + 16] = \frac{64}{3}$. Simpson's rule is a weighted average of T and M.
15. (a) 29 m^3. $\frac{1}{2}[12 + 2(10 + 7 + 5) + 2] = \frac{1}{2}(58)$. (b) $\frac{88}{3} \approx 29.33$ m^3. $\frac{1}{3}[12 + 4(10) + 2(7) + 4(5) + 2] = \frac{88}{3}$.
16. C. Chords lie above the graph when the graph is concave up. Monotonicity or positivity alone does not determine the sign of the trapezoid error.

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17. $T_4 \approx 0.881$, $S_4 \approx 0.882$, and the calculator gives ≈ 0.882. $\Delta x = 0.5$. Simpson is closer: S_4 error ≈ 0.0003, T_4 error ≈ 0.0015.
18. $T_5 \approx 0.786$. $\Delta x = 0.3$. Since $f''(x) = -\frac{1}{x^2} < 0$, the graph is concave down and T_5 underestimates (the integral is about 0.791).
19. $T_4 \approx 17.228$ and the exact value is $\frac{2}{3}(27 - 1) = \frac{52}{3} \approx 17.333$. $\sqrt{x}$ is concave down ($f'' < 0$), so T_4 underestimates.
20. 20.420. $\frac{1}{2}(3.4)(1.2) + \frac{1}{2}(8.5)(1.3) + \frac{1}{2}(9.3)(1.6) + \frac{1}{2}(5.7)(1.9) = 2.04 + 5.525 + 7.44 + 5.415$.
21. $T_6 \approx 1.248$, $S_6 \approx 1.247$, $\arctan 3 \approx 1.249$. $\Delta x = 0.5$. Here the trapezoid error is about 0.001 and the Simpson error is about 0.002; the function changes concavity on $[0, 3]$ so the usual advantage of Simpson's rule is not guaranteed at this small n.

4.7 The Fundamental Theorem of Calculus

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1. $\cos(x^2)$. Part 1: the integrand evaluated at the upper limit.
2. $-\sqrt{x^2 + 1}$. The variable is the lower limit, so there is a minus sign.
3. $9x^2 \ln x$. Chain rule: $\ln(x^3) \cdot 3x^2 = 3 \ln x \cdot 3x^2$.
4. $-2e^{4x^2}$. Reverse limits: $-\int_7^{2x} e^{t^2}\,dt$, derivative $-e^{(2x)^2} \cdot 2$.
5. $2x \sin(x^2) - \sin x$. Upper limit term minus lower limit term.
6. 2. Antiderivative $x^3 - 2x^2 + x$: at 2 it is $8 - 8 + 2 = 2$; at 0 it is 0.
7. $\frac{45}{4}$. Antiderivative $\frac{3}{4}x^{4/3}$: $\frac{3}{4}(16) - \frac{3}{4}(1) = 12 - \frac{3}{4}$.
8. 3. $\ln x\Big|_1^{e^2} = 2$ and $\tan x\Big|_0^{\pi/4} = 1$, total 3.
9. $e - 1 + \frac{\pi}{4}$. Antiderivative $e^x + \arctan x$: $(e + \frac{\pi}{4}) - (1 + 0)$.

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10. Critical points: $g'(x) = x^2 - 4x + 3 = (x - 1)(x - 3)$, so $x = 1$ and $x = 3$. g' changes from positive to negative at $x = 1$: local maximum. g' changes from negative to positive at $x = 3$: local minimum.
11. (a) $\frac{20}{3}$ m. $\int_0^5 (t^2 - 6t + 8)dt = \left(\frac{t^3}{3} - 3t^2 + 8t\right)\Big|_0^5 = \frac{125}{3} - 75 + 40$. (b) $\frac{28}{3}$ m. $v = (t - 2)(t - 4)$ is positive on $[0, 2]$, negative on $(2, 4)$, positive on $(4, 5]$: $\frac{20}{3} + \frac{4}{3} + \frac{4}{3}$.
12. $40 + 6 \ln 3$ cm. $h(4) = 40 + \int_0^4 \frac{6}{2t+1}\,dt = 40 + 3 \ln(2t + 1)\Big|_0^4 = 40 + 3 \ln 9 = 40 + 6 \ln 3$.
13. $y = 3(x - 2)$, and $F(2.1) \approx 0.3$. $F(2) = 0$ and $F'(2) = \sqrt{4 + 5} = 3$ by Part 1. At $x = 2.1$, $y = 3(0.1) = 0.3$.
14. The function $\sec^2 x$ is undefined at $x = \frac{\pi}{2}$, which lies inside $[0, \pi]$, so it is not continuous on the interval and the Fundamental Theorem does not apply. Also the integrand is positive wherever it is defined, so the integral could not be 0.
15. $f(x) = e^x(x^2 + 2x)$. Differentiate both sides using Part 1 on the left and the product rule on the right: $f(x) = 2xe^x + x^2e^x$.
16. D. Part 1 with the chain rule: $\frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}$.

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17. $F'(1.2) = e^{-1.44} \approx 0.237$ by Part 1. Numerical integration gives $F(1.2) \approx 0.807$.
18. $H'(x) = \sqrt{x^2+3} \cdot 2x$, so $H'(1.5) = 3\sqrt{5.25} \approx 6.874$.
19. $G'(x) = \cos(x^2)$, so $G'(1) = \cos 1 \approx 0.540$ and $G(1) \approx 0.905$. Tangent line: $y = 0.905 + 0.540(x-1)$. Then $G(1.2) \approx 0.905 + 0.108 = 1.013$. The numerical integral is 0.974, so the tangent line overestimates slightly.
20. $k \approx 1.962$. $\int_0^k 4e^{-t/2}dt = 8\left(1 - e^{-k/2}\right) = 5$, so $e^{-k/2} = \frac{3}{8}$ and $k = 2\ln\frac{8}{3}$.
21. (a) -5.511. Antiderivative $t\sin t + \cos t$: $(5\sin 5 + \cos 5) - 1 \approx -5.511$. (b) 7.055. The velocity is positive on $[0, \frac{\pi}{2})$, negative on $(\frac{\pi}{2}, \frac{3\pi}{2})$, positive on $(\frac{3\pi}{2}, 5]$; compute $\int_0^5 |v|\,dt \approx 7.055$.

4.8 Accumulation Functions and the FTC Graphically

SET A · SKILLS

1. $g(1) = 1$, $g(2) = 2$, $g(3) = 1$, $g(4) = 0$, $g(5) = 1$, $g(6) = 2$. Areas: triangle $+1$ on $[0,1]$; on $[1,3]$ the graph crosses at $x = 2$ giving $+1$ then -1; on $[3,5]$ it crosses at $x = 4$ giving -1 then $+1$; triangle $+1$ on $[5,6]$.
2. Local maximum at $x = 2$ and local minimum at $x = 4$. Since $g' = f$, g' changes from positive to negative at $x = 2$ and from negative to positive at $x = 4$.
3. Increasing on $(0,2)$ and $(4,6)$ where $f > 0$; decreasing on $(2,4)$ where $f < 0$.
4. $g'' = f'$. Concave up on $(0,1)$ and $(3,5)$, where f is increasing. Concave down on $(1,3)$ and $(5,6)$, where f is decreasing. Inflection points at $x = 1$, $x = 3$ and $x = 5$, where f' changes sign.
5. Candidates: $g(0) = 0$, $g(2) = 2$, $g(4) = 0$, $g(6) = 2$. Absolute maximum value 2 (at $x = 2$ and at $x = 6$); absolute minimum value 0 (at $x = 0$ and at $x = 4$).
6. $g'(4.5) = f(4.5) = 1$ and $g''(4.5) = f'(4.5) = 2$. On $[3,5]$ the graph is a line from $(3,-2)$ to $(5,2)$ with slope 2, so $f(4.5) = -2 + 2(1.5) = 1$.
7. $g(3) = 2 + 5 = 7$ and $g(7) = 2 + 5 - 2 = 5$. Additivity: $\int_0^7 f = \int_0^3 f + \int_3^7 f$.
8. $g(3) = 7$. Since $g'(x) = f(x)$: $g'(2) = 1 > 0$, so g is increasing at $x = 2$; $g'(3) = -2 < 0$, so decreasing at $x = 3$; $g'(5) = 4 > 0$, so increasing at $x = 5$.
9. B. $g'(4) = f(4) = 0$ and g' changes from positive to negative at $x = 4$, so g has a local maximum there. $g(4)$ is not necessarily 0.

SET B · APPLICATIONS AND MIXED

10. $g(0) = 0$, $g(2) = \pi$, $g(4) = 2\pi$, $g(6) = \pi$, $g(8) = 0$. Each semicircle has area 2π; half of the first is π. The lower semicircle subtracts: $2\pi - \pi = \pi$ at $x = 6$ and $2\pi - 2\pi = 0$ at $x = 8$.
11. $g' = f$ changes from positive to negative at $x = 4$, so g has a local maximum there, with $g(4) = 2\pi$. There is no other local extremum in $(0,8)$. Absolute maximum 2π at $x = 4$; absolute minimum 0 at $x = 0$ and $x = 8$.
12. $g'' = f'$. f is decreasing on $(2,6)$ (down the first semicircle to $x = 4$, then continuing down the lower semicircle to $x = 6$). So g is concave down on $(2,6)$. f is increasing on $(0,2)$ and $(6,8)$, so g is concave up there. Inflection points at $x = 2$ and $x = 6$ (concavity changes). At $x = 4$ there is no change of concavity.
13. $k(2) = \int_0^4 f = 2\pi$. $k'(x) = 2f(2x)$, so $k'(1) = 2f(2) = 2(2) = 4$ and $k'(3) = 2f(6) = 2(-2) = -4$.
14. $g(6) = 3$. By trapezoids: $g(1) = 1$, $g(2) = 3.5$, $g(3) = 5.5$, $g(4) = 5.5$, $g(5) = 4$, $g(6) = 3$. The sign of f changes from positive to negative between $x = 3$ and $x = 4$, at $x = 3.5$ (halfway on the line from 1 to -1). The maximum value is $g(3.5) = 5.5 + \frac{1}{2}(0.5)(1) = 5.75$ at $x = 3.5$.
15. $g'(x) = x^2 - 3x + 2 = (x-1)(x-2)$. Critical points $x = 1$ (local maximum, since g' goes from $+$ to $-$) and $x = 2$ (local minimum). $g(2) = \left[\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right]_1^2 = \frac{2}{3} - \frac{5}{6} = -\frac{1}{6}$. $g''(x) = 2x - 3$ changes sign at $x = \frac{3}{2}$: inflection point.
16. C. The region is a trapezoid with parallel sides 2 (top, from $x = 2$ to 4) and 6 (bottom, from 0 to 6) and height 2: $\frac{1}{2}(2+6)(2) = 8$.

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17. $g'(x) = x\cos x$. Local maximum at $x = \frac{\pi}{2} \approx 1.571$ (g' goes $+$ to $-$) and local minimum at $x = \frac{3\pi}{2} \approx 4.712$ ($-$ to $+$). With $g(x) = x\sin x + \cos x - 1$: $g(0) = 0$, $g(\frac{\pi}{2}) \approx 0.571$, $g(\frac{3\pi}{2}) \approx -5.712$, $g(6) \approx -1.716$. Absolute maximum 0.571 at $x \approx 1.571$; absolute minimum -5.712 at $x \approx 4.712$.
18. Solving $\cos x = x\sin x$ gives $x \approx 0.860$ and $x \approx 3.426$ in $(0,6)$. g'' is positive at $x = 0.5$, negative at $x = 2$ and positive at $x = 5$, so the sign changes at both points and each is an inflection point: $x \approx 0.860$ and $x \approx 3.426$.
19. (a) $5 + \int_0^4 F(t)\,dt \approx 20.547$ cubic feet. (b) $F(4) = 3 + 2\sin 1.6 \approx 4.999 > 0$, so the amount is increasing. $F'(t) = \frac{2t}{5}\cos\frac{t^2}{10}$ and $F'(4) = 1.6\cos 1.6 \approx -0.047 < 0$, so the rate of increase is decreasing (the amount is concave down).
20. $x \approx 1.678$. $\int_0^x te^{-t}dt = 1 - (1+x)e^{-x}$, and solving $(1+x)e^{-x} = 0.5$ numerically gives $x \approx 1.678$.
21. $A'(x) = 2x\sqrt{1+x^4}$, so $A'(1.3) \approx 5.106$. $A''(x) = 2\sqrt{1+x^4} + \dfrac{4x^4}{\sqrt{1+x^4}}$, so $A''(1.3) \approx 9.745$.

4.9 Definite Integrals with Substitution and Properties

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1. $\frac{781}{5}$. $u = x^2 + 3$ ($du = 2x\,dx$), limits 3 to 4: $\int_3^4 u^4 du = \frac{4^5 - 3^5}{5}$.
2. $\ln \frac{9}{2}$. $u = x^3 + 1$, limits 2 to 9: $\ln 9 - \ln 2$.
3. $\frac{1}{4}$. $u = \sin x$, limits 0 to 1: $\int_0^1 u^3 du = \frac{1}{4}$.
4. $\frac{e-1}{2}$. $u = x^2$, $du = 2x\,dx$, limits 0 to 1: $\frac{1}{2}\int_0^1 e^u du$.
5. 2. $u = 2x - 1$, $dx = \frac{1}{2}du$, limits 1 to 9: $\frac{1}{2}\int_1^9 u^{-1/2} du = \sqrt{u}\Big|_1^9 = 3 - 1$.
6. 1. $u = x^2 + 16$, $du = 2x\,dx$, limits 16 to 25: $\frac{1}{2}\int_{16}^{25} u^{-1/2} du = \sqrt{u}\Big|_{16}^{25} = 5 - 4$.
7. $\int_5^8 f = 1 - 6 = -5$ and $\int_8^5 f = 5$. Additivity gives $\int_2^8 = \int_2^5 + \int_5^8$; reversing the limits changes the sign.
8. $\frac{16}{3}$. x^3 and x are odd and contribute 0; $\int_{-2}^{2} x^2 dx = 2 \cdot \frac{8}{3}$.
9. 5. On $[0, 3]$ the region is a triangle of area $\frac{9}{2}$; on $[3, 4]$ a triangle of area $\frac{1}{2}$.

SET B · APPLICATIONS AND MIXED

10. $3 \ln 10$ meters. $u = 1 + t^2$, $du = 2t\,dt$, limits 1 to 10: $3\int_1^{10} \frac{du}{u}$.
11. (a) $-\frac{8}{3}$ m: $\left[\frac{t^3}{3} - \frac{3t^2}{2}\right]_0^4 = \frac{64}{3} - 24$. (b) $\frac{19}{3}$ m. $v = t(t-3)$ is negative on $(0, 3)$ and positive on $(3, 4)$: $\left|-\frac{9}{2}\right| + \frac{11}{6} = \frac{19}{3}$.
12. Continuous: $\lim_{x \to 1^-} x^2 = 1$ and $3 - 2(1) = 1 = f(1)$. Split the integral: $\int_0^1 x^2 dx = \frac{1}{3}$ and $\int_1^2 (3 - 2x) dx = \left[3x - x^2\right]_1^2 = 2 - 2 = 0$. Total $\frac{1}{3}$.
13. 34. Even: $\int_{-2}^{2} f = 10$. Odd: $\int_{-2}^{2} g = 0$. Constant: $\int_{-2}^{2} 1\,dx = 4$. Total $3(10) - 2(0) + 4 = 34$.
14. $\frac{1}{2}\ln 2$. $\tan x = \frac{\sin x}{\cos x}$; let $u = \cos x$, $du = -\sin x\,dx$, limits 1 to $\frac{\sqrt{2}}{2}$: $-\int_1^{\sqrt{2}/2} \frac{du}{u} = -\ln \frac{\sqrt{2}}{2} = \ln \sqrt{2}$.
15. (a) 5: $u = x^2$, $x\,dx = \frac{1}{2}du$, limits 0 to 4, so $\frac{1}{2}(10)$. (b) $\frac{5}{2}$: $u = 4x$, $dx = \frac{1}{4}du$, limits 0 to 4, so $\frac{1}{4}(10)$.
16. $a = 2\sqrt{6}$. $\int_0^a \frac{x}{x^2+1} dx = \frac{1}{2}\ln(a^2 + 1) = \ln 5$ gives $a^2 + 1 = 25$, so $a = \sqrt{24} = 2\sqrt{6}$.

SET C · CHALLENGE AND AP-STYLE

17. $\frac{27 - 5\sqrt{5}}{3} \approx 5.273$. $u = x^2 + 5$, $du = 2x\,dx$, limits 5 to 9: $\frac{1}{2} \cdot \frac{2}{3}u^{3/2}\Big|_5^9 = \frac{1}{3}(27 - 5\sqrt{5})$.
18. $e - \frac{1}{e} \approx 2.350$. $u = \cos x$, $du = -\sin x\,dx$, limits 1 to -1: $-\int_1^{-1} e^u du = \int_{-1}^{1} e^u du$.
19. 11.250 (exactly $\frac{45}{4}$). $x^3 - 3x = x(x^2 - 3)$ is negative on $(0, \sqrt{3})$ and positive on $(\sqrt{3}, 3)$. $\int_0^{\sqrt{3}} (3x - x^3) dx = \frac{9}{4}$ and $\int_{\sqrt{3}}^3 (x^3 - 3x) dx = 9$.
20. $\frac{\pi}{2} \approx 1.571$. The odd term $x^5 \cos x$ integrates to 0 over $[-1, 1]$. The term $\sqrt{1 - x^2}$ is a semicircle of radius 1 with area $\frac{\pi}{2}$.
21. $\frac{2\sqrt{2}+2}{15} \approx 0.322$. $u = 1 + x^2$, $x^3 dx = \frac{1}{2}(u - 1) du$, limits 1 to 2: $\frac{1}{2}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^2$.

4.10 Net Change, Tables, and Data

SET A · SKILLS

1. 15. $f(3) = f(1) + \int_1^3 (3x^2 - 4x)\,dx = 5 + [x^3 - 2x^2]_1^3 = 5 + (9 - (-1)) = 15$.
2. -5 meters. $\int_0^5 (4 - 2t)\,dt = [4t - t^2]_0^5 = 20 - 25 = -5$.
3. 13 meters. $v = 0$ at $t = 2$. $\int_0^2 v\,dt = 4$ and $\int_2^5 v\,dt = -9$, so distance $= 4 + 9 = 13$.
4. $4 \ln 4 = 8 \ln 2$ liters. $\int_0^3 \frac{4}{t+1}\,dt = 4 \ln |t + 1|\Big|_0^3 = 4 \ln 4$.
5. $g(6) \approx 26$. Areas: $\frac{2+4}{2}(1) + \frac{4+3}{2}(2) + \frac{3+1}{2}(3) = 3 + 7 + 6 = 16$, so $g(6) \approx 10 + 16 = 26$.
6. Left sum $= 2(1) + 4(2) + 3(3) = 19$. Right sum $= 4(1) + 3(2) + 1(3) = 13$.
7. About 420 people entered the museum during the first 30 minutes after opening.
8. $\int_2^9 h'(t)\,dt$ is in centimeters: the net change in the plant's height from day 2 to day 9. $\frac{1}{7}\int_2^9 h'(t)\,dt$ is in centimeters per day: the average rate of change of height over those 7 days.
9. 17. Areas: triangle $\frac{1}{2}(2)(4) = 4$; rectangle $2(4) = 8$; on $[4, 6]$ the line crosses zero at $x = \frac{16}{3}$, giving $\frac{1}{2} \cdot \frac{4}{3} \cdot 4 - \frac{1}{2} \cdot \frac{2}{3} \cdot 2 = 2$. Net change $= 14$, so $f(6) = 3 + 14 = 17$.

SET B · APPLICATIONS

10. 148 liters. Trapezoids: $\frac{5+8}{2}(3) + \frac{8+10}{2}(3) + \frac{10+7}{2}(4) + \frac{7+4}{2}(5) = 19.5 + 27 + 34 + 27.5 = 108$, so $V(15) \approx 40 + 108 = 148$. The sum estimates the water that flowed into the tank during the 15 minutes.
11. 0.268 meters. $\int_0^4 (5 \sin t - 2)\,dt = 5(1 - \cos 4) - 8 \approx 0.268$.
12. 8.788 meters. $v = 0$ at $t \approx 0.412$ and $t \approx 2.730$, with $v < 0$ on $[0, 0.412)$, $v > 0$ on $(0.412, 2.730)$, $v < 0$ on $(2.730, 4]$. Distance $= \int_0^4 |5 \sin t - 2|\,dt \approx 8.788$.
13. 800.474 liters. Leaked: $\int_0^8 50e^{-0.2t}\,dt = 250(1 - e^{-1.6}) \approx 199.526$. Remaining $= 1000 - 199.526 = 800.474$.
14. Left sum $= 2(12 + 30 + 44 + 38) = 248$. Right sum $= 2(30 + 44 + 38 + 20) = 264$. Average $= 256$. Interpretation: about 256 cars entered the garage during the 8 hours from 6:00 a.m. to 2:00 p.m.
15. (a) $A(6) = 100 + \int_0^6 (3 + 3\sin(0.4t) - 0.5t)\,dt \approx 122.030$ gallons. (b) $A'(t) = 0$ when inflow equals outflow, at $t \approx 7.306$. $A' > 0$ before and $A' < 0$ after (the rates cross once on $(0, 10)$), so A is greatest at $t \approx 7.306$ (about 123.394 gallons, larger than $A(0) = 100$ and $A(10) \approx 117.402$).
16. 7.339 grams. $M(10) = 2 + \int_0^{10} 0.6te^{-0.3t}\,dt \approx 2 + 5.339 = 7.339$.

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SET C · CHALLENGE

17. (a) -3 meters. The first semicircle has area $\frac{1}{2}\pi(2)^2 = 2\pi$ and the second, below the axis, contributes -2π, so displacement is 0 and $s(8) = -3$. (b) 4π meters, since distance is the sum of the two areas, $2\pi + 2\pi$.
18. (a) $A'(t) = 12 - 2t$ changes from positive to negative at $t = 6$, so the maximum is at $t = 6$: $A(6) = 20 + \int_0^6 (12 - 2t)\,dt = 20 + 36 = 56$ liters. (b) $A(t) = 20 + 12t - t^2 = 0$ gives $t^2 - 12t - 20 = 0$, so $t = 6 \pm 2\sqrt{14}$. Only $t = 6 + 2\sqrt{14}$ is positive, so the tank empties at $t = 6 + 2\sqrt{14}$ minutes.
19. $g(5) = 4 + \int_1^5 g'(x)\,dx$, and $-3(4) \le \int_1^5 g' \le 2(4)$. So $-8 \le g(5) \le 12$: smallest possible -8, largest possible 12 (attained when g' is constantly -3 or 2).
20. 0. By the Net Change Theorem, $f(2) - f(-2) = \int_{-2}^{2} x^3 \cos x\,dx$. The integrand is odd (odd times even), and the interval is symmetric, so the integral is 0.
21. Displacement $= \int_0^4 (t^2 - 2t)\,dt = \frac{64}{3} - 16 = \frac{16}{3}$. Since $v < 0$ on $(0, 2)$ and $v > 0$ on $(2, 4)$: $\int_0^2 v\,dt = -\frac{4}{3}$ and $\int_2^4 v\,dt = \frac{20}{3}$. Distance $= \frac{4}{3} + \frac{20}{3} = 8$.

4.11 Average Value and the Mean Value Theorem for Integrals

SET A · SKILLS

1. 9. $\int_0^3 (x^2 + 4x)\,dx = 9 + 18 = 27$, and $27 \div 3 = 9$.
2. $\frac{13}{6}$. $\int_1^9 x^{1/2}\,dx = \frac{2}{3}(27 - 1) = \frac{52}{3}$, and $\frac{52}{3} \div 8 = \frac{13}{6}$.
3. $\frac{2}{e^2 - 1}$. $\int_1^{e^2} \frac{1}{x}\,dx = \ln e^2 = 2$, divided by the length $e^2 - 1$.
4. $\frac{2}{\pi}$. $\int_0^{\pi/2} \cos x\,dx = 1$, divided by $\frac{\pi}{2}$.
5. $\frac{4}{\ln 3}$. $\int_0^{\ln 3} e^{2x}\,dx = \frac{1}{2}(e^{2\ln 3} - 1) = \frac{1}{2}(9 - 1) = 4$, divided by $\ln 3$.
6. $c = \sqrt{\frac{13}{3}} = \frac{\sqrt{39}}{3} \approx 2.082$. The average value is $\frac{1}{2}(27 - 1) = 13$. Solve $3c^2 = 13$ and keep the root in $(1, 3)$.
7. $\frac{11}{3}$. Area: $\frac{1+5}{2}(2) + 5(2) + \frac{5+1}{2}(2) = 6 + 10 + 6 = 22$, and $22 \div 6 = \frac{11}{3}$.
8. $\frac{17}{5}$ (or 3.4). Trapezoids: $\frac{3+5}{2}(2) + \frac{5+2}{2}(4) + \frac{2+4}{2}(4) = 8 + 14 + 12 = 34$, and $34 \div 10 = 3.4$.
9. The average value is 1. $\frac{1}{2}\int_0^2 (3x^2 - 3)\,dx = \frac{1}{2}\left[x^3 - 3x\right]_0^2 = \frac{1}{2}(2) = 1$. It equals the average rate of change $\frac{f(2) - f(0)}{2 - 0} = \frac{2 - 0}{2} = 1$.

SET B · APPLICATIONS

10. 23.820 degrees Celsius. $\int_0^{12} 6\sin\frac{\pi t}{12}\,dt = \frac{144}{\pi}$, so the average is $20 + \frac{12}{\pi} \approx 23.820$.
11. 4.143 moles per liter per minute. $\frac{1}{5}\int_0^5 8e^{-0.3t}\,dt = \frac{1}{5} \cdot \frac{8}{0.3}(1 - e^{-1.5}) \approx 4.143$.
12. Trapezoids: $\frac{1.2+1.6}{2} + \frac{1.6+2.1}{2} + \frac{2.1+1.9}{2} + \frac{1.9+1.4}{2} = 1.4 + 1.85 + 2 + 1.65 = 6.9$. Average $\approx 6.9 \div 4 = 1.725$ cubic meters per hour: the gas flows at an average rate of about 1.725 cubic meters per hour during the 4 hours.
13. Average ≈ 0.297, since $\int_0^2 xe^{-x}\,dx = 1 - 3e^{-2}$. f is continuous, so solutions exist: $f(c) = 0.297$ gives $c \approx 0.480$ and $c \approx 1.804$ (one on each side of the maximum at $x = 1$).
14. Average velocity ≈ -1.568 meters per second: $\frac{1}{4}(3\sin 4 - 4)$. Since $v = 0$ at $t = \arccos\frac{1}{3} \approx 1.231$ and $v < 0$ afterward, average speed $= \frac{1}{4}\int_0^4 |v|\,dt \approx 2.366$ meters per second.
15. 43.582 people per minute. $\frac{1}{10}\int_0^{10} (40 + 25\sin(0.5t))\,dt = 40 + \frac{25}{10} \cdot 2(1 - \cos 5) \approx 43.582$.
16. The integral equals the area of a semicircle of radius 5: $\frac{25\pi}{2}$, so the average is $\frac{25\pi}{2 \cdot 10} = \frac{5\pi}{4} \approx 3.927$. Solve $\sqrt{25 - c^2} = \frac{5\pi}{4}$: $c = \pm 5\sqrt{1 - \frac{\pi^2}{16}} \approx \pm 3.095$.

SET C · CHALLENGE

17. The average value of f is $\frac{1}{5-1}(12) = 3$. Because f is continuous on $[1, 5]$, the Mean Value Theorem for Integrals guarantees a number c in $[1, 5]$ with $f(c) = 3$.
18. $k = 4$. $\int_0^3 kx^2\,dx = 9k$, so the average is $3k$. Setting $3k = 12$ gives $k = 4$.
19. $b = 4$. The average is $\frac{1}{b-1} \cdot \frac{b^3 - 1}{3} = \frac{b^2 + b + 1}{3} = 7$, so $b^2 + b - 20 = 0$ and $b = 4$ (the root -5 is rejected since $b > 1$).
20. (a) Areas: $\frac{1}{2}(2)(4) = 4$; $-\frac{1}{2}(2)(2) = -2$; $-2(2) = -4$; on $[6, 8]$ the areas -1 and $+1$ cancel. Total -2, so the average is $\frac{-2}{8} = -\frac{1}{4}$.
(b) f is continuous on $[0, 8]$ (its graph has no breaks), so by the Mean Value Theorem for Integrals some c has $f(c) = -\frac{1}{4}$.
21. (a) By the Net Change Theorem $\int_2^8 f' = f(8) - f(2) = 18$, so the average value is $\frac{18}{6} = 3$.
(b) f' is continuous on $[2, 8]$ and its average value is 3, so by the Mean Value Theorem for Integrals there is c in $[2, 8]$ with $f'(c) = 3$.

4.12 Integrals Involving Logarithms, Exponentials, and Inverse Trig

SET A · SKILLS

1. $5\ln|x| + C$. The constant 5 factors out and $\int \frac{1}{x}\,dx = \ln|x|$.
2. $\ln(x^4 + 1) + C$. The numerator $4x^3$ is the derivative of $x^4 + 1$, so use $u = x^4 + 1$.
3. $\frac{1}{5}e^{5x} + C$. Let $u = 5x$, $du = 5\,dx$.
4. $\frac{1}{3}e^{x^3} + C$. Let $u = x^3$, $du = 3x^2\,dx$.
5. $\frac{1}{2}$. Let $u = \ln x$, $du = \frac{1}{x}\,dx$; limits 0 to 1: $\int_0^1 u\,du = \frac{1}{2}$.
6. $\frac{\pi}{6}$. $\arcsin x\Big|_0^{1/2} = \arcsin\frac{1}{2} - 0$.
7. $\frac{\pi}{8}$. $\frac{1}{2}\arctan\frac{x}{2}\Big|_0^2 = \frac{1}{2}\arctan 1 = \frac{1}{2} \cdot \frac{\pi}{4}$.
8. $\frac{1}{2}\ln 2$. $\int \tan x\,dx = -\ln|\cos x|$, so the value is $-\ln\frac{\sqrt{2}}{2} + \ln 1 = \ln\sqrt{2} = \frac{1}{2}\ln 2$.
9. $\frac{e^2+3}{2}$. Split: $\int_1^e \left(x + \frac{2}{x}\right)dx = \left[\frac{x^2}{2} + 2\ln x\right]_1^e = \left(\frac{e^2}{2} + 2\right) - \frac{1}{2}$.

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SET B · APPLICATIONS

10. 621.496 grams. $\int_0^{30} 40e^{-0.05t}\,dt = 800\left(1-e^{-1.5}\right) \approx 621.496$.
11. ≈ 460.517 dollars. $\int_0^{45} \frac{200}{q+5}\,dq = 200\ln\frac{50}{5} = 200\ln 10$.
12. $4\arcsin\frac{2}{3} \approx 2.919$ square units. $\int_0^2 \frac{4}{\sqrt{9-x^2}}\,dx = 4\arcsin\frac{x}{3}\Big|_0^2$.
13. 0.324. $\int_1^5 \frac{\ln x}{x}\,dx = \frac{(\ln 5)^2}{2}$, so the average is $\frac{(\ln 5)^2}{8} \approx 0.324$.
14. 10584 individuals. $P(10) = 1000 + \int_0^{10} 300e^{0.2t}\,dt = 1000 + 1500(e^2 - 1) \approx 10583.584$.
15. $12 - 2\ln 5 \approx 8.781$. Rewrite $\frac{3x+1}{x+1} = 3 - \frac{2}{x+1}$ and integrate.
16. ≈ 0.133. Complete the square: $x^2+6x+13 = (x+3)^2+4$, so the value is $\frac{1}{2}\left[\arctan\frac{x+3}{2}\right]_0^3 = \frac{1}{2}(\arctan 3 - \arctan 1.5) \approx 0.133$.

SET C · CHALLENGE

17. $\ln 2$. Let $u = \ln x$, $du = \frac{1}{x}\,dx$. Limits 1 to 2: $\int_1^2 \frac{1}{u}\,du = \ln 2$.
18. $\ln\frac{4}{3}$. $\frac{1}{1+e^x} = \frac{e^{-x}}{e^{-x}+1}$. With $u = e^{-x}+1$, the antiderivative is $-\ln(e^{-x}+1)$. Value: $-\ln\frac{3}{2} + \ln 2 = \ln\frac{4}{3}$.
19. $\frac{3}{2}\ln 2 + \frac{\pi}{2}$. Split: $\frac{3}{2}\int_0^1 \frac{2x}{x^2+1}dx + 2\int_0^1 \frac{dx}{x^2+1} = \frac{3}{2}\ln 2 + 2\cdot\frac{\pi}{4}$.
20. $\frac{\pi}{6}$. $3+2x-x^2 = 4-(x-1)^2$, so the antiderivative is $\arcsin\frac{x-1}{2}$. Value: $\arcsin\frac{1}{2} - \arcsin 0 = \frac{\pi}{6}$.
21. $\frac{7\pi}{3}$. $f(x) = 4\arctan x + C$; $f(1) = \pi + C = 2\pi$ gives $C = \pi$. Then $f(\sqrt{3}) = 4\cdot\frac{\pi}{3} + \pi = \frac{7\pi}{3}$.

4.13 Multiple-Choice Practice on Integrals

SET A · PART A [NO CALCULATOR]

1. C. Each term is raised by one power and divided by the new exponent: $2x^3 - 2x^2 + x$.
2. A. $[x^2+x]_1^3 = 12 - 2 = 10$.
3. E. By the Fundamental Theorem with the chain rule: $(x^6+1)\cdot 2x = 2x^7 + 2x$.
4. B. $g' = f$ is positive on $(0,3)$ and negative on $(3,6)$, so g increases then decreases: $g(3) = 5$ is greater than $g(0) = 0$ and $g(6) = 0$.
5. D. Let $u = x^2+1$, $x\,dx = \frac{1}{2}du$: $\frac{1}{2}\cdot\frac{2}{3}u^{3/2} = \frac{1}{3}u^{3/2}$.
6. B. Let $u = \sin x$: $\int_0^1 u^2\,du = \frac{1}{3}$.
7. D. Width 1, left endpoints $0, 1, 2, 3$: $0+1+4+9 = 14$.
8. A. Use right endpoints: $1(5) + 2(2) + 4(1) = 13$.
9. C. $\int_3^5 f = 7 - 2 = 5$, so $2(5) + \int_3^5 1\,dx = 10 + 2 = 12$.
10. E. $\frac{1}{2}\int_0^2 3x^2\,dx = \frac{1}{2}\cdot 8 = 4$.
11. D. x^3 is odd and integrates to 0; x^2 is even: $2\int_0^2 x^2\,dx = \frac{16}{3}$.
12. B. Let $u = x^2$: $\frac{1}{2}\int_0^1 e^u\,du = \frac{e-1}{2}$.
13. C. $\frac{1}{3}\arctan\frac{x}{3}\Big|_0^3 = \frac{1}{3}\cdot\frac{\pi}{4}$.
14. A. $[3\ln x + x^2]_1^e = (3+e^2) - (0+1) = e^2+2$.
15. E. $v = 3t(t-2)$, so $v < 0$ on $(0,2)$. $\int_0^2 v\,dt = -4$ and $\int_2^3 v\,dt = 4$; distance $= 4 + 4 = 8$. (The displacement is 0.)
16. A. $G'(x) = x^2 - 3x = x(x-3) < 0$ exactly when $0 < x < 3$.
17. E. $f(x) = 2x^3 - 2x + C$; $f(1) = C = 5$. So $f(2) = 16 - 4 + 5 = 17$.
18. B. The numerator is the derivative of the denominator: $\ln(x^2+3x+2)\Big|_0^1 = \ln 6 - \ln 2 = \ln 3$.
19. D. The semicircle (radius 2) has area 2π. The triangle below the axis has area $\frac{1}{2}(2)(3) = 3$ and counts as negative.
20. C. For an increasing function the left sum is low and the right sum is high. For concave up, chords lie above the curve (trapezoids overestimate) and tangent lines lie below (midpoint underestimates).

SET B · PART B [CALCULATOR]

21. A. Use the calculator's numerical integration: 0.88208.
22. D. $\frac{1}{3}\int_0^3 \ln(x^2+1)\,dx \approx \frac{3.406}{3} = 1.135$.
23. B. Trapezoids: $\frac{5+9}{2}(2) + \frac{9+6}{2}(3) + \frac{6+2}{2}(4) = 14 + 22.5 + 16 = 52.5$. Then $30 + 52.5 = 82.5$.
24. E. The velocity changes sign at $t = \pi$, so distance $= \int_0^4 |t\sin t|\,dt \approx 4.425$. The displacement $\int_0^4 v\,dt \approx 1.858$ is not the distance.
25. C. The average is $\frac{1}{4}\int_1^5 \sqrt{x}\,dx \approx 1.6967$. Then $\sqrt{c} = 1.6967$ gives $c = 1.6967^2 \approx 2.879$.
26. E. Leaked: $\int_0^6 8e^{-0.3t}\,dt = \frac{8}{0.3}(1-e^{-1.8}) \approx 22.259$; the volume is $50 - 22.259 \approx 27.741$.
27. B. By the Fundamental Theorem, $F'(x) = \cos(x^2)$, so $F'(2) = \cos 4 \approx -0.654$ (radians).
28. C. $\arctan k = 1$, so $k = \tan 1 \approx 1.557$.
29. A. Width 0.5. $T_4 = 0.5\left[\frac{1}{2}f(0) + f(0.5) + f(1) + f(1.5) + \frac{1}{2}f(2)\right] \approx 3.283$. The value 3.241 is the exact integral, not T_4.
30. D. $100 + \int_0^5 (3 + 4\sin t)\,dt = 100 + 15 + 4(1 - \cos 5) \approx 117.865$.

4.14 Free-Response Practice on Integrals

QUESTION 1 · RATE IN CONTEXT (TABLE)

1. (a) Trapezoids: $\frac{12+18}{2}(4) + \frac{18+22}{2}(5) + \frac{22+16}{2}(4) + \frac{16+10}{2}(7) = 60 + 100 + 76 + 91 = 327$, so $\int_0^{20} r(t)\,dt \approx 327$. The integral is the total amount of fluid, in liters, that enters the tank from $t = 0$ to $t = 20$ minutes.

   (b) $\int_0^{20} L(t)\,dt = \left[3t + 0.2t^2\right]_0^{20} = 60 + 80 = 140$ liters. Amount at $t = 20$: $50 + 327 - 140 = 237$ liters.

   (c) $\frac{1}{20}\int_0^{20} r(t)\,dt \approx \frac{327}{20} = 16.35$ liters per minute.

   (d) Since r is differentiable, it is continuous on $[4, 9]$. We have $r(4) = 18 < 20 < 22 = r(9)$, so by the Intermediate Value Theorem there is a t in $(4, 9)$ with $r(t) = 20$.

   Scoring: 9 points. (a) 3 points: trapezoid setup 1, value 327 1, meaning with units 1. (b) 2 points: $\int L = 140$ 1, answer 237 1. (c) 1 point: 16.35. (d) 3 points: continuity 1, comparison of $r(4)$ and $r(9)$ with 20 1, conclusion citing the Intermediate Value Theorem 1.

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QUESTION 2 · GRAPH AND ACCUMULATION

2. (a) $g(4) = \int_0^4 f = \frac{2+4}{2}(2) + \frac{4+0}{2}(2) = 6 + 4 = 10$. $g(6) = 10 + \int_4^6 f = 10 - \frac{1}{2}(2)(3) = 7$. $g(8) = 7 + 2(-3) = 1$.

(b) $g'(3) = f(3) = 2$ (the segment from $(2, 4)$ to $(4, 0)$ passes through $(3, 2)$). $g''(3) = f'(3) = \frac{0-4}{4-2} = -2$.

(c) $g'' = f'$. Since f is decreasing on $(2, 6)$, $f' < 0$ there, so g is concave down on $(2, 6)$. (On $(0, 2)$, f is increasing, and on $(6, 8)$, f is constant.)

(d) $g' = f$, which is positive on $[0, 4)$ and negative on $(4, 8]$, so g increases then decreases and the only critical point where the sign changes is $x = 4$. Candidates: $g(0) = 0$, $g(4) = 10$, $g(8) = 1$. The absolute maximum value is 10.

Scoring: 9 points. (a) 3 points: 1 for each value. (b) 2 points: $g'(3) = 2$ 1, $g''(3) = -2$ 1. (c) 2 points: interval $(2, 6)$ 1, reason using $f' < 0$ or f decreasing 1. (d) 2 points: justification with sign change of $g' = f$ at $x = 4$ or candidates test 1, answer 10 1.

QUESTION 3 · PARTICLE MOTION

3. (a) $a(t) = v'(t) = 4\cos t - 1$, so $a(2) = 4\cos 2 - 1 \approx -2.665$ meters per second squared.

(b) $x(5) = 2 + \int_0^5 (4\sin t - t)\,dt = 2 + \left[-4\cos t - \frac{t^2}{2}\right]_0^5 = 2 + (4 - 4\cos 5) - 12.5 \approx -7.635$ meters.

(c) $v(t) = 0$ at $t \approx 2.475$; $v > 0$ on $(0, 2.475)$ and $v < 0$ on $(2.475, 5]$. Distance $= \int_0^5 |v(t)|\,dt \approx 4.081 + 13.716 = 17.796$ meters.

(d) $v(2) = 4\sin 2 - 2 \approx 1.637 > 0$ and $a(2) \approx -2.665 < 0$. The velocity and acceleration have opposite signs, so the speed is decreasing at $t = 2$.

Scoring: 9 points. (a) 2 points: v' 1, value with units 1. (b) 3 points: integral setup 1, antiderivative or numerical integration 1, answer 1. (c) 2 points: integral of $|v|$ or split at $t \approx 2.475$ 1, answer 1. (d) 2 points: values of $v(2)$ and $a(2)$ with signs 1, conclusion 1.

QUESTION 4 · RATES AND AVERAGES

4. (a) $C(2) = 3 + \int_0^2 \frac{4t}{t^2+4}\,dt$. With $u = t^2 + 4$, $du = 2t\,dt$: $\int_4^8 \frac{2}{u}\,du = 2\ln u\Big|_4^8 = 2\ln 8 - 2\ln 4 = 2\ln 2$. So $C(2) = 3 + 2\ln 2 \approx 4.386$ milligrams per liter.

(b) $\dfrac{C(2) - C(0)}{2 - 0} = \dfrac{2\ln 2}{2} = \ln 2 \approx 0.693$ milligrams per liter per hour.

(c) $C'(t) = \frac{4t}{t^2+4}$ is continuous on $[0, 2]$ (the denominator is never 0). Its average value on $[0, 2]$ is $\frac{1}{2}\int_0^2 C'(t)\,dt = \ln 2$ by the Net Change Theorem. By the Mean Value Theorem for Integrals (or the Mean Value Theorem for C, which is differentiable on $(0, 2)$ and continuous on $[0, 2]$) there is a c in $(0, 2)$ with $C'(c) = \ln 2$.

(d) $\int_0^2 \frac{6}{t^2+4}\,dt = 6 \cdot \frac{1}{2}\arctan\frac{t}{2}\Big|_0^2 = 3(\arctan 1 - 0) = \frac{3\pi}{4}$ grams (about 2.356 grams).

Scoring: 9 points. (a) 3 points: setup with $C(0)$ 1, substitution or antiderivative 1, answer 1. (b) 1 point: $\ln 2$. (c) 3 points: continuity of C' 1, average value equals $\ln 2$ 1, conclusion citing a correct theorem 1. (d) 2 points: arctangent antiderivative 1, answer $\frac{3\pi}{4}$ 1.

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Chapter 5: Applications of Integrals.

10 sections with explanations and worked examples, then 202 practice problems. Answers and full explanations are at the end of the chapter.

5.1 Area Between Curves

You will use definite integrals to find the area trapped between two curves, choosing vertical or horizontal strips and splitting the region wherever the top curve changes.

A definite integral $\int_a^b f(x)\,dx$ measures signed area between a curve and the x-axis. To find the area of a region trapped between two curves, slice the region into thin strips, find the length of each strip (top minus bottom, or right minus left), and add up the strips with an integral.

Area Between Curves (integrating in x)

Suppose f and g are continuous on $[a, b]$ and $f(x) \geq g(x)$ for every x in $[a, b]$. The area of the region between the curves and between the lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

Each strip has width dx and height (top curve) minus (bottom curve). If the curves cross, use $A = \int_a^b |f(x) - g(x)|\,dx$ and split at each crossing.

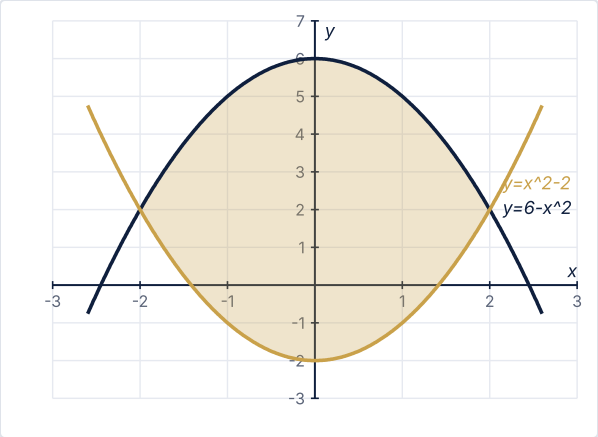
1. **Sketch.** Draw both curves and shade the region. A sketch tells you which curve is on top.
2. **Find the intersections.** Set the two expressions equal and solve. These give the limits of integration when the curves bound the region.
3. **Decide the slice direction.** Use vertical strips (integrate in x) if the top and bottom curves stay the same across the region. Use horizontal strips (integrate in y) if the left and right curves are simpler.
4. **Set up and evaluate.** Write (top) minus (bottom) or (right) minus (left), integrate, and use the Fundamental Theorem of Calculus.

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EXAMPLE 1

Find the area of the region enclosed by $y = 6 - x^2$ and $y = x^2 - 2$.



The region in Example 1.

SOLUTION

Set the curves equal: $6 - x^2 = x^2 - 2$, so $x^2 = 4$ and $x = \pm 2$.

On $[-2, 2]$ the parabola $y = 6 - x^2$ is on top (at $x = 0$ it is 6, while the other is -2).

$A = \int_{-2}^{2} \left[(6 - x^2) - (x^2 - 2)\right] dx = \int_{-2}^{2} (8 - 2x^2)\, dx = \left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$.

$A = \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right) = \frac{64}{3}$.

Area Between Curves (integrating in y)

Suppose u and v are continuous on $[c, d]$ with $u(y) \geq v(y)$ there. The area between the curves $x = u(y)$ and $x = v(y)$ for $c \leq y \leq d$ is

$$A = \int_{c}^{d} \left[u(y) - v(y)\right] dy.$$

Here u is the right curve and v is the left curve. Intersections must be found as y-values.

EXAMPLE 2

Find the area of the region bounded by $x = y^2 - 4$ and $x = 2 - y$.

SOLUTION

Both are given as x in terms of y, so use horizontal strips. Set $y^2 - 4 = 2 - y$: $y^2 + y - 6 = 0$, so $(y + 3)(y - 2) = 0$ and $y = -3$ or $y = 2$.

Test $y = 0$: the line gives $x = 2$ and the parabola gives $x = -4$. The line is on the right.

$A = \int_{-3}^{2} \left[(2 - y) - (y^2 - 4)\right] dy = \int_{-3}^{2} (6 - y - y^2)\, dy = \left[6y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-3}^{2}$.

At $y = 2$ the antiderivative is $12 - 2 - \frac{8}{3} = \frac{22}{3}$. At $y = -3$ it is $-18 - \frac{9}{2} + 9 = -\frac{27}{2}$.

$A = \frac{22}{3} - \left(-\frac{27}{2}\right) = \frac{125}{6}$.

Common mistake

Do not automatically subtract the function that is written first. Always decide which curve is above (or to the right) by testing a point between the intersections. Subtracting in the wrong order gives a negative area, which is a signal to check your work.

When the curves cross inside the interval you are asked about, the top curve changes, and one integral of $f - g$ would let positive and negative parts cancel. Area must be positive everywhere, so split the interval at each crossing and always subtract (top) minus (bottom) on each piece.

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EXAMPLE 3

Find the total area between $y = x^2$ and $y = 3x - 2$ for $0 \le x \le 3$.

SOLUTION

Intersections: $x^2 = 3x - 2$ gives $x^2 - 3x + 2 = 0$, so $x = 1$ and $x = 2$. Both lie inside $[0, 3]$, so the top curve changes.

Test points: at $x = 0.5$, $x^2 = 0.25 > 3x - 2 = -0.5$. At $x = 1.5$, $x^2 = 2.25 < 2.5$. At $x = 2.5$, $x^2 = 6.25 > 5.5$.

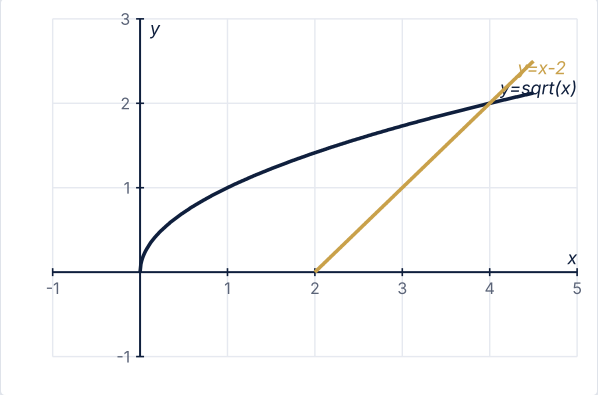
$A = \int_0^1 (x^2 - 3x + 2)\,dx + \int_1^2 (3x - 2 - x^2)\,dx + \int_2^3 (x^2 - 3x + 2)\,dx$.

With $F(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x$ we have $F(0) = 0$, $F(1) = \frac{5}{6}$, $F(2) = \frac{2}{3}$, $F(3) = \frac{3}{2}$.

$A = [F(1) - F(0)] - [F(2) - F(1)] + [F(3) - F(2)] = \frac{5}{6} + \frac{1}{6} + \frac{5}{6} = \frac{11}{6}$.

EXAMPLE 4

Find the area of the region bounded by $y = \sqrt{x}$, $y = x - 2$, and the x-axis.



Region bounded by the curve, the line and the x-axis.

SOLUTION

The line $y = x - 2$ meets $y = \sqrt{x}$ where $x - 2 = \sqrt{x}$. Let $s = \sqrt{x}$: $s^2 - s - 2 = 0$, so $s = 2$ and $x = 4$ (the root $s = -1$ is rejected). The point is $(4, 2)$.

In terms of x, the lower boundary changes (the x-axis, then the line), which would need two integrals. In terms of y, the right curve is always the line $x = y + 2$ and the left curve is always $x = y^2$, for $0 \le y \le 2$.

$A = \int_0^2 [(y + 2) - y^2]\,dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^2 = 2 + 4 - \frac{8}{3} = \frac{10}{3}$.

Choosing the direction

If, with vertical strips, you would need more than one integral because the top or bottom curve changes, try horizontal strips. When both methods give the same answer, you have a built-in check.

Practice

SET A **Skills**

Find the exact area of each region. [No calculator]

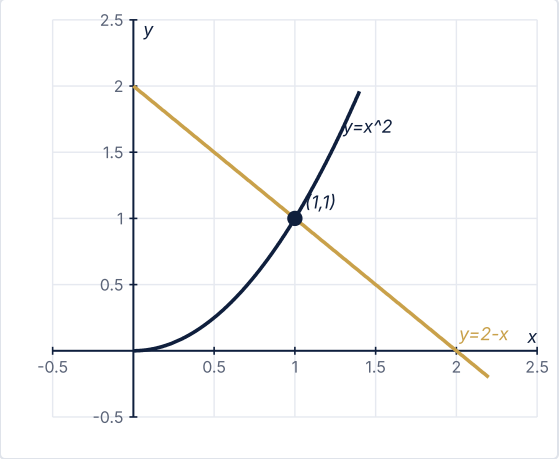
1. The region enclosed by $y = x^2$ and $y = 2x + 3$.
2. The region enclosed by $y = \sqrt{x}$ and $y = \dfrac{x}{2}$.
3. The region bounded by $y = e^x$, $y = 1$, and the line $x = \ln 3$.
4. The region bounded by $y = \cos x$, $y = \sin x$, and the lines $x = 0$ and $x = \dfrac{\pi}{4}$.
5. The region bounded by $y = \dfrac{1}{x}$, $y = \dfrac{1}{x^2}$, and the lines $x = 1$ and $x = e$.
6. The region enclosed by $x = y^2 - 2y$ and $x = y$. (Integrate with respect to y.)
7. The total area between $y = x^3$ and $y = x$ for $-1 \le x \le 2$.

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8. The total area enclosed between $y = \sin x$ and $y = \sin 2x$ for $0 \le x \le \pi$.

9. The region in the first quadrant bounded by $y = x^2$, $y = 2 - x$, and the x-axis is shown. Find its area by integrating with respect to y.

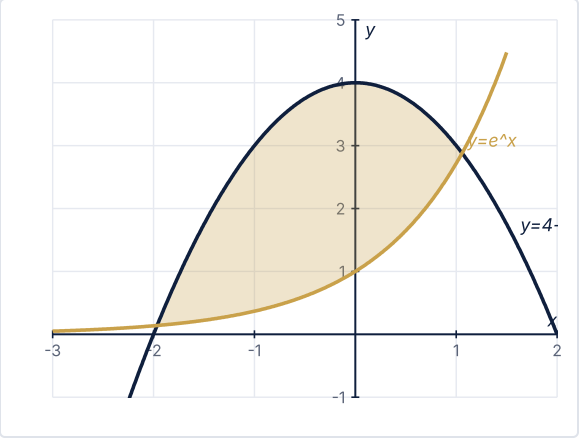


The region lies above the x-axis, right of the parabola and left of the line.

SET B Applications

Give decimal answers correct to three decimal places unless an exact value is requested. [Calculator]

10. Find the area of the region enclosed by the graphs of $y = 4 - x^2$ and $y = e^x$.



The region lies between the two intersection points.

11. Find the area of the region enclosed by $y = \ln x$ and $y = x - 2$.

12. A tank holds 20 liters at $t = 0$. Water enters at rate $A(t) = 12e^{-0.3t}$ liters per minute and leaves at rate $B(t) = 3 + 0.5t$ liters per minute, for $0 \le t \le 8$ minutes.
    (a) Find the time t_1 in $(0, 8)$ at which $A(t) = B(t)$.
    (b) How much water is in the tank at time t_1?
    (c) How much water is in the tank at $t = 8$?

13. Two curves f and g satisfy $f(x) \ge g(x)$ on $[0, 8]$. Values are in the table. Use a trapezoidal sum with the four subintervals shown to approximate the area between the curves.

x	0	2	4	6	8
$f(x)$	10	9	7	4	1
$g(x)$	2	1	1	2	0

14. Which integral gives the area of the region in the first quadrant bounded by $y = x^2$, $y = 8 - x^2$, and the y-axis?
    (A) $\int_0^2 (2x^2 - 8)\,dx$ **(B)** $\int_0^2 (8 - 2x^2)\,dx$ **(C)** $\int_0^4 (8 - 2x^2)\,dx$
    (D) $\int_0^2 (8 - x^2)\,dx$ **(E)** $\int_0^2 8\,dx$

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15. A metal plate has the shape of the region between $y = 9\cos\left(\frac{x}{3}\right)$ and the line $y = \frac{9}{2}$ (units are centimeters), for the part where the cosine curve is above the line. Find the area of the plate in square centimeters.
16. The horizontal line $y = c$ with $c > 0$ and the parabola $y = x^2$ enclose a region of area $\frac{32}{3}$. Find c.

SET C Challenge

Exact answers are required unless a decimal is requested. [No calculator]

17. The region bounded by $y = x^2$, the x-axis, and $x = 2$ is divided into two regions of equal area by the vertical line $x = k$. Find k.
18. The region R is bounded by $y = x^2$ and the line $y = 4$. The horizontal line $y = k$, with $0 < k < 4$, divides R into two regions of equal area. Find k.
19. For $c > 0$, the line $y = cx$ and the parabola $y = x^2$ enclose a region of area 36. Find c.
20. Functions f and g are continuous on $[1, 7]$ with $f(x) \geq g(x)$ on $[1, 4]$ and $f(x) \leq g(x)$ on $[4, 7]$. Also $\int_1^4 f(x)\,dx = 15$, $\int_1^4 g(x)\,dx = 6$, $\int_4^7 f(x)\,dx = 2$, and $\int_4^7 g(x)\,dx = 9$.
    (a) Find the area of the region enclosed by the graphs of f and g on $[1, 7]$.
    (b) Find $\int_1^7 [f(x) - g(x)]\,dx$ and explain why it differs from the area.
21. Find the area of the region that lies above $y = |x|$ and below $y = 6 - x^2$.

5.2 Volumes of Solids with Known Cross-Sections

You will build volumes by adding up thin slices whose area is known, using squares, semicircles, triangles and rectangles standing on a base region.

Imagine slicing a loaf of bread. Each slice is thin and nearly flat, so its volume is about (area of the face) times (thickness). If every slice has a face area we can describe by a formula, adding up all slices with an integral gives the volume of the whole loaf. A solid whose slices share a known shape is said to have **known cross-sections**.

Volume by Cross-Sections

Let a solid lie between $x = a$ and $x = b$. If $A(x)$ is the area of the cross-section perpendicular to the x-axis at x, and A is continuous on $[a, b]$, then

$$V = \int_a^b A(x)\,dx.$$

If the cross-sections are perpendicular to the y-axis, with area $A(y)$ for $c \leq y \leq d$, then $V = \int_c^d A(y)\,dy$.

In almost every problem the **base** is a region in the xy-plane. The length of the base across the region, at a given x, is $s = (\text{top}) - (\text{bottom})$ for vertical slices, or $s = (\text{right}) - (\text{left})$ for horizontal slices. That length s is a side, a diameter, or a leg of the cross-section.

Cross-Section Areas in Terms of the Base Length s

Square with side s: $A = s^2$.
Semicircle with diameter s: $A = \frac{1}{2}\pi\left(\frac{s}{2}\right)^2 = \frac{\pi}{8}s^2$.
Equilateral triangle with side s: $A = \frac{\sqrt{3}}{4}s^2$.
Isosceles right triangle with leg s: $A = \frac{1}{2}s^2$. With hypotenuse s: $A = \frac{1}{4}s^2$.
Rectangle with base s and height h: $A = sh$. If $h = ks$, then $A = ks^2$.

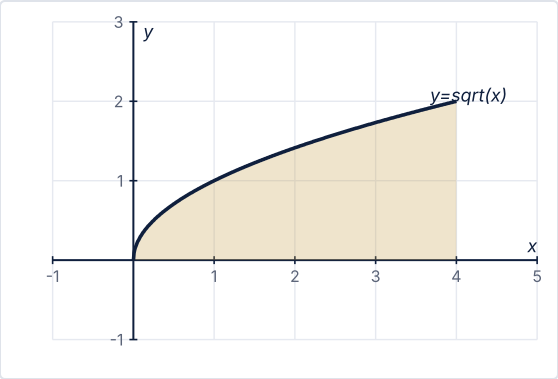
1. **Sketch the base.** Find where the bounding curves meet; these give the limits of integration.
2. **Find s.** Write the length of a slice across the base: $s = f(x) - g(x)$ (top minus bottom) or $s = u(y) - v(y)$ (right minus left).
3. **Write A in terms of s.** Use the correct area formula for the cross-section shape and whatever the problem says about how s is used (side, diameter, leg).
4. **Integrate.** Compute $V = \int A\,dx$ or $\int A\,dy$. Units of volume are cubic units.

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EXAMPLE 1

The base of a solid is the region bounded by $y = \sqrt{x}$, the x-axis, and $x = 4$. Cross-sections perpendicular to the x-axis are squares. Find the volume.



The base region in Example 1.

SOLUTION

At each x in $[0, 4]$ the base extends from $y = 0$ to $y = \sqrt{x}$, so $s = \sqrt{x}$.

$A(x) = s^2 = x$.

$V = \int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4 = 8$ cubic units.

EXAMPLE 2

The base of a solid is the region between $y = x^2$ and $y = 2x$. Cross-sections perpendicular to the x-axis are semicircles whose diameters lie in the base. Find the volume.

SOLUTION

The curves meet where $x^2 = 2x$, so $x = 0$ and $x = 2$. On $(0, 2)$ the line is on top, so the diameter is $s = 2x - x^2$.

$A(x) = \frac{\pi}{8}s^2 = \frac{\pi}{8}(2x - x^2)^2 = \frac{\pi}{8}(4x^2 - 4x^3 + x^4)$.

$V = \frac{\pi}{8}\int_0^2 (4x^2 - 4x^3 + x^4)\,dx = \frac{\pi}{8}\left[\frac{4x^3}{3} - x^4 + \frac{x^5}{5}\right]_0^2 = \frac{\pi}{8}\left(\frac{32}{3} - 16 + \frac{32}{5}\right) = \frac{\pi}{8}\cdot\frac{16}{15} = \frac{2\pi}{15}$.

Common mistake

The side length is the length of the slice, top minus bottom, not just the formula for the top curve. Also remember to square s after you find it: $(2x - x^2)^2$ is not $4x^2 - x^4$.

EXAMPLE 3

The base of a solid is the region bounded by $y = 4 - x^2$ and the x-axis. Cross-sections perpendicular to the x-axis are equilateral triangles with a side in the base. Find the volume.

SOLUTION

The parabola meets the x-axis at $x = \pm 2$. The slice length is $s = 4 - x^2$.

$A(x) = \frac{\sqrt{3}}{4}(4 - x^2)^2 = \frac{\sqrt{3}}{4}(16 - 8x^2 + x^4)$.

$V = \frac{\sqrt{3}}{4}\int_{-2}^2 (16 - 8x^2 + x^4)\,dx = \frac{\sqrt{3}}{4}\cdot 2\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^2 = \frac{\sqrt{3}}{4}\cdot 2\left(32 - \frac{64}{3} + \frac{32}{5}\right)$.

$32 - \frac{64}{3} + \frac{32}{5} = \frac{256}{15}$, so $V = \frac{\sqrt{3}}{4}\cdot\frac{512}{15} = \frac{128\sqrt{3}}{15}$.

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EXAMPLE 4

The base is the region bounded by $y = x^2$ and $y = 9$. Cross-sections perpendicular to the y-axis are squares. Find the volume.

SOLUTION

Slices are perpendicular to the y-axis, so integrate in y from 0 to 9.

At height y the base runs from $x = -\sqrt{y}$ to $x = \sqrt{y}$, so $s = 2\sqrt{y}$.

$A(y) = s^2 = 4y$, and $V = \int_0^9 4y\,dy = \left[2y^2\right]_0^9 = 162$.

EXAMPLE 5

The base is the region under $y = \sin x$ for $0 \le x \le \pi$. Cross-sections perpendicular to the x-axis are rectangles whose height is three times the length of their base in the region. Find the volume.

SOLUTION

The base length of the rectangle is $s = \sin x$ and the height is $3s$, so $A(x) = s \cdot 3s = 3\sin^2 x$.

$V = 3\int_0^{\pi} \sin^2 x\,dx = 3\int_0^{\pi} \frac{1-\cos 2x}{2}\,dx = \frac{3}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{3\pi}{2}$.

Practice

SET A Skills

Find the exact volume of each solid. [No calculator]

1. The base is the region bounded by $y = x^2$, the x-axis, and $x = 2$. Cross-sections perpendicular to the x-axis are squares.
2. The base is the region bounded by $y = e^x$, the x-axis, the y-axis, and $x = \ln 2$. Cross-sections perpendicular to the x-axis are squares.
3. The base is the region between $y = x$ and $y = x^2$. Cross-sections perpendicular to the x-axis are semicircles with diameters in the base.
4. The base is the region between $y = \sqrt{x}$ and $y = x$. Cross-sections perpendicular to the x-axis are equilateral triangles with a side in the base.
5. The base is the region bounded by $y = \cos x$ and the x-axis for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Cross-sections perpendicular to the x-axis are isosceles right triangles with a leg in the base.
6. The base is the region between $y = x^2$ and $y = 2x$. Cross-sections perpendicular to the x-axis are rectangles whose height is half the length of their base in the region.
7. The base is the region bounded by $x = y^2$ and $x = 4$. Cross-sections perpendicular to the y-axis are semicircles with diameters in the base.
8. The base is the region bounded by $x = e^y$, the y-axis, $y = 0$, and $y = 1$. Cross-sections perpendicular to the y-axis are squares.
9. The base is the triangle in the first quadrant bounded by the axes and the line $y = 6 - 3x$. Cross-sections perpendicular to the x-axis are equilateral triangles with a side in the base.

SET B Applications

Give decimal answers correct to three decimal places unless an exact value is requested. [Calculator]

10. The base of a solid is the region bounded by $y = 2 + \sin x$, the x-axis, $x = 0$, and $x = \pi$. Cross-sections perpendicular to the x-axis are squares. Find the volume.
11. The base of a solid is the region bounded by $y = e^{-x^2}$, the x-axis, $x = -1$, and $x = 1$. Cross-sections perpendicular to the x-axis are semicircles with diameters in the base. Find the volume.
12. A solid lies between $x = 0$ and $x = 4$ (centimeters). The area $A(x)$, in square centimeters, of its cross-section perpendicular to the x-axis is given in the table. Use a trapezoidal sum with four subintervals to approximate the volume, and include units.

x (cm)	0	1	2	3	4
$A(x)$	0	3.1	4.8	4.2	2.0

13. A sculpture sits on a base region bounded by $y = 1 + x^2$ and $y = 5 - x$. Cross-sections perpendicular to the x-axis are equilateral triangles with a side in the base. Find the volume of the sculpture.

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14. The base of a solid is the region bounded by $y = x^3$ and $y = x^2$. Cross-sections perpendicular to the x-axis are semicircles with diameters in the base. Which integral gives the volume?

(A) $\int_0^1 \frac{\pi}{8}(x^2 - x^3)^2\,dx$ (B) $\int_0^1 \frac{\pi}{2}(x^2 - x^3)^2\,dx$ (C) $\int_0^1 \frac{\pi}{4}(x^2 - x^3)^2\,dx$

(D) $\int_0^1 \frac{\pi}{8}(x^2 - x^3)^2\,dx$ (E) $\int_0^1 \pi(x^2 - x^3)^2\,dx$

15. A solid has as its base the disk $x^2 + y^2 \le 9$ (lengths in meters). Cross-sections perpendicular to the x-axis are squares. Find the volume in cubic meters.

16. The base of a solid is the region bounded by the x-axis, $x = 4$, and $y = k\sqrt{x}$, where $k > 0$. Cross-sections perpendicular to the x-axis are squares. The volume is 72. Find k.

SET C Challenge

Exact answers are required unless a decimal is requested. [No calculator]

17. The base is the region between $y = x^2$ and $y = 8 - x^2$. Cross-sections perpendicular to the x-axis are squares whose diagonal lies in the base. Find the volume. (A square with diagonal d has area $\frac{1}{2}d^2$.)

18. The base is the region bounded by $y = 0$ and $y = 2\sqrt{x}$ for $0 \le x \le 4$. Cross-sections perpendicular to the x-axis are rectangles whose height is $x + 1$. Find the volume.

19. The base is the region bounded by $y = \sqrt{x}$, the x-axis, and $x = k$ with $k > 0$. Cross-sections perpendicular to the x-axis are semicircles with diameters in the base. The volume is 4π. Find k.

20. The volume of a solid is given by $V = \int_{-1}^{1} \sqrt{3}\,(1 - x^2)\,dx$. Describe a base region and cross-sections that produce this integral, and evaluate it.

21. A solid has square cross-sections perpendicular to the x-axis. Its base is the region bounded by $y = 2\sqrt{x}$, the x-axis, and the vertical line at $x = b$. The part of the solid from $x = 0$ to $x = b$ has volume $V(b)$.

(a) Find $V(b)$.

(b) If b increases at 3 units per second, how fast is V changing when $b = 5$?

5.3 Volumes of Revolution: Disks and Washers

You will find the volume of a solid formed by rotating a region about a horizontal or vertical line, using circular disks and washers whose radii are distances from the axis.

Rotate a region in the plane about a line, and it sweeps out a solid. If the region touches the axis of rotation, thin slices perpendicular to the axis are **disks**. If there is a gap between the region and the axis, the slices are **washers**, which are disks with a hole in the middle. In both cases each slice has a circular shape, so the cross-section method of the previous lesson applies with $A = \pi(\text{radius})^2$.

Disk and Washer Methods

Let a region be rotated about a horizontal line. If the region lies between $x = a$ and $x = b$, has outer radius $R(x)$ and inner radius $r(x)$ (distances from the axis), with R and r continuous and $R(x) \ge r(x) \ge 0$, then

$$V = \pi \int_a^b \left[R(x)^2 - r(x)^2\right] dx.$$

When the region touches the axis, $r(x) = 0$ and this is the **disk method**: $V = \pi \int_a^b R(x)^2\,dx$.

For a vertical axis, use horizontal slices and integrate in y: $V = \pi \int_c^d \left[R(y)^2 - r(y)^2\right] dy$.

Finding the Radii

A radius is a **distance from the axis of rotation** to a boundary of the region.

Axis $y = k$: a boundary curve $y = f(x)$ has radius $|f(x) - k|$. Axis $x = h$: a boundary curve $x = u(y)$ has radius $|u(y) - h|$.

The region must stay on one side of the axis. The outer radius belongs to the boundary farther from the axis, the inner radius to the boundary closer to it.

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1. **Sketch and shade.** Mark the axis of rotation and decide which side of it the region lies on.
2. **Choose the slice direction.** Slices must be perpendicular to the axis: vertical slices for a horizontal axis (integrate in x), horizontal slices for a vertical axis (integrate in y).
3. **Find R and r.** Measure the distance from the axis to the far and near boundaries of a typical slice.
4. **Integrate.** Compute $\pi \int (R^2 - r^2)$ over the right limits, taking care to square each radius first, then subtract.

EXAMPLE 1: DISKS ABOUT THE X-AXIS

The region under $y = 2\sqrt{x}$ from $x = 0$ to $x = 3$ is rotated about the x-axis. Find the volume.

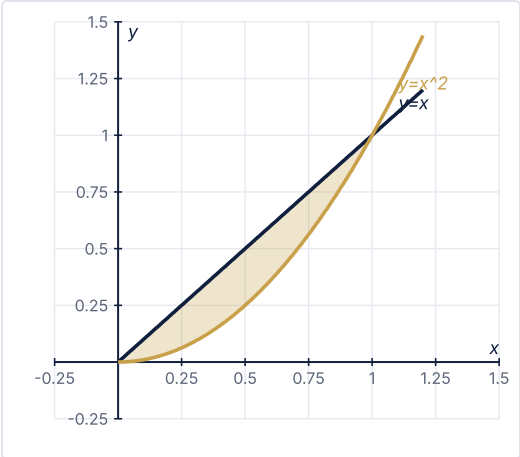
SOLUTION

The region touches the axis, so each slice is a disk of radius $R = 2\sqrt{x}$.

$V = \pi \int_0^3 (2\sqrt{x})^2 dx = \pi \int_0^3 4x\, dx = \pi \left[2x^2\right]_0^3 = 18\pi.$

EXAMPLE 2: WASHERS ABOUT THE X-AXIS

The region between $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume.



The region in Example 2.

SOLUTION

The curves meet at $x = 0$ and $x = 1$. On $(0, 1)$, $x \geq x^2$, so $R = x$ and $r = x^2$.

$V = \pi \int_0^1 \left(x^2 - x^4\right) dx = \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi \left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}.$

Common mistake

The washer formula is $\pi \int (R^2 - r^2)\, dx$, not $\pi \int (R - r)^2\, dx$. The second expression is the area of a disk whose radius is the thickness of the washer, which is a different shape. Square each radius first, then subtract.

EXAMPLE 3: WASHERS ABOUT A LINE Y = K

The region between $y = x$ and $y = x^2$ is rotated about the line $y = 2$. Find the volume.

SOLUTION

The region lies below the line $y = 2$, so radii are $2 - y$. The farther boundary from $y = 2$ is the lower curve $y = x^2$, so $R = 2 - x^2$. The nearer boundary is $y = x$, so $r = 2 - x$.

$V = \pi \int_0^1 \left[(2 - x^2)^2 - (2 - x)^2\right] dx = \pi \int_0^1 \left(4x - 5x^2 + x^4\right) dx.$

$V = \pi \left[2x^2 - \frac{5x^3}{3} + \frac{x^5}{5}\right]_0^1 = \pi \left(2 - \frac{5}{3} + \frac{1}{5}\right) = \frac{8\pi}{15}.$

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EXAMPLE 4: ROTATING ABOUT THE Y-AXIS

The region bounded by $y = x^2$ (for $x \geq 0$), the y-axis, and $y = 4$ is rotated about the y-axis. Find the volume.

SOLUTION

The axis is vertical, so use horizontal slices and write the curve as $x = \sqrt{y}$.

The region touches the axis. At height y, $R = \sqrt{y}$, for $0 \leq y \leq 4$.

$V = \pi \int_0^4 (\sqrt{y})^2 dy = \pi \int_0^4 y\, dy = \pi \left[\frac{y^2}{2}\right]_0^4 = 8\pi$.

EXAMPLE 5: ROTATING ABOUT A VERTICAL LINE X = H

The region bounded by $x = y^2$ and $x = 4$ is rotated about the line $x = 6$. Find the volume.

SOLUTION

The curves meet where $y^2 = 4$, so $y = \pm 2$. At height y the region runs from $x = y^2$ to $x = 4$, all to the left of $x = 6$.

The farther boundary from $x = 6$ is $x = y^2$, so $R = 6 - y^2$. The nearer boundary is $x = 4$, so $r = 6 - 4 = 2$.

$V = \pi \int_{-2}^{2} \left[(6 - y^2)^2 - 4\right] dy = \pi \int_{-2}^{2} \left(32 - 12y^2 + y^4\right) dy = 2\pi \left[32y - 4y^3 + \frac{y^5}{5}\right]_0^2$.

$V = 2\pi \left(64 - 32 + \frac{32}{5}\right) = \frac{384\pi}{5}$.

Practice

SET A Skills

Find the exact volume of the solid generated by rotating each region. [No calculator]

1. The region under $y = x^2$ from $x = 0$ to $x = 2$, about the x-axis.
2. The region under $y = e^x$ from $x = 0$ to $x = \ln 2$, about the x-axis.
3. The region under $y = \sin x$ from $x = 0$ to $x = \pi$, about the x-axis.
4. The region between $y = 2x$ and $y = x^2$, about the x-axis.
5. The region between $y = \sqrt{x}$ and $y = x$, about the x-axis.
6. The region bounded by $y = x^3$, $y = 8$, and the y-axis, about the y-axis.
7. The region bounded by $y = x^2$, the x-axis, and $x = 2$, about the line $y = -1$.
8. The region bounded by $y = \sqrt{x}$, the x-axis, and $x = 4$, about the line $y = 2$.
9. The region between $y = x$ and $y = x^2$, about the line $x = 2$. (Use horizontal slices.)

SET B Applications

Give decimal answers correct to three decimal places unless an exact value is requested. [Calculator]

10. The region under $y = e^{-x^2}$ from $x = 0$ to $x = 1$ is rotated about the x-axis. Find the volume.
11. The region bounded by $y = \ln x$, the x-axis, and $x = e$ is rotated about the x-axis. Find the exact volume and a decimal approximation.
12. A vase is formed by rotating the region under $y = 2 + \sin x$ for $0 \leq x \leq \pi$ about the x-axis (lengths in centimeters). Find its volume, and give its capacity in milliliters (1 cubic centimeter holds 1 milliliter).
13. A solid sphere of radius 5 centimeters is generated by rotating the region under $y = \sqrt{25 - x^2}$ about the x-axis. Use the disk method to show that its volume is $\frac{500\pi}{3}$, and give the decimal value.
14. The region R is bounded by $y = \sqrt{x}$, the line $y = 1$, and the line $x = 4$. Which integral gives the volume of the solid generated when R is rotated about the line $y = -1$?

(A) $\pi \int_1^4 (x - 1)\, dx$

(B) $\pi \int_1^4 \left[(\sqrt{x} + 1)^2 - 4\right] dx$

(C) $\pi \int_1^4 (\sqrt{x} + 1 - 2)^2 \, dx$

(D) $\pi \int_1^4 (\sqrt{x} + 1)^2 \, dx$

(E) $\pi \int_0^4 \left[(\sqrt{x} + 1)^2 - 1\right] dx$

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15. A shaft is turned from a rod. The radius $r(x)$, in centimeters, at distance x centimeters from one end is measured as shown. Use a trapezoidal sum with four subintervals of $r(x)^2$ to approximate the volume of the shaft in cubic centimeters.

x (cm)	0	1	2	3	4
$r(x)$	2	2.4	3	2.8	1.5

16. The region in the first quadrant bounded by $y = 2\sin x$ and $y = x^2$ is rotated about the x-axis. Find the volume.

SET C Challenge

Exact answers are required unless a decimal is requested. [No calculator]

17. The region between $y = x^2$ and $y = x + 2$ is rotated about the x-axis. Find the volume.
18. The region bounded by $y = x^2$ and $y = 4$ is rotated about the line $y = 4$. Find the volume.
19. Let R be the region between $y = x^2$ and $y = 2x$.
    (a) Find the volume when R is rotated about the x-axis.
    (b) Find the volume when R is rotated about the line $y = 4$.
    (c) Which volume is larger, and why does that make sense geometrically?
20. A student computes the volume of the solid formed by rotating the region between $y = x$ and $y = x^2$ about the x-axis as $\pi \int_0^1 (x - x^2)^2 dx$. Find this value, find the correct volume, and explain the error.
21. The region bounded by $y = \ln x$, the x-axis, and $x = e$ is rotated about the y-axis. Find the volume.

5.4 Separable Differential Equations

You will solve first-order equations of the form dy/dx = g(x)h(y) by separating variables, apply initial conditions, and decide on what interval a solution is valid.

A **differential equation** relates a function to its derivatives. In this lesson we solve the simplest useful type: equations where, after algebra, all the y's can be collected on one side with dy and all the x's on the other side with dx. These appear whenever a rate of change depends separately on the input and on the current value.

Separable Differential Equation

A first-order differential equation is **separable** if it can be written

$$\frac{dy}{dx} = g(x)\,h(y).$$

To solve, where $h(y) \neq 0$, separate and integrate:

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx.$$

Include a single constant of integration C, on one side only.

1. **Separate.** Write $\frac{1}{h(y)}\,dy = g(x)\,dx$. Only y and dy on the left, only x and dx on the right.
2. **Integrate both sides.** Add $+C$ on one side immediately after integrating, before doing any other algebra.
3. **Use the initial condition.** Substitute the given point (x_0, y_0) to find C. Do this as soon as possible; it often simplifies the algebra.
4. **Solve for y.** Isolate y if possible. When the problem gives $y(x_0)$, use it to choose the correct sign or branch.
5. **State the domain.** Give the largest open interval containing x_0 on which the formula is defined and satisfies the differential equation.

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EXAMPLE 1

Solve $\frac{dy}{dx} = 3x^2 y$ with $y(0) = 2$.

SOLUTION

Separate: $\frac{dy}{y} = 3x^2\,dx$.

Integrate: $\ln|y| = x^3 + C$. So $|y| = e^C e^{x^3}$, and $y = Ae^{x^3}$ for a constant A.

Use $y(0) = 2$: $2 = Ae^0$, so $A = 2$.

$y = 2e^{x^3}$, defined for all real x.

EXAMPLE 2

Solve $\frac{dy}{dx} = x(y - 3)$ with $y(0) = 5$.

SOLUTION

The constant function $y = 3$ solves the equation but does not meet $y(0) = 5$. For $y \neq 3$, separate: $\frac{dy}{y-3} = x\,dx$.

Integrate: $\ln|y - 3| = \frac{x^2}{2} + C$.

Use $y(0) = 5$: $\ln|2| = C$, so $\ln|y - 3| = \frac{x^2}{2} + \ln 2$.

Exponentiate: $|y - 3| = 2e^{x^2/2}$. Since $y - 3 = 2 > 0$ at $x = 0$ and $2e^{x^2/2}$ never reaches 0, $y - 3 = 2e^{x^2/2}$.

$y = 3 + 2e^{x^2/2}$ for all real x.

Common mistakes

(1) Forgetting $+C$ costs most of the credit on a differential equation problem. (2) Putting $+C$ in after solving for y changes the answer: from $\ln|y| = x^2 + C$ you get $y = Ae^{x^2}$, not $y = e^{x^2} + C$. (3) Writing $y = \sqrt{\cdots}$ when the initial condition says $y < 0$, or the reverse. (4) Not checking the domain.

Domain of a Particular Solution

The domain of a particular solution is the largest open interval containing the initial x-value on which the formula for y is defined and is a differentiable solution of the equation.

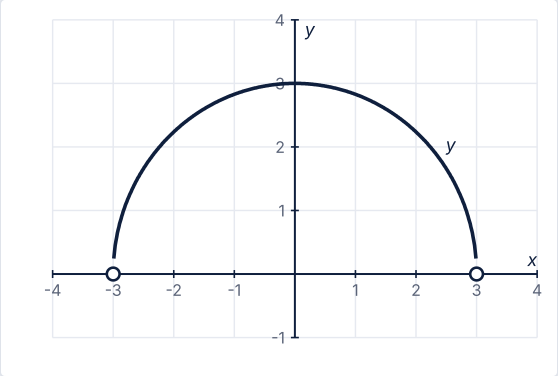
Watch for denominators that become 0, square roots of negative numbers, and logarithms of non-positive numbers. Do not extend the interval past such a point, even if the formula could be evaluated beyond it.

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EXAMPLE 3

Solve $\frac{dy}{dx} = -\frac{x}{y}$ with $y(0) = 3$, and state the domain of the solution.



The solution is the upper half of a circle, without its endpoints.

SOLUTION

Separate: $y\,dy = -x\,dx$. Integrate: $\frac{y^2}{2} = -\frac{x^2}{2} + C$, so $x^2 + y^2 = 2C$.

Use $y(0) = 3$: $0 + 9 = 2C$, so $x^2 + y^2 = 9$.

Solve for y. Because $y(0) = 3 > 0$ we take the positive root: $y = \sqrt{9 - x^2}$.

At $x = \pm 3$ the value is $y = 0$ and the equation $\frac{dy}{dx} = -\frac{x}{y}$ is undefined (the tangent is vertical). So the domain is $-3 < x < 3$.

EXAMPLE 4

Solve $\frac{dy}{dx} = xy^2$ with $y(0) = 1$, and state the domain.

SOLUTION

Separate: $y^{-2}\,dy = x\,dx$. Integrate: $-\frac{1}{y} = \frac{x^2}{2} + C$.

Use $y(0) = 1$: $-1 = C$, so $-\frac{1}{y} = \frac{x^2}{2} - 1$, and $y = \frac{1}{1 - \frac{x^2}{2}} = \frac{2}{2 - x^2}$.

The formula fails at $x = \pm\sqrt{2}$. The domain is the interval containing $x_0 = 0$: $-\sqrt{2} < x < \sqrt{2}$.

Notice the solution grows without bound as $x \to \sqrt{2}$, even though the equation looks harmless.

EXAMPLE 5

Solve $\frac{dy}{dx} = e^{x-y}$ with $y(0) = 1$.

SOLUTION

Rewrite $e^{x-y} = e^x e^{-y}$. Separate: $e^y\,dy = e^x\,dx$.

Integrate: $e^y = e^x + C$. Use $y(0) = 1$: $e = 1 + C$, so $C = e - 1$.

Take logarithms: $y = \ln(e^x + e - 1)$. The argument is always positive, so the domain is all real numbers.

Practice

SET A Skills

Solve each differential equation with the given initial condition. State the domain where asked. [No calculator]

1. $\frac{dy}{dx} = 5y$, $y(0) = -2$.
2. $\frac{dy}{dx} = \frac{x}{y^2}$, $y(0) = 2$.
3. $\frac{dy}{dx} = y\cos x$, $y(0) = e$.
4. $\frac{dy}{dx} = 2x(y + 1)$, $y(0) = 3$.
5. $\frac{dy}{dx} = \frac{y}{x}$, $x > 0$, $y(1) = 3$.
6. $\frac{dy}{dx} = \frac{2x}{3y^2}$, $y(1) = 2$.
7. $\frac{dy}{dx} = xe^y$, $y(0) = 0$. State the domain.
8. $\frac{dy}{dx} = x(1 + y^2)$, $y(0) = 0$. State the domain.

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9. $\dfrac{dy}{dx} = \dfrac{y-2}{x+1}$, $y(0) = 5$. State the domain.

SET B Applications

Give decimal answers correct to three decimal places unless an exact value is requested. [Calculator]

10. Water drains from a tank. The volume V, in liters, satisfies $\dfrac{dV}{dt} = -\dfrac{1}{2}\sqrt{V}$ with $V(0) = 64$, where t is in minutes.
    (a) Solve for $V(t)$ while the tank contains water.
    (b) Find $V(10)$.
    (c) When does the tank become empty?

11. A tank holds 100 liters of liquid. A salt solution enters so that the mass S of salt, in kilograms, satisfies $\dfrac{dS}{dt} = 0.3 - 0.02S$, with $S(0) = 0$ and t in minutes.
    (a) Solve for $S(t)$.
    (b) Find $S(30)$.
    (c) What value does S approach as $t \to \infty$?

12. A falling object has speed v in meters per second, with $\dfrac{dv}{dt} = 9.8 - 0.5v$ and $v(0) = 0$.
    (a) Solve for $v(t)$.
    (b) Find $v(2)$.
    (c) Find the limiting (terminal) speed.

13. A curve passes through $(2, 1)$, and at every point (x, y) on it with $y > 0$ its slope is $-\dfrac{x}{4y}$. Find an equation of the curve in the form $y = f(x)$ and state its domain.

14. Which function is a solution of $\dfrac{dy}{dx} = x^2y^2$ for every constant C (on an interval where it is defined)?

    (A) $y = Ce^{x^3/3}$ **(B)** $y = \dfrac{3}{x^3 + C}$ **(C)** $y = -\dfrac{3}{x^3 + C}$

    (D) $y = \dfrac{x^3}{3} + C$ **(E)** $y = \sqrt{\dfrac{2x^3}{3} + C}$

15. Solve $\dfrac{dy}{dx} = e^x y^2$ with $y(0) = 1$, and find the domain of the solution as an interval. Give the right endpoint to three decimal places.

16. The function $y = f(x)$ satisfies $\dfrac{dy}{dx} = y \sin x$ and $f(0) = 2$. Find $f(1)$.

SET C Challenge

Give exact answers. [No calculator]

17. Solve $\dfrac{dy}{dx} = -2xy^2$ with $y(0) = -1$. State the domain, and explain why the domain is not all real numbers.

18. Each curve $y = Cx^2$ (with $C \neq 0$) satisfies $\dfrac{dy}{dx} = \dfrac{2y}{x}$. Find the curve that crosses every parabola $y = Cx^2$ at a right angle (slope $-\dfrac{x}{2y}$ at each point) and passes through $(2, 1)$.

19. Solve $\dfrac{dy}{dx} = 6x\sqrt{y}$ with $y(0) = 4$.

20. The function $y = f(x)$ satisfies $\dfrac{dy}{dx} = 2xy$ and $f(1) = e$. Find $f(2)$.

21. Solve $\dfrac{dy}{dx} = \dfrac{y}{x^2}$ with $y(1) = 2$, and state the domain.

5.5 Exponential Growth, Decay, and the Logistic Model

You will model change proportional to the amount present (growth, decay, cooling) and growth limited by a carrying capacity, solving each model and interpreting its parameters.

Many quantities change at a rate that depends on how much of the quantity is present: a population grows faster when it is larger, a radioactive sample decays faster when there is more of it, and a hot object cools faster when it is much hotter than its surroundings. Each of these gives a differential equation we can solve by separating variables.

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Exponential Growth and Decay

If a quantity y changes at a rate proportional to itself,

$$\frac{dy}{dt} = ky,$$

then $y(t) = y_0 e^{kt}$, where $y_0 = y(0)$.

If $k > 0$ the quantity grows (**doubling time** $T = \dfrac{\ln 2}{k}$). If $k < 0$ it decays (**half-life** $T = \dfrac{\ln 2}{|k|}$). Equivalently $y = y_0 \cdot 2^{t/T}$ for doubling and $y = y_0 \left(\frac{1}{2}\right)^{t/T}$ for half-life.

To find k you need one extra piece of data besides y_0: a second measurement such as $y(4) = 1500$, a doubling time, or a half-life. Always write the model first, plug in the data, solve for k, and only then answer the question.

EXAMPLE 1

A culture has 500 cells at $t = 0$ and 1500 cells after 4 hours. Assuming $\dfrac{dP}{dt} = kP$, find $P(t)$, the population after 6 hours, and the time to reach 4500 cells.

SOLUTION

$P = 500e^{kt}$. From $P(4) = 1500$: $e^{4k} = 3$, so $e^k = 3^{1/4}$ and $P(t) = 500 \cdot 3^{t/4}$.

At $t = 6$: $P = 500 \cdot 3^{3/2} = 1500\sqrt{3} \approx 2598$.

$500 \cdot 3^{t/4} = 4500$ gives $3^{t/4} = 9 = 3^2$, so $t = 8$ hours.

EXAMPLE 2

A sample of 80 milligrams of a radioactive substance has a half-life of 12 days. Find the amount after 36 days and the initial rate of change.

SOLUTION

$y = 80\left(\frac{1}{2}\right)^{t/12}$, equivalently $y = 80e^{kt}$ with $k = -\dfrac{\ln 2}{12}$.

At $t = 36$: $y = 80\left(\frac{1}{2}\right)^3 = 10$ milligrams.

$\dfrac{dy}{dt} = ky$, so at $t = 0$: $\dfrac{dy}{dt} = -\dfrac{\ln 2}{12} \cdot 80 = -\dfrac{20 \ln 2}{3} \approx -4.621$ milligrams per day.

Newton's Law of Cooling (Rate Toward a Constant)

If the temperature T of an object changes at a rate proportional to the difference between T and the constant surrounding temperature T_s,

$$\frac{dT}{dt} = -k\,(T - T_s), \quad k > 0,$$

then

$$T(t) = T_s + (T_0 - T_s)\,e^{-kt}.$$

The same method solves any equation $\dfrac{dy}{dt} = k(y - c)$: let $u = y - c$, so $u' = ku$ and $y = c + (y_0 - c)e^{kt}$. The difference from the constant is what grows or decays exponentially.

EXAMPLE 3 [CALCULATOR]

Coffee at 90°C is placed in a 20°C room. After 5 minutes it is 70°C. Find $T(t)$ and the time when it reaches 40°C.

SOLUTION

$T = 20 + 70e^{-kt}$. At $t = 5$: $50 = 70e^{-5k}$, so $e^{-5k} = \frac{5}{7}$ and $k = \frac{1}{5}\ln\frac{7}{5} \approx 0.0673$.

Set $T = 40$: $20 = 70e^{-kt}$, so $e^{-kt} = \frac{2}{7}$ and $t = \frac{\ln(7/2)}{k} = \frac{5 \ln 3.5}{\ln 1.4} \approx 18.616$ minutes.

Common mistakes

(1) Writing $T = T_0 e^{-kt}$ for cooling: the quantity that decays exponentially is $T - T_s$, not T. (2) Using the half-life as k: the half-life is $\frac{\ln 2}{|k|}$. (3) Mixing time units, such as a half-life in years and a time in months.

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The Logistic Model

Growth limited by a **carrying capacity** L is modeled by

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right), \quad k > 0.$$

With $P(0) = P_0 > 0$ the solution is

$$P(t) = \frac{L}{1 + Ae^{-kt}}, \qquad A = \frac{L - P_0}{P_0}.$$

$P = 0$ and $P = L$ are constant solutions. If $0 < P_0 < L$, P increases toward L; if $P_0 > L$, P decreases toward L. In either case $\lim_{t\to\infty} P = L$.

The growth rate $\frac{dP}{dt}$ is largest when $P = \frac{L}{2}$, where $\frac{dP}{dt} = \frac{kL}{4}$. The graph of P has its inflection point at that height.

You can see why the inflection point is at $P = \frac{L}{2}$ without solving. Differentiate $\frac{dP}{dt} = kP - \frac{k}{L}P^2$ with respect to t:

$$\frac{d^2P}{dt^2} = \left(k - \frac{2k}{L}P\right)\frac{dP}{dt} = \frac{k}{L}(L - 2P)\frac{dP}{dt}.$$

For $0 < P < L$ we have $\frac{dP}{dt} > 0$, so the concavity is determined by $L - 2P$: concave up for $P < \frac{L}{2}$ and concave down for $P > \frac{L}{2}$.

EXAMPLE 4

A population satisfies $\frac{dP}{dt} = 0.4P\left(1 - \frac{P}{1000}\right)$ with $P(0) = 100$. Find $P(t)$, the time when growth is fastest, and the maximum growth rate.

SOLUTION

$L = 1000$, $k = 0.4$, $A = \frac{1000-100}{100} = 9$. So $P(t) = \dfrac{1000}{1 + 9e^{-0.4t}}$.

Growth is fastest when $P = 500$: $9e^{-0.4t} = 1$, so $t = \dfrac{\ln 9}{0.4} \approx 5.493$.

Maximum rate $= \dfrac{kL}{4} = \dfrac{0.4 \cdot 1000}{4} = 100$ per unit time.

EXAMPLE 5

A quantity satisfies $\frac{dy}{dt} = 0.02\,y\,(50 - y)$ with $y(0) = 10$. Without solving, find $\lim_{t\to\infty} y$ and the value of y at which the graph changes concavity. Justify.

SOLUTION

Write $0.02y(50 - y) = y\left(1 - \frac{y}{50}\right)$, so $k = 1$ and $L = 50$. Since $0 < y(0) < 50$, y increases toward L: $\lim_{t\to\infty} y = 50$.

$\dfrac{d^2y}{dt^2} = 0.02(50 - 2y)\dfrac{dy}{dt}$. Because $\frac{dy}{dt} > 0$ for $0 < y < 50$, $\frac{d^2y}{dt^2}$ changes from positive to negative exactly when $y = 25$.

Justification: the graph is concave up while $y < 25$ and concave down while $y > 25$, so there is an inflection point where $y = 25$.

Practice

SET A **Skills**

Answer each question. [No calculator]

1. A quantity satisfies $\frac{dy}{dt} = ky$ with $y(0) = 5$ and $y(2) = 20$. Find $y(t)$ and $y(3)$.
2. Solve $\frac{dy}{dt} = -3y$ with $y(0) = 12$, and find the time when $y = 6$.
3. A substance has a half-life of 8 years. Starting with 64 grams, how much remains after 24 years, and when will 1 gram remain?
4. A bacterial population doubles every 5 hours. If it starts at 300, find the population after 15 hours and write the population as P_0e^{kt}, giving k exactly.
5. A population satisfies $\frac{dP}{dt} = 0.1P$. Find its doubling time exactly.
6. Newton's law of cooling gives $\frac{dT}{dt} = -0.2(T - 10)$ with $T(0) = 50$. Find $T(t)$ and $T(5\ln 2)$.
7. Solve $\frac{dy}{dt} = \frac{1}{2}(y - 8)$ with $y(0) = 12$.

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8. A population satisfies $\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{600}\right)$ with $0 < P(0) < 600$. Find the carrying capacity, the population when growth is fastest, and the maximum growth rate.

9. A quantity satisfies $\frac{dy}{dt} = 2y\left(1 - \frac{y}{200}\right)$. Find $\lim_{t\to\infty} y(t)$ if (a) $y(0) = 20$ and (b) $y(0) = 300$.

SET B Applications

Give decimal answers correct to three decimal places unless an exact value is requested. [Calculator]

10. A bacterial population is modeled by $\frac{dP}{dt} = kP$. It is 2000 at $t = 0$ hours and 3500 at $t = 3$ hours. Find k, the population at $t = 8$, and the time when it reaches 10000.

11. A radioactive isotope has a half-life of 28 years. A sample begins with 150 grams. How much remains after 50 years, and how long until only 20% of the original sample remains?

12. Soup at $95°$C is placed in a room at $22°$C. After 10 minutes it is $70°$C. Assume $\frac{dT}{dt} = -k(T - 22)$. (a) Find k. (b) Find when the soup reaches $40°$C. (c) Find $\frac{dT}{dt}$ at $t = 10$, with units.

13. A medication leaves the bloodstream so that its concentration C satisfies $\frac{dC}{dt} = -0.12C$, with t in hours. If $C(0) = 8$ milligrams per liter, when does C equal 1 milligram per liter?

14. A population satisfies $\frac{dP}{dt} = 0.05P\left(1 - \frac{P}{800}\right)$ with $P(0) = 100$. At what value of P is the population growing most rapidly?

(A) $P = 0$ **(B)** $P = 100$ **(C)** $P = 200$ **(D)** $P = 800$ **(E)** $P = 400$

15. A fish population in a lake satisfies $\frac{dP}{dt} = 0.8P\left(1 - \frac{P}{5000}\right)$, with t in years and $P(0) = 500$.

(a) Find $P(t)$.

(b) Find $P(3)$.

(c) Find when $P = 2500$ and the maximum growth rate.

16. The table shows a quantity that appears to grow exponentially, $y = y_0 e^{kt}$. Use the first and last measurements to find k, and then predict $y(5)$.

t	0	1	2	3
y	100	134	180	241

SET C Challenge

Answer each question with full justification. [No calculator]

17. Show that for positive constants L, A, k, the function $y = \frac{L}{1 + Ae^{-kt}}$ satisfies $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$.

18. A population satisfies $\frac{dP}{dt} = 0.02\,P\,(30 - P)$ with $P(0) = 5$.

(a) Find $\lim_{t\to\infty} P(t)$.

(b) Find $\frac{d^2P}{dt^2}$ in terms of P, and use it to find the value of P at which the graph has an inflection point.

(c) Describe the concavity of the graph when $P < 15$ and when $P > 15$.

19. The temperature T of a cooling object, in degrees Celsius, satisfies $\frac{dT}{dt} = -k(T - T_s)$ for constants $k > 0$ and T_s. Measurements are $T(0) = 100$, $T(10) = 60$, and $T(20) = 40$. Find T_s, then find k and the time when $T = 25$.

20. A fish population y, in thousands, with harvesting satisfies $\frac{dy}{dt} = 0.1y - 5$, with $y(0) = 40$ and t in years. Solve for $y(t)$ and find exactly when the population reaches zero.

21. Culture A starts with 800 cells and doubles every 2 hours. Culture B starts with 100 cells and quadruples every 3 hours. Find exactly when they are equal in size.

5.6 Slope Fields and Euler's Method

You will read and match slope fields, sketch solution curves from them, and use Euler's method to approximate solutions, deciding whether each estimate is too large or too small.

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Some differential equations cannot be solved with a formula, or you may only need to know what solutions look like. Two tools give you that information: a **slope field**, which shows the direction of every solution at once, and **Euler's method**, which follows the slope field step by step to compute approximate values.

Slope Fields

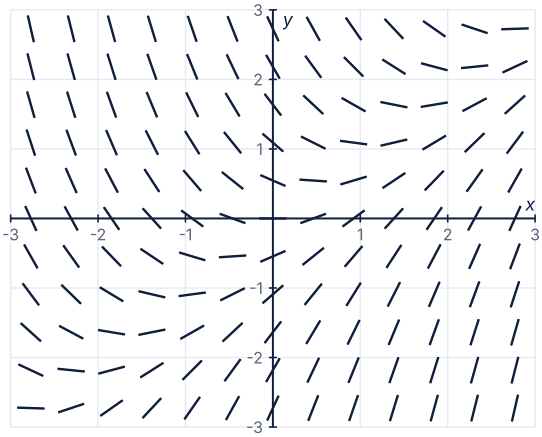
For the differential equation $\frac{dy}{dx} = f(x, y)$, a **slope field** (direction field) draws, at many grid points (x, y), a short segment with slope $f(x, y)$.

A solution curve $y = y(x)$ is tangent to the segment at each of its points. Starting at an initial point, you can sketch the solution by following the segments.

- **Horizontal segments** occur where $f(x, y) = 0$. If $f(x, y) = 0$ for every x at a fixed $y = c$, then $y = c$ is a constant (equilibrium) solution.
- If f depends only on x, as in $\frac{dy}{dx} = x^2$, the segments are identical along each vertical line, and solution curves are vertical shifts of one another.
- If f depends only on y, as in $\frac{dy}{dx} = y(2 - y)$, the segments are identical along each horizontal line, and solution curves are horizontal shifts of one another.
- Positive slope means the solution rises at that point; negative slope means it falls; steepness shows how fast.

EXAMPLE 1

For $\frac{dy}{dx} = x - y$, find the slope of the field at $(-1, -1)$, $(0, 1)$, $(1, 0)$, $(2, 1)$ and $(1, 3)$. Then describe where the segments are horizontal.



Slope field for dy/dx = x - y.

SOLUTION

Substitute: at $(-1, -1)$ the slope is $-1 - (-1) = 0$; at $(0, 1)$ it is $0 - 1 = -1$; at $(1, 0)$ it is 1; at $(2, 1)$ it is 1; at $(1, 3)$ it is -2.

The slope is 0 exactly when $x - y = 0$, so the segments are horizontal along the line $y = x$.

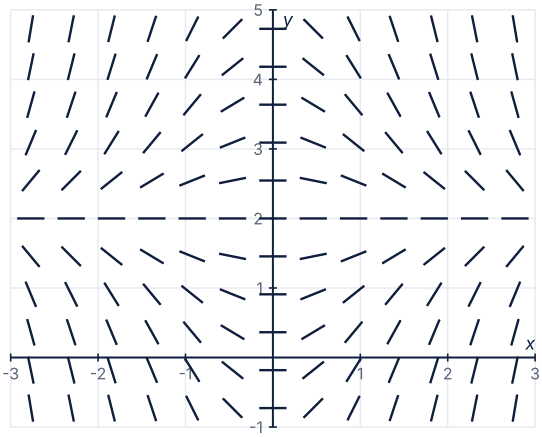
Above that line ($y > x$) the slope is negative; below it the slope is positive.

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EXAMPLE 2: MATCHING AN EQUATION

A slope field has horizontal segments along the vertical line $x = 0$ and along the horizontal line $y = 2$. At $(1, 3)$ the slope is negative, and at $(1, 1)$ the slope is positive. Which equation matches: (A) $\frac{dy}{dx} = x(y-2)$, (B) $\frac{dy}{dx} = x(2-y)$, (C) $\frac{dy}{dx} = y - 2$, (D) $\frac{dy}{dx} = xy$?



Slope field for the matching in Example 2.

SOLUTION

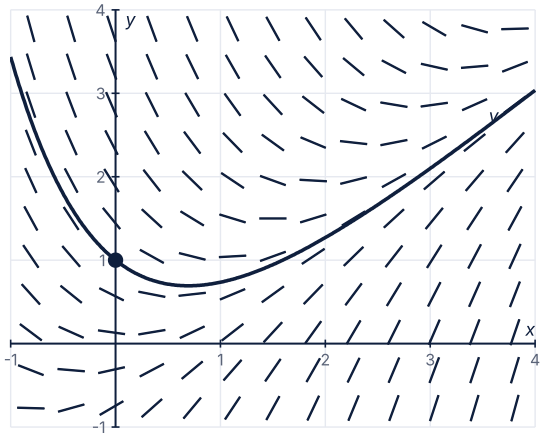
Horizontal segments along $x = 0$ and $y = 2$ means $f(x, y) = 0$ whenever $x = 0$ or $y = 2$. This fits (A) and (B), which are 0 on both lines. (C) is zero only when $y = 2$ and (D) only on the axes.

Test $(1, 3)$: (A) gives $1(3-2) = 1 > 0$, (B) gives $1(2-3) = -1 < 0$. The slope must be negative, so (B).

Check $(1, 1)$ with (B): $1(2-1) = 1 > 0$, as required. The answer is (B).

EXAMPLE 3: A SOLUTION CURVE ON THE FIELD

Solve $\frac{dy}{dx} = x - y$ with $y(0) = 1$ by checking the proposed solution $y = x - 1 + 2e^{-x}$, and explain how it fits the slope field.



The solution through (0,1) follows the field.

SOLUTION

Check the initial condition: $y(0) = 0 - 1 + 2 = 1$.

Differentiate: $y' = 1 - 2e^{-x}$. Compute $x - y = x - (x - 1 + 2e^{-x}) = 1 - 2e^{-x}$. They match.

At $(0, 1)$ the slope is $0 - 1 = -1$. The curve starts sloping down, levels off at $x = \ln 2 \approx 0.693$ (where $y = x$, so the slope is 0) and then rises along the line $y = x - 1$, which the solution approaches. This is just what the field shows.

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Euler's Method

To approximate the solution of $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$, using step size h, repeat

$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + h\, f(x_n, y_n).$$

Each step moves along the tangent line (the slope field segment) at the current point. The slope is always computed at the **old** point (x_n, y_n).

EXAMPLE 4

Use Euler's method with two steps of size $h = 0.5$ to approximate $y(1)$, where $\frac{dy}{dx} = x - y$ and $y(0) = 1$.

SOLUTION

Step 1: slope at $(0, 1)$ is $0 - 1 = -1$. $y_1 = 1 + 0.5(-1) = 0.5$ at $x_1 = 0.5$.

Step 2: slope at $(0.5, 0.5)$ is 0. $y_2 = 0.5 + 0.5(0) = 0.5$ at $x_2 = 1$.

So $y(1) \approx 0.5$. The true value from Example 3 is $y(1) = 2e^{-1} \approx 0.736$, so Euler's method underestimates here.

Over- or underestimate?

Euler's method follows tangent lines. If the solution is concave up on the interval, the tangent lines lie below the curve and the estimate is too small. If the solution is concave down, the tangent lines lie above the curve and the estimate is too large. To justify, find $\frac{d^2y}{dx^2}$ by differentiating the equation (using the chain rule on y) and determine its sign.

EXAMPLE 5: JUSTIFYING OVER- OR UNDERESTIMATE

Use Euler's method with $h = 0.5$ to approximate $y(1)$ for $\frac{dy}{dx} = -xy$, $y(0) = 2$. Is the approximation an over- or underestimate?

SOLUTION

Step 1: slope at $(0, 2)$ is 0, so $y_1 = 2$ at $x_1 = 0.5$.

Step 2: slope at $(0.5, 2)$ is -1, so $y_2 = 2 + 0.5(-1) = 1.5$. Thus $y(1) \approx 1.5$.

Concavity: $\frac{d^2y}{dx^2} = -y - x\frac{dy}{dx} = -y + x^2y = y(x^2 - 1)$. For $0 \le x < 1$ and $y > 0$, this is negative, so the solution is concave down and the tangent-line estimates are too large.

Justification: the approximation 1.5 is an overestimate. (The true value is $2e^{-1/2} \approx 1.213$.)

Practice

SET A Skills

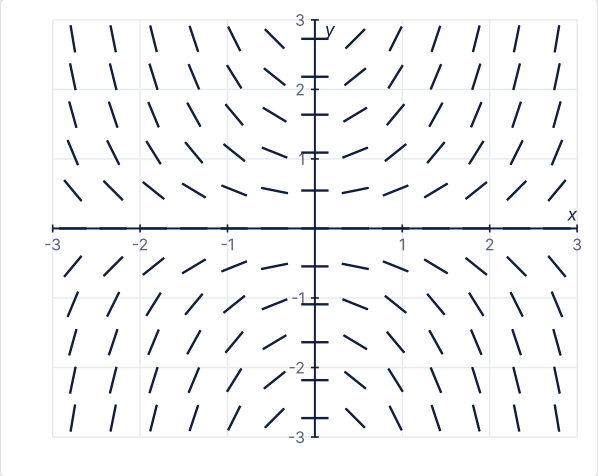
Answer each question. [No calculator]

1. For $\frac{dy}{dx} = 2x - y$, find the slope of the field at $(0, 0)$, $(1, 3)$, $(-1, 2)$, and $(2, 4)$.
2. For $\frac{dy}{dx} = y - x$, describe the set of points where the slope field has horizontal segments, and find the slope at $(0, 2)$.
3. Slopes in the field for $\frac{dy}{dx} = \frac{x}{2}$ depend only on x. What do the segments look like along a vertical line? Find the solution with $y(0) = 1$.
4. Find all constant (equilibrium) solutions of $\frac{dy}{dx} = (y - 1)(y + 2)$.

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5. The slope field shown belongs to which differential equation?



A slope field.

(A) $\frac{dy}{dx} = xy$ **(B)** $\frac{dy}{dx} = x + y$ **(C)** $\frac{dy}{dx} = \frac{x}{y}$ **(D)** $\frac{dy}{dx} = \frac{y}{x}$ **(E)** $\frac{dy}{dx} = x^2 y$

6. Use Euler's method with two steps of size $h = 0.5$ to approximate $y(1)$ for $\frac{dy}{dx} = x + y$, $y(0) = 1$.

7. Use Euler's method with two steps of size $h = 0.25$ to approximate $y(0.5)$ for $\frac{dy}{dx} = y^2$, $y(0) = 1$. Give an exact fraction.

8. Use Euler's method with two steps of size $h = 0.5$ to approximate $y(2)$ for $\frac{dy}{dx} = x^2 - y$, $y(1) = 2$.

9. Use Euler's method with three steps of size $h = 1$ to approximate $y(3)$ for $\frac{dy}{dx} = \frac{x}{y}$, $y(0) = 2$.

SET B Applications

Give decimal answers correct to three decimal places unless an exact value is requested. [Calculator]

10. Use Euler's method with four steps of size $h = 0.25$ to approximate $y(1)$ for $\frac{dy}{dx} = xe^{-y}$, $y(0) = 0$. (The exact solution is $y = \ln\left(1 + \frac{x^2}{2}\right)$.) Compare with $y(1)$.

11. A cup of tea has temperature $T(t)$ in degrees Celsius with $\frac{dT}{dt} = -0.1(T - 20)$ and $T(0) = 90$, where t is in minutes. Use Euler's method with three steps of size 2 minutes to estimate $T(6)$, and compare with the exact value $20 + 70e^{-0.6}$. Is the estimate too large or too small, and why?

12. A function $y = f(x)$ satisfies $\frac{dy}{dx} = 3x - y$ and $f(2) = 1$. (a) Write the tangent line at $x = 2$ and use it to approximate $f(2.1)$. (b) Find $\frac{d^2y}{dx^2}$ at $(2, 1)$ and decide whether the approximation is an over- or underestimate.

13. A population satisfies $\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{100}\right)$ with $P(0) = 20$. Use Euler's method with three steps of size $h = 1$ to estimate $P(3)$.

14. Euler's method with step size 0.5 is used to approximate $y(1)$, where $\frac{dy}{dx} = y - x$ and $y(0) = 2$. The approximation is

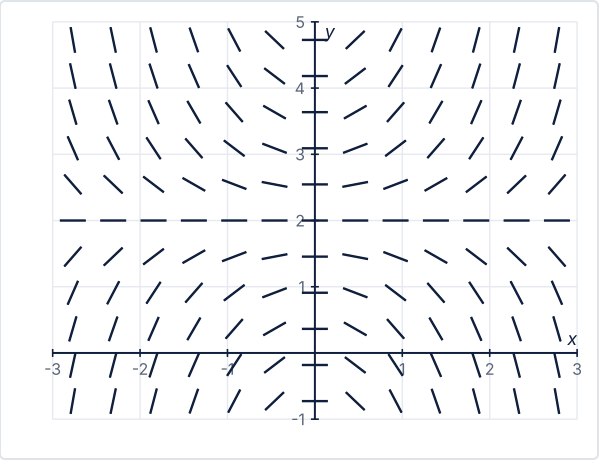
(A) 3 **(B)** 3.625 **(C)** 4.25 **(D)** 4.5 **(E)** 4.718

15. Use Euler's method with five steps of size $h = 0.2$ to approximate $y(1)$ for $\frac{dy}{dx} = x - y$, $y(0) = 1$. Compare with the exact value $2e^{-1}$ and with the two-step result from the lesson, and comment.

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16. The slope field for $\frac{dy}{dx} = x(y-2)$ is shown. Let $y = f(x)$ be the solution with $f(0) = 1$. Determine, with justification, whether f has a relative maximum, relative minimum, or neither at $x = 0$.



Slope field for dy/dx = x(y-2).

SET C Challenge

Answer each question with justification. [No calculator]

17. Consider $\frac{dy}{dx} = y(y-4)$ with $y(0) = 2$.

(a) Find the constant solutions.

(b) Explain why $\lim_{x\to\infty} y = 0$.

(c) Show that $\frac{d^2y}{dx^2} = (2y-4)\,y(y-4)$ and use it to show that the graph of the solution has an inflection point at $(0, 2)$.

18. Consider $\frac{dy}{dx} = \frac{2y}{x}$ for $x > 0$, with $y(1) = 1$.

(a) Use Euler's method with two steps of size 0.5 to approximate $y(2)$.

(b) Solve the equation exactly and find $y(2)$.

(c) Is the Euler approximation an over- or underestimate? Justify.

19. The differential equation $\frac{dy}{dx} = x + y$ has the solution $y = -x - 1$.

(a) Verify this.

(b) The solution with $y(0) = 0$ is $y = e^x - x - 1$. Use Euler's method with two steps of size 0.25 to approximate $y(0.5)$, and compare with the true value in terms of e.

(c) Explain why Euler's method underestimates here.

20. Euler's method with n equal steps on $[0, 1]$ is applied to $\frac{dy}{dx} = y$, $y(0) = 1$. (a) Show that the approximation to $y(1)$ is $\left(1 + \frac{1}{n}\right)^n$. (b) Evaluate it for $n = 4$. (c) What does the approximation approach as $n \to \infty$?

21. Consider $\frac{dy}{dx} = y^2$, $y(0) = 1$. (a) Use Euler's method with two steps of size 0.5 to approximate $y(1)$. (b) Solve the equation exactly. (c) Explain why Euler's method gives a finite answer even though the exact solution is not defined at $x = 1$.

5.7 Accumulation and Rates in Context

Learn to turn a rate into an amount, a total distance or a net change, and to explain what an integral means in context, with units.

Many AP questions give you a **rate** (liters per minute, cars per hour, people per year) and ask for an **amount**. The Fundamental Theorem of Calculus is the bridge: accumulating a rate over an interval gives the net change in the quantity, and adding the starting value gives the amount at the end.

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Amount = Initial + Accumulated Rate

If $Q'(t)$ is the rate of change of a quantity $Q(t)$ and Q' is continuous on $[a, b]$, then

$$Q(b) = Q(a) + \int_a^b Q'(t)\,dt.$$

The integral $\int_a^b Q'(t)\,dt$ is the **net change** in Q from $t = a$ to $t = b$. Its units are (units of Q') times (units of t), which are the units of Q. Quantity increases where the rate is positive and decreases where the rate is negative.

EXAMPLE 1: A FILLING TANK

A tank holds 50 gallons at $t = 0$ minutes. Water enters at the rate $r(t) = 12t - 3t^2$ gallons per minute for $0 \le t \le 6$. (a) How much water is in the tank at $t = 3$? (b) At what time is the amount of water greatest, and how much is there then?

SOLUTION

(a) $\int_0^3 (12t - 3t^2)\,dt = \left[6t^2 - t^3\right]_0^3 = 54 - 27 = 27$, so the tank holds $50 + 27 = 77$ gallons.

(b) $r(t) = 3t(4 - t)$ is positive on $(0, 4)$ and negative on $(4, 6)$, so the amount increases until $t = 4$ and then decreases.

$\int_0^4 (12t - 3t^2)\,dt = 96 - 64 = 32$, so the greatest amount is $50 + 32 = 82$ gallons at $t = 4$ minutes.

Justification: the amount is greatest at $t = 4$ because r changes from positive to negative there.

Displacement and Total Distance

For a particle moving on a line with velocity $v(t)$:

$$\text{displacement} = \int_a^b v(t)\,dt, \qquad \text{total distance} = \int_a^b |v(t)|\,dt.$$

To find total distance, split $[a, b]$ at every zero of v where the sign changes, integrate on each piece, and add the absolute values.

Final position: $s(b) = s(a) + \int_a^b v(t)\,dt$.

EXAMPLE 2: DISPLACEMENT VERSUS DISTANCE

A particle has velocity $v(t) = t^2 - 4t + 3$ for $0 \le t \le 4$. Find the displacement and the total distance traveled.

SOLUTION

An antiderivative is $F(t) = \frac{t^3}{3} - 2t^2 + 3t$. Since $v(t) = (t - 1)(t - 3)$, $v > 0$ on $(0, 1)$, $v < 0$ on $(1, 3)$, $v > 0$ on $(3, 4)$.

$F(0) = 0$, $F(1) = \frac{4}{3}$, $F(3) = 0$, $F(4) = \frac{4}{3}$.

Displacement $= F(4) - F(0) = \frac{4}{3}$.

Distance $= \left|\frac{4}{3} - 0\right| + \left|0 - \frac{4}{3}\right| + \left|\frac{4}{3} - 0\right| = 4$.

EXAMPLE 3: A POPULATION MODEL [CALCULATOR]

Bacteria grow at the rate $P'(t) = 500e^{-0.2t}$ per hour, and $P(0) = 1000$. Find the population at $t = 5$ hours.

SOLUTION

$P(5) = 1000 + \int_0^5 500e^{-0.2t}\,dt$.

$\int_0^5 500e^{-0.2t}\,dt = 2500\left(1 - e^{-1}\right) \approx 1580.301$.

$P(5) \approx 1000 + 1580.301 = 2580.301$, so about 2580 bacteria (units: bacteria).

Common mistake

Do not forget the initial amount. The integral gives only the **change**. Also, an answer to "how much water is in the tank" needs units, and "total distance" requires $|v|$, not v.

When a rate is given only in a table, estimate the accumulation with a Riemann sum or the trapezoid rule, using the actual widths of the subintervals.

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EXAMPLE 4: TRAFFIC FROM A TABLE

The rate $C(t)$ at which cars pass a checkpoint, in cars per minute, is recorded.

t (min)	0	5	10	15	20
$C(t)$	4	10	14	9	6

SOLUTION

Use the trapezoid rule on four subintervals of width 5.

$\int_0^{20} C(t)\,dt \approx \frac{5}{2}\left[(4+10)+(10+14)+(14+9)+(9+6)\right] = \frac{5}{2}(76) = 190.$

About 190 cars passed the checkpoint during the 20 minutes.

Interpreting an integral

Always answer in four parts: what quantity, over what interval, with what units, and net or total. For example, if $R(t)$ is the rate water flows out of a pipe in liters per minute, then $\int_2^6 R(t)\,dt$ is the number of liters that flow out between $t = 2$ and $t = 6$ minutes.

Practice

SET A Skills

Answer each question. Include units where appropriate. [No calculator]

1. Water flows into a tank at $r(t) = 4t + 1$ liters per minute. The tank holds 20 liters at $t = 0$ minutes. How many liters are in the tank at $t = 5$?
2. A particle moves with velocity $v(t) = 3t^2 - 12$ and has position $s(0) = 5$. Find $s(3)$.
3. A particle has velocity $v(t) = 2t - 6$ for $0 \le t \le 5$. Find (a) the displacement and (b) the total distance traveled.
4. The rate of change of the volume of water in a basin is $f(t)$ liters per minute. Explain in words the meaning of $\int_0^8 f(t)\,dt$, including units.
5. A deer population grows at $P'(t) = 12\sqrt{t}$ deer per year, and $P(0) = 100$. Find $P(4)$.
6. Cars enter a parking garage at the rate $r(t) = 3t^2$ cars per minute for $0 \le t \le 2$. Find the average rate of entry over this interval.
7. The volume of oil in a drum changes at the rate $V'(t) = (t-2)(t-5)$ liters per hour for $0 \le t \le 8$. At what time is there a local maximum of V? Justify your answer.
8. For the drum in the previous problem, find the net change in volume from $t = 2$ to $t = 5$.
9. A pipe delivers water at the rate $r(t)$ gallons per minute, with values in the table. Use the trapezoid rule with the three subintervals shown to approximate the gallons delivered from $t = 0$ to $t = 8$.

t (min)	0	2	6	8
$r(t)$	5	9	11	6

SET B Applications

Give decimal answers correct to three decimal places. [Calculator]

10. Cars enter a highway at the rate $E(t) = 40 + 25\sin\left(\frac{\pi t}{6}\right)$ cars per minute, $0 \le t \le 12$. How many cars enter during the first 5 minutes?
11. A tank holds 200 gallons at $t = 0$. Water enters at $20e^{-0.15t}$ gallons per minute and leaves at a constant 6 gallons per minute, $0 \le t \le 12$. (a) At what time is the amount of water in the tank greatest? (b) How many gallons are in the tank at that time?
12. A culture has 800 cells at $t = 0$ hours and grows at the rate $P'(t) = 120e^{0.3t}$ cells per hour. At what time does the culture reach 2000 cells?
13. A particle moves along a line with velocity $v(t) = t\cos t$ for $0 \le t \le 4$ and has $s(0) = 2$. Find (a) the position at $t = 4$ and (b) the total distance traveled.
14. Water flows into a reservoir at the rate $f(t) = 5 + 3\ln(t+1)$ thousand liters per hour. Find the average rate of flow over $0 \le t \le 6$.
15. A drug is absorbed into the blood at the rate $A(t) = 8te^{-0.5t}$ milligrams per hour for $0 \le t \le 10$. (a) How many milligrams are absorbed in the 10 hours? (b) At what time is the absorption rate greatest?

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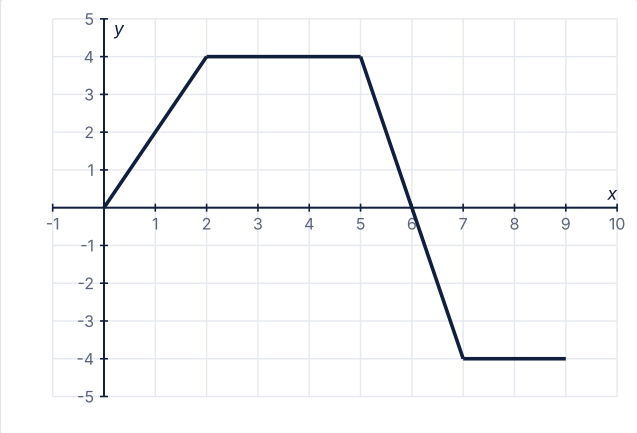
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16. A bucket holds 12 liters of water at $t = 0$ minutes and leaks at the rate $L(t) = 0.8\sqrt{t+1}$ liters per minute. At what time does the bucket become empty?

SET C Challenge

Justify your answers. [No calculator]

17. A particle moves on a line with velocity $v(t)$ in meters per second, shown in the graph (piecewise linear), and $s(0) = 0$.
    (a) Find the displacement from $t = 0$ to $t = 9$.
    (b) Find the total distance traveled.
    (c) At what time is the particle farthest in the positive direction from the start? Justify.



Graph of v(t), velocity in meters per second.

18. Water enters a tank at $3t$ gallons per minute and leaves at t^2 gallons per minute for $0 \le t \le 5$. The tank holds 10 gallons at $t = 0$. Find the absolute maximum and absolute minimum amounts of water in the tank on $[0, 5]$. Justify your answers.
19. Water flows into a pool at the rate $kt + 2$ gallons per minute, where k is a constant. In the first 4 minutes, exactly 40 gallons enter. Find k.
20. Sugar is added to a solution at the rate $S(t)$ grams per minute and removed at the rate $D(t)$ grams per minute. There are 30 grams at $t = 0$. (a) Write an expression for the grams present at $t = 10$. (b) Explain the meaning of $\int_2^5 D(t)\,dt$.
21. A quantity W has $W'(t) = 6t^2$ and $W(0) = 1$. Find the average value of W on $[0, 2]$.

5.8 Arc Length

Learn the arc length formula for y=f(x), how to recognize curves whose root simplifies, and how to evaluate a length numerically.

A curve has a length, and calculus lets you find it. Zoom in on a smooth curve and a tiny piece looks like a straight segment. The length of that piece is given by the Pythagorean Theorem, and adding up all of the pieces gives an integral. (Arc length is a BC topic.)

Arc Length Formula

If f' is continuous on $[a, b]$, the length of the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

Origin of the formula: a small piece has horizontal change dx and vertical change $dy = f'(x)\,dx$, so its length is $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (f')^2}\,dx$.

The hypothesis matters. If f' fails to exist or is unbounded somewhere on the interval, the formula cannot be applied directly. For example $y = x^{2/3}$ has no derivative at $x = 0$, so on $[-1, 1]$ you should solve for x in terms of y and integrate with respect to y, using $L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$.

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EXAMPLE 1: A DERIVATIVE THAT MAKES THE ROOT DISAPPEAR

Find the length of $y = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 3$. [No calculator]

SOLUTION

$f'(x) = x^{1/2}$, which is continuous on $[0, 3]$. Then $1 + (f')^2 = 1 + x$.

$L = \int_0^3 \sqrt{1+x}\,dx = \frac{2}{3}(1+x)^{3/2}\Big|_0^3 = \frac{2}{3}(8-1)$.

$L = \frac{14}{3}$.

EXAMPLE 2: SET UP AND USE A CALCULATOR

Find the length of $y = x^2$ from $x = 0$ to $x = 2$. [Calculator]

SOLUTION

$f'(x) = 2x$, so $L = \int_0^2 \sqrt{1+4x^2}\,dx$.

This integral has no simple elementary antiderivative for this course, so evaluate it numerically.

$L \approx 4.647$.

When the Radicand Is a Perfect Square

Test makers choose curves for which $1 + (f')^2$ is a perfect square, so that the square root can be removed. Expand $1 + (f')^2$ and look for the identity

$$(a-b)^2 + 4ab = (a+b)^2,$$

which gives $1 + (a-b)^2 = (a+b)^2$ whenever $4ab = 1$.

A typical case is $f'(x) = \frac{1}{2}\left(g - \frac{1}{g}\right)$, which gives $1 + (f')^2 = \frac{1}{4}\left(g + \frac{1}{g}\right)^2$.

EXAMPLE 3: A PERFECT SQUARE

Find the length of $y = \frac{x^3}{6} + \frac{1}{2x}$ from $x = 1$ to $x = 2$. [No calculator]

SOLUTION

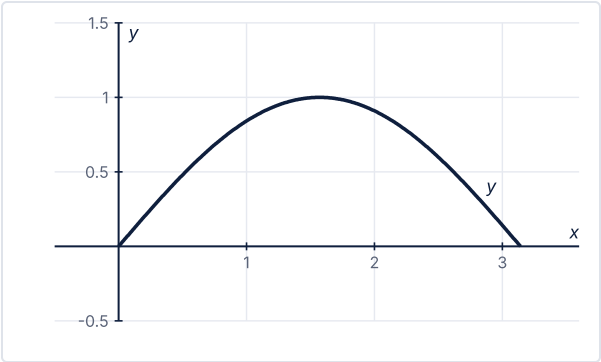
$f'(x) = \frac{x^2}{2} - \frac{1}{2x^2}$, continuous on $[1, 2]$.

$1 + (f')^2 = 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4} = \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$.

$L = \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^2 = \left(\frac{4}{3} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{17}{12}$.

EXAMPLE 4: A CURVE WITH A CALCULATOR

Find the length of one arch of $y = \sin x$, from $x = 0$ to $x = \pi$. [Calculator]



SOLUTION

$f'(x) = \cos x$, so $L = \int_0^\pi \sqrt{1 + \cos^2 x}\,dx$.

$L \approx 3.820$. This is a little more than the straight-line distance $\pi \approx 3.142$ between the endpoints, as it must be.

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Common mistakes

(1) Forgetting to square the derivative, or writing $\sqrt{1+f'(x)}$. (2) Writing $\sqrt{1+f(x)^2}$ with f instead of f'. (3) Simplifying $\sqrt{1+(f')^2}$ as $1+f'$, which is false. (4) A length is never negative, and it is always at least the straight-line distance between the endpoints, so use that as a quick check.

Practice

SET A Skills

Find or set up the arc length as indicated. [No calculator]

1. Write, but do not evaluate, an integral for the length of $y = e^{2x}$ from $x = 0$ to $x = 1$.
2. Find the length of $y = \frac{2}{3}(x-1)^{3/2}$ from $x = 1$ to $x = 5$.
3. Use the arc length formula to find the length of $y = 2x + 3$ from $x = 1$ to $x = 4$, and check with the distance formula.
4. Find the length of $y = x^{3/2}$ from $x = 0$ to $x = \frac{4}{9}$.
5. Find the length of $y = \frac{x^2}{4} - \frac{\ln x}{2}$ from $x = 1$ to $x = e$.
6. Find the length of $y = \frac{1}{3}\left(x^2+2\right)^{3/2}$ from $x = 0$ to $x = 3$.
7. Find the length of $y = \frac{x^4}{8} + \frac{1}{4x^2}$ from $x = 1$ to $x = 2$.
8. Write, but do not evaluate, an integral for the length of $y = \ln x$ from $x = 1$ to $x = 3$.
9. Find the length of $y = \frac{1}{2}\left(e^x + e^{-x}\right)$ from $x = 0$ to $x = \ln 2$.

SET B Applications

Give decimal answers correct to three decimal places. [Calculator]

10. Find the length of $y = x^2$ from $x = 0$ to $x = 3$.
11. Find the length of $y = e^x$ from $x = 0$ to $x = 1$.
12. Find the length of $y = \ln x$ from $x = 1$ to $x = 4$.
13. A hanging cable follows $y = 10\left(e^{x/20} + e^{-x/20}\right)$ meters, for $-10 \le x \le 10$. Find its length.
14. A roller coaster hill follows $y = 4\sin\left(\frac{\pi x}{10}\right)$, with x and y in meters, for $0 \le x \le 10$. How long is the track along the hill?
15. An arch is described by $y = 9 - \frac{x^2}{4}$ (meters), $-6 \le x \le 6$, with the ground at $y = 0$. Find the length of the arch (the curved part only).
16. A path follows $y = \frac{x^3}{10}$ for $0 \le x \le 5$ (kilometers). How long is the path?

SET C Challenge

Show your reasoning. [No calculator]

17. Find the length of $y = \frac{x^2}{2} - \frac{\ln x}{4}$ from $x = 1$ to $x = 2$.
18. A curve passes through $(0, 1)$ with $\frac{dy}{dx} = \frac{1}{2}\left(e^x - e^{-x}\right)$. Find the length of the curve from $x = 0$ to $x = \ln 3$.
19. A bead moves along $y = \frac{2}{3}x^{3/2}$ so that $\frac{dx}{dt} = 5$ units per second. Find the rate $\frac{ds}{dt}$ at which arc length is being traced out when $x = 3$. (Use $\frac{ds}{dt} = \sqrt{1+(f')^2}\,\frac{dx}{dt}$.)
20. Let L be the length of $y = x^2$ on $[0, 1]$. Show that $1 \le L \le 2$ by comparing $\sqrt{1+4x^2}$ with 1 and with $1+2x$ on $[0, 1]$.
21. A smooth curve $y = f(x)$ has $f'(x) \ge 0$ and increasing, with the values below. Use a left Riemann sum with three subintervals of equal width to approximate the length of the curve from $x = 0$ to $x = 6$. Is it an underestimate or an overestimate? Explain.

x	0	2	4	6
$f'(x)$	0	$\frac{3}{4}$	$\frac{4}{3}$	$\frac{12}{5}$

5.9 Multiple-Choice Practice on Applications of Integrals

Thirty multiple-choice questions covering area, volume, accumulation, differential equations, slope fields and arc length.

Work Part A without a calculator, then Part B with one. Each question has exactly one correct choice.

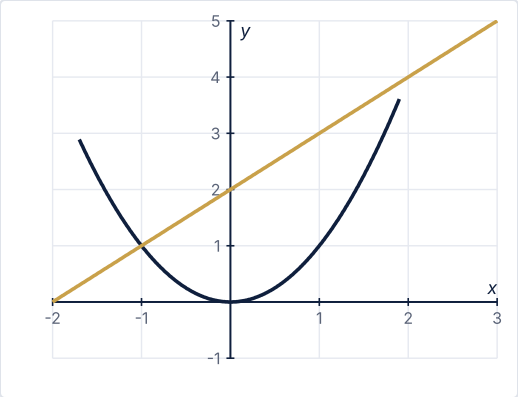
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SET A Part A [No calculator]

Choose the best answer for each question. [No calculator]

1. What is the area of the region enclosed by the graphs of $y = x^2$ and $y = x + 2$? [No calculator]



(A) 9 (B) $\frac{7}{2}$ (C) $\frac{9}{2}$ (D) $\frac{11}{2}$ (E) $\frac{13}{6}$

2. What is the area of the region bounded by $y = \sqrt{x}$ and $y = \frac{x}{2}$? [No calculator]

(A) $\frac{8}{3}$ (B) $\frac{16}{3}$ (C) 4 (D) 2 (E) $\frac{4}{3}$

3. What is the area of the region bounded by $y = e^x$, $y = 1$ and $x = \ln 3$? [No calculator]

(A) $2 - \ln 3$ (B) $3 - \ln 3$ (C) $\ln 3 - 2$ (D) 2 (E) $2 + \ln 3$

4. The region bounded by $y = \sqrt{x}$, the x-axis and $x = 4$ is revolved about the x-axis. What is the volume of the solid? [No calculator]

(A) 16π (B) 8π (C) 4π (D) $\frac{32\pi}{5}$ (E) $\frac{16\pi}{3}$

5. The region bounded by $y = x$ and $y = x^2$ is revolved about the x-axis. What is the volume of the solid? [No calculator]

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{30}$ (C) $\frac{4\pi}{15}$ (D) $\frac{2\pi}{15}$ (E) $\frac{\pi}{15}$

6. The region bounded by $y = x^2$, the x-axis and $x = 1$ is revolved about the y-axis. What is the volume of the solid? [No calculator]

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{5}$ (C) $\frac{\pi}{2}$ (D) π (E) $\frac{\pi}{5}$

7. The base of a solid is the region between $y = x^2$ and $y = 2x$. Cross sections perpendicular to the x-axis are squares. What is the volume of the solid? [No calculator]

(A) $\frac{8}{15}$ (B) $\frac{32}{15}$ (C) $\frac{4}{3}$ (D) $\frac{2}{3}$ (E) $\frac{16}{15}$

8. What is the average value of $f(x) = 3x^2 - 2x$ on $[1, 3]$? [No calculator]

(A) 9 (B) 18 (C) 8 (D) 6 (E) 12

9. A particle moves along a line with velocity $v(t) = t^2 - 3t$ for $0 \le t \le 4$. What is the total distance traveled? [No calculator]

(A) $\frac{8}{3}$ (B) $-\frac{8}{3}$ (C) $\frac{11}{6}$ (D) $\frac{19}{3}$ (E) $\frac{9}{2}$

10. The rate $r(t)$ at which water flows into a tank, in liters per hour, is given in the table. Using a trapezoidal sum with the three subintervals given by the table, what is the approximate number of liters that flow in from $t = 0$ to $t = 9$? [No calculator]

t (hours)	0	3	5	9
$r(t)$	4	6	7	3

(A) 52 (B) 48 (C) 44 (D) 24 (E) 96

11. If $\frac{dy}{dx} = \frac{2x}{y}$ with $y(0) = 3$ and $y > 0$, what is $y(2)$? [No calculator]

(A) 5 (B) $\sqrt{13}$ (C) $\sqrt{17}$ (D) 17 (E) $\sqrt{11}$

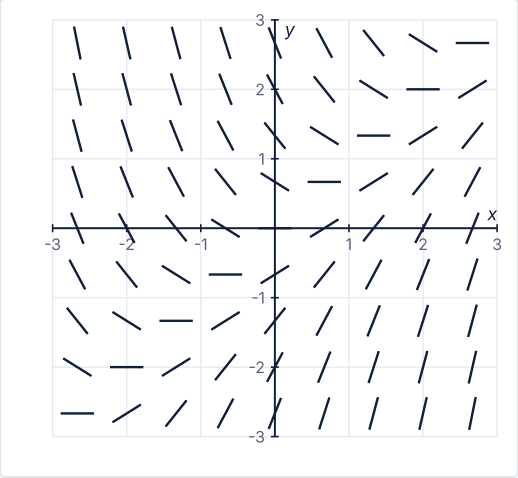
12. A quantity satisfies $\frac{dy}{dt} = ky$ with $y(0) = 5$ and $y(2) = 20$. What is $y(5)$? [No calculator]

(A) 100 (B) 80 (C) 320 (D) 125 (E) 160

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13. The slope field shown could be a graph of solutions to which differential equation? [No calculator]



(A) $\frac{dy}{dx} = x - y$ **(B)** $\frac{dy}{dx} = y - x$ **(C)** $\frac{dy}{dx} = x + y$ **(D)** $\frac{dy}{dx} = xy$ **(E)** $\frac{dy}{dx} = -y$

14. If $\frac{dy}{dx} = x + y$ and $y(0) = 1$, what is the approximation of $y(1)$ using Euler's method with two steps of equal size? [No calculator]

(A) 2 **(B)** 3 **(C)** 2.25 **(D)** 2.5 **(E)** 1.5

15. Water in a tank holds 40 liters at $t = 0$ hours and flows in at the rate $r(t)$ liters per hour. Which of the following is the meaning of $\int_0^5 r(t)\,dt$? [No calculator]

(A) The average rate of flow, in liters per hour, over the first 5 hours

(B) The number of liters that flow into the tank from $t = 0$ to $t = 5$

(C) The rate of flow, in liters per hour, at $t = 5$

(D) The number of liters in the tank at $t = 5$

(E) The time at which the flow rate is greatest

16. The population of a colony changes at the rate $P'(t) = 100e^{0.1t}$ individuals per day, and $P(0) = 500$. What is $P(10)$? [No calculator]

(A) $1000e$ **(B)** $1000e + 500$ **(C)** $1000e - 500$ **(D)** $100e + 400$ **(E)** $500e$

17. What is the area of the region bounded by $y = \ln x$, the x-axis and the line $x = e$? [No calculator]

(A) $e - 1$ **(B)** e **(C)** $e - 2$ **(D)** 1 **(E)** $\frac{1}{e}$

18. What is the length of the curve $y = \frac{x^3}{3} + \frac{1}{4x}$ from $x = 1$ to $x = 2$? [No calculator]

(A) $\frac{7}{3}$ **(B)** $\frac{13}{6}$ **(C)** $\frac{5}{2}$ **(D)** $\frac{17}{12}$ **(E)** $\frac{59}{24}$

19. The region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 1$ is revolved about the line $y = -1$. What is the volume of the solid? [No calculator]

(A) $\frac{11\pi}{6}$ **(B)** $\frac{\pi}{2}$ **(C)** $\frac{5\pi}{6}$ **(D)** $\frac{4\pi}{3}$ **(E)** $\frac{7\pi}{6}$

20. A tank holds 30 gallons at $t = 0$ minutes. Water enters at 12 gallons per minute and leaves at $2t$ gallons per minute. What is the greatest amount of water in the tank for $t \geq 0$? [No calculator]

(A) 62 gallons **(B)** 66 gallons **(C)** 78 gallons

(D) 54 gallons **(E)** 42 gallons

SET B Part B [Calculator]

Choose the best answer for each question. Decimal answers are given to three decimal places. [Calculator]

21. What is the area of the region enclosed by the graphs of $y = 4 - x^2$ and $y = e^x$? [Calculator]

(A) 7.713 **(B)** 3.214 **(C)** 1.957 **(D)** 6.428 **(E)** 7.428

22. The region bounded by $y = e^{-x^2}$, the x-axis, $x = 0$ and $x = 1$ is revolved about the x-axis. What is the volume of the solid? [Calculator]

(A) 1.879 **(B)** 0.598 **(C)** 2.346 **(D)** 3.758 **(E)** 1.986

23. What is the average value of $f(x) = \ln(x^2 + 1)$ on $[0, 3]$? [Calculator]

(A) 3.406 **(B)** 2.303 **(C)** 1.151 **(D)** 0.568 **(E)** 1.135

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24. A tank holds 100 liters at $t = 0$ minutes. Water enters at the rate $r(t) = 6 + 4\cos\left(\frac{t^2}{8}\right)$ liters per minute. How many liters are in the tank at $t = 5$? [Calculator]
(A) 37.553 **(B)** 130.000 **(C)** 137.553 **(D)** 147.553 **(E)** 102.001

25. A particle moves along a line with velocity $v(t) = \sin(t^2)$ for $0 \le t \le 2$. What is the total distance traveled? [Calculator]
(A) 0.805 **(B)** 0.985 **(C)** 1.485 **(D)** 0.492 **(E)** 1.970

26. What is the length of the curve $y = \frac{x^3}{4}$ from $x = 0$ to $x = 2$? [Calculator]
(A) 3.096 **(B)** 2.000 **(C)** 2.411 **(D)** 2.236 **(E)** 3.496

27. If $\frac{dy}{dx} = x^2 - y$ and $y(0) = 1$, what is the Euler approximation of $y(1)$ using four steps of equal size? [Calculator]
(A) 0.563 **(B)** 0.463 **(C)** 0.750 **(D)** 0.513 **(E)** 0.813

28. A culture satisfies $\frac{dP}{dt} = 0.35P$ with $P(0) = 200$, where t is in hours. At what time does P reach 1000? [Calculator]
(A) 2.299 **(B)** 19.736 **(C)** 14.286 **(D)** 3.961 **(E)** 4.598

29. The base of a solid is the region enclosed by $y = x^2$ and $y = \cos x$. Cross sections perpendicular to the x-axis are squares. What is the volume of the solid? [Calculator]
(A) 0.687 **(B)** 1.095 **(C)** 0.874 **(D)** 1.749 **(E)** 0.437

30. The amount of oil in a tank changes at the rate $A'(t) = 5te^{-t/3} - 4$ liters per hour for $0 \le t \le 12$, and $A(0) = 10$. What is the greatest amount of oil in the tank on this interval? [Calculator]
(A) 2.879 **(B)** 12.732 **(C)** 7.970 **(D)** 10.000 **(E)** 15.732

5.10 Free-Response Practice on Applications of Integrals

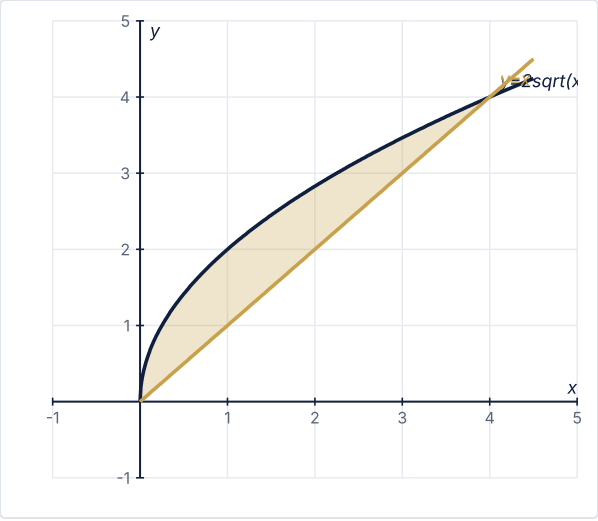
Four multi-part free-response questions on area, volume, rates in context, differential equations and slope fields.

Write complete justifications. Question 2 is a calculator question; the others are no-calculator questions.

QUESTION 1 Free Response

Show your work. [No calculator]

1. Let R be the region in the first quadrant bounded by the graphs of $y = 2\sqrt{x}$ and $y = x$, as shown in the figure.
(a) Find the area of R.
(b) Find the volume of the solid generated when R is revolved about the x-axis.
(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of the solid.
(d) Write an expression involving integrals for the volume of the solid generated when R is revolved about the line $y = -1$, and evaluate it.



Region R.

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QUESTION 2 **Free Response**

Show your work. [Calculator]

2. A tank contains 300 gallons of water at time $t = 0$ hours. For $0 \le t \le 12$, water flows into the tank at the rate $R(t) = 15 + 8\sin\left(\frac{\pi t}{6}\right)$ gallons per hour and flows out at the constant rate of 12 gallons per hour. Let $W(t)$ be the number of gallons in the tank at time t.
   (a) How many gallons of water flow into the tank during the first 4 hours? Give your answer to three decimal places.
   (b) Is the amount of water in the tank increasing or decreasing at $t = 9$? Give a reason.
   (c) At what time t in $0 \le t \le 12$ is $W(t)$ greatest? Justify your answer.
   (d) Find the greatest amount of water in the tank, to three decimal places, and the average rate of inflow over $0 \le t \le 12$.

QUESTION 3 **Free Response**

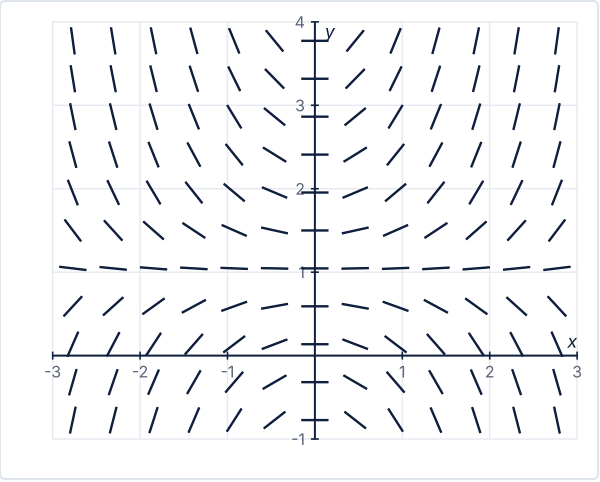
Show your work. [No calculator]

3. Consider the differential equation $\frac{dy}{dx} = xe^{-y}$. Let $y = f(x)$ be the particular solution with $f(2) = 0$.
   (a) Write an equation for the line tangent to the graph of f at $x = 2$ and use it to approximate $f(2.2)$.
   (b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Is the approximation in part (a) an overestimate or an underestimate of $f(2.2)$? Give a reason.
   (c) Find $f(x)$ by solving the differential equation with the given initial condition.
   (d) For what value of x does $f(x) = \ln 7$?

QUESTION 4 **Free Response**

Show your work. [No calculator]

4. The slope field for the differential equation $\frac{dy}{dx} = x(y - 1)$ is shown. Let $y = g(x)$ be the particular solution with $g(0) = 2$.
   (a) The slope field shows a horizontal row of zero slopes. Find all points where the slope is zero, and find the slope at $(1, 3)$ and at $(-1, 0)$.
   (b) Does g have a relative minimum, a relative maximum or neither at $x = 0$? Justify.
   (c) Use Euler's method with two steps of equal size, starting at $(0, 2)$, to approximate $g(1)$.
   (d) Find $g(x)$ by solving the differential equation with $g(0) = 2$.



Slope field for dy/dx = x(y-1).

Answers Chapter 5: Answers and Explanations

Answers for every practice problem, numbered to match. Read the explanation even when your answer is correct.

5.1 Area Between Curves

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SET A · SKILLS

1. $\frac{32}{3}$. The curves meet where $x^2 = 2x + 3$, so $x = -1, 3$. $A = \int_{-1}^{3}(2x + 3 - x^2)\,dx = \left[x^2 + 3x - \frac{x^3}{3}\right]_{-1}^{3} = 9 - \left(-\frac{5}{3}\right) = \frac{32}{3}$.
2. $\frac{4}{3}$. Intersections: $\sqrt{x} = \frac{x}{2}$ gives $x = 0, 4$. $A = \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)dx = \frac{16}{3} - 4 = \frac{4}{3}$.
3. $2 - \ln 3$. $A = \int_0^{\ln 3}(e^x - 1)\,dx = [e^x - x]_0^{\ln 3} = (3 - \ln 3) - 1 = 2 - \ln 3$.
4. $\sqrt{2} - 1$. On $[0, \pi/4]$, $\cos x \geq \sin x$. $A = [\sin x + \cos x]_0^{\pi/4} = \sqrt{2} - 1$.
5. $\frac{1}{e}$. On $[1, e]$, $\frac{1}{x} \geq \frac{1}{x^2}$. $A = \left[\ln x + \frac{1}{x}\right]_1^e = \left(1 + \frac{1}{e}\right) - 1 = \frac{1}{e}$.
6. $\frac{9}{2}$. Set $y^2 - 2y = y$: $y = 0, 3$. At $y = 1$, the line gives 1 and the parabola gives -1, so the line is on the right. $A = \int_0^3 (3y - y^2)\,dy = \frac{27}{2} - 9 = \frac{9}{2}$.
7. $\frac{11}{4}$. The curves cross at $x = -1, 0, 1$. $A = \int_{-1}^{0}(x^3 - x)\,dx + \int_0^1 (x - x^3)\,dx + \int_1^2 (x^3 - x)\,dx = \frac{1}{4} + \frac{1}{4} + \frac{9}{4} = \frac{11}{4}$.
8. $\frac{5}{2}$. Curves meet where $\sin x(2\cos x - 1) = 0$: $x = 0, \frac{\pi}{3}, \pi$. $\sin 2x$ is on top on $[0, \frac{\pi}{3}]$, $\sin x$ on top on $[\frac{\pi}{3}, \pi]$. $A = \frac{1}{4} + \frac{9}{4} = \frac{5}{2}$.
9. $\frac{5}{6}$. Solving for x: right curve $x = 2 - y$, left curve $x = \sqrt{y}$, for $0 \leq y \leq 1$. $A = \int_0^1 (2 - y - \sqrt{y})\,dy = 2 - \frac{1}{2} - \frac{2}{3} = \frac{5}{6}$.

SET B · APPLICATIONS

10. 6.428. The curves meet at $x \approx -1.965$ and $x \approx 1.058$ (use the calculator's solver). The parabola is on top between them, so $A = \int_{-1.965}^{1.058}\left(4 - x^2 - e^x\right)dx \approx 6.428$.
11. 1.949. Solving $\ln x = x - 2$ gives $x \approx 0.159$ and $x \approx 3.146$. The logarithm is on top between them: $A = \int_{0.159}^{3.146}(\ln x - x + 2)\,dx \approx 1.949$.
12. (a) $t_1 \approx 3.197$ minutes, from solving $12e^{-0.3t} = 3 + 0.5t$.
    (b) $20 + \int_0^{t_1}(A - B)\,dt \approx 32.524$ liters.
    (c) $20 + \int_0^{8}(A - B)\,dt \approx 16.371$ liters (the tank loses water after t_1).
13. 41. The differences $f - g$ are $8, 8, 6, 2, 1$. Trapezoidal sum with width 2: $\frac{2}{2}[8 + 2(8) + 2(6) + 2(2) + 1] = 8 + 16 + 12 + 4 + 1 = 41$.
14. B. The curves meet where $x^2 = 8 - x^2$, so $x = 2$. For $0 \leq x \leq 2$, $8 - x^2 \geq x^2$, so the integrand is $(8 - x^2) - x^2 = 8 - 2x^2$. Choice A has the wrong sign, C goes past the intersection, D forgets to subtract the lower curve, and E ignores both curves' shapes.
15. 18.491 square centimeters. The curves meet where $\cos(x/3) = \frac{1}{2}$, so $x = \pm\pi$. $A = \int_{-\pi}^{\pi}\left(9\cos\frac{x}{3} - \frac{9}{2}\right)dx = 27\sqrt{3} - 9\pi \approx 18.491$.
16. $c = 4$. Area $= 2\int_0^{\sqrt{c}}(c - x^2)\,dx = \frac{4}{3}c^{3/2}$. Setting $\frac{4}{3}c^{3/2} = \frac{32}{3}$ gives $c^{3/2} = 8$, so $c = 4$.

SET C · CHALLENGE

17. $k = \sqrt[3]{4}$. Total area $\int_0^2 x^2 dx = \frac{8}{3}$, so half is $\frac{4}{3}$. Solve $\int_0^k x^2 dx = \frac{k^3}{3} = \frac{4}{3}$: $k^3 = 4$.
18. $k = 2\sqrt[3]{2} \approx 2.520$. Area of R: $\int_{-2}^{2}(4 - x^2)\,dx = \frac{32}{3}$. The part of R below $y = k$ lies between the parabola and the line $y = k$ and has area $2\int_0^{\sqrt{k}}(k - x^2)\,dx = \frac{4}{3}k^{3/2}$. Set it equal to $\frac{16}{3}$: $k^{3/2} = 4$, so $k = 4^{2/3} = 2\sqrt[3]{2}$.
19. $c = 6$. The curves meet at $x = 0$ and $x = c$, with the line on top. $\int_0^c (cx - x^2)\,dx = \frac{c^3}{2} - \frac{c^3}{3} = \frac{c^3}{6} = 36$, so $c^3 = 216$.
20. (a) 16. Area $= \int_1^4 (f - g)\,dx + \int_4^7 (g - f)\,dx = (15 - 6) + (9 - 2) = 9 + 7 = 16$.
    (b) $\int_1^7 (f - g)\,dx = (15 + 2) - (6 + 9) = 2$. It differs because on $[4, 7]$ the integrand $f - g$ is negative, so that part (worth -7) subtracts instead of adding.
21. $\frac{44}{3}$. The curves meet where $|x| = 6 - x^2$: for $x \geq 0$, $x^2 + x - 6 = 0$ gives $x = 2$, and by symmetry $x = -2$. $A = 2\int_0^2 (6 - x^2 - x)\,dx = 2\left[6x - \frac{x^3}{3} - \frac{x^2}{2}\right]_0^2 = 2\left(12 - \frac{8}{3} - 2\right) = \frac{44}{3}$.

5.2 Volumes of Solids with Known Cross-Sections

SET A · SKILLS

1. $\frac{32}{5}$. $s = x^2$, $A = x^4$, $V = \int_0^2 x^4\,dx = \frac{32}{5}$.
2. $\frac{3}{2}$. $s = e^x$, $A = e^{2x}$, $V = \int_0^{\ln 2} e^{2x}\,dx = \frac{e^{2\ln 2} - 1}{2} = \frac{4 - 1}{2} = \frac{3}{2}$.
3. $\frac{\pi}{240}$. Curves meet at $x = 0, 1$; $s = x - x^2$. $V = \frac{\pi}{8}\int_0^1 (x - x^2)^2 dx = \frac{\pi}{8}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{\pi}{8} \cdot \frac{1}{30}$.
4. $\frac{\sqrt{3}}{120}$. Curves meet at $x = 0, 1$; $s = \sqrt{x} - x$. $V = \frac{\sqrt{3}}{4}\int_0^1 (x - 2x^{3/2} + x^2)\,dx = \frac{\sqrt{3}}{4}\left(\frac{1}{2} - \frac{4}{5} + \frac{1}{3}\right) = \frac{\sqrt{3}}{4} \cdot \frac{1}{30}$.
5. $\frac{\pi}{4}$. $s = \cos x$, $A = \frac{1}{2}\cos^2 x$. $V = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos^2 x\,dx = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$.
6. $\frac{8}{15}$. $s = 2x - x^2$ on $[0, 2]$, height $\frac{s}{2}$, so $A = \frac{s^2}{2}$. $V = \frac{1}{2}\int_0^2 (2x - x^2)^2 dx = \frac{1}{2} \cdot \frac{16}{15} = \frac{8}{15}$.
7. $\frac{64\pi}{15}$. For $-2 \leq y \leq 2$, $s = 4 - y^2$. $V = \frac{\pi}{8}\int_{-2}^{2}(4 - y^2)^2 dy = \frac{\pi}{8} \cdot \frac{512}{15} = \frac{64\pi}{15}$.
8. $\frac{e^2 - 1}{2}$. $s = e^y$ (distance from the y-axis), $A = e^{2y}$. $V = \int_0^1 e^{2y} dy = \frac{e^2 - 1}{2}$.
9. $6\sqrt{3}$. The line meets the axes at $x = 2$ and $y = 6$. $s = 6 - 3x$ for $0 \leq x \leq 2$. $V = \frac{\sqrt{3}}{4}\int_0^2 (6 - 3x)^2 dx = \frac{9\sqrt{3}}{4}\int_0^2 (2 - x)^2 dx = \frac{9\sqrt{3}}{4} \cdot \frac{8}{3} = 6\sqrt{3}$.

SET B · APPLICATIONS

10. 22.137. $s = 2 + \sin x$, so $V = \int_0^{\pi}(2 + \sin x)^2 dx = \int_0^{\pi}\left(4 + 4\sin x + \sin^2 x\right)dx = 4\pi + 8 + \frac{\pi}{2} = 8 + \frac{9\pi}{2} \approx 22.137$.
11. 0.470. $s = e^{-x^2}$, so $V = \frac{\pi}{8}\int_{-1}^{1} e^{-2x^2}\,dx \approx 0.470$ (evaluated numerically).
12. About 13.1 cubic centimeters. Trapezoidal sum with width 1: $\frac{1}{2}[0 + 2(3.1) + 2(4.8) + 2(4.2) + 2.0] = \frac{1}{2}(26.2) = 13.1$ cm^3.
13. 17.199. The curves meet where $1 + x^2 = 5 - x$, so $x = \frac{-1 \pm \sqrt{17}}{2} \approx -2.562,\ 1.562$. $s = 4 - x - x^2$ and $V = \frac{\sqrt{3}}{4}\int_{-2.562}^{1.562}(4 - x - x^2)^2 dx \approx 17.199$.
14. D. The curves meet at $x = 0, 1$ with $y = x^2$ on top, so the diameter is $s = x^2 - x^3$. A semicircle with diameter s has area $\frac{\pi}{8}s^2$. Choice A forgot to square, B uses $\frac{\pi}{2}$ (radius s), C uses $\frac{\pi}{4}$, and E uses πs^2, the area of a full circle of radius s.
15. 144 cubic meters. At x the slice of the disk has length $s = 2\sqrt{9 - x^2}$, so $A = 4(9 - x^2)$. $V = 4\int_{-3}^{3}(9 - x^2)\,dx = 4 \cdot 36 = 144$.
16. $k = 3$. $s = k\sqrt{x}$, $A = k^2 x$, $V = k^2\int_0^4 x\,dx = 8k^2$. Setting $8k^2 = 72$ gives $k^2 = 9$, so $k = 3$.

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SET C · CHALLENGE

17. $\frac{1024}{15}$. The curves meet at $x=\pm 2$, and $d=8-2x^2$. $V=\frac{1}{2}\int_{-2}^{2}(8-2x^2)^2dx=2\int_{-2}^{2}(4-x^2)^2dx=2\cdot\frac{512}{15}=\frac{1024}{15}$.
18. $\frac{544}{15}$. $s=2\sqrt{x}$ and the height is $x+1$ (not a multiple of s), so $A=2\sqrt{x}(x+1)$. $V=2\int_0^4\left(x^{3/2}+x^{1/2}\right)dx=2\left(\frac{64}{5}+\frac{16}{3}\right)=\frac{544}{15}$.
19. $k=8$. $s=\sqrt{x}$, so $A=\frac{\pi}{8}x$. $V=\frac{\pi}{8}\int_0^k x\,dx=\frac{\pi k^2}{16}$. Setting $\frac{\pi k^2}{16}=4\pi$ gives $k^2=64$, so $k=8$.
20. $\frac{4\sqrt{3}}{3}$. Since $\frac{\sqrt{3}}{4}s^2=\sqrt{3}(1-x^2)$ means $s=2\sqrt{1-x^2}$, one possibility is the unit disk $x^2+y^2\le 1$ as base, with equilateral triangles perpendicular to the x-axis having a side as the chord in the base. Evaluate: $\sqrt{3}\left[x-\frac{x^3}{3}\right]_{-1}^{1}=\sqrt{3}\cdot\frac{4}{3}$.
21. (a) $s=2\sqrt{x}$, so $A(x)=4x$ and $V(b)=\int_0^b 4x\,dx=2b^2$.
(b) $\frac{dV}{dt}=\frac{dV}{db}\cdot\frac{db}{dt}=4b\cdot 3$. At $b=5$ this is $4(5)(3)=60$ cubic units per second.

5.3 Volumes of Revolution: Disks and Washers

SET A · SKILLS

1. $\frac{32\pi}{5}$. Disks with $R=x^2$: $V=\pi\int_0^2 x^4dx=\frac{32\pi}{5}$.
2. $\frac{3\pi}{2}$. $V=\pi\int_0^{\ln 2}e^{2x}dx=\pi\cdot\frac{e^{2\ln 2}-1}{2}=\frac{3\pi}{2}$.
3. $\frac{\pi^2}{2}$. $V=\pi\int_0^{\pi}\sin^2 x\,dx=\pi\int_0^{\pi}\frac{1-\cos 2x}{2}dx=\pi\cdot\frac{\pi}{2}$.
4. $\frac{64\pi}{15}$. The curves meet at $x=0,2$; $R=2x$, $r=x^2$. $V=\pi\int_0^2(4x^2-x^4)dx=\pi\left(\frac{32}{3}-\frac{32}{5}\right)=\frac{64\pi}{15}$.
5. $\frac{\pi}{6}$. The curves meet at $x=0,1$; $R=\sqrt{x}$, $r=x$. $V=\pi\int_0^1(x-x^2)dx=\pi\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{\pi}{6}$.
6. $\frac{96\pi}{5}$. Write $x=y^{1/3}$; the region touches the axis. $V=\pi\int_0^8 y^{2/3}dy=\pi\cdot\frac{3}{5}\cdot 8^{5/3}=\pi\cdot\frac{3}{5}\cdot 32=\frac{96\pi}{5}$.
7. $\frac{176\pi}{15}$. Distances from $y=-1$: $R=x^2+1$, $r=1$. $V=\pi\int_0^2\left[(x^2+1)^2-1\right]dx=\pi\int_0^2(x^4+2x^2)dx=\pi\left(\frac{32}{5}+\frac{16}{3}\right)$.
8. $\frac{40\pi}{3}$. The region lies below the line $y=2$: $R=2-0=2$, $r=2-\sqrt{x}$. $V=\pi\int_0^4\left[4-(2-\sqrt{x})^2\right]dx=\pi\int_0^4(4\sqrt{x}-x)\,dx=\pi\left(\frac{64}{3}-8\right)$.
9. $\frac{\pi}{2}$. For $0\le y\le 1$ the region runs from $x=y$ to $x=\sqrt{y}$. Distances from $x=2$: $R=2-y$, $r=2-\sqrt{y}$. $V=\pi\int_0^1\left[(2-y)^2-(2-\sqrt{y})^2\right]dy=\pi\int_0^1\left(4\sqrt{y}-5y+y^2\right)dy=\pi\left(\frac{8}{3}-\frac{5}{2}+\frac{1}{3}\right)=\frac{\pi}{2}$.

SET B · APPLICATIONS

10. 1.879. Disks with $R=e^{-x^2}$: $V=\pi\int_0^1 e^{-2x^2}dx\approx 1.879$.
11. $\pi(e-2)\approx 2.257$. $V=\pi\int_1^e(\ln x)^2dx=\pi\left[x(\ln x)^2-2x\ln x+2x\right]_1^e=\pi\left[(e-2e+2e)-2\right]=\pi(e-2)$.
12. $V=\pi\int_0^{\pi}(2+\sin x)^2dx=\pi\left(4\pi+8+\frac{\pi}{2}\right)=\pi\left(8+\frac{9\pi}{2}\right)\approx 69.546$ cubic centimeters, so about 69.546 milliliters.
13. The cross-section at x is a disk of radius $\sqrt{25-x^2}$, so $V=\pi\int_{-5}^{5}(25-x^2)\,dx=\pi\left[25x-\frac{x^3}{3}\right]_{-5}^{5}=\pi\left(\frac{250}{3}+\frac{250}{3}\right)=\frac{500\pi}{3}\approx 523.599$ cubic centimeters.
14. B. The region lies above $y=1$, which is above the axis $y=-1$. The outer radius is the distance to $y=\sqrt{x}$, namely $\sqrt{x}+1$, and the inner radius is the distance to $y=1$, namely 2. Choice A uses the wrong axis, C squares $R-r$, D omits the hole, and E integrates over the wrong interval with the wrong inner radius.
15. About 80.817 cubic centimeters. Values of r^2 are $4,\ 5.76,\ 9,\ 7.84,\ 2.25$. $V\approx\pi\cdot\frac{1}{2}\left[4+2(5.76)+2(9)+2(7.84)+2.25\right]=25.725\pi\approx 80.817$.
16. 4.365. The curves meet at $x=0$ and $x\approx 1.404$, with $y=2\sin x$ above. $V=\pi\int_0^{1.404}\left(4\sin^2 x-x^4\right)dx\approx 4.365$.

SET C · CHALLENGE

17. $\frac{72\pi}{5}$. The curves meet at $x=-1,2$ and the line is above the parabola, with both above the axis. $V=\pi\int_{-1}^{2}\left[(x+2)^2-x^4\right]dx=\pi\left[\frac{(x+2)^3}{3}-\frac{x^5}{5}\right]_{-1}^{2}=\pi\left(\frac{64}{3}-\frac{32}{5}-\frac{1}{3}-\frac{1}{5}\right)=\frac{72\pi}{5}$.
18. $\frac{512\pi}{15}$. The region touches the axis $y=4$. Radius is $4-x^2$ on $[-2,2]$. $V=\pi\int_{-2}^{2}(4-x^2)^2dx=2\pi\left(32-\frac{64}{3}+\frac{32}{5}\right)=\frac{512\pi}{15}$.
19. (a) $\pi\int_0^2(4x^2-x^4)dx=\frac{64\pi}{15}$.
(b) $R=4-x^2$, $r=4-2x$: $V=\pi\int_0^2\left[(4-x^2)^2-(4-2x)^2\right]dx=\pi\int_0^2(16x-12x^2+x^4)dx=\pi\left(32-32+\frac{32}{5}\right)=\frac{32\pi}{5}$.
(c) $\frac{32\pi}{5}=\frac{96\pi}{15}>\frac{64\pi}{15}$. Rotating about $y=4$ puts the region farther from the axis, so each slice has a larger radius and the solid is bigger.
20. The student's value is $\pi\int_0^1(x^2-2x^3+x^4)dx=\frac{\pi}{30}$. The correct volume is $\pi\int_0^1(x^2-x^4)dx=\frac{2\pi}{15}$. The error is using $(R-r)^2$ instead of R^2-r^2: the area of a washer is $\pi R^2-\pi r^2$, not the area of a disk whose radius is the width $R-r$.
21. $\frac{\pi(e^2+1)}{2}$. Use horizontal slices; the region runs from $x=e^y$ to $x=e$ for $0\le y\le 1$. Axis is the y-axis, so $R=e$, $r=e^y$. $V=\pi\int_0^1(e^2-e^{2y})\,dy=\pi\left(e^2-\frac{e^2-1}{2}\right)=\frac{\pi(e^2+1)}{2}$.

5.4 Separable Differential Equations

SET A · SKILLS

1. $y=-2e^{5x}$. Separate $\frac{dy}{y}=5\,dx$: $\ln|y|=5x+C$, so $y=Ae^{5x}$. $y(0)=-2$ gives $A=-2$.
2. $y=\left(\frac{3}{2}x^2+8\right)^{1/3}$. $y^2dy=x\,dx$ gives $\frac{y^3}{3}=\frac{x^2}{2}+C$. $y(0)=2$: $\frac{8}{3}=C$. So $y^3=\frac{3}{2}x^2+8$.
3. $y=e^{1+\sin x}$. $\frac{dy}{y}=\cos x\,dx$ gives $\ln|y|=\sin x+C$. $y(0)=e$: $1=C$. So $y=e^{\sin x+1}$.
4. $y=4e^{x^2}-1$. $\frac{dy}{y+1}=2x\,dx$ gives $\ln|y+1|=x^2+C$, so $y+1=Ae^{x^2}$. $y(0)=3$: $A=4$.
5. $y=3x$. $\frac{dy}{y}=\frac{dx}{x}$ gives $\ln y=\ln x+C$, so $y=Ax$. $y(1)=3$: $A=3$.
6. $y=\sqrt[3]{x^2+7}$. $3y^2dy=2x\,dx$ gives $y^3=x^2+C$. $y(1)=2$: $8=1+C$, $C=7$.
7. $y=-\ln\left(1-\frac{x^2}{2}\right)$, domain $-\sqrt{2}<x<\sqrt{2}$. $e^{-y}dy=x\,dx$ gives $-e^{-y}=\frac{x^2}{2}+C$. $y(0)=0$: $C=-1$. So $e^{-y}=1-\frac{x^2}{2}$, which requires $1-\frac{x^2}{2}>0$.
8. $y=\tan\left(\frac{x^2}{2}\right)$, domain $-\sqrt{\pi}<x<\sqrt{\pi}$. $\frac{dy}{1+y^2}=x\,dx$ gives $\arctan y=\frac{x^2}{2}+C$, and $C=0$. The tangent is defined while $\frac{x^2}{2}<\frac{\pi}{2}$, that is $|x|<\sqrt{\pi}$.

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9. $y = 3x + 5$, domain $x > -1$. $\frac{dy}{y-2} = \frac{dx}{x+1}$ gives $\ln|y-2| = \ln|x+1| + C$, so $y - 2 = A(x+1)$. $y(0) = 5$: $A = 3$. The equation is undefined at $x = -1$, so the domain is the interval containing 0 that stops there.

SET B · APPLICATIONS

10. (a) Separate: $V^{-1/2}dV = -\frac{1}{2}dt$. Integrate: $2\sqrt{V} = -\frac{t}{2} + C$. $V(0) = 64$ gives $C = 16$, so $\sqrt{V} = 8 - \frac{t}{4}$ and $V = \left(8 - \frac{t}{4}\right)^2$.
(b) $V(10) = (8 - 2.5)^2 = 30.25$ liters.
(c) $V = 0$ when $8 - \frac{t}{4} = 0$, so $t = 32$ minutes.

11. (a) $\frac{dS}{dt} = 0.02(15 - S)$. $\frac{dS}{15-S} = 0.02\,dt$ gives $-\ln|15 - S| = 0.02t + C$. $S(0) = 0$ gives $C = -\ln 15$. So $15 - S = 15e^{-0.02t}$ and $S = 15\left(1 - e^{-0.02t}\right)$.
(b) $S(30) = 15(1 - e^{-0.6}) \approx 6.768$ kilograms.
(c) $S \to 15$ kilograms.

12. (a) $\frac{dv}{9.8-0.5v} = dt$ gives $-2\ln|9.8 - 0.5v| = t + C$. $v(0) = 0$ gives $C = -2\ln 9.8$. So $9.8 - 0.5v = 9.8e^{-t/2}$ and $v = 19.6\left(1 - e^{-t/2}\right)$.
(b) $v(2) = 19.6(1 - e^{-1}) \approx 12.390$ meters per second.
(c) As $t \to \infty$, $v \to 19.6$ meters per second.

13. $y = \dfrac{\sqrt{8 - x^2}}{2}$, domain $-2\sqrt{2} < x < 2\sqrt{2}$. $4y\,dy = -x\,dx$ gives $2y^2 = -\frac{x^2}{2} + C$. At $(2, 1)$: $2 = -2 + C$, $C = 4$. So $y^2 = 2 - \frac{x^2}{4}$ and (since $y > 0$) $y = \frac{\sqrt{8-x^2}}{2}$. The slope is undefined where $y = 0$, at $x = \pm 2\sqrt{2}$.

14. C. Separating gives $-\frac{1}{y} = \frac{x^3}{3} + C_1$, so $y = -\frac{1}{x^3/3 + C_1} = -\frac{3}{x^3 + C}$. Check: $y' = \frac{9x^2}{(x^3+C)^2} = x^2y^2$. Choice B has the opposite sign and gives $y' = -x^2y^2$. A satisfies $y' = x^2y$, not x^2y^2. D and E fail by direct differentiation.

15. $y = \dfrac{1}{2 - e^x}$, domain $x < \ln 2 \approx 0.693$, that is $(-\infty, 0.693)$. $y^{-2}dy = e^x\,dx$ gives $-\frac{1}{y} = e^x + C$. $y(0) = 1$: $C = -2$. So $y = \frac{1}{2-e^x}$, which is undefined at $x = \ln 2$.

16. $f(1) = 2e^{1-\cos 1} \approx 3.167$. $\frac{dy}{y} = \sin x\,dx$ gives $\ln|y| = -\cos x + C$, so $y = Ae^{-\cos x}$. $f(0) = 2$ gives $Ae^{-1} = 2$, so $A = 2e$ and $f(1) = 2e^{1-\cos 1}$.

SET C · CHALLENGE

17. $y = \dfrac{1}{x^2 - 1}$, domain $-1 < x < 1$. $y^{-2}dy = -2x\,dx$ gives $-\frac{1}{y} = -x^2 + C$. $y(0) = -1$: $1 = C$. So $\frac{1}{y} = x^2 - 1$. The formula fails at $x = \pm 1$, and the domain must be the interval containing 0, so it is $(-1, 1)$. The solution decreases without bound as $x \to \pm 1$.

18. $x^2 + 2y^2 = 6$, so $y = \sqrt{3 - \frac{x^2}{2}}$ near $(2, 1)$. $2y\,dy = -x\,dx$ gives $y^2 = -\frac{x^2}{2} + K$. At $(2, 1)$: $1 = -2 + K$, $K = 3$. Thus $y^2 = 3 - \frac{x^2}{2}$, i.e. $x^2 + 2y^2 = 6$.

19. $y = \dfrac{(3x^2 + 4)^2}{4}$. Since $y(0) = 4 > 0$ we can divide by $\sqrt{y}$: $y^{-1/2}dy = 6x\,dx$ gives $2\sqrt{y} = 3x^2 + C$. $y(0) = 4$: $4 = C$. So $\sqrt{y} = \frac{3x^2+4}{2}$, and $y = \frac{(3x^2+4)^2}{4}$.

20. e^4. $\frac{dy}{y} = 2x\,dx$ gives $y = Ae^{x^2}$. $f(1) = e$ gives $Ae = e$, so $A = 1$ and $f(2) = e^4$.

21. $y = 2e^{1-1/x}$, domain $x > 0$. $\frac{dy}{y} = x^{-2}dx$ gives $\ln|y| = -\frac{1}{x} + C$. $y(1) = 2$: $\ln 2 = -1 + C$. So $y = 2e^{1-1/x}$. The equation is undefined at $x = 0$, so the domain is the interval containing 1: $(0, \infty)$.

5.5 Exponential Growth, Decay, and the Logistic Model

SET A · SKILLS

1. $y = 5 \cdot 2^t$, so $y(3) = 40$. $y = 5e^{kt}$ and $20 = 5e^{2k}$ gives $e^{2k} = 4$, so $e^k = 2$ and $k = \ln 2$.

2. $y = 12e^{-3t}$; $y = 6$ when $t = \frac{\ln 2}{3}$. Setting $12e^{-3t} = 6$ gives $e^{-3t} = \frac{1}{2}$.

3. 8 grams after 24 years (three half-lives: $64 \to 32 \to 16 \to 8$). 1 gram remains after six half-lives, which is 48 years.

4. $P(15) = 300 \cdot 2^3 = 2400$. Doubling time 5 gives $e^{5k} = 2$, so $k = \frac{\ln 2}{5}$ and $P = 300e^{(\ln 2/5)t}$.

5. $10\ln 2$. $e^{0.1T} = 2$ gives $T = \frac{\ln 2}{0.1} = 10\ln 2$.

6. $T = 10 + 40e^{-0.2t}$, and $T(5\ln 2) = 10 + 40e^{-\ln 2} = 10 + 20 = 30$. Let $u = T - 10$; then $u' = -0.2u$ with $u(0) = 40$.

7. $y = 8 + 4e^{t/2}$. Let $u = y - 8$: $u' = \frac{1}{2}u$, $u(0) = 4$, so $u = 4e^{t/2}$.

8. Carrying capacity 600. Growth is fastest when $P = \frac{600}{2} = 300$, and the maximum rate is $\frac{kL}{4} = \frac{0.3 \cdot 600}{4} = 45$.

9. (a) 200. (b) 200. The equation is logistic with $k = 2$ and $L = 200$. A population starting below L increases toward L, and one starting above L decreases toward L.

SET B · APPLICATIONS

10. $k = \frac{\ln 1.75}{3} \approx 0.187$ per hour. $P(8) \approx 8894.703$. $P = 10000$ when $e^{kt} = 5$, so $t = \frac{\ln 5}{k} \approx 8.628$ hours. Here $2000e^{3k} = 3500$ gives $e^{3k} = 1.75$.

11. After 50 years: $150\left(\frac{1}{2}\right)^{50/28} \approx 43.505$ grams. For 20% remaining, $\left(\frac{1}{2}\right)^{t/28} = 0.2$ gives $t = 28\log_2 5 = \frac{28\ln 5}{\ln 2} \approx 65.014$ years.

12. (a) $T = 22 + 73e^{-kt}$. $48 = 73e^{-10k}$ gives $k = \frac{1}{10}\ln\frac{73}{48} \approx 0.042$ per minute.
(b) $18 = 73e^{-kt}$ gives $t = \frac{\ln(73/18)}{k} \approx 33.394$ minutes.
(c) $\frac{dT}{dt} = -k(70 - 22) = -48k \approx -2.012$ degrees Celsius per minute.

13. $C = 8e^{-0.12t}$, and $C = 1$ when $e^{-0.12t} = \frac{1}{8}$, so $t = \frac{\ln 8}{0.12} \approx 17.329$ hours.

14. E. The equation is logistic with $L = 800$. The growth rate $kP\left(1 - \frac{P}{L}\right)$ is a downward parabola in P with maximum at $P = \frac{L}{2} = 400$. Since $P(0) = 100 < 400$, the population reaches 400 as it grows.

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15. (a) $L = 5000$, $A = \frac{5000-500}{500} = 9$. So $P(t) = \dfrac{5000}{1+9e^{-0.8t}}$.

    (b) $P(3) = \frac{5000}{1+9e^{-2.4}} \approx 2752.604$.

    (c) $P = 2500$ when $9e^{-0.8t} = 1$, so $t = \frac{\ln 9}{0.8} \approx 2.747$ years. Maximum rate $\frac{kL}{4} = \frac{0.8 \cdot 5000}{4} = 1000$ fish per year.

16. $k = \frac{\ln 2.41}{3} \approx 0.293$ and $y(5) = 100e^{5k} \approx 433.207$. The ratios $1.34, 1.343, 1.339$ are nearly constant, which supports an exponential model.

SET C · CHALLENGE

17. Write $y = L(1 + Ae^{-kt})^{-1}$. Then $\frac{dy}{dt} = \frac{LAke^{-kt}}{(1+Ae^{-kt})^2}$. On the other side, $1 - \frac{y}{L} = 1 - \frac{1}{1+Ae^{-kt}} = \frac{Ae^{-kt}}{1+Ae^{-kt}}$, so $ky\left(1 - \frac{y}{L}\right) = k \cdot \frac{L}{1+Ae^{-kt}} \cdot \frac{Ae^{-kt}}{1+Ae^{-kt}} = \frac{LAke^{-kt}}{(1+Ae^{-kt})^2}$. The two sides are equal.

18. (a) Rewrite as $0.6P\left(1 - \frac{P}{30}\right)$, so $L = 30$. Since $0 < P(0) < 30$, P increases toward 30: the limit is 30.

    (b) $\frac{dP}{dt} = 0.6P - 0.02P^2$, so $\frac{d^2P}{dt^2} = (0.6 - 0.04P)\frac{dP}{dt} = 0.02(30 - 2P)\frac{dP}{dt}$. For $0 < P < 30$, $\frac{dP}{dt} > 0$, so the sign of $\frac{d^2P}{dt^2}$ is the sign of $30 - 2P$. It changes sign at $P = 15$, which gives the inflection point.

    (c) Concave up when $P < 15$ (since $30 - 2P > 0$) and concave down when $P > 15$ (since $30 - 2P < 0$).

19. $T_s = 20$, $k = \frac{\ln 2}{10}$, and $t = 40$. Since $T - T_s = (100 - T_s)e^{-kt}$, the ratios satisfy $\frac{60-T_s}{100-T_s} = \frac{40-T_s}{60-T_s}$. Cross-multiplying, $(60 - T_s)^2 = (100 - T_s)(40 - T_s)$, so $3600 - 120T_s = 4000 - 140T_s$, giving $T_s = 20$. Then $T - 20$ takes the values $80, 40, 20$: it halves every 10 minutes, so $e^{-10k} = \frac{1}{2}$, $k = \frac{\ln 2}{10}$. For $T = 25$: $80\left(\frac{1}{2}\right)^{t/10} = 5$, so $\left(\frac{1}{2}\right)^{t/10} = \frac{1}{16}$ and $t = 40$ minutes.

20. $y = 50 - 10e^{0.1t}$. Write $\frac{dy}{dt} = 0.1(y - 50)$, so $y - 50 = (40 - 50)e^{0.1t}$. Then $y = 0$ when $e^{0.1t} = 5$, so $t = 10\ln 5 \approx 16.094$ years.

21. $t = 18$ hours. $A(t) = 800 \cdot 2^{t/2}$ and $B(t) = 100 \cdot 4^{t/3} = 100 \cdot 2^{2t/3}$. Equal when $8 = 2^{2t/3 - t/2} = 2^{t/6}$, so $\frac{t}{6} = 3$ and $t = 18$.

5.6 Slope Fields and Euler's Method

SET A · SKILLS

1. Slopes: $0, -1, -4, 0$. Substitute into $2x - y$: $0 - 0 = 0$; $2 - 3 = -1$; $-2 - 2 = -4$; $4 - 4 = 0$.

2. Horizontal segments occur where $y - x = 0$, i.e. along the line $y = x$. At $(0, 2)$ the slope is $2 - 0 = 2$.

3. The segments are all parallel (identical) along each vertical line. Since $\frac{dy}{dx} = \frac{x}{2}$, $y = \frac{x^2}{4} + C$, and $y(0) = 1$ gives $y = \frac{x^2}{4} + 1$.

4. $y = 1$ and $y = -2$. These are the values where $(y - 1)(y + 2) = 0$ for all x.

5. A. The field is horizontal on both axes and, for $x < 0 < y$ (upper left), slopes are negative. For $\frac{dy}{dx} = xy$ the sign is negative in quadrants II and IV and positive in I and III. Choice B is 0 along $y = -x$. Choices C and D are undefined on an axis. Choice E is positive whenever $y > 0$, which would make the upper-left slopes positive.

6. $y(1) \approx 2.5$. Step 1: slope at $(0, 1)$ is 1, $y_1 = 1 + 0.5(1) = 1.5$. Step 2: slope at $(0.5, 1.5)$ is 2, $y_2 = 1.5 + 0.5(2) = 2.5$.

7. $\frac{105}{64}$. Step 1: slope at $(0, 1)$ is 1, $y_1 = 1 + \frac{1}{4} = \frac{5}{4}$. Step 2: slope at $(0.25, \frac{5}{4})$ is $\frac{25}{16}$, $y_2 = \frac{5}{4} + \frac{1}{4} \cdot \frac{25}{16} = \frac{80+25}{64} = \frac{105}{64}$.

8. $\frac{15}{8}$. Step 1: slope at $(1, 2)$ is $1 - 2 = -1$, $y_1 = 2 + 0.5(-1) = 1.5$. Step 2: slope at $(1.5, 1.5)$ is $2.25 - 1.5 = 0.75$, $y_2 = 1.5 + 0.5(0.75) = 1.875$.

9. 3.3. Step 1: slope at $(0, 2)$ is 0, $y_1 = 2$. Step 2: slope at $(1, 2)$ is 0.5, $y_2 = 2.5$. Step 3: slope at $(2, 2.5)$ is 0.8, $y_3 = 2.5 + 0.8 = 3.3$.

SET B · APPLICATIONS

10. $y(1) \approx 0.337$ (the successive values are $y_1 = 0$, $y_2 = 0.0625$, $y_3 \approx 0.1799$, $y_4 \approx 0.3366$). The exact value is $\ln 1.5 \approx 0.405$, so the estimate is too small.

11. Estimates: $T(2) \approx 76.000$, $T(4) \approx 64.800$, $T(6) \approx 55.840$. The exact value is 58.417. The estimate is too small because $T - 20$ is decaying, so T is concave up ($T'' = 0.01(T - 20) > 0$) and tangent lines lie below the curve.

12. (a) $f'(2) = 3(2) - 1 = 5$. Tangent: $y = 1 + 5(x - 2)$, so $f(2.1) \approx 1 + 5(0.1) = 1.5$.

    (b) $\frac{d^2y}{dx^2} = 3 - \frac{dy}{dx} = 3 - 5 = -2 < 0$, so the graph is concave down near $x = 2$ and the tangent line lies above it: the approximation 1.5 is an overestimate.

13. $P(1) \approx 24.8000$, $P(2) \approx 30.3949$, $P(3) \approx 36.742$. Each step is $P_{n+1} = P_n + 0.3P_n(1 - P_n/100)$.

14. C. Step 1: slope at $(0, 2)$ is 2, $y_1 = 2 + 0.5(2) = 3$. Step 2: slope at $(0.5, 3)$ is 2.5, $y_2 = 3 + 0.5(2.5) = 4.25$. Choice A stops after one step; B uses the new x in the slope at each step; D forgets to subtract x; E is the exact value $2 + e$ of the solution, not an Euler estimate.

15. $y(1) \approx 0.655$ (values: 0.800, 0.680, 0.624, 0.619, 0.655). The exact value is $2/e \approx 0.736$. The two-step estimate was 0.5. Smaller steps give a better approximation, but both are underestimates because the solution is concave up.

16. Relative maximum at $x = 0$. At $x = 0$, $\frac{dy}{dx} = 0 \cdot (1 - 2) = 0$, so $x = 0$ is critical. Differentiating, $\frac{d^2y}{dx^2} = (y - 2) + x\frac{dy}{dx}$, which at $(0, 1)$ equals $-1 < 0$. By the second derivative test, f has a relative maximum at $x = 0$.

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17. (a) $y = 0$ and $y = 4$.
(b) For $0 < y < 4$ we have $y(y-4) < 0$, so y decreases. A solution cannot cross the constant solution $y = 0$, so it is bounded below by 0 and decreasing, and it approaches $y = 0$.
(c) $\frac{d^2y}{dx^2} = \frac{d}{dx}(y^2 - 4y) = (2y-4)\frac{dy}{dx} = (2y-4)y(y-4)$. At $(0,2)$ this is 0. For y slightly above 2 (with $y < 4$), $2y - 4 > 0$ and $y(y-4) < 0$, so $y'' < 0$; for y slightly below 2, $y'' > 0$. Since y is decreasing, $y > 2$ for $x < 0$ and $y < 2$ for $x > 0$, so y'' changes sign from negative to positive at $x = 0$ (concave down, then concave up), and $(0,2)$ is an inflection point.

18. (a) Step 1: slope at $(1,1)$ is 2, $y_1 = 1 + 0.5(2) = 2$ at $x = 1.5$. Step 2: slope at $(1.5, 2)$ is $\frac{4}{1.5} = \frac{8}{3}$, $y_2 = 2 + 0.5 \cdot \frac{8}{3} = \frac{10}{3}$. So $y(2) \approx \frac{10}{3}$.
(b) $\frac{dy}{y} = \frac{2\,dx}{x}$ gives $y = Ax^2$, and $y(1) = 1$ gives $y = x^2$. So $y(2) = 4$.
(c) Underestimate: $y = x^2$ is concave up ($y'' = 2 > 0$), so tangent lines lie below it. $\frac{10}{3} < 4$.

19. (a) $y' = -1$ and $x + y = x + (-x - 1) = -1$. They agree.
(b) Step 1: slope at $(0,0)$ is 0, $y_1 = 0$. Step 2: slope at $(0.25, 0)$ is 0.25, $y_2 = 0.25(0.25) = 0.0625 = \frac{1}{16}$. The true value is $e^{0.5} - 1.5 \approx 0.1487$.
(c) $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx} = 1 + x + y > 0$ near $(0,0)$, so the solution is concave up and tangent-line steps fall below the curve.

20. (a) Each step multiplies by $1 + h \cdot 1 = 1 + \frac{1}{n}$, because the slope equals the current y-value. After n steps, $y_n = \left(1 + \frac{1}{n}\right)^n$.
(b) $\left(\frac{5}{4}\right)^4 = \frac{625}{256} \approx 2.441$.
(c) It approaches $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$, the exact value $y(1)$ of the solution $y = e^x$.

21. (a) Step 1: slope at $(0,1)$ is 1, $y_1 = 1 + 0.5 = 1.5$. Step 2: slope at $(0.5, 1.5)$ is 2.25, $y_2 = 1.5 + 0.5(2.25) = 2.625 = \frac{21}{8}$.
(b) $y^{-2}dy = dx$ gives $-\frac{1}{y} = x + C$, and $y(0) = 1$ gives $C = -1$. So $y = \frac{1}{1-x}$ for $x < 1$.
(c) Euler's method only uses a finite number of tangent-line steps and never checks that the solution exists. The exact solution grows without bound as $x \to 1^-$, so the true solution has a vertical asymptote at $x = 1$ and $y(1)$ does not exist; the finite Euler value 2.625 is meaningless there.

5.7 Accumulation and Rates in Context

SET A · SKILLS

1. 75 liters. $\int_0^5 (4t+1)\,dt = [2t^2 + t]_0^5 = 55$, and $20 + 55 = 75$.
2. $s(3) = -4$. $s(3) = 5 + \int_0^3 (3t^2 - 12)\,dt = 5 + [t^3 - 12t]_0^3 = 5 + (27 - 36) = -4$.
3. (a) -5. $\int_0^5 (2t-6)\,dt = [t^2 - 6t]_0^5 = 25 - 30 = -5$.
(b) 13. $v < 0$ on $(0,3)$ and $v > 0$ on $(3,5)$. $|\int_0^3| = 9$ and $\int_3^5 = 4$, so the distance is $9 + 4 = 13$.
4. It is the net change in the volume of water in the basin, in liters, from $t = 0$ to $t = 8$ minutes (the number of liters gained, with outflow counted negatively).
5. $P(4) = 164$. $\int_0^4 12t^{1/2}\,dt = 8t^{3/2}\Big|_0^4 = 64$, so $P(4) = 100 + 64 = 164$ deer.
6. 4 cars per minute. Average $= \frac{1}{2}\int_0^2 3t^2\,dt = \frac{1}{2}[t^3]_0^2 = \frac{1}{2}(8) = 4$.
7. Local maximum at $t = 2$. $V'(t)$ changes from positive to negative at $t = 2$ ($V' > 0$ for $t < 2$, $V' < 0$ on $(2,5)$), so V has a local maximum there by the First Derivative Test.
8. $-\frac{9}{2}$ liters. $\int_2^5 (t^2 - 7t + 10)\,dt = \left[\frac{t^3}{3} - \frac{7t^2}{2} + 10t\right]_2^5 = \frac{25}{6} - \frac{26}{3} = -\frac{9}{2}$.
9. 71 gallons. $\frac{2}{2}(5+9) + \frac{4}{2}(9+11) + \frac{2}{2}(11+6) = 14 + 40 + 17 = 71$.

SET B · APPLICATIONS

10. 289.096 cars. $\int_0^5 \left(40 + 25\sin\frac{\pi t}{6}\right)dt = 200 + \frac{150}{\pi}\left(1 - \cos\frac{5\pi}{6}\right) \approx 289.096$.
11. (a) $t \approx 8.026$ minutes. The net rate $20e^{-0.15t} - 6$ changes from positive to negative when $e^{-0.15t} = 0.3$.
(b) About 245.174 gallons. $V = 200 + \int_0^{8.026}\left(20e^{-0.15t} - 6\right)dt$.
12. $t \approx 4.621$ hours. Solve $800 + 400\left(e^{0.3t} - 1\right) = 2000$, so $e^{0.3t} = 4$ and $t = \frac{\ln 4}{0.3}$.
13. (a) $s(4) = 2 + \int_0^4 t\cos t\,dt = 2 + (4\sin 4 + \cos 4 - 1) \approx -2.681$.
(b) About 5.822. $v > 0$ on $(0, \frac{\pi}{2})$ and $v < 0$ on $(\frac{\pi}{2}, 4]$, so distance $= \int_0^{\pi/2} v\,dt - \int_{\pi/2}^4 v\,dt$.
14. About 8.811 thousand liters per hour. Average $= \frac{1}{6}\int_0^6 (5 + 3\ln(t+1))\,dt \approx 8.811$.
15. (a) About 30.706 mg. $\int_0^{10} 8te^{-0.5t}dt = 32 - 192e^{-5}$.
(b) $t = 2$ hours. $A'(t) = 8e^{-0.5t}(1 - 0.5t)$ changes from positive to negative at $t = 2$.
16. $t \approx 7.204$ minutes. Solve $0.8\int_0^T \sqrt{t+1}\,dt = \frac{1.6}{3}\left[(T+1)^{3/2} - 1\right] = 12$, so $(T+1)^{3/2} = 23.5$.

SET C · CHALLENGE

17. (a) 8 meters. Areas: $4 + 12 + 2 - 2 - 8 = 8$.
(b) 28 meters. $4 + 12 + 2 + 2 + 8 = 28$ (v is zero at $t = 6$).
(c) $t = 6$. s increases while $v > 0$ and decreases afterward since $v < 0$ on $(6,9]$, and $s(6) = 18 > s(0)$; $s(9) = 8$.
18. Absolute maximum 14.5 gallons at $t = 3$; absolute minimum $\frac{35}{6}$ gallons at $t = 5$. $V'(t) = 3t - t^2$ is positive on $(0,3)$ and negative on $(3,5)$. $V(0) = 10$, $V(3) = 10 + \frac{27}{2} - 9 = 14.5$, $V(5) = 10 + \frac{75}{2} - \frac{125}{3} = \frac{35}{6}$. Compare candidates: max at $t = 3$, min at the endpoint $t = 5$.
19. $k = 4$. $\int_0^4 (kt + 2)\,dt = 8k + 8 = 40$, so $k = 4$.
20. (a) $30 + \int_0^{10}(S(t) - D(t))\,dt$ grams.
(b) It is the total number of grams of sugar removed from the solution between $t = 2$ and $t = 5$ minutes.

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21. 5. $W(t) = 1 + 2t^3$. Average $= \frac{1}{2}\int_0^2 (1+2t^3)\,dt = \frac{1}{2}\left[t + \frac{t^4}{2}\right]_0^2 = \frac{1}{2}(10) = 5$.

5.8 Arc Length

SET A · SKILLS

1. $L = \int_0^1 \sqrt{1+4e^{4x}}\,dx$. Here $f'(x) = 2e^{2x}$, so $(f')^2 = 4e^{4x}$.
2. $\frac{2}{3}\left(5\sqrt{5} - 1\right)$. $f' = \sqrt{x-1}$, $1 + (f')^2 = x$, so $L = \int_1^5 \sqrt{x}\,dx = \frac{2}{3}\left[x^{3/2}\right]_1^5$.
3. $3\sqrt{5}$. $f' = 2$, $L = \int_1^4 \sqrt{5}\,dx = 3\sqrt{5}$. Distance between $(1,5)$ and $(4,11)$: $\sqrt{9+36} = 3\sqrt{5}$.
4. $\frac{8}{27}\left(2\sqrt{2} - 1\right)$. $f' = \frac{3}{2}x^{1/2}$, $1 + (f')^2 = 1 + \frac{9x}{4}$, $L = \frac{4}{9}\cdot\frac{2}{3}\left(1 + \frac{9x}{4}\right)^{3/2}\Big|_0^{4/9} = \frac{8}{27}\left(2^{3/2} - 1\right)$.
5. $\frac{e^2+1}{4}$. $f' = \frac{x}{2} - \frac{1}{2x}$, so $1 + (f')^2 = \left(\frac{x}{2} + \frac{1}{2x}\right)^2$, $L = \left[\frac{x^2}{4} + \frac{\ln x}{2}\right]_1^e = \frac{e^2}{4} + \frac{1}{2} - \frac{1}{4}$.
6. 12. $f' = x\sqrt{x^2+2}$, $1 + (f')^2 = x^4 + 2x^2 + 1 = (x^2+1)^2$, $L = \int_0^3 (x^2 + 1)\,dx = 9 + 3 = 12$.
7. $\frac{33}{16}$. $f' = \frac{x^3}{2} - \frac{1}{2x^3}$, $1 + (f')^2 = \left(\frac{x^3}{2} + \frac{1}{2x^3}\right)^2$, $L = \left[\frac{x^4}{8} - \frac{1}{4x^2}\right]_1^2 = \frac{31}{16} + \frac{1}{8} = \frac{33}{16}$.
8. $L = \int_1^3 \sqrt{1 + \frac{1}{x^2}}\,dx$. Here $f'(x) = \frac{1}{x}$.
9. $\frac{3}{4}$. $f' = \frac{1}{2}(e^x - e^{-x})$, $1 + (f')^2 = \frac{1}{4}(e^x + e^{-x})^2$, $L = \frac{1}{2}\left[e^x - e^{-x}\right]_0^{\ln 2} = \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{3}{4}$.

SET B · APPLICATIONS

10. 9.747. $L = \int_0^3 \sqrt{1+4x^2}\,dx$.
11. 2.003. $L = \int_0^1 \sqrt{1+e^{2x}}\,dx$.
12. 3.343. $L = \int_1^4 \sqrt{1 + \frac{1}{x^2}}\,dx$.
13. 20.844 meters. $y' = \frac{1}{2}\left(e^{x/20} - e^{-x/20}\right)$ and $L = \int_{-10}^{10} \sqrt{1 + (y')^2}\,dx = \int_{-10}^{10} \frac{1}{2}\left(e^{x/20} + e^{-x/20}\right)dx$.
14. 13.207 meters. $y' = \frac{2\pi}{5}\cos\frac{\pi x}{10}$, $L = \int_0^{10} \sqrt{1 + (y')^2}\,dx$.
15. 22.611 meters. $y' = -\frac{x}{2}$, $L = \int_{-6}^{6} \sqrt{1 + \frac{x^2}{4}}\,dx$.
16. 14.424 km. $y' = \frac{3x^2}{10}$, $L = \int_0^5 \sqrt{1 + \frac{9x^4}{100}}\,dx$.

SET C · CHALLENGE

17. $\frac{3}{2} + \frac{\ln 2}{4}$. $f' = x - \frac{1}{4x}$, $1 + (f')^2 = \left(x + \frac{1}{4x}\right)^2$, $L = \left[\frac{x^2}{2} + \frac{\ln x}{4}\right]_1^2 = 2 + \frac{\ln 2}{4} - \frac{1}{2}$.
18. $\frac{4}{3}$. $1 + (y')^2 = \frac{1}{4}\left(e^x + e^{-x}\right)^2$, $L = \frac{1}{2}\left[e^x - e^{-x}\right]_0^{\ln 3} = \frac{1}{2}\left(3 - \frac{1}{3}\right) = \frac{4}{3}$.
19. 10 units per second. $f' = \sqrt{x}$, so $\frac{ds}{dx} = \sqrt{1+x}$, which is 2 at $x = 3$. Then $\frac{ds}{dt} = 2(5) = 10$.
20. On $[0,1]$, $1 \le \sqrt{1+4x^2}$, and $1 + 4x^2 \le (1+2x)^2 = 1 + 4x + 4x^2$, so $\sqrt{1+4x^2} \le 1 + 2x$. Integrating, $1 \le L \le \int_0^1 (1+2x)\,dx = 2$.
21. $\frac{47}{6}$, an underestimate. Integrand $\sqrt{1+(f')^2}$ has values $1, \frac{5}{4}, \frac{5}{3}, \frac{13}{5}$ at $x = 0, 2, 4, 6$. Left sum $= 2\left(1 + \frac{5}{4} + \frac{5}{3}\right) = \frac{47}{6}$. The integrand is increasing (since $f' \ge 0$ is increasing), so left sums are underestimates.

5.9 Multiple-Choice Practice on Applications of Integrals

SET A · PART A [NO CALCULATOR]

1. C. The curves meet at $x = -1$ and $x = 2$. Area $= \int_{-1}^2 (x + 2 - x^2)\,dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^2 = \frac{9}{2}$.
2. E. They meet where $\sqrt{x} = \frac{x}{2}$, at $x = 0$ and $x = 4$, with $\sqrt{x}$ on top. Area $= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)dx = \frac{16}{3} - 4 = \frac{4}{3}$.
3. A. Area $= \int_0^{\ln 3} (e^x - 1)\,dx = [e^x - x]_0^{\ln 3} = (3 - \ln 3) - 1 = 2 - \ln 3$.
4. B. Disks: $V = \pi\int_0^4 (\sqrt{x})^2 dx = \pi\int_0^4 x\,dx = \pi \cdot 8 = 8\pi$.
5. D. Washers: $V = \pi\int_0^1 \left(x^2 - x^4\right)dx = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$.
6. C. Shells: $V = 2\pi\int_0^1 x \cdot x^2\,dx = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$.
7. E. The curves meet at $x = 0$ and $x = 2$. Side $= 2x - x^2$, so $V = \int_0^2 (2x - x^2)^2 dx = \int_0^2 \left(4x^2 - 4x^3 + x^4\right)dx = \frac{32}{3} - 16 + \frac{32}{5} = \frac{16}{15}$.
8. A. $\frac{1}{3-1}\int_1^3 (3x^2 - 2x)\,dx = \frac{1}{2}\left[x^3 - x^2\right]_1^3 = \frac{1}{2}(18 - 0) = 9$.
9. D. $v < 0$ on $(0,3)$ and $v > 0$ on $(3,4)$. $\left|\int_0^3 v\right| = \frac{9}{2}$ and $\int_3^4 v = \frac{11}{6}$, so the distance is $\frac{9}{2} + \frac{11}{6} = \frac{19}{3}$. (The net change $-\frac{8}{3}$ is the displacement.)
10. B. Trapezoids: $\frac{3}{2}(4+6) + \frac{2}{2}(6+7) + \frac{4}{2}(7+3) = 15 + 13 + 20 = 48$. (52 is a left sum and 44 a right sum.)
11. C. Separate: $y\,dy = 2x\,dx$, so $\frac{y^2}{2} = x^2 + C$. With $y(0) = 3$: $y^2 = 2x^2 + 9$. Then $y(2)^2 = 17$.
12. E. $y = 5e^{kt}$ and $e^{2k} = 4$, so $e^k = 2$ and $y(5) = 5 \cdot 2^5 = 160$.
13. A. Slopes are zero along the line $y = x$, positive below it and negative above it. That matches $x - y$. At $(0,1)$ the slope is -1, which rules out $y - x$ and $x + y$; xy and $-y$ are not zero along $y = x$.
14. D. Step size 0.5. $y_1 = 1 + 0.5(0+1) = 1.5$. $y_2 = 1.5 + 0.5(0.5 + 1.5) = 2.5$.
15. B. The integral of a rate is the net change in the quantity. The amount in the tank at $t = 5$ would be $40 + \int_0^5 r(t)\,dt$.
16. C. $P(10) = 500 + \int_0^{10} 100e^{0.1t}dt = 500 + 1000\,(e - 1) = 1000e - 500$.
17. D. Area $= \int_1^e \ln x\,dx = [x\ln x - x]_1^e = (e - e) - (0 - 1) = 1$.
18. E. $y' = x^2 - \frac{1}{4x^2}$ and $1 + (y')^2 = \left(x^2 + \frac{1}{4x^2}\right)^2$. $L = \int_1^2 \left(x^2 + \frac{1}{4x^2}\right)dx = \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^2 = \frac{59}{24}$.
19. A. Washers with outer radius $\sqrt{x} + 1$ and inner radius 1: $V = \pi\int_0^1 \left[(\sqrt{x}+1)^2 - 1\right]dx = \pi\int_0^1 (x + 2\sqrt{x})\,dx = \pi\left(\frac{1}{2} + \frac{4}{3}\right) = \frac{11\pi}{6}$. (The value $\frac{\pi}{2}$ ignores the shift of the axis.)
20. B. The net rate $12 - 2t$ changes from positive to negative at $t = 6$, so the amount is greatest then: $30 + \int_0^6 (12 - 2t)\,dt = 30 + 36 = 66$.

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SET B · PART B [CALCULATOR]

21. D. The curves meet at $x \approx -1.965$ and $x \approx 1.058$, with $4 - x^2$ above. Area $= \int (4 - x^2 - e^x)\,dx \approx 6.428$.

22. A. Disks: $V = \pi \int_0^1 \left(e^{-x^2}\right)^2 dx = \pi \int_0^1 e^{-2x^2} dx \approx 1.879$.

23. E. $\frac{1}{3}\int_0^3 \ln(x^2+1)\,dx \approx 1.135$.

24. C. $100 + \int_0^5 \left(6 + 4\cos\frac{t^2}{8}\right) dt \approx 137.553$. The number 37.553 is only the change.

25. B. v changes sign at $t = \sqrt{\pi} \approx 1.772$. Distance $= \int_0^{\sqrt{\pi}} \sin(t^2)\,dt - \int_{\sqrt{\pi}}^2 \sin(t^2)\,dt \approx 0.985$. (The displacement is 0.805.)

26. A. $y' = \frac{3x^2}{4}$, so $L = \int_0^2 \sqrt{1 + \frac{9x^4}{16}}\,dx \approx 3.096$.

27. D. Step size 0.25. $y_1 = 0.7500$, $y_2 = 0.5781$, $y_3 = 0.4961$, $y_4 = 0.5127$, so $y(1) \approx 0.513$.

28. E. $P = 200e^{0.35t}$. Set $200e^{0.35t} = 1000$, so $t = \frac{\ln 5}{0.35} \approx 4.598$.

29. C. The curves meet at $x = \pm 0.824$. Side $= \cos x - x^2$, so $V = \int_{-0.824}^{0.824} (\cos x - x^2)^2 dx \approx 0.874$.

30. B. $A' = 0$ at $t \approx 1.189$ and $t \approx 6.089$. A decreases, then increases to $t \approx 6.089$, then decreases. Candidates: $A(0) = 10$, $A(6.089) \approx 12.732$, $A(12) \approx 2.879$. The greatest is about 12.732.

5.10 Free-Response Practice on Applications of Integrals

QUESTION 1 · FREE RESPONSE

1. (a) $\int_0^4 (2\sqrt{x} - x)\,dx = \left[\frac{4}{3}x^{3/2} - \frac{x^2}{2}\right]_0^4 = \frac{32}{3} - 8 = \frac{8}{3}$. (The curves meet where $2\sqrt{x} = x$, at $x = 0$ and $x = 4$.)

(b) Washers: $V = \pi \int_0^4 \left[(2\sqrt{x})^2 - x^2\right] dx = \pi \int_0^4 (4x - x^2)\,dx = \pi\left(32 - \frac{64}{3}\right) = \frac{32\pi}{3}$.

(c) Side $= 2\sqrt{x} - x$, so $V = \int_0^4 (2\sqrt{x} - x)^2\,dx = \int_0^4 (4x - 4x^{3/2} + x^2)\,dx = 32 - \frac{256}{5} + \frac{64}{3} = \frac{32}{15}$.

(d) $V = \pi \int_0^4 \left[(2\sqrt{x}+1)^2 - (x+1)^2\right] dx = \pi \int_0^4 (4\sqrt{x} + 2x - x^2)\,dx = \pi\left(\frac{64}{3} + 16 - \frac{64}{3}\right) = 16\pi$.

Scoring: 9 points. (a) 2 points: integrand 1, answer 1. (b) 2 points: integral setup 1, answer 1. (c) 2 points: integral setup 1, answer 1. (d) 3 points: radii 2, evaluation 1.

QUESTION 2 · FREE RESPONSE

2. (a) $\int_0^4 \left(15 + 8\sin\frac{\pi t}{6}\right) dt = 60 + \frac{48}{\pi}\left(1 - \cos\frac{2\pi}{3}\right) \approx 82.918$ gallons.

(b) $W'(9) = R(9) - 12 = 15 - 8 - 12 = -5 < 0$, so the amount is decreasing at $t = 9$ ($\sin\frac{3\pi}{2} = -1$).

(c) $W'(t) = 3 + 8\sin\frac{\pi t}{6} = 0$ gives $t \approx 6.734$ in $(0, 12)$ with sign change from positive to negative, and $t \approx 11.266$ with sign change from negative to positive. Candidates: $W(0) = 300$, $W(6.734) \approx 349.645$, $W(12) = 336$. The greatest is at $t \approx 6.734$ hours.

(d) Greatest amount ≈ 349.645 gallons. Average inflow $= \frac{1}{12}\int_0^{12} R(t)\,dt = \frac{180}{12} = 15$ gallons per hour.

Scoring: 9 points. (a) 2 points: integral 1, answer 1. (b) 2 points: $W'(9)$ 1, conclusion with reason 1. (c) 3 points: $W' = 0$ 1, sign analysis 1, candidates compared 1. (d) 2 points: maximum 1, average 1.

QUESTION 3 · FREE RESPONSE

3. (a) $\left.\frac{dy}{dx}\right|_{(2,0)} = 2e^0 = 2$. Tangent line: $y = 2(x - 2)$. $f(2.2) \approx 0.4$.

(b) $\frac{d^2y}{dx^2} = e^{-y} + x\left(-e^{-y}\right)\frac{dy}{dx} = e^{-y} - x^2e^{-2y}$. At $(2, 0)$: $1 - 4 = -3 < 0$, so the graph is concave down near $x = 2$ and lies below its tangent line. The approximation 0.4 is an overestimate.

(c) $e^y\,dy = x\,dx$, so $e^y = \frac{x^2}{2} + C$. With $f(2) = 0$: $1 = 2 + C$, $C = -1$. Thus $e^y = \frac{x^2}{2} - 1$ and $f(x) = \ln\left(\frac{x^2}{2} - 1\right)$, for $x > \sqrt{2}$.

(d) $\frac{x^2}{2} - 1 = 7$ gives $x^2 = 16$, and $x = 4$ since $x > \sqrt{2}$.

Scoring: 9 points. (a) 2 points: tangent line 1, approximation 1. (b) 2 points: second derivative 1, conclusion with reason 1. (c) 4 points: separation 1, antiderivatives 1, constant 1, solution with domain 1. (d) 1 point.

QUESTION 4 · FREE RESPONSE

4. (a) $x(y-1) = 0$ when $x = 0$ or $y = 1$, so the slope is zero along the y-axis and along the line $y = 1$. At $(1, 3)$: $1(2) = 2$. At $(-1, 0)$: $(-1)(-1) = 1$.

(b) $g'(0) = 0 \cdot (2 - 1) = 0$. $\frac{d^2y}{dx^2} = (y-1) + x\frac{dy}{dx}$, so $g''(0) = 1 > 0$. Since $g'(0) = 0$ and $g''(0) > 0$, g has a relative minimum at $x = 0$.

(c) $h = 0.5$. $y_1 = 2 + 0.5 \cdot g'(0) = 2 + 0.5(0) = 2$. At $(0.5, 2)$ the slope is $0.5(1) = 0.5$, so $y_2 = 2 + 0.5(0.5) = 2.25$. $g(1) \approx 2.25$.

(d) $\frac{dy}{y-1} = x\,dx$, so $\ln|y-1| = \frac{x^2}{2} + C$. With $g(0) = 2$: $\ln 1 = C$, so $C = 0$ and $y - 1 = e^{x^2/2}$. $g(x) = 1 + e^{x^2/2}$.

Scoring: 9 points. (a) 3 points: zero-slope locations 1, two slopes 2. (b) 2 points: $g'(0) = 0$ and $g''(0) > 0$ with conclusion. (c) 2 points: first step 1, second step 1. (d) 2 points: separation and antiderivatives 1, constant and solution 1.

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Chapter 6: Techniques of Integration.

9 sections with explanations and worked examples, then 181 practice problems. Answers and full explanations are at the end of the chapter.

6.1 Integration by Parts

Learn to reverse the Product Rule: choose u and dv wisely, apply the formula once or repeatedly, and handle definite and cyclic integrals.

Some integrals are products that the Substitution Rule cannot handle, such as $\int xe^x\,dx$ or $\int x\cos x\,dx$. Integration by parts reverses the Product Rule. Since $\frac{d}{dx}[uv] = u\,v' + v\,u'$, integrating both sides and rearranging gives the formula below.

Integration by Parts

If u and v are differentiable functions of x with continuous derivatives,

$$\int u\,dv = uv - \int v\,du.$$

For definite integrals,

$$\int_a^b u\,dv = \Big[uv\Big]_a^b - \int_a^b v\,du.$$

Here $du = u'(x)\,dx$ and $dv = v'(x)\,dx$. The goal is to trade a hard integral $\int u\,dv$ for an easier integral $\int v\,du$.

Choosing u and dv: LIATE

Choose u as the function that comes first in this list: **L**ogarithmic, **I**nverse trigonometric, **A**lgebraic (polynomial), **T**rigonometric, **E**xponential. Let dv be everything else.

Requirements: you must be able to integrate dv to get v, and $v\,du$ should be simpler than $u\,dv$. LIATE is a guide, not a law: always check that the new integral is easier.

EXAMPLE 1: POLYNOMIAL TIMES EXPONENTIAL

Find $\int xe^{3x}\,dx$.

SOLUTION

Algebraic comes before exponential, so $u = x$ and $dv = e^{3x}dx$. Then $du = dx$ and $v = \frac{1}{3}e^{3x}$.

$\int xe^{3x}dx = \frac{x}{3}e^{3x} - \int \frac{1}{3}e^{3x}dx$.

$= \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C$. Check by differentiating: $\frac{1}{3}e^{3x} + xe^{3x} - \frac{1}{3}e^{3x} = xe^{3x}$.

EXAMPLE 2: A LOGARITHM AND A DEFINITE INTEGRAL

Evaluate $\int_1^e x\ln x\,dx$. [No calculator]

SOLUTION

Logarithmic comes first: $u = \ln x$, $dv = x\,dx$, so $du = \frac{1}{x}\,dx$ and $v = \frac{x^2}{2}$.

$\int_1^e x\ln x\,dx = \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e$.

$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4}$.

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A single logarithm

To integrate $\ln x$ alone, take $u = \ln x$ and $dv = dx$. Then $\int \ln x\,dx = x\ln x - \int x \cdot \frac{1}{x}\,dx = x\ln x - x + C$. The same idea works for inverse trigonometric functions: for $\int \arctan x\,dx$ take $u = \arctan x$, $dv = dx$.

EXAMPLE 3: REPEATED PARTS

Find $\int x^2 \sin x\,dx$.

SOLUTION

First pass: $u = x^2$, $dv = \sin x\,dx$, so $du = 2x\,dx$, $v = -\cos x$. $\int x^2 \sin x\,dx = -x^2\cos x + \int 2x\cos x\,dx$.

Second pass on $\int 2x\cos x\,dx$: $u = 2x$, $dv = \cos x\,dx$, so $du = 2\,dx$, $v = \sin x$. $\int 2x\cos x\,dx = 2x\sin x - \int 2\sin x\,dx = 2x\sin x + 2\cos x$.

$\int x^2 \sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C$.

Cyclic Integrals

For $\int e^x \sin x\,dx$ or $\int e^{ax}\cos bx\,dx$, apply parts twice. The original integral reappears on the right side. Treat it as an unknown I, then solve the equation for I.

Keep the same type of choice both times: if u was the exponential in the first pass, make u the exponential in the second pass too, otherwise you undo your work.

EXAMPLE 4: A CYCLIC INTEGRAL

Find $I = \int e^x \sin x\,dx$.

SOLUTION

Let $u = \sin x$, $dv = e^x dx$: $I = e^x \sin x - \int e^x \cos x\,dx$.

Again let $u = \cos x$, $dv = e^x dx$: $\int e^x \cos x\,dx = e^x\cos x + \int e^x \sin x\,dx = e^x\cos x + I$.

So $I = e^x\sin x - e^x\cos x - I$, hence $2I = e^x(\sin x - \cos x)$.

$I = \frac{e^x}{2}(\sin x - \cos x) + C$.

Common mistakes

(1) Dropping the minus sign in $uv - \int v\,du$. (2) Forgetting v is an antiderivative of dv, not dv itself. (3) In a definite integral, forgetting to evaluate uv at both limits. (4) Choosing u so that the new integral is harder, for example $u = e^x$ and $dv = x\,dx$ in $\int xe^x dx$ produces $\int x^2 e^x\,dx$. If it gets worse, swap your choices.

Practice

SET A Skills

Evaluate each integral. [No calculator]

1. Find $\int xe^{-2x}\,dx$.
2. Find $\int x\cos(3x)\,dx$.
3. Find $\int x^2 \ln x\,dx$.
4. Find $\int \frac{\ln x}{x^2}\,dx$.
5. Find $\int x^2 e^x\,dx$.
6. Find $\int \arctan x\,dx$.
7. Find $\int x\sec^2 x\,dx$ on an interval where $\cos x \neq 0$.
8. Find $\int x\sqrt{x+1}\,dx$.
9. Evaluate $\int_0^{\pi} x\sin x\,dx$.

SET B Applications

Give decimal answers correct to three decimal places. [Calculator]

10. Find the area of the region bounded by $y = xe^{-x}$, the x-axis and $x = 4$.
11. The region under $y = \sin x$ for $0 \le x \le \pi$ is revolved about the y-axis. Find the volume of the solid generated.
12. A particle has velocity $v(t) = te^{-t/2}$ meters per second. Find its displacement from $t = 0$ to $t = 6$ seconds.
13. Find the average value of $f(x) = \ln x$ on $[1, 5]$.

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14. A factory releases a pollutant at the rate $r(t) = 3te^{-t/2}$ kilograms per hour. How many kilograms are released in the first 8 hours?
15. Evaluate $\int_0^1 \arcsin x\,dx$.
16. Evaluate $\int_0^{\pi/2} x^2\cos x\,dx$.

SET C Challenge

Show all steps. [No calculator]

17. Find $\int e^{2x}\cos x\,dx$.
18. Find $\int (\ln x)^2\,dx$.
19. Find $\int x^3 e^{x^2}\,dx$. (Hint: substitute first.)
20. Evaluate $\int_0^1 x\arctan x\,dx$.
21. A function f is twice differentiable with $f(1) = 3$, $f(3) = 5$, $f'(1) = 2$ and $f'(3) = 4$. If f'' is continuous, find $\int_1^3 xf''(x)\,dx$.

6.2 Partial Fractions

Learn to split a rational function into simple fractions so that it integrates to logarithms and powers. This unlocks logistic growth and many separable equations.

A rational function is a quotient of polynomials. Some, like $\frac{7x+4}{(x+1)(x-2)}$, have no obvious antiderivative, yet each can be rewritten as a sum of simple fractions that integrate to logarithms and powers. This rewriting is called **partial fraction decomposition**, and it is the reverse of adding fractions over a common denominator.

Partial Fractions: Distinct and Repeated Linear Factors

Let $\frac{P(x)}{Q(x)}$ be a rational function in lowest terms with $\deg P < \deg Q$ and Q factored into linear factors. Each factor of Q contributes terms as follows.

A distinct factor $(x - r)$ contributes $\frac{A}{x-r}$. A repeated factor $(x - r)^m$ contributes $\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}$.

The constants are found by clearing denominators and then substituting convenient values of x (the roots) or comparing coefficients.

Improper Rational Functions and Basic Antiderivatives

If $\deg P \geq \deg Q$, divide first: $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$ where S is the quotient and $\deg R < \deg Q$. Then decompose $\frac{R}{Q}$.

$$\int \frac{A}{x-r}\,dx = A\ln|x-r| + C, \qquad \int \frac{A}{(x-r)^m}\,dx = \frac{A\,(x-r)^{1-m}}{1-m} + C \ \ (m \geq 2).$$

1. **Check the degrees.** If the numerator's degree is not less than the denominator's, do long division first.
2. **Factor the denominator** completely into linear factors (and irreducible quadratics, if any).
3. **Write the template** with one unknown constant for each factor and each power of a repeated factor.
4. **Clear denominators and solve** for the constants, using the roots of the factors to make terms vanish.
5. **Integrate term by term,** and check by differentiating or by recombining the fractions.

EXAMPLE 1: DISTINCT FACTORS

Find $\int \frac{7x+4}{(x+1)(x-2)}\,dx$.

SOLUTION

Write $\frac{7x+4}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$. Multiply by $(x+1)(x-2)$: $7x + 4 = A(x-2) + B(x+1)$.

Let $x = -1$: $-3 = -3A$, so $A = 1$. Let $x = 2$: $18 = 3B$, so $B = 6$.

$\int \left(\frac{1}{x+1} + \frac{6}{x-2}\right)dx = \ln|x+1| + 6\ln|x-2| + C$.

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EXAMPLE 2: A REPEATED FACTOR

Find $\int \frac{3x^2+18x+11}{(x-1)(x+3)^2}\,dx$.

SOLUTION

The template is $\frac{A}{x-1}+\frac{B}{x+3}+\frac{C}{(x+3)^2}$, so $3x^2+18x+11=A(x+3)^2+B(x-1)(x+3)+C(x-1)$.

Let $x=1$: $32=16A$, so $A=2$. Let $x=-3$: $-16=-4C$, so $C=4$. Compare the x^2 coefficients: $3=A+B$, so $B=1$.

$\int\left(\frac{2}{x-1}+\frac{1}{x+3}+\frac{4}{(x+3)^2}\right)dx=2\ln|x-1|+\ln|x+3|-\frac{4}{x+3}+C$.

Common mistake

For a repeated factor $(x+3)^2$ you need both $\frac{B}{x+3}$ and $\frac{C}{(x+3)^2}$. Using only $\frac{C}{(x+3)^2}$ leaves too few constants and the equation cannot be solved. Also, $\int\frac{1}{(x+3)^2}dx$ is $-\frac{1}{x+3}$, not a logarithm.

EXAMPLE 3: LONG DIVISION FIRST

Find $\int \frac{x^3+3x^2+2x+3}{x^2+x-2}\,dx$.

SOLUTION

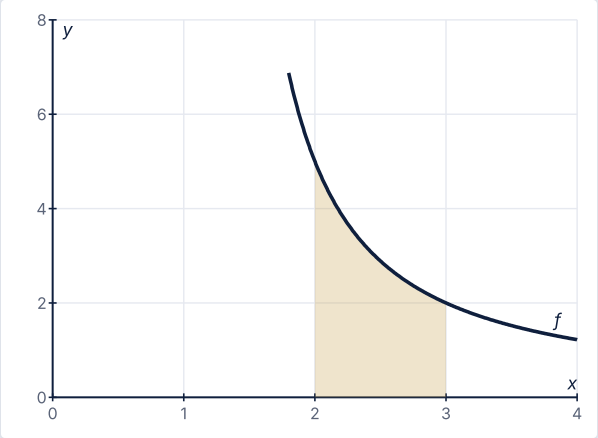
The numerator has higher degree, so divide: $x^3+3x^2+2x+3=(x+2)(x^2+x-2)+(2x+7)$. Thus the integrand is $x+2+\frac{2x+7}{(x-1)(x+2)}$.

Set $\frac{2x+7}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}$. At $x=1$: $9=3A$, so $A=3$. At $x=-2$: $3=-3B$, so $B=-1$.

$\int\left(x+2+\frac{3}{x-1}-\frac{1}{x+2}\right)dx=\frac{x^2}{2}+2x+3\ln|x-1|-\ln|x+2|+C$.

EXAMPLE 4: A DEFINITE INTEGRAL AND ITS AREA

Find the area of the region under $f(x)=\frac{3x-1}{(x-1)^2}$ from $x=2$ to $x=3$.



The shaded region lies under f from x = 2 to x = 3.

SOLUTION

On $[2,3]$ the function is continuous and positive. Decompose: $\frac{3x-1}{(x-1)^2}=\frac{3}{x-1}+\frac{2}{(x-1)^2}$, since $3(x-1)+2=3x-1$.

$\int_2^3\left(\frac{3}{x-1}+\frac{2}{(x-1)^2}\right)dx=\left[3\ln(x-1)-\frac{2}{x-1}\right]_2^3=(3\ln 2-1)-(0-2)=3\ln 2+1$.

The area is $3\ln 2+1$ square units.

Irreducible quadratics. A quadratic factor such as x^2+1 that has no real roots contributes a term $\frac{Bx+C}{x^2+1}$. Such a term splits into a logarithm (from $\frac{2Bx}{x^2+1}$) and an inverse tangent (from $\frac{1}{x^2+1}$).

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Practice

SET A Skills

Find each integral or constant. [No calculator]

1. Find $\int \frac{3}{(x-1)(x+2)}\,dx$.
2. Find $\int \frac{x+11}{(x-1)(x+3)}\,dx$.
3. Find $\int \frac{2x-1}{x^2-x-6}\,dx$.
4. Find constants A and B so that $\frac{5x+11}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4}$.
5. Find $\int \frac{6}{x^2-9}\,dx$.
6. Find $\int \frac{x+4}{(x+1)^2}\,dx$.
7. Find $\int \frac{3x-2}{(x-2)^2}\,dx$.
8. Find $\int \frac{x^2+3}{x+1}\,dx$.
9. Find $\int \frac{x^2}{x^2-1}\,dx$.

SET B Applications and Mixed Problems

Give exact answers where asked; otherwise give decimals to three places. [Calculator]

10. Find the area of the region under $y = \frac{6}{(x+1)(x+4)}$ from $x = 0$ to $x = 2$. Give a decimal and the exact form.
11. A population satisfies $\frac{dy}{dt} = 0.2\,y\,(5-y)$ with $y(0) = 1$. Separate variables and use partial fractions to show that $y = \frac{5e^t}{4+e^t}$, then find when $y = 4$.
12. In a reaction, the amount x (in grams) of product obeys $\frac{dx}{dt} = 0.02(10-x)(5-x)$ for $0 \le x < 5$, with $x(0) = 0$ (t in minutes). Find the time when $x = 4$.
13. A particle moves on a line with velocity $v(t) = \frac{8t+6}{(t+1)(t+2)}$ meters per second for $t \ge 0$. Find the distance traveled from $t = 0$ to $t = 3$.
14. Find the average value of $f(x) = \frac{5x+3}{x(x+1)}$ on $[1, 4]$.
15. A student writes $\int \frac{1}{x^2-4}\,dx = \ln|x^2-4| + C$. Explain what is wrong and give the correct antiderivative.
16. Evaluate $\int_1^2 \frac{2x+3}{x(x+3)}\,dx$ and give a decimal.

SET C Challenge

Show the decomposition. [No calculator]

17. Find $\int \frac{x^2+2}{x(x-1)^2}\,dx$.
18. Find $\int \frac{x^3}{x^2-4}\,dx$.
19. Evaluate $\int_3^4 \frac{4}{x^2-x-2}\,dx$.
20. Use the substitution $u = e^x$ and then partial fractions to find $\int \frac{e^x}{(e^x-1)(e^x+2)}\,dx$.
21. Find $\int \frac{\cos x}{\sin^2 x - 3\sin x + 2}\,dx$.

6.3 Trigonometric Substitution and Trig Integrals

Master integrals of products of trig powers and remove square roots by a trigonometric substitution. The trig substitution part extends beyond the exam but builds real fluency.

Integrals built from trigonometric functions come in families, and each family has a standard attack. Square roots of the form $\sqrt{a^2 \pm x^2}$ or $\sqrt{x^2 - a^2}$ can be removed by substituting a trigonometric function for x. Note: trigonometric substitution is an extension beyond what the AP exam requires, but the identities and substitution skills here appear throughout calculus.

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Powers of Sine and Cosine

Odd power of $\sin x$ or $\cos x$: peel off one factor and convert the rest with $\sin^2 x + \cos^2 x = 1$, then let u be the other function. For example $\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx$ with $u = \cos x$.

Even powers of both: use the half-angle identities

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2},$$

repeating if needed, together with $\sin x \cos x = \frac{1}{2}\sin 2x$.

Powers of Secant and Tangent

Use $\tan^2 x = \sec^2 x - 1$ and the derivatives $\frac{d}{dx}\tan x = \sec^2 x$, $\frac{d}{dx}\sec x = \sec x \tan x$.

Even power of $\sec x$: peel off $\sec^2 x$, convert the rest to $\tan x$, and let $u = \tan x$. **Odd power of $\tan x$ (with at least one $\sec x$):** peel off $\sec x \tan x$, convert the rest to $\sec x$, and let $u = \sec x$.

EXAMPLE 1: ODD POWER OF COSINE

Find $\int \sin^3 x \cos^2 x\,dx$.

SOLUTION

Sine has an odd power. Write $\sin^3 x = \sin^2 x\,\sin x = (1-\cos^2 x)\sin x$.

Let $u = \cos x$, $du = -\sin x\,dx$: $\int (1-u^2)u^2(-du) = -\int (u^2-u^4)\,du = -\frac{u^3}{3} + \frac{u^5}{5} + C$.

Answer: $-\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$.

EXAMPLE 2: EVEN POWERS

Evaluate $\int_0^{\pi/2} \cos^4 x\,dx$.

SOLUTION

Write $\cos^4 x = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1}{4}\left(1 + 2\cos 2x + \cos^2 2x\right)$ and use $\cos^2 2x = \frac{1+\cos 4x}{2}$. This gives $\cos^4 x = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$.

Antiderivative: $\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}$. At $x = \frac{\pi}{2}$ the sine terms are 0, and at 0 everything is 0.

Value: $\frac{3}{8}\cdot\frac{\pi}{2} = \frac{3\pi}{16}$.

EXAMPLE 3: TANGENT AND SECANT

Find $\int \tan^3 x \sec x\,dx$.

SOLUTION

Tangent has an odd power: $\tan^3 x \sec x = \tan^2 x \cdot \sec x \tan x = (\sec^2 x - 1)\sec x \tan x$.

Let $u = \sec x$, $du = \sec x \tan x\,dx$: $\int (u^2-1)\,du = \frac{u^3}{3} - u + C$.

Answer: $\frac{\sec^3 x}{3} - \sec x + C$.

Trigonometric Substitution

For $a > 0$, these substitutions use $\sin^2 t + \cos^2 t = 1$ and $1 + \tan^2 t = \sec^2 t$ to remove the root:

$\sqrt{a^2 - x^2}$: let $x = a\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, so $\sqrt{a^2-x^2} = a\cos t$. $\sqrt{a^2 + x^2}$: let $x = a\tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$, so $\sqrt{a^2+x^2} = a\sec t$. $\sqrt{x^2 - a^2}$: let $x = a\sec t$, $0 \le t < \frac{\pi}{2}$ (for $x \ge a$), so $\sqrt{x^2-a^2} = a\tan t$.

Always replace dx too. For an indefinite integral, draw a right triangle to convert back to x. For a definite integral, convert the limits to t and skip the back-substitution.

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EXAMPLE 4: SINE SUBSTITUTION

Find $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$ for $0 < x < 2$.

SOLUTION

Let $x = 2\sin t$. Then $dx = 2\cos t\,dt$ and $\sqrt{4-x^2} = 2\cos t$ (positive since $\cos t > 0$ here).

$\int \frac{2\cos t\,dt}{4\sin^2 t \cdot 2\cos t} = \frac{1}{4}\int \csc^2 t\,dt = -\frac{1}{4}\cot t + C$.

From the triangle with hypotenuse 2 and opposite side x, the adjacent side is $\sqrt{4-x^2}$, so $\cot t = \frac{\sqrt{4-x^2}}{x}$. Answer: $-\frac{\sqrt{4-x^2}}{4x} + C$.

Common mistake

Forgetting dx in a trig substitution, or writing $\sqrt{a^2-x^2} = a\cos t$ without checking the range of t. Also, do not mix the variables: when you change the limits of a definite integral to t-values, you must not convert back to x afterward.

EXAMPLE 5: TANGENT SUBSTITUTION, DEFINITE

Evaluate $\displaystyle\int_0^2 \frac{dx}{(x^2+4)^{3/2}}$.

SOLUTION

Let $x = 2\tan t$, so $dx = 2\sec^2 t\,dt$ and $(x^2+4)^{3/2} = (4\sec^2 t)^{3/2} = 8\sec^3 t$. Limits: $x = 0 \to t = 0$, $x = 2 \to t = \frac{\pi}{4}$.

$\int_0^{\pi/4} \frac{2\sec^2 t}{8\sec^3 t}\,dt = \frac{1}{4}\int_0^{\pi/4} \cos t\,dt = \frac{1}{4}\sin t\Big|_0^{\pi/4}$.

Value: $\frac{1}{4}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{8} \approx 0.177$.

Practice

SET A Skills

Find each integral. [No calculator]

1. Find $\displaystyle\int \cos^3 x\,dx$.
2. Find $\displaystyle\int \sin^3 x\,dx$.
3. Find $\displaystyle\int \sin^2 x\,dx$.
4. Evaluate $\displaystyle\int_0^{\pi} \cos^2 x\,dx$.
5. Find $\displaystyle\int \sin^2 x\cos^3 x\,dx$.
6. Find $\displaystyle\int \sec^2 x\tan^4 x\,dx$.
7. Find $\displaystyle\int \tan^2 x\,dx$.
8. Find $\displaystyle\int \sec^3 x\tan x\,dx$.
9. Use the substitution $x = 3\sin t$ to evaluate $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$, and check with geometry.

SET B Applications and Mixed Problems

Give decimals to three places unless an exact form is asked for. [Calculator]

10. Evaluate $\displaystyle\int_0^{\pi/2} \sin^4 x\,dx$ and give a decimal.
11. An alternating current is $I(\theta) = 4\sin\theta$ amperes. The root mean square current is $\sqrt{\text{average of } I^2 \text{ over one period}}$. Find it over $[0, 2\pi]$.
12. The region under $y = \sin^2 x$ for $0 \le x \le \pi$ is revolved about the x-axis. Find the volume.
13. The ellipse $\frac{x^2}{4} + y^2 = 1$ bounds a region. Use $x = 2\sin t$ to find its area.
14. Evaluate $\displaystyle\int_0^1 \frac{x^2}{(1+x^2)^{3/2}}\,dx$ and give a decimal.
15. Evaluate $\displaystyle\int_0^{\pi/4} \tan^3 x\,dx$ and give a decimal.
16. A circle $x^2 + y^2 = 9$ is cut by the line $x = 1$. Find the area of the part of the disk to the right of $x = 1$.

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SET C **Challenge**

Exact answers only. [No calculator]

17. Find $\int \sin^4 x\,dx$.

18. Find $\int \sec^4 x\,dx$.

19. Find $\int \tan^3 x \sec^3 x\,dx$.

20. Find $\int \frac{dx}{x^2\sqrt{x^2+4}}$.

21. Find $\int \frac{\sqrt{x^2-1}}{x}\,dx$ for $x > 1$.

6.4 L'Hopital's Rule

Use derivatives to evaluate limits of the form 0/0 and infinity/infinity, and convert other indeterminate forms into these. You will also learn when the rule does not apply.

Many limits produce a form such as $\frac{0}{0}$ whose value cannot be read off directly. L'Hopital's Rule turns such a limit into a limit of derivatives, which is often easy. It is also the main tool for comparing how fast different functions grow.

L'Hopital's Rule

Suppose f and g are differentiable near c (except possibly at c), with $g'(x) \neq 0$ near c. If $\lim_{x\to c} \frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$

provided the right-hand limit exists (or is $\pm\infty$).

The rule holds for c finite, $c = \pm\infty$, and one-sided limits. Differentiate the numerator and denominator **separately**; do not use the quotient rule. If the new limit is again indeterminate, apply the rule again.

Other Indeterminate Forms

$0 \cdot \infty$: write $f \cdot g$ as $\frac{f}{1/g}$ or $\frac{g}{1/f}$. $\infty - \infty$: combine into a single fraction (common denominator) or factor.

1^∞, 0^0, ∞^0: if $y = f(x)^{g(x)}$, compute $\lim \ln y = \lim g(x) \ln f(x)$, which is a $0 \cdot \infty$ form. If $\lim \ln y = L$, then $\lim y = e^L$.

EXAMPLE 1: FORM 0/0

Find $\lim_{x\to 0} \frac{e^{3x}-1}{\sin 2x}$.

SOLUTION

At $x = 0$ the numerator is $e^0 - 1 = 0$ and the denominator is $\sin 0 = 0$, so the form is $\frac{0}{0}$ and the rule applies.

$\lim_{x\to 0} \frac{3e^{3x}}{2\cos 2x} = \frac{3\cdot 1}{2\cdot 1} = \frac{3}{2}$.

EXAMPLE 2: FORM INFINITY OVER INFINITY, REPEATED

Find $\lim_{x\to\infty} \frac{x^2}{e^{3x}}$.

SOLUTION

Both x^2 and e^{3x} tend to ∞, so the form is $\frac{\infty}{\infty}$. Differentiate: $\lim \frac{2x}{3e^{3x}}$, still $\frac{\infty}{\infty}$.

Differentiate again: $\lim \frac{2}{9e^{3x}} = 0$. Hence $\lim_{x\to\infty} \frac{x^2}{e^{3x}} = 0$: the exponential grows faster than any polynomial.

EXAMPLE 3: APPLYING THE RULE THREE TIMES

Find $\lim_{x\to 0} \frac{x - \sin x}{x^3}$.

SOLUTION

Form $\frac{0}{0}$. First application: $\lim \frac{1-\cos x}{3x^2}$, still $\frac{0}{0}$.

Second: $\lim \frac{\sin x}{6x}$, still $\frac{0}{0}$. Third: $\lim \frac{\cos x}{6} = \frac{1}{6}$.

Answer: $\frac{1}{6}$. (Each step was justified because the previous limit was indeterminate.)

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When the rule does not apply

Check the form first. The limit $\lim_{x\to 2}\frac{x+1}{x^2}=\frac{3}{4}$ is not indeterminate, and using the rule would wrongly give $\lim\frac{1}{2x}=\frac{1}{4}$. Also, the rule needs $\lim\frac{f'}{g'}$ to exist: $\lim_{x\to\infty}\frac{x+\sin x}{x}=1$ by dividing through by x, but $\frac{1+\cos x}{1}$ has no limit, so the rule gives no information there.

EXAMPLE 4: FORM 0 TIMES INFINITY

Find $\lim_{x\to 0^+}\sqrt{x}\ln x$.

SOLUTION

As $x\to 0^+$, $\sqrt{x}\to 0$ and $\ln x\to -\infty$. Rewrite as $\frac{\ln x}{x^{-1/2}}$, which has the form $\frac{-\infty}{\infty}$.

$\lim\frac{1/x}{-\frac{1}{2}x^{-3/2}}=\lim\left(-2x^{1/2}\right)=0$. Answer: 0.

EXAMPLE 5: FORM 1 TO THE INFINITY

Find $\lim_{x\to 0}(1+3x)^{1/x}$.

SOLUTION

The form is 1^∞. Let $y=(1+3x)^{1/x}$, so $\ln y=\frac{\ln(1+3x)}{x}$, which is $\frac{0}{0}$ as $x\to 0$.

$\lim\frac{3/(1+3x)}{1}=3$. So $\ln y\to 3$ and $y\to e^3$. Answer: e^3.

Practice

SET A Skills

Evaluate each limit. [No calculator]

1. Find $\lim_{x\to 0}\dfrac{e^{4x}-1}{2x}$.
2. Find $\lim_{x\to 2}\dfrac{x^3-8}{x^2-4}$.
3. Find $\lim_{x\to 0}\dfrac{1-\cos 3x}{x^2}$.
4. Find $\lim_{x\to\infty}\dfrac{5x^2+3x}{2x^2-7}$.
5. Find $\lim_{x\to\infty}\dfrac{\ln x}{\sqrt{x}}$.
6. Find $\lim_{x\to\infty}\dfrac{x^3}{e^{2x}}$.
7. Find $\lim_{x\to\infty}x\sin\dfrac{3}{x}$.
8. Find $\lim_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)$.
9. Find $\lim_{x\to 0^+}x^{2x}$.

SET B Applications and Mixed Problems

Give decimals to three places where a decimal is requested. [Calculator]

10. Functions f and g are differentiable at $x=2$, with values given in the table. Find $\lim_{x\to 2}\dfrac{f(x)}{g(x)}$ and justify. (The functions are continuous at 2.)

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	0	6	0	-4

11. Let $h(x)=\dfrac{\sin x-x\cos x}{x^3}$. (a) Evaluate $h(0.1)$ to three decimal places. (b) Use the rule to find $\lim_{x\to 0}h(x)$.
12. An account holds 1000 dollars at an annual rate of 6%, compounded n times per year, so after one year it holds $A_n=1000\left(1+\frac{0.06}{n}\right)^n$. Find $\lim_{n\to\infty}A_n$.
13. A concentration is modeled by $R(t)=\dfrac{t^2}{e^{0.5t}}$ for $t\ge 0$. (a) Find $\lim_{t\to\infty}R(t)$. (b) Find $R(10)$.
14. A circular arc of radius 1 subtends a central angle θ (in radians). The area between the arc and its chord is $\frac{1}{2}(\theta-\sin\theta)$. Find $\lim_{\theta\to 0}\dfrac{\text{area}}{\theta^3}$.
15. A student evaluates $\lim_{x\to 1}\dfrac{x^2+3}{x+1}$ with L'Hopital's Rule and gets $\lim\frac{2x}{1}=2$. Is the answer right? Explain.

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16. Find $\lim_{x\to 0}(\cos x)^{1/x^2}$. Give an exact value and a decimal.

SET C **Challenge**

Show the form at each step. [No calculator]

17. Find $\lim_{x\to 0}\dfrac{\tan x - x}{x^3}$.

18. Find $\lim_{x\to 0^+}(\sin x)^x$.

19. Find $\lim_{x\to\infty}(\ln x)^{1/x}$.

20. Find $\lim_{x\to \pi/2^-}(\sec x - \tan x)$.

21. Explain why L'Hopital's Rule cannot be used to evaluate $\lim_{x\to 0}\dfrac{x^2\sin(1/x)}{\sin x}$, then find the limit.

6.5 Improper Integrals

Extend integration to infinite intervals and unbounded integrands using limits, and decide whether such integrals converge or diverge. This leads to infinite series and probability.

A definite integral $\int_a^b f(x)\,dx$ assumes a finite interval and a function that is continuous on it. An **improper integral** relaxes one of these conditions: either an endpoint is infinite, or f blows up at some point of the interval. We define such integrals using limits, and they either converge to a number or diverge.

Improper Integrals of Type 1: Infinite Limits

If f is continuous on $[a,\infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

Likewise $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$.

For $\int_{-\infty}^\infty f(x)\,dx$, choose any number c and split into $\int_{-\infty}^c f + \int_c^\infty f$. The integral converges only if **both** pieces converge. If a limit is finite, the integral **converges** to it; otherwise it **diverges**.

Improper Integrals of Type 2: Infinite Discontinuities

If f is continuous on $[a,b)$ and unbounded near b, then $\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$. If the trouble is at a, use $\lim_{c\to a^+}\int_c^b f$.

If f is unbounded at an interior point d, split at d: $\int_a^b f = \int_a^d f + \int_d^b f$. Both pieces must converge. Always scan the interval for points where the integrand is undefined before applying the Fundamental Theorem.

EXAMPLE 1: INFINITE UPPER LIMIT

Evaluate $\displaystyle\int_2^\infty \frac{6}{x^3}\,dx$.

SOLUTION

$\int_2^b 6x^{-3}dx = \left[-3x^{-2}\right]_2^b = -\frac{3}{b^2}+\frac{3}{4}$.

As $b\to\infty$, $-\frac{3}{b^2}\to 0$. The integral converges to $\frac{3}{4}$.

EXAMPLE 2: INTEGRATION BY PARTS WITH A LIMIT

Evaluate $\displaystyle\int_0^\infty xe^{-2x}\,dx$.

SOLUTION

By parts with $u=x$, $dv=e^{-2x}dx$: $\int xe^{-2x}dx = -\frac{x}{2}e^{-2x}-\frac{1}{4}e^{-2x}$.

Evaluate from 0 to b: $-\frac{b}{2}e^{-2b}-\frac{1}{4}e^{-2b}+\frac{1}{4}$. Since $be^{-2b}=\frac{b}{e^{2b}}\to 0$ (L'Hopital's Rule), the limit is $\frac{1}{4}$. The integral converges to $\frac{1}{4}$.

EXAMPLE 3: UNBOUNDED INTEGRAND AT AN ENDPOINT

Evaluate $\displaystyle\int_0^4 \frac{dx}{\sqrt{x}}$.

SOLUTION

The integrand is unbounded as $x\to 0^+$, so use $\lim_{c\to 0^+}\int_c^4 x^{-1/2}dx = \lim_{c\to 0^+}\left[2\sqrt{x}\right]_c^4$.

$= \lim_{c\to 0^+}(4-2\sqrt{c}) = 4$. The integral converges to 4.

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Common mistake

Applying the Fundamental Theorem across a discontinuity. For $\int_0^2 \frac{dx}{(x-1)^2}$, the antiderivative $-\frac{1}{x-1}$ gives $-1-1=-2$, which is absurd because the integrand is positive. The integrand is unbounded at $x=1$, inside the interval, so you must split there. Here $\int_0^1 \frac{dx}{(x-1)^2}$ diverges, so the whole integral diverges.

The p-Integrals and the Comparison Test

$\int_1^\infty \frac{dx}{x^p}$ converges (to $\frac{1}{p-1}$) if $p>1$ and diverges if $p\le 1$. $\int_0^1 \frac{dx}{x^p}$ converges (to $\frac{1}{1-p}$) if $p<1$ and diverges if $p\ge 1$.

Comparison. If $0\le f(x)\le g(x)$ for all $x\ge a$, then: $\int_a^\infty g$ converges $\Rightarrow$ $\int_a^\infty f$ converges; $\int_a^\infty f$ diverges $\Rightarrow$ $\int_a^\infty g$ diverges. The inequality must go the right way, and both functions must be nonnegative.

EXAMPLE 4: COMPARISON

Decide whether $\int_1^\infty e^{-x^2}\,dx$ converges, and bound its value.

SOLUTION

For $x\ge 1$, $x^2\ge x$, so $e^{-x^2}\le e^{-x}$, and both are positive.

$\int_1^\infty e^{-x}dx=\lim_{b\to\infty}\left(e^{-1}-e^{-b}\right)=e^{-1}$, which converges. By comparison, $\int_1^\infty e^{-x^2}dx$ converges and its value is at most e^{-1}.

Contrast: $\int_1^\infty \frac{2+\sin x}{x}\,dx$ diverges, because $\frac{2+\sin x}{x}\ge\frac{1}{x}>0$ and $\int_1^\infty \frac{dx}{x}$ diverges.

Practice

SET A Skills

Evaluate each improper integral, or state that it diverges. [No calculator]

1. Evaluate $\int_1^\infty \frac{dx}{x^3}$.
2. Determine whether $\int_1^\infty \frac{dx}{\sqrt{x}}$ converges. Justify.
3. Evaluate $\int_0^\infty e^{-2x}\,dx$.
4. Evaluate $\int_0^1 x^{-1/3}\,dx$.
5. Determine whether $\int_0^1 \frac{dx}{x}$ converges.
6. Evaluate $\int_{-\infty}^0 e^x\,dx$.
7. Evaluate $\int_0^\infty \frac{dx}{1+x^2}$.
8. Evaluate $\int_e^\infty \frac{dx}{x(\ln x)^2}$.
9. Evaluate $\int_0^1 \ln x\,dx$.

SET B Applications and Mixed Problems

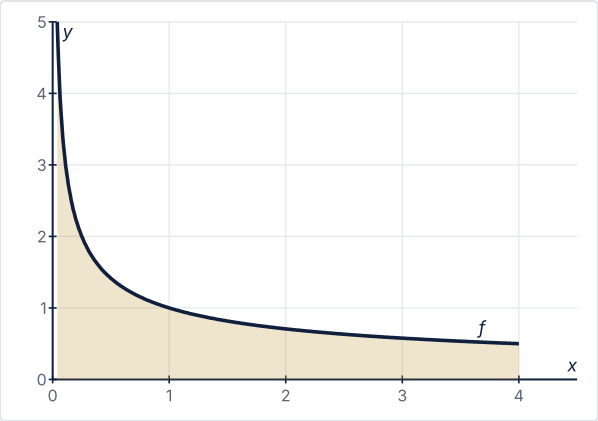
Give decimals to three places where a decimal is requested. [Calculator]

10. A radioactive sample emits at the rate $r(t)=5e^{-0.4t}$ millicuries per hour. Find the total amount emitted over all time, $\int_0^\infty r(t)\,dt$.
11. The attractive force on a probe is $F(r)=\frac{36}{r^2}$ newtons at distance r meters. Find the work needed to move it from $r=3$ to infinity.
12. Find the area of the region between $y=\frac{3}{x^2+9}$ and the x-axis over the whole real line.

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13. The graph shows $f(x) = \dfrac{1}{\sqrt{x}}$ for $0 < x \le 4$. The region between the graph and the x-axis is unbounded vertically. Does it have finite area? Find it.



Graph of f(x) = 1/sqrt(x), with the region from x near 0 to x = 4 shaded.

14. Use a calculator to evaluate $\displaystyle\int_1^{\infty} e^{-x^2}\,dx$ numerically, and compare it with the bound e^{-1} found by comparison.

15. The region under $y = \dfrac{1}{x}$ for $x \ge 1$ is revolved about the x-axis. Find the volume.

16. For which values of p does $\displaystyle\int_0^1 x^{-p}\,dx$ converge? Evaluate it for $p = 0.6$.

SET C Challenge

Use limits in every solution. [No calculator]

17. Evaluate $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$.

18. Evaluate $\displaystyle\int_0^{\infty} xe^{-x^2}\,dx$.

19. Evaluate $\displaystyle\int_0^2 \frac{dx}{(x-1)^{2/3}}$ (take real cube roots).

20. Evaluate $\displaystyle\int_2^{\infty} \frac{dx}{x^2-1}$.

21. Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{e^x + e^{-x}}$.

6.6 Functions Defined by Integrals

Use the Fundamental Theorem with the Chain Rule to differentiate accumulation functions, analyze their graphs, and work with functions that have no elementary antiderivative.

Many important functions are defined as an integral with a variable upper limit, such as $F(x) = \int_a^x f(t)\,dt$. They record accumulated area, and the Fundamental Theorem of Calculus tells you their derivative without finding any antiderivative. This is how we study functions like $\int_0^x e^{-t^2}dt$ that have no elementary formula.

Fundamental Theorem of Calculus, Part 1 (with the Chain Rule)

If f is continuous on an interval containing a, then $F(x) = \int_a^x f(t)\,dt$ is differentiable there and $F'(x) = f(x)$.

If $u(x)$ and $v(x)$ are differentiable,

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))\,u'(x), \qquad \frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = f(u(x))\,u'(x) - f(v(x))\,v'(x).$$

Swapping the limits changes the sign: $\frac{d}{dx}\int_x^a f(t)\,dt = -f(x)$.

Analyzing a Function Defined by an Integral

For $g(x) = \int_a^x f(t)\,dt$ with f continuous: $g(a) = 0$; $g' = f$; $g'' = f'$ (where it exists); and $g(b) = g(a) + \int_a^b f(t)\,dt$.

Justification templates. g has a local maximum at $x = c$ because $g'(x) = f(x)$ changes from positive to negative at c. g has a point of inflection at $x = c$ because $g'' = f'$ changes sign at c (equivalently, f changes from increasing to decreasing, or the reverse). g is concave down on an interval because f is decreasing there.

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EXAMPLE 1: CHAIN RULE

Find $\frac{d}{dx}\int_1^{x^3} \ln(1+t^2)\,dt$.

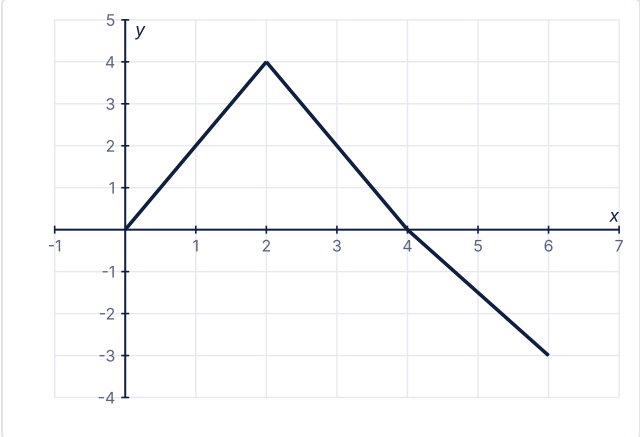
SOLUTION

The upper limit is $u(x) = x^3$ with $u'(x) = 3x^2$, and $f(t) = \ln(1+t^2)$.

The derivative is $f(x^3)\cdot 3x^2 = 3x^2\ln\left(1+x^6\right)$.

EXAMPLE 2: READING A GRAPH

The graph shows f, made of line segments. Let $g(x) = \int_0^x f(t)\,dt$. Find $g(2)$, $g(4)$, $g(6)$, the location of the maximum of g on $[0,6]$, and any inflection point.



Graph of f, made of three line segments.

SOLUTION

Areas: $\int_0^2 f = \frac{1}{2}(2)(4) = 4$, $\int_2^4 f = \frac{1}{2}(2)(4) = 4$, $\int_4^6 f = -\frac{1}{2}(2)(3) = -3$. So $g(2) = 4$, $g(4) = 8$, $g(6) = 5$.

$g' = f$ changes from positive to negative at $x = 4$, so g has a local maximum there. Compare with the endpoints: $g(0) = 0$, $g(4) = 8$, $g(6) = 5$, so the absolute maximum on $[0,6]$ is 8 at $x = 4$.

$g'' = f'$ is positive on $(0,2)$ and negative on $(2,6)$, so g has a point of inflection at $x = 2$.

Common mistake

Forgetting the factor $u'(x)$ when the upper limit is not just x, or forgetting the minus sign when the variable is the lower limit. Also, the variable of integration is a dummy: in $\frac{d}{dx}\int_0^{x^2}\sin(t)\,dt$ the answer is $\sin(x^2)\cdot 2x$, with x (not t) in the result.

EXAMPLE 3: NO ELEMENTARY ANTIDERIVATIVE

Let $F(x) = \int_0^x e^{-t^2}\,dt$. Find $F'(x)$, show that F is increasing and concave down for $x > 0$, and find $F(1)$.

SOLUTION

By FTC Part 1, $F'(x) = e^{-x^2} > 0$, so F is increasing everywhere. Then $F''(x) = -2xe^{-x^2} < 0$ for $x > 0$, so F is concave down for $x > 0$.

No elementary antiderivative exists, but a calculator integrates numerically: $F(1) \approx 0.747$.

EXAMPLE 4: A LIMIT OF AN INTEGRAL FUNCTION

Find $\lim_{x\to 0}\frac{1}{x^3}\int_0^x \sin(t^2)\,dt$.

SOLUTION

As $x \to 0$ the integral tends to 0 and $x^3 \to 0$, so the form is $\frac{0}{0}$. By L'Hopital's Rule and FTC Part 1, the limit equals $\lim \frac{\sin(x^2)}{3x^2}$.

Since $\lim_{u\to 0}\frac{\sin u}{u} = 1$ (with $u = x^2$), the limit is $\frac{1}{3}$.

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EXAMPLE 5: RECOVERING F

A continuous function f and a constant c satisfy $\int_c^x f(t)\,dt = 2x^2 - 8$ for all x. Find $f(x)$ and c.

SOLUTION

Differentiate both sides with respect to x: $f(x) = 4x$.

Put $x = c$: the left side is 0, so $2c^2 - 8 = 0$ and $c = 2$ or $c = -2$. Both values work.

Practice

SET A Skills

Answer each question. [No calculator]

1. Let $G(x) = \int_2^x \sqrt{t^2 + 5}\,dt$. Find $G'(x)$ and $G'(2)$.
2. Find $\frac{d}{dx}\int_0^{2x} e^{t^2}\,dt$.
3. Find $\frac{d}{dx}\int_x^5 \sqrt{1 + t^4}\,dt$.
4. Find $\frac{d}{dx}\int_x^{x^2} t^3\,dt$ and check it by integrating first.
5. Let $F(x) = \int_0^x (t^2 - 4)\,dt$. Find the critical points of F and classify them using F''.
6. Let $G(x) = 5 + \int_2^x t^3\,dt$. Find $G(2)$, $G'(2)$ and $G(3)$.
7. Find $\lim_{x \to 0} \frac{1}{x}\int_0^x \cos(t^2)\,dt$.
8. Find $\lim_{x \to 0} \frac{\int_0^x \sin t\,dt}{x^2}$.
9. A continuous function satisfies $\int_a^x f(t)\,dt = x^3 - 27$ for all x. Find $f(x)$ and a.

SET B Applications and Mixed Problems

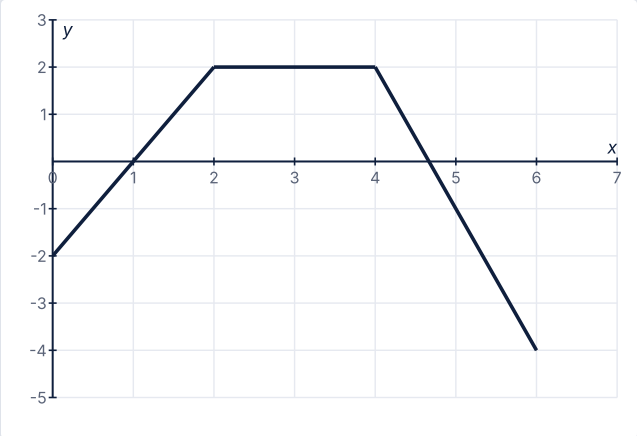
Give decimals to three places. [Calculator]

10. The graph shows f on $[0, 6]$, made of line segments. Let $g(x) = \int_0^x f(t)\,dt$.

    (a) Find $g(2)$, $g(4)$ and $g(6)$.

    (b) Find the absolute minimum and absolute maximum values of g on $[0, 6]$.

    (c) On what interval is g concave down?



Graph of f.

11. Let $F(x) = \int_0^x e^{-t^2}\,dt$. Find $F(2)$ and $F'(2)$ to three decimal places, and write the tangent line at $x = 2$.
12. The sine integral is $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$. Find $\mathrm{Si}(\pi)$ and $\mathrm{Si}(2\pi)$, and justify that Si has a local maximum at $x = \pi$.
13. The volume $V(t)$ of liquid in a tank (liters) satisfies $V'(t) = 6\sin(0.4t^2)$ for $0 \le t \le 3$ minutes, with $V(0) = 40$. Find $V(3)$, and the time when V is greatest on $[0, 3]$ and that greatest value.

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14. Let $G(x) = \int_0^{x^2} \sqrt{1+t^3}\,dt$. Find $G'(1.5)$ to three decimal places.

15. Evaluate $\lim_{x\to 0} \frac{1}{x^3}\int_0^x \ln(1+t^2)\,dt$. Then check by computing the quotient at $x = 0.1$ to three decimal places.

16. A continuous function f has the values in the table, and $g(x) = \int_0^x f(t)\,dt$. (a) Use trapezoids with subintervals of width 1 to estimate $g(2)$ and $g(4)$. (b) Explain why g has a critical point between $x = 2$ and $x = 3$. If f has only one zero in that interval, what kind of critical point is it?

t	0	1	2	3	4
$f(t)$	2	5	3	-1	-4

SET C Challenge

Justify each conclusion. [No calculator]

17. Find $\frac{d}{dx}\int_{\sin x}^{\cos x} \sqrt{1+t^2}\,dt$.

18. A continuous function f satisfies $\int_0^{x^2} f(t)\,dt = x^4 + x^3$ for $x \geq 0$. Find $f(4)$.

19. Find $\lim_{x\to 0} \frac{1}{x^3}\int_0^x \left(e^{t^2} - 1\right)dt$.

20. Find $\lim_{x\to\infty} \frac{\int_0^x e^{t^2}\,dt}{e^{x^2}}$.

21. Let $F(x) = 1 + \int_0^x e^{-t^2}\,dt$. Without a calculator, show that $F(0.5) < 1.5$.

6.7 Choosing the Right Technique

Practice deciding among rewriting, substitution, parts, partial fractions and geometry. Pick the method before you start computing, and save time and errors.

Knowing every integration technique is only half the skill. The other half is choosing the right one quickly. In this section you practice reading an integrand, recognizing its structure, and deciding what to try first. Often the first move is not integration at all but a rewrite.

A Checklist for Choosing a Technique

Work down this list and stop at the first technique that fits. (1) **Simplify or rewrite:** expand, divide term by term, use a trig identity, or complete the square. (2) **Basic formula:** is it already a standard antiderivative? (3) **Substitution:** is there an inner function whose derivative (up to a constant) also appears?

(4) **Parts:** is it a product of unrelated types (polynomial times exponential or trig, or a log or inverse trig alone)? (5) **Partial fractions:** is it a proper rational function with a factorable denominator (divide first if not proper)? (6) **Geometry or symmetry:** does a definite integral describe an area you know, such as a circle, or is the integrand odd on a symmetric interval?

Cue in the integrand	First move	Example
Inner function and its derivative	Substitution	$\int xe^{x^2}dx$
Polynomial times e^x, $\sin x$, $\cos x$	Parts, polynomial as u	$\int xe^{3x}dx$
$\ln x$ or an inverse trig function	Parts, that function as u	$\int \ln x\,dx$
Proper rational, factorable denominator	Partial fractions	$\int \frac{1}{(x-1)(x+3)}dx$
Degree of top is at least degree of bottom	Divide, then another method	$\int \frac{x^2+3}{x+1}dx$
Quadratic under a root or in a denominator	Complete the square, or trig	$\int \frac{dx}{x^2+4x+13}$
Definite integral of $\sqrt{r^2-x^2}$	Geometry (circle)	$\int_{-2}^{2}\sqrt{4-x^2}\,dx$

Antiderivatives Worth Knowing

$\int \tan x\,dx = -\ln|\cos x| + C$, $\int \ln x\,dx = x\ln x - x + C$, $\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C$.

$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C$, $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$. Integration by parts: $\int u\,dv = uv - \int v\,du$.

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EXAMPLE 1: REWRITE FIRST

Find $\int \frac{x^3 + 4x^2 - 5}{x^2}\,dx$.

SOLUTION

Divide each term by x^2: $x + 4 - 5x^{-2}$.

$\int \left(x + 4 - 5x^{-2}\right) dx = \frac{x^2}{2} + 4x + \frac{5}{x} + C$.

EXAMPLE 2: SUBSTITUTION

Find $\int \frac{(\ln x)^3}{x}\,dx$.

SOLUTION

The inner function $\ln x$ has derivative $\frac{1}{x}$, which appears. Let $u = \ln x$, $du = \frac{dx}{x}$.

$\int u^3\,du = \frac{u^4}{4} + C = \frac{(\ln x)^4}{4} + C$.

EXAMPLE 3: PARTS

Find $\int x^2 \ln x\,dx$.

SOLUTION

A logarithm times a power: let $u = \ln x$, $dv = x^2 dx$, so $du = \frac{dx}{x}$ and $v = \frac{x^3}{3}$.

$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$.

EXAMPLE 4: PARTIAL FRACTIONS

Find $\int \frac{5x + 7}{x^2 + 3x + 2}\,dx$.

SOLUTION

The denominator factors as $(x+1)(x+2)$ and the degrees are fine. Write $5x + 7 = A(x+2) + B(x+1)$. At $x = -1$: $2 = A$. At $x = -2$: $-3 = -B$, so $B = 3$.

$\int \left(\frac{2}{x+1} + \frac{3}{x+2}\right) dx = 2\ln|x+1| + 3\ln|x+2| + C$. (Substitution on the whole denominator does not work, because $5x + 7$ is not a multiple of $2x + 3$.)

Common mistake

Using a technique because it worked last time. Compare $\int \frac{x}{x^2+4}dx$ (substitution, giving $\frac{1}{2}\ln(x^2+4)$) with $\int \frac{1}{x^2+4}dx$ (the arctangent formula, giving $\frac{1}{2}\arctan\frac{x}{2}$). Always check the numerator against the derivative of the denominator before choosing.

EXAMPLE 5: SUBSTITUTION, THEN PARTS

Find $\int e^{\sqrt{x}}\,dx$.

SOLUTION

The inner function is $\sqrt{x}$, but its derivative $\frac{1}{2\sqrt{x}}$ is missing. Try $u = \sqrt{x}$, so $x = u^2$ and $dx = 2u\,du$: the integral becomes $2\int ue^u\,du$.

By parts (with u for the polynomial): $\int ue^u du = ue^u - e^u$. So the result is $2(u-1)e^u + C$.

Answer: $2\left(\sqrt{x} - 1\right)e^{\sqrt{x}} + C$.

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Practice

SET A Skills

Name the technique, then find each integral. [No calculator]

1. Find $\int \dfrac{x^4 - 3x^2 + 2}{x^2}\,dx$.
2. Find $\int \dfrac{x}{x^2 + 4}\,dx$.
3. Find $\int xe^{3x}\,dx$.
4. Find $\int \dfrac{dx}{(x-1)(x+3)}$.
5. Find $\int \cos x\, e^{\sin x}\,dx$.
6. Find $\int \dfrac{2x+1}{x^2+x+5}\,dx$.
7. Find $\int x \sin x\,dx$.
8. Find $\int \ln x\,dx$.
9. Evaluate $\int_{-2}^{2} \sqrt{4 - x^2}\,dx$ using geometry.

SET B Applications and Mixed Problems

Give exact values where asked; otherwise decimals to three places. [Calculator]

10. Find the area of the region under $y = x \ln x$ from $x = 1$ to $x = e$. Give an exact value and a decimal.
11. Find the average value of $f(x) = xe^{-x}$ on $[0, 3]$.
12. The region under $y = \dfrac{1}{\sqrt{x^2+1}}$ from $x = 0$ to $x = 1$ is revolved about the x-axis. Find the volume.
13. A current is $i(t) = 3t\sqrt{t^2+1}$ amperes. Find the charge passing from $t = 0$ to $t = 2$ seconds.
14. Evaluate $\int_0^2 \dfrac{3x+5}{(x+1)(x+3)}\,dx$ and give a decimal.
15. Evaluate $\int_1^{\infty} \dfrac{\ln x}{x^3}\,dx$.
16. Evaluate $\int_0^1 x^3\sqrt{1-x^2}\,dx$ and give a decimal.

SET C Challenge

Exact answers only. [No calculator]

17. Find $\int e^x \sin x\,dx$.
18. Find $\int x^3 e^{x^2}\,dx$.
19. Find $\int \arctan x\,dx$.
20. Find $\int \dfrac{dx}{x^2 + 4x + 13}$.
21. Find $\int \sqrt{x}\,\ln x\,dx$.

6.8 Multiple-Choice Practice on Techniques of Integration

Thirty multiple-choice problems mixing integration by parts, partial fractions, substitution, and improper integrals, in the style of the AP exam.

Part A is done without a calculator and Part B allows one. Work each problem on paper first, then check the explanation. In Part B, give decimal answers to three places.

SET A Part A [No calculator]

Choose the best answer. [No calculator]

1. Find $\int xe^{2x}\,dx$.
   (A) $\frac{1}{2}xe^{2x} + C$ (B) $\frac{1}{2}(x-1)e^{2x} + C$ (C) $\frac{1}{4}(2x-1)e^{2x} + C$
   (D) $\frac{1}{4}(2x+1)e^{2x} + C$ (E) $(2x-1)e^{2x} + C$
2. $\int_1^e x \ln x\,dx =$
   (A) $\frac{e^2+1}{4}$ (B) $\frac{e^2-1}{4}$ (C) $\frac{e^2}{2}$ (D) $\frac{e^2+1}{2}$ (E) $\frac{e^2}{4}$
3. $\int_2^3 \frac{dx}{x^2-1} =$
   (A) $\ln \frac{3}{2}$ (B) $\frac{1}{2}\ln 2$ (C) $\frac{1}{2}\ln \frac{2}{3}$ (D) $2\ln \frac{3}{2}$ (E) $\frac{1}{2}\ln \frac{3}{2}$
4. $\int_9^{\infty} \frac{6}{x^{3/2}}\,dx =$
   (A) 2 (B) 4 (C) 6
   (D) 12 (E) The integral diverges.

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5. Which of the following improper integrals converges?

(A) $\int_1^\infty x^{-1/2}\,dx$ (B) $\int_1^\infty \frac{1}{x}\,dx$ (C) $\int_0^1 \frac{1}{x^{3/2}}\,dx$

(D) $\int_0^1 x^{-1/3}\,dx$ (E) $\int_1^\infty \frac{3}{\sqrt{x}}\,dx$

6. $\int_0^1 \frac{x}{(x^2+1)^2}\,dx =$

(A) $\frac{1}{4}$ (B) $\frac{1}{8}$ (C) $\frac{1}{2}$ (D) $\frac{\ln 2}{2}$ (E) $\frac{3}{4}$

7. Find $\int x^2 \ln x\,dx$.

(A) $\frac{x^3}{3}\ln x - \frac{x^3}{3} + C$ (B) $\frac{x^3}{3}\ln x + \frac{x^3}{9} + C$ (C) $\frac{x^3}{3}\ln x - \frac{x^3}{9} + C$

(D) $x^2 \ln x - \frac{x^2}{3} + C$ (E) $\frac{x^3}{3}\ln x - \frac{x^2}{9} + C$

8. Find $\int \frac{5x+1}{(x-1)(x+2)}\,dx$.

(A) $3\ln|x-1| + 2\ln|x+2| + C$ (B) $2\ln|x-1| + 3\ln|x+2| + C$ (C) $2\ln|x-1| - 3\ln|x+2| + C$

(D) $5\ln|x^2 + x - 2| + C$ (E) $-\frac{2}{x-1} - \frac{3}{x+2} + C$

9. Find $\int x^2 \sin x\,dx$.

(A) $-x^2\cos x + 2x\sin x - 2\cos x + C$ (B) $-x^2\cos x - 2x\sin x + 2\cos x + C$ (C) $x^2\cos x - 2x\sin x - 2\cos x + C$

(D) $-x^2\cos x + 2x\sin x + 2\cos x + C$ (E) $-\frac{x^3}{3}\cos x + C$

10. $\int_0^\infty xe^{-2x}\,dx =$

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{1}{8}$

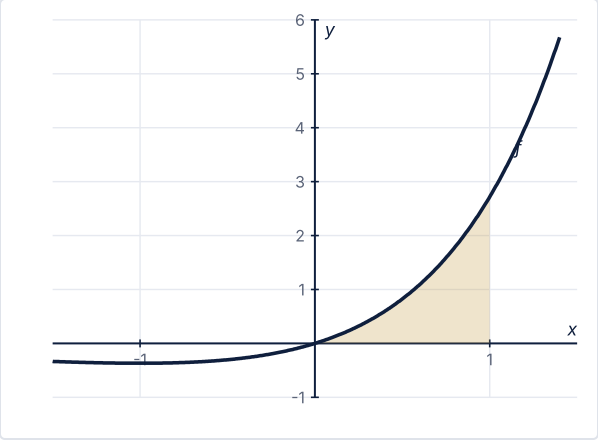
(D) The integral diverges. (E) $\frac{1}{4}$

11. The function f is differentiable on $[0,2]$ with the values shown, and $\int_0^2 f(x)\,dx = 7$. What is $\int_0^2 xf'(x)\,dx$?

x	0	2
$f(x)$	1	5

(A) -3 (B) 3 (C) 10 (D) 17 (E) 5

12. The graph of $f(x) = xe^x$ is shown. What is the area of the shaded region?



Graph of f(x) = x e^x; the shaded region lies between the graph and the x-axis for 0 <= x <= 1.

(A) 1 (B) $e - 1$ (C) e (D) 2 (E) $\frac{e}{2}$

13. Which of the following is a correct conclusion with a valid justification for $\int_1^\infty \frac{dx}{\sqrt{x^2+1}}$?

(A) It converges, because $\frac{1}{\sqrt{x^2+1}} < \frac{1}{x^2}$ for $x \geq 1$.

(B) It converges, because the integrand approaches 0 as $x \to \infty$.

(C) It diverges, because $\frac{1}{\sqrt{x^2+1}} \geq \frac{1}{\sqrt{2}\,x}$ for $x \geq 1$ and $\int_1^\infty \frac{dx}{\sqrt{2}\,x}$ diverges.

(D) It diverges, because $\frac{1}{\sqrt{x^2+1}} \leq \frac{1}{x}$ for $x \geq 1$ and $\int_1^\infty \frac{dx}{x}$ diverges.

(E) It converges, because $\frac{1}{\sqrt{x^2+1}} \leq \frac{1}{x}$ for $x \geq 1$.

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14. $\int_0^1 \frac{2x+3}{x^2+3x+2}\,dx =$
(A) $\ln 6$ (B) $\ln 2$ (C) 1 (D) $\ln 3$ (E) $\frac{3}{2}$

15. Find $\int \arctan x\,dx$.
(A) $x\arctan x - \ln(1+x^2) + C$ (B) $x\arctan x + \frac{1}{2}\ln(1+x^2) + C$ (C) $\frac{1}{1+x^2} + C$
(D) $\frac{x^2}{2}\arctan x + C$ (E) $x\arctan x - \frac{1}{2}\ln(1+x^2) + C$

16. $\int_0^{\pi/2} x\cos x\,dx =$
(A) $\frac{\pi}{2} - 1$ (B) $\frac{\pi}{2} + 1$ (C) $1 - \frac{\pi}{2}$ (D) $\frac{\pi}{2}$ (E) 1

17. $\int_0^3 (x-1)^{-2/3}\,dx =$
(A) $3\left(\sqrt[3]{2} - 1\right)$ (B) $3\left(1 + \sqrt[3]{2}\right)$ (C) $3\sqrt[3]{2}$
(D) 0 (E) The integral diverges.

18. $\int_1^e \frac{\ln x}{x^2}\,dx =$
(A) $1 - \frac{1}{e}$ (B) $\frac{2}{e} - 1$ (C) $1 - \frac{2}{e}$ (D) $1 + \frac{2}{e}$ (E) 1

19. Which is the correct form of the partial fraction decomposition of $\frac{3x+1}{x^2(x-2)}$?
(A) $\frac{A}{x} + \frac{B}{x-2}$ (B) $\frac{A}{x^2} + \frac{B}{x-2}$ (C) $\frac{A}{x} + \frac{B}{x^2} + \frac{Cx}{x-2}$
(D) $\frac{A}{x^3} + \frac{B}{x-2}$ (E) $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$

20. $\int_0^\infty \frac{6}{x^2+9}\,dx =$
(A) $\frac{\pi}{2}$ (B) 2π (C) 6
(D) π (E) The integral diverges.

SET B Part B [Calculator]

Choose the best answer. [Calculator]

21. $\int_0^2 x^2 e^{-x}\,dx =$
(A) 1.323 (B) 0.647 (C) 1.729 (D) 0.981 (E) 2.000

22. $\int_1^\infty \frac{dx}{x(x+1)^2} =$
(A) 0.693 (B) 0.500 (C) 1.193 (D) 0.193 (E) 0.307

23. $\int_0^3 \ln(x^2+1)\,dx =$
(A) 3.406 (B) 0.908 (C) 6.908 (D) 2.498 (E) 2.157

24. $\int_0^2 x^2\ln(x+1)\,dx =$
(A) 3.296 (B) 4.185 (C) 0.210 (D) 2.930 (E) 2.407

25. $\int_2^\infty \frac{dx}{x(\ln x)^2} =$
(A) 0.693 (B) 2.081 (C) 1.443 (D) 1.000 (E) 0.721

26. The average value of $f(x) = xe^{-x/2}$ on $[0,4]$ is
(A) 2.376 (B) 1.188 (C) 2.917 (D) 0.594 (E) 0.500

27. $\int_2^5 \frac{3x+1}{(x-1)(x+3)}\,dx =$
(A) 1.856 (B) 2.326 (C) 3.243 (D) 0.446 (E) 2.749

28. Medicine is absorbed at the rate $r(t) = te^{-0.4t}$ milligrams per hour for $t \ge 0$. The total amount absorbed over all time is
(A) 2.500 (B) 3.125 (C) 6.250 (D) 5.000 (E) 12.500

29. $\int_0^1 x\arctan x\,dx =$
(A) 0.285 (B) 0.785 (C) 1.285 (D) 0.215 (E) 0.393

30. A drug is absorbed at the rate $r(t) = te^{-t/2}$ milligrams per hour. The amount absorbed during the first 6 hours is
(A) 4.000 (B) 5.500 (C) 3.403 (D) 0.797 (E) 3.203

6.9 Free-Response Practice on Techniques of Integration

Four free-response questions on parts, partial fractions and improper integrals, with scoring guides in the answers.

Each question is worth 9 points. Write complete solutions with correct setups, and give a justification wherever the question asks for one.

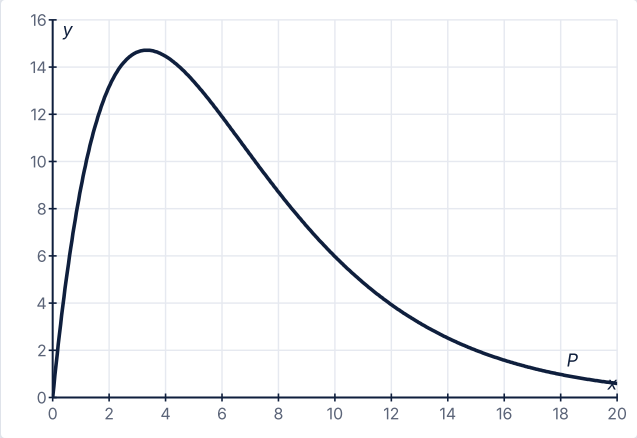
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QUESTION 1 Pollutant entering a lake [Calculator]

Show your work. A calculator may be used; give decimals to three places.

1. Pollutant flows into a small lake at a rate of $P(t) = 12te^{-0.3t}$ kilograms per day, where t is measured in days and $t \geq 0$. The graph of P is shown. A calculator may be used.
   (a) Find the amount of pollutant that enters the lake during the first 10 days. Give your answer to three decimal places.
   (b) Write an improper integral for the total amount of pollutant that ever enters the lake, and evaluate it exactly.
   (c) At what time t is the pollutant entering the lake at the greatest rate? Justify your answer.
   (d) Find the time k at which exactly 100 kilograms of pollutant have entered the lake.

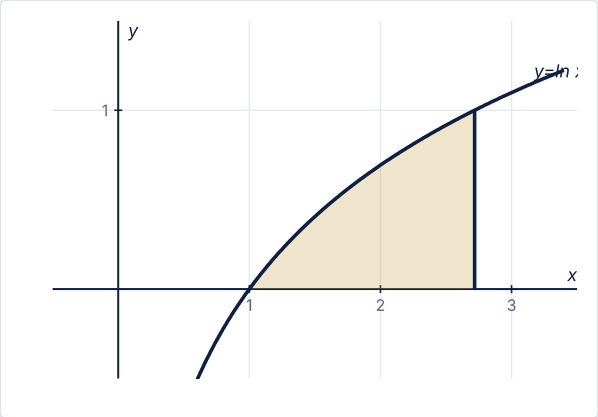


Graph of $y = P(t)$, kilograms per day, against t in days.

QUESTION 2 Region under a logarithm [No calculator]

Show your work. No calculator is allowed; give exact answers.

2. Let R be the region bounded by the graph of $y = \ln x$, the x-axis, and the vertical line $x = e$, as shown. [No calculator]
   (a) Find the area of R.
   (b) Find the volume of the solid generated when R is revolved about the x-axis.
   (c) Find the volume of the solid generated when R is revolved about the y-axis.
   (d) Find the average value of $y = \ln x$ on the interval $[1, e]$.



Region R lies under $y = \ln x$ for $1 \leq x \leq e$.

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QUESTION 3 An improper integral [No calculator]

Show your work. No calculator is allowed; give exact answers.

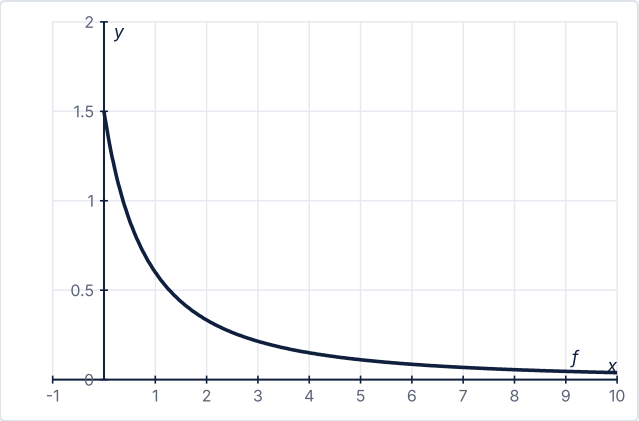
3. Let $f(x) = \frac{6}{(x+1)(x+4)}$ for $x \geq 0$. [No calculator]

(a) Find constants A and B such that $f(x) = \frac{A}{x+1} + \frac{B}{x+4}$.

(b) Determine whether $\int_0^\infty f(x)\,dx$ converges. If it does, find its value.

(c) Determine whether $\int_0^\infty x f(x)\,dx$ converges. Justify your answer.

(d) The region between the graph of f and the x-axis for $x \geq 0$ is revolved about the x-axis. Write an integral for the volume and show that it equals $\pi\left(5 - \frac{16}{3}\ln 2\right)$.



Graph of $y = f(x)$ for $x \geq 0$.

QUESTION 4 A particle on a line [Calculator]

Show your work. A calculator may be used; give decimals to three places.

4. A particle moves along the x-axis with velocity $v(t) = t\ln(t+1)$ for $0 \leq t \leq 4$, where t is in seconds and v is in meters per second. The particle is at position $x = 2$ at time $t = 0$. Selected values of v are shown. A calculator may be used.

(a) Find the acceleration of the particle at $t = 2$. Is the speed of the particle increasing or decreasing at $t = 2$? Give a reason.

(b) Use integration by parts to find the exact value of $\int_0^4 v(t)\,dt$.

(c) Use a trapezoidal sum with the four subintervals of length 1 given by the table to approximate $\int_0^4 v(t)\,dt$. Is this approximation an overestimate or an underestimate? Justify your answer using v''.

(d) Find the time t at which the particle is at position $x = 8$.

t	0	1	2	3	4
$v(t)$	0	0.693	2.197	4.159	6.438

Answers Chapter 6: Answers and Explanations

Answers for every practice problem, numbered to match. Read the explanation even when your answer is correct.

6.1 Integration by Parts

SET A · SKILLS

1. $-\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C$. Let $u = x$, $dv = e^{-2x}dx$; then $v = -\frac{1}{2}e^{-2x}$ and $\int xe^{-2x}dx = -\frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}dx$.

2. $\frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x + C$. $u = x$, $dv = \cos 3x\,dx$, $v = \frac{1}{3}\sin 3x$; then $-\frac{1}{3}\int \sin 3x\,dx = \frac{1}{9}\cos 3x$.

3. $\frac{x^3}{3}\ln x - \frac{x^3}{9} + C$. $u = \ln x$, $dv = x^2dx$; the new integral is $\int \frac{x^2}{3}dx$.

4. $-\frac{\ln x}{x} - \frac{1}{x} + C$. $u = \ln x$, $dv = x^{-2}dx$, $v = -\frac{1}{x}$; then $\int \frac{1}{x^2}dx = -\frac{1}{x}$.

5. $e^x\left(x^2 - 2x + 2\right) + C$. Parts twice: $x^2e^x - 2\int xe^x dx = x^2e^x - 2\left(xe^x - e^x\right)$.

6. $x\arctan x - \frac{1}{2}\ln\left(1 + x^2\right) + C$. $u = \arctan x$, $dv = dx$; then $\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2)$.

7. $x\tan x + \ln|\cos x| + C$. $u = x$, $dv = \sec^2 x\,dx$, $v = \tan x$; $\int \tan x\,dx = -\ln|\cos x|$.

8. $\frac{2}{3}x(x+1)^{3/2} - \frac{4}{15}(x+1)^{5/2} + C$. $u = x$, $dv = \sqrt{x+1}\,dx$, $v = \frac{2}{3}(x+1)^{3/2}$; then $-\frac{2}{3}\int (x+1)^{3/2}dx$.

9. π. $\Big[-x\cos x\Big]_0^\pi + \int_0^\pi \cos x\,dx = \pi + 0 = \pi$.

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SET B · APPLICATIONS

10. 0.908. $\int_0^4 xe^{-x}dx = \Big[-xe^{-x}-e^{-x}\Big]_0^4 = 1-5e^{-4}$.
11. 19.739 cubic units. Shells: $2\pi\int_0^{\pi} x\sin x\,dx = 2\pi\cdot\pi = 2\pi^2$.
12. 3.203 meters. $\int_0^6 te^{-t/2}dt = \Big[-2te^{-t/2}-4e^{-t/2}\Big]_0^6 = 4-16e^{-3}$.
13. 1.012. $\frac{1}{4}\int_1^5 \ln x\,dx = \frac{1}{4}\Big[x\ln x - x\Big]_1^5 = \frac{5\ln 5-4}{4}$.
14. 10.901 kg. $3\Big[-2te^{-t/2}-4e^{-t/2}\Big]_0^8 = 3\left(4-20e^{-4}\right)$.
15. 0.571. $u=\arcsin x$, $dv=dx$: $\Big[x\arcsin x\Big]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx = \frac{\pi}{2}-1$.
16. 0.467. Parts twice: $\Big[x^2\sin x + 2x\cos x - 2\sin x\Big]_0^{\pi/2} = \frac{\pi^2}{4}-2$.

SET C · CHALLENGE

17. $\frac{e^{2x}}{5}(2\cos x+\sin x)+C$. With $u=\cos x$ twice: $I = e^{2x}\cdot\frac{1}{2}\cos x + \frac{1}{2}\int e^{2x}\sin x\,dx$ and $\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}I$. Then $\frac{5}{4}I = \frac{e^{2x}}{4}(2\cos x+\sin x)$.
18. $x(\ln x)^2 - 2x\ln x + 2x + C$. $u=(\ln x)^2$, $dv=dx$: $x(\ln x)^2 - 2\int \ln x\,dx$, and $\int \ln x\,dx = x\ln x - x$.
19. $\frac{1}{2}\left(x^2-1\right)e^{x^2}+C$. Let $w=x^2$, $dw=2x\,dx$: $\frac{1}{2}\int we^w dw = \frac{1}{2}\left(we^w-e^w\right)$.
20. $\frac{\pi}{4}-\frac{1}{2}$. $u=\arctan x$, $dv=x\,dx$ gives $\Big[\frac{x^2}{2}\arctan x\Big]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}dx = \frac{\pi}{8} - \frac{1}{2}\left(1-\frac{\pi}{4}\right)$.
21. 8. $u=x$, $dv=f''(x)dx$: $\Big[xf'(x)\Big]_1^3 - \int_1^3 f'(x)\,dx = (12-2)-(5-3) = 8$.

6.2 Partial Fractions

SET A · SKILLS

1. Write $\frac{3}{(x-1)(x+2)} = \frac{1}{x-1} - \frac{1}{x+2}$ (at $x=1$, $A=3/3=1$; at $x=-2$, $B=3/(-3)=-1$). Answer: $\ln|x-1| - \ln|x+2| + C$.
2. $x+11 = A(x+3)+B(x-1)$. At $x=1$: $12=4A$, so $A=3$. At $x=-3$: $8=-4B$, so $B=-2$. Answer: $3\ln|x-1| - 2\ln|x+3| + C$.
3. Factor $x^2-x-6=(x-3)(x+2)$. Then $2x-1 = A(x+2)+B(x-3)$ gives $A=1$ (at $x=3$) and $B=1$ (at $x=-2$). Answer: $\ln|x-3| + \ln|x+2| + C = \ln|x^2-x-6| + C$. (The numerator is the derivative of the denominator, so a substitution also works.)
4. $5x+11 = A(x+4)+B(x+1)$. At $x=-1$: $6=3A$, so $A=2$. At $x=-4$: $-9=-3B$, so $B=3$. Answer: $A=2$, $B=3$.
5. $\frac{6}{(x-3)(x+3)} = \frac{1}{x-3} - \frac{1}{x+3}$. Answer: $\ln|x-3| - \ln|x+3| + C = \ln\left|\frac{x-3}{x+3}\right| + C$.
6. Write $x+4 = (x+1)+3$, so the integrand is $\frac{1}{x+1} + \frac{3}{(x+1)^2}$. Answer: $\ln|x+1| - \frac{3}{x+1} + C$.
7. Write $3x-2 = 3(x-2)+4$, so the integrand is $\frac{3}{x-2} + \frac{4}{(x-2)^2}$. Answer: $3\ln|x-2| - \frac{4}{x-2} + C$.
8. Divide: $x^2+3 = (x+1)(x-1)+4$, so the integrand is $x-1+\frac{4}{x+1}$. Answer: $\frac{x^2}{2} - x + 4\ln|x+1| + C$.
9. Divide: $\frac{x^2}{x^2-1} = 1 + \frac{1}{x^2-1} = 1 + \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$. Answer: $x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$.

SET B · APPLICATIONS AND MIXED PROBLEMS

10. $\frac{6}{(x+1)(x+4)} = \frac{2}{x+1} - \frac{2}{x+4}$. Area $= \left[2\ln\frac{x+1}{x+4}\right]_0^2 = 2\ln\frac{1}{2} - 2\ln\frac{1}{4} = 2\ln 2 \approx 1.386$.
11. $\frac{1}{y(5-y)} = \frac{1}{5}\left(\frac{1}{y} + \frac{1}{5-y}\right)$, so $\frac{1}{5}\ln\frac{y}{5-y} = 0.2t + C$, i.e. $\ln\frac{y}{5-y} = t + C'$. With $y(0)=1$: $C' = \ln\frac{1}{4}$, so $\frac{y}{5-y} = \frac{e^t}{4}$ and $y = \frac{5e^t}{4+e^t}$. Setting $y=4$: $\frac{4}{1} = \frac{e^t}{4}$, so $e^t = 16$ and $t = \ln 16 \approx 2.773$.
12. $\frac{1}{(10-x)(5-x)} = \frac{1}{5}\left(\frac{1}{5-x} - \frac{1}{10-x}\right)$. Integrating: $\frac{1}{5}\ln\frac{10-x}{5-x} = 0.02t + C$. At $t=0$: $C = \frac{1}{5}\ln 2$. So $\ln\frac{10-x}{5-x} = 0.1t + \ln 2$. For $x=4$: $\ln 6 = 0.1t + \ln 2$, so $t = 10\ln 3 \approx 10.986$ minutes.
13. $v(t) > 0$ for $t \geq 0$, so distance equals displacement. $\frac{8t+6}{(t+1)(t+2)} = \frac{-2}{t+1} + \frac{10}{t+2}$. Distance $= \left[-2\ln(t+1) + 10\ln(t+2)\right]_0^3 = 10\ln 5 - 14\ln 2 \approx 6.390$ meters.
14. $f(x) = \frac{3}{x} + \frac{2}{x+1}$. $\int_1^4 f\,dx = \left[3\ln x + 2\ln(x+1)\right]_1^4 = 3\ln 4 + 2\ln 5 - 2\ln 2 = 4\ln 2 + 2\ln 5$. Divide by 3: $\frac{4\ln 2 + 2\ln 5}{3} \approx 1.997$.
15. Differentiating gives $\frac{2x}{x^2-4}$, not $\frac{1}{x^2-4}$; the numerator is not the derivative of the denominator. Correct: $\frac{1}{(x-2)(x+2)} = \frac{1}{4}\left(\frac{1}{x-2} - \frac{1}{x+2}\right)$, so the answer is $\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$.
16. $\frac{2x+3}{x(x+3)} = \frac{1}{x} + \frac{1}{x+3}$ (at $x=0$: $A=1$; at $x=-3$: $B=1$). Value: $\left[\ln x + \ln(x+3)\right]_1^2 = \ln 10 - \ln 4 = \ln\frac{5}{2} \approx 0.916$.

SET C · CHALLENGE

17. Template $\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$. At $x=0$: $A=2$. At $x=1$: $C=3$. Matching x^2 coefficients, $A+B=1$, so $B=-1$. Answer: $2\ln|x| - \ln|x-1| - \frac{3}{x-1} + C$.
18. Divide: $\frac{x^3}{x^2-4} = x + \frac{4x}{x^2-4} = x + \frac{2}{x-2} + \frac{2}{x+2}$. Answer: $\frac{x^2}{2} + 2\ln|x-2| + 2\ln|x+2| + C = \frac{x^2}{2} + 2\ln|x^2-4| + C$.
19. $\frac{4}{(x-2)(x+1)} = \frac{4}{3}\left(\frac{1}{x-2} - \frac{1}{x+1}\right)$. Value: $\frac{4}{3}\left[\ln\frac{x-2}{x+1}\right]_3^4 = \frac{4}{3}\left(\ln\frac{2}{5} - \ln\frac{1}{4}\right) = \frac{4}{3}\ln\frac{8}{5}$.
20. With $u=e^x$, $du=e^x\,dx$: $\int \frac{du}{(u-1)(u+2)} = \frac{1}{3}\int\left(\frac{1}{u-1} - \frac{1}{u+2}\right)du = \frac{1}{3}\ln\left|\frac{u-1}{u+2}\right| + C$. Answer: $\frac{1}{3}\ln\left|\frac{e^x-1}{e^x+2}\right| + C$.
21. Let $u=\sin x$, $du=\cos x\,dx$: $\int \frac{du}{(u-1)(u-2)} = \int\left(\frac{1}{u-2} - \frac{1}{u-1}\right)du = \ln\left|\frac{u-2}{u-1}\right| + C$. Answer: $\ln\left|\frac{\sin x-2}{\sin x-1}\right| + C$.

6.3 Trigonometric Substitution and Trig Integrals

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SET A · SKILLS

1. $\cos^3 x = (1-\sin^2 x)\cos x$; let $u = \sin x$. Answer: $\sin x - \frac{\sin^3 x}{3} + C$.
2. $\sin^3 x = (1-\cos^2 x)\sin x$; let $u = \cos x$. Answer: $-\cos x + \frac{\cos^3 x}{3} + C$.
3. Use $\sin^2 x = \frac{1-\cos 2x}{2}$. Answer: $\frac{x}{2} - \frac{\sin 2x}{4} + C$.
4. $\int_0^{\pi} \frac{1+\cos 2x}{2}\,dx = \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi} = \frac{\pi}{2}$.
5. Cosine has the odd power: $\cos^3 x = (1-\sin^2 x)\cos x$, $u = \sin x$: $\int (u^2 - u^4)du$. Answer: $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$.
6. Let $u = \tan x$, $du = \sec^2 x\,dx$. Answer: $\frac{\tan^5 x}{5} + C$.
7. Use $\tan^2 x = \sec^2 x - 1$. Answer: $\tan x - x + C$.
8. Write it as $\sec^2 x \cdot \sec x \tan x$ and let $u = \sec x$. Answer: $\frac{\sec^3 x}{3} + C$.
9. $dx = 3\cos t\,dt$, $\sqrt{9-x^2} = 3\cos t$, limits 0 to $\frac{\pi}{2}$: $9\int_0^{\pi/2} \cos^2 t\,dt = 9 \cdot \frac{\pi}{4} = \frac{9\pi}{4}$. Geometry: this is the area of a quarter of a circle of radius 3, which is $\frac{1}{4}\pi \cdot 9$.

SET B · APPLICATIONS AND MIXED PROBLEMS

10. $\sin^4 x = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$. The cosine terms vanish over $[0, \frac{\pi}{2}]$, leaving $\frac{3}{8} \cdot \frac{\pi}{2} = \frac{3\pi}{16} \approx 0.589$.
11. Average of $I^2 = \frac{1}{2\pi}\int_0^{2\pi} 16\sin^2\theta\,d\theta = \frac{16}{2\pi} \cdot \pi = 8$. So the RMS current is $\sqrt{8} = 2\sqrt{2} \approx 2.828$ amperes.
12. $V = \pi\int_0^{\pi} \sin^4 x\,dx = \pi \cdot \frac{3\pi}{8} = \frac{3\pi^2}{8} \approx 3.701$ cubic units.
13. Area $= 2\int_{-2}^{2} \sqrt{1 - \frac{x^2}{4}}\,dx = 2\int_{-\pi/2}^{\pi/2} \cos t \cdot 2\cos t\,dt = 4 \cdot \frac{\pi}{2} = 2\pi \approx 6.283$.
14. Let $x = \tan t$: the integral is $\int_0^{\pi/4} \frac{\tan^2 t \sec^2 t}{\sec^3 t}dt = \int_0^{\pi/4} (\sec t - \cos t)\,dt = \left[\ln|\sec t + \tan t| - \sin t\right]_0^{\pi/4} = \ln(1+\sqrt{2}) - \frac{\sqrt{2}}{2} \approx 0.174$.
15. $\tan^3 x = \tan x(\sec^2 x - 1)$, so an antiderivative is $\frac{\tan^2 x}{2} + \ln|\cos x|$. Value: $\frac{1}{2} + \ln\frac{\sqrt{2}}{2} = \frac{1}{2} - \frac{1}{2}\ln 2 \approx 0.153$.
16. Area $= \int_1^3 2\sqrt{9-x^2}\,dx = \left[x\sqrt{9-x^2} + 9\arcsin\frac{x}{3}\right]_1^3 = \frac{9\pi}{2} - \sqrt{8} - 9\arcsin\frac{1}{3} \approx 8.250$ square units.

SET C · CHALLENGE

17. $\sin^4 x = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$. Answer: $\frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$.
18. Peel off $\sec^2 x$: $\sec^4 x = (1+\tan^2 x)\sec^2 x$; let $u = \tan x$. Answer: $\tan x + \frac{\tan^3 x}{3} + C$.
19. $\tan^3 x \sec^3 x = \tan^2 x \sec^2 x \cdot \sec x \tan x = (\sec^2 x - 1)\sec^2 x \cdot \sec x \tan x$; let $u = \sec x$: $\int (u^4 - u^2)du$. Answer: $\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$.
20. Let $x = 2\tan t$: $\int \frac{2\sec^2 t}{4\tan^2 t \cdot 2\sec t}dt = \frac{1}{4}\int \frac{\cos t}{\sin^2 t}dt = -\frac{1}{4\sin t} + C$. Since $\sin t = \frac{x}{\sqrt{x^2+4}}$, the answer is $-\frac{\sqrt{x^2+4}}{4x} + C$.
21. Let $x = \sec t$: $\int \frac{\tan t}{\sec t}\sec t \tan t\,dt = \int \tan^2 t\,dt = \tan t - t + C$. Answer: $\sqrt{x^2-1} - \operatorname{arcsec} x + C$.

6.4 L'Hopital's Rule

SET A · SKILLS

1. Form $\frac{0}{0}$. $\lim \frac{4e^{4x}}{2} = 2$.
2. Form $\frac{0}{0}$. $\lim \frac{3x^2}{2x} = \frac{12}{4} = 3$.
3. Form $\frac{0}{0}$. $\lim \frac{3\sin 3x}{2x}$ is still $\frac{0}{0}$; again: $\lim \frac{9\cos 3x}{2} = \frac{9}{2}$.
4. Form $\frac{\infty}{\infty}$. $\lim \frac{10x+3}{4x}$ is still $\frac{\infty}{\infty}$; again: $\lim \frac{10}{4} = \frac{5}{2}$.
5. Form $\frac{\infty}{\infty}$. $\lim \frac{1/x}{\frac{1}{2\sqrt{x}}} = \lim \frac{2}{\sqrt{x}} = 0$.
6. Apply the rule three times: $\frac{3x^2}{2e^{2x}} \to \frac{6x}{4e^{2x}} \to \frac{6}{8e^{2x}} \to 0$. Answer: 0.
7. Form $\infty \cdot 0$. Rewrite as $\frac{\sin(3/x)}{1/x}$ (form $\frac{0}{0}$): $\lim \frac{\cos(3/x)\cdot(-3/x^2)}{-1/x^2} = \lim 3\cos\frac{3}{x} = 3$.
8. Combine: $\frac{e^x - 1 - x}{x(e^x-1)}$, form $\frac{0}{0}$. Rule: $\frac{e^x-1}{e^x-1+xe^x}$, still $\frac{0}{0}$. Rule: $\frac{e^x}{2e^x+xe^x} \to \frac{1}{2}$.
9. Form 0^0. $\ln y = 2x\ln x = \frac{2\ln x}{1/x} \to \lim \frac{2/x}{-1/x^2} = \lim(-2x) = 0$. So $y \to e^0 = 1$.

SET B · APPLICATIONS AND MIXED PROBLEMS

10. Both $f(2) = g(2) = 0$, so the form is $\frac{0}{0}$. Write $\frac{f(x)}{g(x)} = \frac{f(x)/(x-2)}{g(x)/(x-2)}$; the numerator tends to $f'(2) = 6$ and the denominator to $g'(2) = -4$. The limit is $-\frac{3}{2} = -1.500$. (The rule gives the same result when f' and g' are continuous.)
11. (a) $h(0.1) = \frac{0.099833 - 0.099500}{0.001} \approx 0.333$. (b) Form $\frac{0}{0}$; the first derivative step gives $\frac{x\sin x}{3x^2} = \frac{\sin x}{3x}$, which tends to $\frac{1}{3}$. The limit is $\frac{1}{3} \approx 0.333$, consistent with part (a).
12. Let $y = \left(1 + \frac{0.06}{n}\right)^n$; $\ln y = \frac{\ln(1+0.06/n)}{1/n} \to 0.06$ by the rule (form $\frac{0}{0}$). So $y \to e^{0.06}$ and the limit is $1000e^{0.06} \approx 1061.837$ dollars.
13. (a) Form $\frac{\infty}{\infty}$. Rule twice: $\frac{2t}{0.5e^{0.5t}} \to \frac{2}{0.25e^{0.5t}} \to 0$. (b) $R(10) = \frac{100}{e^5} \approx 0.674$.
14. $\lim \frac{\theta - \sin\theta}{2\theta^3}$ has the form $\frac{0}{0}$; three applications give $\frac{1-\cos\theta}{6\theta^2} \to \frac{\sin\theta}{12\theta} \to \frac{\cos\theta}{12} \to \frac{1}{12} \approx 0.083$.
15. The value 2 is correct only by coincidence, and the method is invalid. At $x = 1$ the form is $\frac{4}{2}$, not indeterminate, so the rule does not apply (for example, it would wrongly give $\frac{1}{4}$ for $\lim_{x\to 2}\frac{x+1}{x^2} = \frac{3}{4}$). Correct justification: the function is continuous at 1, so substitute to get $\frac{4}{2} = 2$.
16. Form 1^{∞}. $\ln y = \frac{\ln\cos x}{x^2}$ is $\frac{0}{0}$; rule: $\frac{-\tan x}{2x}$, still $\frac{0}{0}$; rule: $\frac{-\sec^2 x}{2} \to -\frac{1}{2}$. So $y \to e^{-1/2} \approx 0.607$.

SET C · CHALLENGE

17. Form $\frac{0}{0}$. $\frac{\sec^2 x - 1}{3x^2} = \frac{\tan^2 x}{3x^2} \to \frac{1}{3}\left(\lim \frac{\tan x}{x}\right)^2 = \frac{1}{3}$ (or use the rule again: $\frac{2\sec^2 x \tan x}{6x} \to \frac{1}{3}$). Answer: $\frac{1}{3}$.
18. Form 0^0. $\ln y = x\ln\sin x = \frac{\ln\sin x}{1/x}$; rule: $\frac{\cot x}{-1/x^2} = -\frac{x^2\cos x}{\sin x} = -x\cos x \cdot \frac{x}{\sin x} \to 0$. So $y \to e^0 = 1$.
19. Form ∞^0. $\ln y = \frac{\ln\ln x}{x}$, form $\frac{\infty}{\infty}$; rule: $\frac{1/(x\ln x)}{1} \to 0$. So $y \to 1$.
20. Form $\infty - \infty$. Combine: $\frac{1-\sin x}{\cos x}$, form $\frac{0}{0}$. Rule: $\frac{-\cos x}{-\sin x} \to \frac{0}{1} = 0$.
21. The numerator's derivative $2x\sin\frac{1}{x} - \cos\frac{1}{x}$ oscillates, so $\lim \frac{f'}{g'}$ does not exist and the rule gives no conclusion. Instead write the expression as $x\sin\frac{1}{x} \cdot \frac{x}{\sin x}$. Since $\left|x\sin\frac{1}{x}\right| \le |x| \to 0$ and $\frac{x}{\sin x} \to 1$, the limit is 0.

6.5 Improper Integrals

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SET A · SKILLS

1. $\left[-\frac{1}{2x^2}\right]_1^b = \frac{1}{2} - \frac{1}{2b^2} \to \frac{1}{2}$. Converges to $\frac{1}{2}$.
2. This is a p-integral with $p = \frac{1}{2} \le 1$. Directly: $\left[2\sqrt{x}\right]_1^b = 2\sqrt{b} - 2 \to \infty$. It diverges.
3. $\left[-\frac{1}{2}e^{-2x}\right]_0^b = \frac{1}{2} - \frac{1}{2}e^{-2b} \to \frac{1}{2}$. Converges to $\frac{1}{2}$.
4. The integrand is unbounded at 0. $\lim_{c\to 0^+}\left[\frac{3}{2}x^{2/3}\right]_c^1 = \frac{3}{2}$. Converges to $\frac{3}{2}$.
5. $\lim_{c\to 0^+}\left[\ln x\right]_c^1 = \lim_{c\to 0^+}(-\ln c) = \infty$. It diverges.
6. $\lim_{a\to -\infty}\left[e^x\right]_a^0 = 1 - 0 = 1$. Converges to 1.
7. $\lim_{b\to\infty}\arctan b - \arctan 0 = \frac{\pi}{2}$. Converges to $\frac{\pi}{2}$.
8. Let $u = \ln x$: $\int_1^\infty u^{-2}du = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1$. Converges to 1.
9. By parts, $\int \ln x\,dx = x\ln x - x$. Then $(0-1) - \lim_{c\to 0^+}(c\ln c - c) = -1 - 0 = -1$ (since $c\ln c \to 0$). Converges to -1.

SET B · APPLICATIONS AND MIXED PROBLEMS

10. $\lim_{b\to\infty}\left[-12.5e^{-0.4t}\right]_0^b = 12.5$. The total is 12.500 millicurie-hours.
11. $W = \int_3^\infty 36r^{-2}dr = \lim_{b\to\infty}\left[-\frac{36}{r}\right]_3^b = 12$ joules, i.e. 12.000.
12. $\int_{-\infty}^{\infty}\frac{3}{x^2+9}dx = \left[\arctan\frac{x}{3}\right]_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi \approx 3.142$.
13. Yes, the area is finite: $\lim_{c\to 0^+}\left[2\sqrt{x}\right]_c^4 = 4 = 4.000$ square units, even though the graph is unbounded.
14. The value is about 0.139 (it converges, by comparison with e^{-x}). The bound is $e^{-1} \approx 0.368$, and $0.139 \le 0.368$, consistent.
15. $V = \pi\int_1^\infty x^{-2}dx = \pi\lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = \pi \approx 3.142$ cubic units.
16. It converges exactly when $p < 1$, with value $\frac{1}{1-p}$. For $p = 0.6$: $\frac{1}{0.4} = 2.500$.

SET C · CHALLENGE

17. By parts, $\int \frac{\ln x}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x}$. At b: $-\frac{\ln b}{b} - \frac{1}{b} \to 0$ (L'Hopital). At 1: -1. Value: $0 - (-1) = 1$.
18. Let $u = -x^2$: antiderivative $-\frac{1}{2}e^{-x^2}$. $\lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2} + \frac{1}{2}\right) = \frac{1}{2}$.
19. The integrand is unbounded at $x = 1$, inside the interval. Split: $\int_0^1 + \int_1^2$. An antiderivative is $3(x-1)^{1/3}$. First piece: $3\cdot 0 - 3(-1) = 3$. Second: $3(1) - 0 = 3$. Both converge; total 6.
20. $\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$, antiderivative $\frac{1}{2}\ln\frac{x-1}{x+1}$. As $b\to\infty$ this tends to 0; at 2: $\frac{1}{2}\ln\frac{1}{3}$. Value: $0 - \frac{1}{2}\ln\frac{1}{3} = \frac{1}{2}\ln 3$.
21. Multiply top and bottom by e^x: $\frac{e^x}{e^{2x}+1}$, with antiderivative $\arctan(e^x)$. Value: $\lim_{b\to\infty}\arctan(e^b) - \arctan(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$.

6.6 Functions Defined by Integrals

SET A · SKILLS

1. $G'(x) = \sqrt{x^2+5}$, so $G'(2) = \sqrt{9} = 3$.
2. Chain rule: $e^{(2x)^2}\cdot 2 = 2e^{4x^2}$.
3. Reverse the limits: $-\int_5^x \sqrt{1+t^4}dt$, so the derivative is $-\sqrt{1+x^4}$.
4. $f(x^2)\cdot 2x - f(x)\cdot 1 = x^6\cdot 2x - x^3 = 2x^7 - x^3$. Check: $\int_x^{x^2} t^3dt = \frac{x^8-x^4}{4}$, derivative $2x^7 - x^3$.
5. $F'(x) = x^2 - 4 = 0$ at $x = \pm 2$. $F''(x) = 2x$: $F''(-2) = -4 < 0$ gives a local maximum at $x = -2$; $F''(2) = 4 > 0$ gives a local minimum at $x = 2$.
6. $G(2) = 5$. $G'(x) = x^3$, so $G'(2) = 8$. $G(3) = 5 + \left[\frac{t^4}{4}\right]_2^3 = 5 + \frac{81-16}{4} = \frac{85}{4}$.
7. Form $\frac{0}{0}$. The rule gives $\lim\cos(x^2) = 1$.
8. Form $\frac{0}{0}$. The rule: $\lim\frac{\sin x}{2x} = \frac{1}{2}$.
9. Differentiate: $f(x) = 3x^2$. At $x = a$ the integral is 0: $a^3 = 27$, so $a = 3$.

SET B · APPLICATIONS AND MIXED PROBLEMS

10. (a) $\int_0^2 f = -1 + 1 = 0$ (triangles of area 1 each, below then above the axis), so $g(2) = 0$; $\int_2^4 f = 4$ so $g(4) = 4$; on $[4,6]$ f crosses zero at $x = \frac{14}{3}$: area $+\frac{2}{3} - \frac{8}{3} = -2$, so $g(6) = 2$.
(b) $g' = f$ is negative on $(0,1)$, positive on $(1,\frac{14}{3})$, negative after. Candidates: $g(0) = 0$, $g(1) = -1$, $g\left(\frac{14}{3}\right) = 4 + \frac{2}{3} = \frac{14}{3} \approx 4.667$, $g(6) = 2$. Minimum -1, maximum $\frac{14}{3} \approx 4.667$.
(c) $g'' = f' < 0$ only on $(4,6)$, so g is concave down on $(4,6)$.
11. $F(2) \approx 0.882$ (numerical integration). $F'(2) = e^{-4} \approx 0.018$. Tangent line: $y \approx 0.882 + 0.018(x-2)$.
12. $\mathrm{Si}(\pi) \approx 1.852$ and $\mathrm{Si}(2\pi) \approx 1.418$. Since $\mathrm{Si}'(x) = \frac{\sin x}{x}$ changes from positive to negative at $x = \pi$ (on $(0,\pi)$ the sine is positive, on $(\pi,2\pi)$ negative), Si has a local maximum at π.
13. $V(3) = 40 + \int_0^3 6\sin(0.4t^2)dt \approx 48.225$ liters. V' is positive until $0.4t^2 = \pi$, i.e. $t = \sqrt{\pi/0.4} \approx 2.802$, and negative after, so the maximum is at $t \approx 2.802$ with $V \approx 48.489$ liters (the value at $t = 0$ is 40 and at $t = 3$ is 48.225, both smaller).
14. $G'(x) = \sqrt{1+x^6}\cdot 2x$. At $x = 1.5$: $3\sqrt{1+11.390625} = 3\sqrt{12.390625} \approx 10.560$.
15. Form $\frac{0}{0}$; the rule gives $\lim\frac{\ln(1+x^2)}{3x^2} = \frac{1}{3}$ (since $\ln(1+u) \approx u$ for small u; or apply the rule again: $\frac{2x/(1+x^2)}{6x} \to \frac{1}{3}$). At $x = 0.1$ the quotient is $\frac{0.000332}{0.001} \approx 0.332$, consistent with $\frac{1}{3} \approx 0.333$.
16. (a) $g(2) \approx \frac{2+5}{2} + \frac{5+3}{2} = 7.5$. $g(4) \approx 7.5 + \frac{3-1}{2} + \frac{-1-4}{2} = 7.5 + 1 - 2.5 = 6.0$. (b) $f(2) = 3 > 0$ and $f(3) = -1 < 0$, and f is continuous, so by the Intermediate Value Theorem $f = g' = 0$ somewhere in $(2,3)$, which is a critical point of g. If this is the only zero of f there, then $g' = f$ changes from positive to negative and g has a local maximum.

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SET C · CHALLENGE

17. $\sqrt{1+\cos^2 x}\cdot(-\sin x)-\sqrt{1+\sin^2 x}\cdot\cos x$.

18. Differentiate: $f(x^2)\cdot 2x = 4x^3+3x^2$, so $f(x^2)=2x^2+\frac{3x}{2}$ for $x>0$. Put $x=2$: $f(4)=8+3=11$.

19. Form $\frac{0}{0}$. The rule: $\lim\frac{e^{x^2}-1}{3x^2}$, still $\frac{0}{0}$. Again: $\frac{2xe^{x^2}}{6x}=\frac{e^{x^2}}{3}\to\frac{1}{3}$.

20. The numerator tends to ∞ (the integrand is at least 1), so the form is $\frac{\infty}{\infty}$. The rule: $\lim\frac{e^{x^2}}{2xe^{x^2}}=\lim\frac{1}{2x}=0$.

21. $F(0)=1$, $F'(0)=1$, and $F''(x)=-2xe^{-x^2}<0$ for $x>0$, so F is concave down on $[0,0.5]$ and lies below its tangent line $y=1+x$. Hence $F(0.5)<1.5$. (Alternatively, $e^{-t^2}<1$ for $t>0$.)

6.7 Choosing the Right Technique

SET A · SKILLS

1. Rewrite as x^2-3+2x^{-2}. Answer: $\frac{x^3}{3}-3x-\frac{2}{x}+C$.

2. Substitution $u=x^2+4$, $du=2x\,dx$. Answer: $\frac{1}{2}\ln(x^2+4)+C$.

3. Parts: $u=x$, $dv=e^{3x}dx$, $v=\frac{1}{3}e^{3x}$. $\frac{x}{3}e^{3x}-\frac{1}{3}\int e^{3x}dx$. Answer: $\left(\frac{x}{3}-\frac{1}{9}\right)e^{3x}+C$.

4. Partial fractions: $\frac{1}{4}\left(\frac{1}{x-1}-\frac{1}{x+3}\right)$. Answer: $\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right|+C$.

5. Let $u=\sin x$, $du=\cos x\,dx$. Answer: $e^{\sin x}+C$.

6. The numerator is the derivative of the denominator. Let $u=x^2+x+5$. Answer: $\ln(x^2+x+5)+C$ (the quantity is always positive).

7. Parts: $u=x$, $dv=\sin x\,dx$. $-x\cos x+\int\cos x\,dx$. Answer: $-x\cos x+\sin x+C$.

8. Parts with $u=\ln x$, $dv=dx$: $x\ln x-\int 1\,dx$. Answer: $x\ln x-x+C$.

9. The graph of $y=\sqrt{4-x^2}$ is the upper half of a circle of radius 2, so the area is $\frac{1}{2}\pi(2)^2=2\pi$.

SET B · APPLICATIONS AND MIXED PROBLEMS

10. Parts: $\int x\ln x\,dx=\frac{x^2}{2}\ln x-\frac{x^2}{4}$. From 1 to e: $\left(\frac{e^2}{2}-\frac{e^2}{4}\right)-\left(0-\frac{1}{4}\right)=\frac{e^2+1}{4}\approx 2.097$.

11. Parts: $\int_0^3 xe^{-x}dx=\left[-xe^{-x}-e^{-x}\right]_0^3=1-4e^{-3}$. Divide by 3: $\frac{1-4e^{-3}}{3}\approx 0.267$.

12. $V=\pi\int_0^1\frac{dx}{x^2+1}=\pi\arctan x\Big|_0^1=\frac{\pi^2}{4}\approx 2.467$ cubic units.

13. Let $u=t^2+1$, $du=2t\,dt$: $\int_0^2 3t\sqrt{t^2+1}\,dt=\left[(t^2+1)^{3/2}\right]_0^2=5^{3/2}-1\approx 10.180$ coulombs.

14. Partial fractions: $\frac{1}{x+1}+\frac{2}{x+3}$. Value: $\left[\ln(x+1)+2\ln(x+3)\right]_0^2=\ln 3+2\ln 5-2\ln 3=2\ln 5-\ln 3\approx 2.120$.

15. Parts: $\int\frac{\ln x}{x^3}dx=-\frac{\ln x}{2x^2}-\frac{1}{4x^2}$. As $b\to\infty$ the antiderivative tends to 0; at 1 it is $-\frac{1}{4}$. Value: $0-\left(-\frac{1}{4}\right)=0.250$.

16. Let $u=1-x^2$, $x\,dx=-\frac{du}{2}$, $x^2=1-u$: $\frac{1}{2}\int_0^1(1-u)\sqrt{u}\,du=\frac{1}{2}\left(\frac{2}{3}-\frac{2}{5}\right)=\frac{2}{15}\approx 0.133$.

SET C · CHALLENGE

17. Parts twice: $I=e^x\sin x-\int e^x\cos x\,dx=e^x\sin x-e^x\cos x-I$. So $2I=e^x(\sin x-\cos x)$. Answer: $\frac{e^x(\sin x-\cos x)}{2}+C$.

18. Let $u=x^2$, $du=2x\,dx$: $\frac{1}{2}\int ue^u\,du=\frac{1}{2}(u-1)e^u$. Answer: $\frac{(x^2-1)e^{x^2}}{2}+C$.

19. Parts: $u=\arctan x$, $dv=dx$, $du=\frac{dx}{1+x^2}$. $x\arctan x-\int\frac{x}{1+x^2}dx$. Answer: $x\arctan x-\frac{1}{2}\ln(1+x^2)+C$.

20. Complete the square: $x^2+4x+13=(x+2)^2+9$. Then $\int\frac{dx}{(x+2)^2+3^2}=\frac{1}{3}\arctan\frac{x+2}{3}+C$.

21. Parts: $u=\ln x$, $dv=x^{1/2}dx$, $v=\frac{2}{3}x^{3/2}$. $\frac{2}{3}x^{3/2}\ln x-\frac{2}{3}\int x^{1/2}dx$. Answer: $\frac{2}{3}x^{3/2}\ln x-\frac{4}{9}x^{3/2}+C$.

6.8 Multiple-Choice Practice on Techniques of Integration

SET A · PART A [NO CALCULATOR]

1. C. Parts with $u=x$, $dv=e^{2x}dx$: $\frac{1}{2}xe^{2x}-\frac{1}{2}\int e^{2x}dx=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C$.

2. A. Parts: $\left[\frac{x^2}{2}\ln x-\frac{x^2}{4}\right]_1^e=\frac{e^2}{2}-\frac{e^2}{4}+\frac{1}{4}=\frac{e^2+1}{4}$.

3. E. $\frac{1}{x^2-1}=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$, so the integral is $\frac{1}{2}\left[\ln\frac{x-1}{x+1}\right]_2^3=\frac{1}{2}\left(\ln\frac{1}{2}-\ln\frac{1}{3}\right)=\frac{1}{2}\ln\frac{3}{2}$.

4. B. $\lim_{b\to\infty}\left[-12x^{-1/2}\right]_9^b=0+\frac{12}{3}=4$.

5. D. Near a single bad endpoint, $\int x^{-p}dx$ converges at $x=0$ only if $p<1$ (here $p=\frac{1}{3}$, value $\frac{3}{2}$). The others have $p\geq 1$ at 0 or $p\leq 1$ at ∞, so they diverge.

6. A. Let $u=x^2+1$: $\frac{1}{2}\int_1^2 u^{-2}du=\frac{1}{2}\left[-\frac{1}{u}\right]_1^2=\frac{1}{2}\left(1-\frac{1}{2}\right)=\frac{1}{4}$.

7. C. Parts with $u=\ln x$, $dv=x^2dx$: $\frac{x^3}{3}\ln x-\int\frac{x^2}{3}dx=\frac{x^3}{3}\ln x-\frac{x^3}{9}+C$.

8. B. Write $\frac{5x+1}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}$. At $x=1$: $A=\frac{6}{3}=2$. At $x=-2$: $B=\frac{-9}{-3}=3$.

9. D. Apply parts twice. First: $-x^2\cos x+\int 2x\cos x\,dx$. Then $\int 2x\cos x\,dx=2x\sin x-\int 2\sin x\,dx=2x\sin x+2\cos x$.

10. E. Antiderivative: $-\frac{x}{2}e^{-2x}-\frac{1}{4}e^{-2x}$. As $x\to\infty$ it tends to 0; at 0 it is $-\frac{1}{4}$. So the value is $\frac{1}{4}$.

11. B. Parts with $u=x$, $dv=f'(x)dx$: $\left[xf(x)\right]_0^2-\int_0^2 f(x)dx=2(5)-0-7=3$.

12. A. Parts: $\int_0^1 xe^x dx=\left[(x-1)e^x\right]_0^1=0-(-1)=1$.

13. C. For $x\geq 1$, $\sqrt{x^2+1}\leq\sqrt{2x^2}=\sqrt{2}\,x$, so the integrand is at least $\frac{1}{\sqrt{2}x}$, a divergent bound from below. A bound from above by a divergent integral (choice D) proves nothing.

14. D. The numerator is the derivative of the denominator, so the integral is $\left[\ln(x^2+3x+2)\right]_0^1=\ln 6-\ln 2=\ln 3$.

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15. E. Parts with $u = \arctan x$, $dv = dx$: $x\arctan x - \int \frac{x}{1+x^2}dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C$.
16. A. Parts: $[x\sin x + \cos x]_0^{\pi/2} = \left(\frac{\pi}{2} + 0\right) - (0 + 1) = \frac{\pi}{2} - 1$.
17. B. The integrand is unbounded at $x = 1$, inside the interval, so split there. With antiderivative $3(x-1)^{1/3}$: $\left[3(x-1)^{1/3}\right]_0^1 + \left[3(x-1)^{1/3}\right]_1^3 = 3 + 3\sqrt[3]{2}$. Each piece converges because $p = \frac{2}{3} < 1$.
18. C. Parts with $u = \ln x$, $dv = x^{-2}dx$: $\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^e = \left(-\frac{1}{e} - \frac{1}{e}\right) - (0 - 1) = 1 - \frac{2}{e}$.
19. E. A repeated linear factor x^2 needs a term for each power, $\frac{A}{x}$ and $\frac{B}{x^2}$, and the simple factor $x - 2$ needs one constant over it.
20. D. $\int \frac{6}{x^2+9}dx = 2\arctan\frac{x}{3}$. As $x \to \infty$ this tends to $2 \cdot \frac{\pi}{2} = \pi$, and it is 0 at $x = 0$.

SET B · PART B [CALCULATOR]

21. B. Parts twice gives $\left[-(x^2 + 2x + 2)e^{-x}\right]_0^2 = 2 - 10e^{-2} \approx 0.647$.
22. D. $\frac{1}{x(x+1)^2} = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}$, so the integral is $\left[\ln\frac{x}{x+1} + \frac{1}{x+1}\right]_1^\infty = 0 - \left(-\ln 2 + \frac{1}{2}\right) = \ln 2 - \frac{1}{2} \approx 0.193$.
23. A. Parts: $x\ln(x^2+1) - \int \frac{2x^2}{x^2+1}dx = x\ln(x^2+1) - 2x + 2\arctan x$. At $x = 3$ this is $3\ln 10 - 6 + 2\arctan 3 \approx 3.406$.
24. E. Parts with $u = \ln(x+1)$, $dv = x^2dx$: $\frac{8}{3}\ln 3 - \frac{1}{3}\int_0^2 \frac{x^3}{x+1}dx$, and $\frac{x^3}{x+1} = x^2 - x + 1 - \frac{1}{x+1}$. The result is $3\ln 3 - \frac{8}{9} \approx 2.407$.
25. C. Let $u = \ln x$: $\int_{\ln 2}^\infty u^{-2}du = \frac{1}{\ln 2} \approx 1.443$.
26. D. $\int_0^4 xe^{-x/2}dx = \left[-2xe^{-x/2} - 4e^{-x/2}\right]_0^4 = 4 - 12e^{-2}$. Divide by the length 4: $1 - 3e^{-2} \approx 0.594$.
27. B. $\frac{3x+1}{(x-1)(x+3)} = \frac{1}{x-1} + \frac{2}{x+3}$, so the integral is $[\ln|x-1| + 2\ln|x+3|]_2^5 = \ln 4 + 2\ln\frac{8}{5} \approx 2.326$.
28. C. $\int_0^\infty te^{-0.4t}dt = \frac{1}{(0.4)^2} = 6.25$ milligrams (parts, with the boundary terms vanishing at ∞).
29. A. Parts: $\frac{x^2}{2}\arctan x - \frac{1}{2}\int_0^1 \left(1 - \frac{1}{1+x^2}\right)dx = \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{1}{2} \approx 0.285$.
30. E. $\int_0^6 te^{-t/2}dt = \left[-2te^{-t/2} - 4e^{-t/2}\right]_0^6 = 4 - 16e^{-3} \approx 3.203$ milligrams.

6.9 Free-Response Practice on Techniques of Integration

QUESTION 1 · POLLUTANT ENTERING A LAKE [CALCULATOR]

1. (a) $\int_0^{10} 12te^{-0.3t}dt = \left[-40te^{-0.3t} - \frac{400}{3}e^{-0.3t}\right]_0^{10} = \frac{400}{3} - \frac{1600}{3}e^{-3} \approx 106.780$ kilograms.
   (b) Total $= \int_0^\infty 12te^{-0.3t}dt = \lim_{b\to\infty}\left[-40te^{-0.3t} - \frac{400}{3}e^{-0.3t}\right]_0^b = 0 - \left(-\frac{400}{3}\right) = \frac{400}{3} \approx 133.333$ kilograms (the terms with b tend to 0).
   (c) $P'(t) = 12e^{-0.3t}(1 - 0.3t)$. This is 0 only at $t = \frac{10}{3}$. P' is positive for $t < \frac{10}{3}$ and negative for $t > \frac{10}{3}$, so P has an absolute maximum at $t = \frac{10}{3} \approx 3.333$ days (maximum rate $\frac{40}{e} \approx 14.715$ kilograms per day).
   (d) Solve $\frac{400}{3} - \left(40k + \frac{400}{3}\right)e^{-0.3k} = 100$ on a calculator: $k \approx 8.975$ days.
   Scoring: 9 points. (a) 2 points: 1 for the correct integral, 1 for the value. (b) 3 points: 1 for the limit form, 1 for the antiderivative, 1 for the exact value. (c) 2 points: 1 for $P'(t) = 0$ at $\frac{10}{3}$, 1 for the sign-change justification. (d) 2 points: 1 for the equation, 1 for the value.

QUESTION 2 · REGION UNDER A LOGARITHM [NO CALCULATOR]

2. (a) $\int_1^e \ln x\,dx = [x\ln x - x]_1^e = (e - e) - (0 - 1) = 1$.
   (b) Disks: $V = \pi\int_1^e (\ln x)^2dx$. By parts twice, $\int(\ln x)^2dx = x(\ln x)^2 - 2x\ln x + 2x$. So $V = \pi\left[(e - 2e + 2e) - (0 - 0 + 2)\right] = \pi(e - 2)$.
   (c) Shells: $V = 2\pi\int_1^e x\ln x\,dx = 2\pi\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_1^e = 2\pi\cdot\frac{e^2+1}{4} = \frac{\pi(e^2+1)}{2}$.
   (d) $\frac{1}{e-1}\int_1^e \ln x\,dx = \frac{1}{e-1}$.
   Scoring: 9 points. (a) 2 points: 1 for the integral, 1 for the value. (b) 3 points: 1 for the disk setup, 1 for the antiderivative, 1 for the value. (c) 2 points: 1 for the shell setup, 1 for the value. (d) 2 points: 1 for the average value formula, 1 for the value.

QUESTION 3 · AN IMPROPER INTEGRAL [NO CALCULATOR]

3. (a) $6 = A(x+4) + B(x+1)$. At $x = -1$: $A = 2$. At $x = -4$: $B = -2$.
   (b) $\int_0^b f = \left[2\ln\frac{x+1}{x+4}\right]_0^b = 2\ln\frac{b+1}{b+4} - 2\ln\frac{1}{4}$. As $b \to \infty$ the first term tends to $2\ln 1 = 0$. The integral converges to $2\ln 4 = 4\ln 2$.
   (c) $xf(x) = \frac{6x}{(x+1)(x+4)}$ is positive on $[1, \infty)$ and $\lim_{x\to\infty}\frac{xf(x)}{1/x} = \lim\frac{6x^2}{x^2+5x+4} = 6$, a positive finite number. Since $\int_1^\infty \frac{dx}{x}$ diverges, the limit comparison test shows $\int_0^\infty xf(x)dx$ diverges (the part on $[0, 1]$ is a finite proper integral).
   (d) $V = \pi\int_0^\infty f(x)^2dx$. Since $f = \frac{2}{x+1} - \frac{2}{x+4}$, $f^2 = \frac{4}{(x+1)^2} + \frac{4}{(x+4)^2} - \frac{8}{(x+1)(x+4)}$, and $\frac{1}{(x+1)(x+4)} = \frac{1}{3}\left(\frac{1}{x+1} - \frac{1}{x+4}\right)$. Integrating from 0 to ∞: $4 + 1 - \frac{8}{3}\left(0 - \ln\frac{1}{4}\right) = 5 - \frac{16}{3}\ln 2$. So $V = \pi\left(5 - \frac{16}{3}\ln 2\right)$.
   Scoring: 9 points. (a) 2 points: 1 for the setup, 1 for both constants. (b) 3 points: 1 for the limit form, 1 for the antiderivative, 1 for the value. (c) 2 points: 1 for a valid comparison, 1 for the conclusion. (d) 2 points: 1 for the volume integral, 1 for the evaluation.

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QUESTION 4 · A PARTICLE ON A LINE [CALCULATOR]

4. (a) $a(t) = v'(t) = \ln(t+1) + \frac{t}{t+1}$, so $a(2) = \ln 3 + \frac{2}{3} \approx 1.765$ meters per second squared. Since $v(2) = 2\ln 3 > 0$ and $a(2) > 0$, the velocity and acceleration have the same sign, so the speed is increasing.

(b) Parts with $u = \ln(t+1)$, $dv = t\,dt$: $\int_0^4 t\ln(t+1)dt = \left[\frac{t^2}{2}\ln(t+1)\right]_0^4 - \frac{1}{2}\int_0^4 \frac{t^2}{t+1}dt$. Since $\frac{t^2}{t+1} = t - 1 + \frac{1}{t+1}$, the last integral is $\left[\frac{t^2}{2} - t + \ln(t+1)\right]_0^4 = 4 + \ln 5$. The result is $8\ln 5 - 2 - \frac{1}{2}\ln 5 = \frac{15}{2}\ln 5 - 2 \approx 10.071$.

(c) $T = \frac{1}{2}[0 + 2(0.693) + 2(2.197) + 2(4.159) + 6.438] \approx 10.268$ (using unrounded values). $v''(t) = \frac{1}{t+1} + \frac{1}{(t+1)^2} > 0$ on $[0,4]$, so v is concave up and the trapezoids lie above the graph. The approximation is an overestimate (consistent with $10.268 > 10.071$).

(d) $x(t) = 2 + \int_0^t s\ln(s+1)ds = 2 + \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\left(\frac{t^2}{2} - t + \ln(t+1)\right)$. Since $v \geq 0$, x is increasing, so there is exactly one solution. Solving $x(t) = 8$ on a calculator gives $t \approx 3.271$ seconds.

Scoring: 9 points. (a) 2 points: 1 for the acceleration, 1 for the conclusion with reason. (b) 3 points: 1 for the parts setup, 1 for the antiderivative, 1 for the value. (c) 2 points: 1 for the trapezoidal sum, 1 for the concavity justification. (d) 2 points: 1 for the position equation, 1 for the correct conclusion.

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Chapter 7: Series, Parametrics, Vectors, and Polar.

13 sections with explanations and worked examples, then 265 practice problems. Answers and full explanations are at the end of the chapter.

7.1 Sequences

Learn to find limits of sequences, test them for monotonicity and boundedness, and handle recursive definitions. Sequences are the foundation for infinite series.

A **sequence** is an ordered list of numbers $a_1, a_2, a_3, \ldots$ written $\{a_n\}$. It can be given by an **explicit formula**, such as $a_n = \frac{n}{n+1}$, which lets you compute any term directly, or by a **recursive formula**, which gives the first term (or first few) and a rule that builds each term from earlier ones, such as $a_1 = 2$, $a_{n+1} = 3a_n - 1$. Because a sequence is a function whose inputs are the positive integers, we can study what happens to a_n as n grows without bound.

Limit of a Sequence

$\lim_{n\to\infty} a_n = L$ means that a_n gets arbitrarily close to the number L and stays close for all sufficiently large n. Then the sequence **converges** to L. If there is no such finite number L (the terms grow without bound or keep oscillating), the sequence **diverges**.

Link to functions. If $\lim_{x\to\infty} f(x) = L$ and $a_n = f(n)$, then $\lim_{n\to\infty} a_n = L$. So L'Hopital's Rule may be used on $f(x)$, but never directly on a_n (a sequence has no derivative).

Limit laws (sum, product, quotient with nonzero denominator limit, constant multiple) hold for convergent sequences. **Squeeze:** if $b_n \le a_n \le c_n$ and $b_n \to L$, $c_n \to L$, then $a_n \to L$. In particular $|a_n| \to 0$ implies $a_n \to 0$.

Growth Rates and Geometric Sequences

For large n the following list is in increasing order of growth: $\ln n \ll n^p \ll b^n \ll n! \ll n^n$ (here $p > 0$ and $b > 1$). A ratio of two of these tends to 0 if the slower one is on top.

The geometric sequence r^n converges to 0 if $|r| < 1$, converges to 1 if $r = 1$, and diverges if $r > 1$ or $r \le -1$.

EXAMPLE 1: RATIONAL AND EXPONENTIAL LIMITS

Find each limit. (a) $\lim_{n\to\infty} \frac{3n^2+1}{2n^2-n}$ (b) $\lim_{n\to\infty} \frac{n^2}{e^n}$ (c) $\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n$

SOLUTION

(a) Divide numerator and denominator by n^2: $\frac{3+1/n^2}{2-1/n} \to \frac{3}{2}$.

(b) Use $f(x) = \frac{x^2}{e^x}$, which has form $\frac{\infty}{\infty}$. L'Hopital twice: $\frac{2x}{e^x} \to \frac{2}{e^x} \to 0$. So the limit is 0 (exponentials beat powers).

(c) This has form 1^∞. Let $y = (1+3/x)^x$. Then $\ln y = x\ln(1+3/x) = \frac{\ln(1+3/x)}{1/x} \to \frac{3}{1} = 3$ by L'Hopital. So the limit is e^3.

Common mistake

Do not apply L'Hopital's Rule to a_n itself. Replace n by a continuous variable x, differentiate $f(x)$, and then conclude about the sequence. Also, a limit of ∞ means the sequence diverges.

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Bounded and Monotonic Sequences

$\{a_n\}$ is **increasing** if $a_{n+1} \geq a_n$ for all n and **decreasing** if $a_{n+1} \leq a_n$ for all n. It is **bounded** if there are numbers m, M with $m \leq a_n \leq M$ for all n.

Monotonic Convergence Theorem. Every sequence that is bounded **and** monotonic (increasing or decreasing) converges.
To test monotonicity, check the sign of $a_{n+1} - a_n$, compare $\frac{a_{n+1}}{a_n}$ with 1 for positive terms, or use the sign of $f'(x)$ when $a_n = f(n)$.

EXAMPLE 2: MONOTONIC AND BOUNDED

Show that $a_n = \frac{2n}{n+1}$ converges and find its limit.

SOLUTION

Monotonic: $a_{n+1} - a_n = \frac{2(n+1)}{n+2} - \frac{2n}{n+1} = \frac{2(n+1)^2 - 2n(n+2)}{(n+1)(n+2)} = \frac{2}{(n+1)(n+2)} > 0$, so the sequence is increasing.

Bounded: $a_n = 2 - \frac{2}{n+1} < 2$ and $a_n \geq a_1 = 1$, so $1 \leq a_n < 2$.

An increasing sequence that is bounded converges (Monotonic Convergence Theorem). Its limit is $\lim \frac{2n}{n+1} = 2$.

EXAMPLE 3: A RECURSIVE SEQUENCE

Let $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n + 3}$. Show the sequence converges and find its limit.

SOLUTION

Bounded: $a_1 = 1 < 3$. If $a_n < 3$, then $a_{n+1} = \sqrt{2a_n + 3} < \sqrt{9} = 3$. By induction, $0 < a_n < 3$ for all n.

Increasing: $a_{n+1}^2 - a_n^2 = 2a_n + 3 - a_n^2 = (3 - a_n)(a_n + 1) > 0$ because $a_n < 3$. Since all terms are positive, $a_{n+1} > a_n$.

The sequence is increasing and bounded, so it converges to some L. Taking the limit of both sides of $a_{n+1} = \sqrt{2a_n + 3}$ gives $L = \sqrt{2L + 3}$, so $L^2 - 2L - 3 = 0$ and $L = 3$ or $L = -1$. The terms are positive, so $L = 3$.

Common mistake

Finding L from $L = g(L)$ is valid only after you know the limit exists. For $a_1 = 1$, $a_{n+1} = 2a_n + 1$, solving $L = 2L + 1$ gives $L = -1$, yet the terms $1, 3, 7, 15, \ldots$ grow without bound and the sequence diverges. Also remember: a bounded sequence need not converge (for example $(-1)^n$), and a monotonic sequence need not converge (for example n).

EXAMPLE 4: WRITING FORMULAS

Give an explicit and a recursive formula for the sequence $5, 8, 11, 14, \ldots$ and for $3, 6, 12, 24, \ldots$

SOLUTION

First sequence: each term adds 3. Recursive: $a_1 = 5$, $a_{n+1} = a_n + 3$. Explicit: $a_n = 5 + 3(n-1) = 3n + 2$.

Second sequence: each term is multiplied by 2. Recursive: $b_1 = 3$, $b_{n+1} = 2b_n$. Explicit: $b_n = 3 \cdot 2^{n-1}$. This one diverges to ∞.

Practice

SET A Skills

Work each problem. [No calculator]

1. Write the first four terms of $a_n = \frac{(-1)^n n}{n+1}$.
2. Write the first five terms of the sequence $a_1 = 3$, $a_{n+1} = 2a_n - 1$.
3. Find $\lim_{n\to\infty} \frac{5n^2 - 3}{2n^2 + n + 4}$.
4. Find $\lim_{n\to\infty} \frac{4n+1}{\sqrt{n^2+9}}$.
5. Find $\lim_{n\to\infty} \frac{\ln n}{n}$.
6. Find $\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n$.
7. Find an explicit formula for $4, 7, 10, 13, \ldots$ and a recursive formula for it.
8. Find an explicit formula for $2, 6, 18, 54, \ldots$ Does the sequence converge?
9. Does the sequence $a_n = \cos(n\pi)$ converge? Explain.

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SET B Applications and Mixed Practice

Give decimals to three places where needed. [Calculator]

10. A patient receives 50 mg of a drug each hour, and between doses the body removes 20% of what is present. The amount just after the nth dose satisfies $d_1 = 50$, $d_{n+1} = 0.8d_n + 50$. Find d_5 and the limit of the sequence.
11. A ball dropped from 10 m rebounds to 60% of its previous height each time. The peak height after the nth bounce is $h_n = 10(0.6)^n$ meters. After how many bounces is the peak height first below 0.1 m?
12. An account starts with 1000 dollars and earns 6% annual interest compounded n times per year, so the year-end balance is $a_n = 1000\left(1 + \frac{0.06}{n}\right)^n$ dollars. Find a_1, a_4, a_{12}, and $\lim_{n\to\infty} a_n$.
13. A population satisfies $P_1 = 100$ and $P_{n+1} = P_n + 0.1P_n\left(1 - \frac{P_n}{500}\right)$. Find P_2 and P_3. If the sequence converges to a positive limit, what is it?
14. A student tabulates $a_n = n\sin\left(\frac{1}{n}\right)$. Use the table to guess the limit, then confirm it with a limit calculation.

n	1	10	100
a_n	0.841	0.998	1.000

15. Newton's method gives the recursion $a_1 = 3$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{7}{a_n}\right)$. Find a_2, a_3 and the limit of the sequence.
16. For $a_n = \frac{2n+1}{n+3}$ the limit is 2. Find the smallest N such that $|a_n - 2| < 0.001$ for all $n \geq N$.

SET C Challenge and AP-Style Problems

Justify carefully. [No calculator]

17. Show that $a_n = \frac{n}{n^2+1}$ is decreasing for $n \geq 1$, bounded, and find its limit.
18. Let $a_1 = 1$ and $a_{n+1} = \sqrt{6 + a_n}$. Prove that $\{a_n\}$ converges and find its limit.
19. Find $\lim_{n\to\infty}\left(1 - \frac{1}{2n}\right)^{3n}$.
20. Use the Squeeze Theorem to show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.
21. If a_n converges, then $a_{n+1} - a_n \to 0$. Show that the converse is false by considering $a_n = \sqrt{n}$.

7.2 Infinite Series: Geometric and Telescoping

Define the sum of an infinite series through partial sums, then master geometric and telescoping series and the nth-term test for divergence.

An **infinite series** is a sum of infinitely many terms, $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$. We cannot add infinitely many numbers one by one, so we define the sum through **partial sums** $S_N = a_1 + a_2 + \cdots + a_N$. The series **converges** to S if the sequence of partial sums $\{S_N\}$ converges to S, that is $\lim_{N\to\infty} S_N = S$. Otherwise the series **diverges**.

Geometric Series

A geometric series has the form $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$ with a constant ratio r. Its partial sums are $S_N = a\,\frac{1-r^N}{1-r}$ (for N terms, $r \neq 1$).

If $|r| < 1$ the series converges and

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} = \frac{\text{first term}}{1-r}.$$

If $|r| \geq 1$ (and $a \neq 0$) the series diverges.

The formula uses the **first term of the sum as written**, whatever the starting index: $\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}$.

EXAMPLE 1: GEOMETRIC SERIES

Find the sum of each series, or explain why it diverges. (a) $\sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n$ (b) $\sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^n$ (c) $\sum_{n=1}^{\infty} \frac{3^{n+1}}{5^n}$ (d) $\sum_{n=0}^{\infty}\left(-\frac{5}{4}\right)^n$

SOLUTION

(a) $a = 5$, $r = \frac{2}{3}$, $|r| < 1$: sum $= \frac{5}{1-2/3} = 15$.

(b) The first term is $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ and $r = \frac{1}{2}$: sum $= \frac{1/8}{1/2} = \frac{1}{4}$.

(c) Rewrite $\frac{3^{n+1}}{5^n} = 3 \cdot \left(\frac{3}{5}\right)^n$. The first term (at $n = 1$) is $\frac{9}{5}$ and $r = \frac{3}{5}$: sum $= \frac{9/5}{2/5} = \frac{9}{2}$.

(d) $|r| = \frac{5}{4} \geq 1$, so the series diverges.

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The nth-Term Test for Divergence

If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Equivalently: if $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then $\sum a_n$ **diverges**.

The converse is false: $a_n \to 0$ does **not** guarantee convergence.

Common mistake

The nth-term test can only prove divergence. The series $\sum \frac{1}{n}$ has terms tending to 0 yet diverges (you will prove this in the next section). Never conclude that a series converges just because $a_n \to 0$.

EXAMPLE 2: REPEATING DECIMALS AND THE NTH-TERM TEST

(a) Write $0.2\overline{7} = 0.27777\ldots$ as a fraction. (b) Does $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ converge?

SOLUTION

(a) $0.2\overline{7} = 0.2 + 0.07 + 0.007 + \cdots = \frac{2}{10} + \frac{0.07}{1-0.1} = \frac{1}{5} + \frac{7}{90} = \frac{18}{90} + \frac{7}{90} = \frac{25}{90} = \frac{5}{18}$.

(b) $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$. By the nth-term test, the series diverges.

Telescoping Series

A series **telescopes** when partial fractions (or algebra) write each term as a difference $b_n - b_{n+k}$, so that most terms in S_N cancel and only a few remain.

Then S_N equals the first few b's minus the last few, and the sum is $\lim_{N\to\infty} S_N$.

EXAMPLE 3: A TELESCOPING SERIES

Find $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

SOLUTION

Partial fractions: $\frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$.

Write out $S_N = \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+2}\right)\right]$. Everything cancels except 1, $\frac{1}{2}$, $-\frac{1}{N+1}$, $-\frac{1}{N+2}$.

So $S_N = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right) \to \frac{3}{4}$. The sum is $\frac{3}{4}$.

EXAMPLE 4: VALUES OF X

For which values of x does $\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$ converge? Find the sum.

SOLUTION

This is geometric with $r = \frac{x}{3}$. It converges when $\left|\frac{x}{3}\right| < 1$, that is $-3 < x < 3$.

For those x the sum is $\frac{1}{1 - x/3} = \frac{3}{3-x}$.

Practice

SET A **Skills**

Find each sum or decide that it diverges. Justify. [No calculator]

1. Find the partial sum S_4 of $\sum_{n=1}^{\infty}(2n-1)$, and guess a formula for S_N.
2. Find $\sum_{n=0}^{\infty} 7\left(\frac{1}{4}\right)^n$.
3. Find $\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n$.
4. Find $\sum_{n=2}^{\infty} 4\left(\frac{1}{5}\right)^n$.
5. Find $\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}}$.
6. Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.
7. Write the repeating decimal $0.\overline{36}$ as a fraction in lowest terms.
8. Does $\sum_{n=1}^{\infty} \frac{n+1}{2n+5}$ converge? Give a reason.
9. Does $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$ converge? Give a reason.

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SET B Applications and Mixed Practice

Give decimals to three places where needed. [Calculator]

10. A patient takes 20 mg of a medicine each day, and just before each new dose 30% of the previous amount is still in the body. The amount just after the nth dose is $20(1+0.3+0.3^2+\cdots+0.3^{n-1})$ mg. Find the amount after 5 doses and the long-term amount.
11. A ball is dropped from 8 m. Each time it hits the ground it rebounds to 70% of its previous height. Find the total distance the ball travels.
12. In a simple economy, 85% of every dollar received is spent again. If 2 million dollars is newly injected, the total spending is $2+2(0.85)+2(0.85)^2+\cdots$ million dollars. Find the total.
13. In each stage, one third of the area that is still unshaded in a unit square is shaded. After N stages the shaded area is $1-\left(\frac{2}{3}\right)^N$. What is the smallest N for which more than 99% of the square is shaded?
14. A pendulum's first swing covers 50 cm and each later swing covers 92% of the one before. Find the total distance it covers before stopping, and the number of the first swing shorter than 1 cm.
15. Let S_N be the Nth partial sum of $\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$. Find S_{10} and the sum of the series.
16. Write $2.\overline{045}=2.045045045\ldots$ as a ratio of integers.

SET C Challenge and AP-Style Problems

Justify carefully. [No calculator]

17. Find $\sum_{n=1}^{\infty}\frac{2^n+3^n}{6^n}$.
18. Show that $\sum_{n=2}^{\infty}\ln\left(1-\frac{1}{n^2}\right)$ converges and find its sum.
19. Find $\sum_{n=1}^{\infty}\frac{2}{4n^2-1}$.
20. For which x does $\sum_{n=0}^{\infty}\left(\frac{x}{x+2}\right)^n$ converge? Find the sum for those x.
21. Show that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges even though its terms tend to 0, by showing $S_{2^k}\geq 1+\frac{k}{2}$.

7.3 Convergence Tests I

Decide whether series with positive terms converge using the divergence test, p-series, the integral test, and direct and limit comparison.

Most series cannot be summed exactly, so the first question is always: **does it converge?** The tests in this section decide convergence for series with positive terms by comparing a series with something we already understand, either an improper integral or a simpler series. Start every problem with the nth-term (divergence) test: if $\lim a_n\neq 0$, the series diverges and you are done.

p-Series and the Integral Test

p-series. $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p>1$ and diverges if $p\leq 1$. The case $p=1$ is the **harmonic series**, which diverges.

Integral Test. Suppose f is positive, continuous and decreasing on $[1,\infty)$ (or on $[k,\infty)$) and $a_n=f(n)$. Then $\sum a_n$ and $\int_1^{\infty}f(x)\,dx$ either both converge or both diverge.

The integral test tells you only whether the series converges, **not** what it sums to. For a convergent series, the error after N terms satisfies $\int_{N+1}^{\infty}f(x)dx\leq R_N\leq\int_N^{\infty}f(x)dx$, where $R_N=S-S_N$.

EXAMPLE 1: P-SERIES AND THE INTEGRAL TEST

Decide whether each series converges. (a) $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ (b) $\sum_{n=1}^{\infty}\frac{5}{\sqrt{n}}$ (c) $\sum_{n=1}^{\infty}ne^{-n^2}$ (d) $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$

SOLUTION

(a) p-series with $p=\frac{3}{2}>1$: converges.

(b) $5\sum n^{-1/2}$ is a constant times a p-series with $p=\frac{1}{2}\leq 1$: diverges.

(c) Let $f(x)=xe^{-x^2}$. It is positive and continuous. $f'(x)=e^{-x^2}(1-2x^2)<0$ for $x\geq 1$, so f is decreasing. $\int_1^{\infty}xe^{-x^2}dx=\lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^b=\frac{1}{2e}$ converges, so the series converges.

(d) $f(x)=\frac{1}{x\ln x}$ is positive, continuous and decreasing for $x\geq 2$. $\int_2^{\infty}\frac{dx}{x\ln x}=\lim_{b\to\infty}[\ln(\ln x)]_2^b=\infty$, so the series diverges.

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Common mistake

You must check that f is positive, continuous and decreasing before using the integral test, and write that you checked. Also do not confuse the p-series test (for $\frac{1}{n^p}$ only) with the geometric series test (for r^n).

Comparison Tests

Direct Comparison. Suppose $0 \le a_n \le b_n$ for all large n. If $\sum b_n$ converges, then $\sum a_n$ converges. If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Limit Comparison. Suppose $a_n > 0$, $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where $0 < c < \infty$. Then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Good comparison series are p-series $\frac{1}{n^p}$ and geometric series r^n. To pick one, keep only the dominant term in the numerator and denominator.

EXAMPLE 2: DIRECT COMPARISON

Decide whether (a) $\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ and (b) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ converge.

SOLUTION

(a) For $n \ge 1$, $n^3 + n > n^3$, so $0 < \frac{1}{n^3+n} < \frac{1}{n^3}$. The p-series $\sum \frac{1}{n^3}$ converges ($p = 3 > 1$), so by direct comparison the given series converges.

(b) For $n \ge 2$, $0 < \sqrt{n} - 1 < \sqrt{n}$, so $\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}} > 0$. The p-series $\sum \frac{1}{\sqrt{n}}$ diverges ($p = \frac{1}{2}$), so by direct comparison the given series diverges.

EXAMPLE 3: LIMIT COMPARISON

Decide whether (a) $\sum_{n=1}^{\infty} \frac{2n^2+1}{n^4+3n}$ and (b) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converge.

SOLUTION

(a) The dominant behavior is $\frac{2n^2}{n^4} = \frac{2}{n^2}$. Compare with $b_n = \frac{1}{n^2}$: $\lim \frac{(2n^2+1)/(n^4+3n)}{1/n^2} = \lim \frac{2n^4+n^2}{n^4+3n} = 2$. Since $0 < 2 < \infty$ and $\sum \frac{1}{n^2}$ converges, the series converges.

(b) Compare with $b_n = \left(\frac{1}{2}\right)^n$: $\lim \frac{1/(2^n-1)}{1/2^n} = \lim \frac{2^n}{2^n-1} = 1$. The geometric series $\sum \left(\frac{1}{2}\right)^n$ converges, so the series converges.

Common mistake

In direct comparison the inequality must point the right way. Showing $a_n \le b_n$ with $\sum b_n$ **diverging** proves nothing about $\sum a_n$. For example $\frac{1}{n^2} < \frac{1}{n}$, but $\sum \frac{1}{n^2}$ converges. In limit comparison, a limit of 0 or ∞ is inconclusive.

EXAMPLE 4: CHOOSE THE RIGHT TEST

Decide whether $\sum_{n=1}^{\infty} \frac{n}{n+1}$ and $\sum_{n=1}^{\infty} \frac{3n+1}{n^2+4}$ converge.

SOLUTION

First series: $\lim \frac{n}{n+1} = 1 \ne 0$, so it diverges by the nth-term test.

Second series: terms tend to 0, so the nth-term test is inconclusive. Compare with $\frac{1}{n}$: $\lim \frac{(3n+1)/(n^2+4)}{1/n} = \lim \frac{3n^2+n}{n^2+4} = 3$. The harmonic series diverges, so this series diverges by limit comparison.

Practice

SET A **Skills**

Decide whether each series converges. Name the test and justify. [No calculator]

1. Determine whether $\sum_{n=1}^{\infty} \frac{n}{n+4}$ converges.
2. Determine whether $\sum_{n=1}^{\infty} \frac{7}{n^{4/3}}$ converges.
3. Determine whether $\sum_{n=1}^{\infty} \frac{2}{\sqrt[3]{n}}$ converges.
4. Use the integral test to decide whether $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converges.

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5. Use direct comparison to decide whether $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ converges.
6. Use limit comparison with $\left(\frac{1}{3}\right)^n$ to decide whether $\sum_{n=1}^{\infty} \frac{1}{3^n-2}$ converges.
7. Decide whether $\sum_{n=1}^{\infty} \frac{n+2}{n^3+5}$ converges.
8. Decide whether $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+3}$ converges.
9. Decide whether $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ converges.

SET B Applications and Mixed Practice

Give decimals to three places where needed. [Calculator]

10. Let S_{10} be the 10th partial sum of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Use S_{10} and the integral test remainder bounds to find an interval containing the sum.
11. How many terms of $\sum_{n=1}^{\infty} \frac{1}{n^3}$ guarantee that the partial sum is within 0.001 of the full sum (using the integral test remainder bound)?
12. A stack of identical blocks can be arranged so that the top N blocks overhang the edge by $\frac{1}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right)$ block lengths. What is the smallest N for which the overhang exceeds 2 block lengths? What does the divergence of the harmonic series imply?
13. Use S_5 of $\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ and the comparison $\frac{1}{n^3+n} < \frac{1}{n^3}$ to give an interval that contains the sum.
14. Show that $\ln(1001) \le H_{1000} \le 1+\ln(1000)$ gives bounds for $H_{1000} = \sum_{n=1}^{1000} \frac{1}{n}$, and state the bounds to three places.
15. Decide whether $\sum_{n=1}^{\infty} \frac{n+1}{n^{5/2}+3}$ converges, and justify with a comparison.
16. Show that $\sum_{n=1}^{\infty} ne^{-n/2}$ converges using the integral test.

SET C Challenge and AP-Style Problems

Justify carefully. [No calculator]

17. Show that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if and only if $p > 1$.
18. Show that $\sum_{n=1}^{\infty} \frac{2^n+n}{3^n}$ converges by direct comparison.
19. Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$. Then $\lim \frac{a_n}{b_n} = 0$ and $\sum b_n$ diverges. Does this prove that $\sum a_n$ diverges? Explain.
20. Show that $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$ converges.
21. Show that $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ diverges.

7.4 Convergence Tests II

Learn the Ratio Test and the Alternating Series Test, tell absolute from conditional convergence, bound the error of an alternating sum, and choose the right test for any series.

In Convergence Tests I you met the nth-term test, geometric series, p-series, the integral test and comparison tests. This section adds the tools you need for series with factorials, exponentials and alternating signs: the **Ratio Test**, the **Alternating Series Test**, and the ideas of absolute and conditional convergence.

The Ratio Test

Let $\sum a_n$ be a series with nonzero terms and suppose

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists (or is ∞).

If $L < 1$, the series converges absolutely. If $L > 1$ (including $L = \infty$), the series diverges. If $L = 1$, the test gives **no information**.

Use it when terms contain factorials, or powers like c^n. It compares the series to a geometric series with ratio L.

EXAMPLE 1

Determine whether $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ converges.

SOLUTION

Form the ratio: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1}$.

$L = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$.

By the Ratio Test, the series converges (absolutely).

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EXAMPLE 2

Determine whether $\sum_{n=1}^{\infty} \frac{n^2}{4^n}$ converges.

SOLUTION

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{4^{n+1}} \cdot \frac{4^n}{n^2} = \frac{1}{4}\left(\frac{n+1}{n}\right)^2$.

As $n \to \infty$, $\left(\frac{n+1}{n}\right)^2 \to 1$, so $L = \frac{1}{4} < 1$.

The series converges by the Ratio Test.

The Alternating Series Test

Let $b_n > 0$. The alternating series $\sum(-1)^n b_n$ (or $\sum(-1)^{n+1} b_n$) converges if both conditions hold: (1) $b_{n+1} \le b_n$ for all sufficiently large n (the terms are eventually decreasing), and (2) $\lim_{n\to\infty} b_n = 0$.

Alternating Series Error Bound. If the series satisfies these conditions and S_N is the sum of the first N terms, then

$$|S - S_N| \le b_{N+1}.$$

The error is no larger than the first omitted term, and S lies between S_N and S_{N+1}.

EXAMPLE 3

Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ converges. Does it converge absolutely?

SOLUTION

Here $b_n = \frac{1}{2n+1} > 0$. Since $2n+3 > 2n+1$, we have $b_{n+1} < b_n$, and $\lim b_n = 0$.

By the Alternating Series Test, the series converges.

The absolute series is $\sum \frac{1}{2n+1}$. Since $\frac{1}{2n+1} \ge \frac{1}{3n}$ for $n \ge 1$ and $\sum \frac{1}{3n}$ diverges (harmonic), the absolute series diverges by direct comparison.

The original series converges but not absolutely.

Absolute and Conditional Convergence

$\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges. $\sum a_n$ **converges conditionally** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Absolute convergence implies convergence. So if $\sum |a_n|$ converges, you may conclude that $\sum a_n$ converges.

EXAMPLE 4

Use the first four terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ to approximate the sum. Give an error bound.

SOLUTION

$S_4 = 1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} = \frac{1549}{1728} \approx 0.8964$.

The series satisfies the Alternating Series Test ($b_n = 1/n^3$ decreases to 0), so $|S - S_4| \le b_5 = \frac{1}{125} = 0.008$.

Therefore $0.8964 - 0.008 \le S \le 0.8964 + 0.008$. In fact S lies between S_4 and S_5.

EXAMPLE 5

Classify each series as absolutely convergent, conditionally convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+2}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ (c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

SOLUTION

(a) $|a_n| = \frac{n}{n+2} \to 1 \neq 0$, so $a_n \not\to 0$. Diverges by the nth-term test.

(b) $b_n = n^{-1/2}$ decreases to 0, so it converges. $\sum n^{-1/2}$ is a p-series with $p = \frac{1}{2} \le 1$, so it diverges. Conditionally convergent.

(c) $\sum \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, so the series converges absolutely.

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1. **Choosing a test.** Look at the form of a_n first.
2. **Does $a_n \to 0$?** If not, the series diverges (nth-term test). Stop.
3. **Factorials or exponentials like c^n?** Try the Ratio Test.
4. **Alternating signs?** Check the Alternating Series Test, then test $\sum |a_n|$ to decide between absolute and conditional.
5. **Rational or radical function of n?** Compare with a p-series (direct or limit comparison).
6. **Easy to integrate?** Try the integral test.

Common mistakes

(1) When the Ratio Test gives $L = 1$ it says nothing: both $\sum \frac{1}{n}$ (diverges) and $\sum \frac{1}{n^2}$ (converges) give $L = 1$. (2) Do not use the Alternating Series Test to conclude divergence. If $b_n \not\to 0$, use the nth-term test instead. (3) The error bound b_{N+1} applies only to alternating series that satisfy the test. (4) If $|a_{n+1}/a_n| < 1$ for every n but the limit is 1, the series can still diverge.

Practice

SET A Skills

Decide convergence or divergence, showing the test you use. [No calculator]

1. Use the Ratio Test to decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{n}{5^n}$ converges.
2. Use the Ratio Test to decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!}$ converges.
3. Use the Ratio Test to decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}$ converges.
4. Use the Ratio Test to decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ converges.
5. Use the Ratio Test to decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2}$ converges.
6. (a) Show that the Ratio Test gives $L = 1$ for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

   (b) Use another test to decide whether the series converges.
7. Does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+4}$ converge? If so, is the convergence absolute or conditional?
8. Show that $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$ diverges.
9. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3+1}$ converges absolutely.

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SET B Applications and Mixed

Answer each question. [Calculator]

10. Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$.
    (a) Compute S_5.
    (b) Give an upper bound for $|S - S_5|$.

11. What is the least N for which the Alternating Series Error Bound guarantees that S_N approximates $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1}$ with error less than 0.001?

12. Let $a_n = n^2(0.9)^n$.
    (a) Find $\frac{a_{11}}{a_{10}}$.
    (b) Find $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ and decide whether $\sum a_n$ converges.

13. Classify $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ as absolutely convergent, conditionally convergent or divergent. Justify.

14. A student claims: \"Since $\left|\frac{a_{n+1}}{a_n}\right| < 1$ for every n, the series $\sum a_n$ must converge.\" Give a counterexample and explain the error.

15. Approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,2^n}$ by S_4 and state the maximum error.

16. For which values of $c > 0$ does $\sum_{n=1}^{\infty} \frac{c^n}{n^2}$ converge?

SET C Challenge

Justify your answers fully. [No calculator]

17. For a series of positive terms, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{3}{2}$. What can you conclude about $\sum a_n$ and about $\sum (-1)^n a_n$?

18. For $k > 0$ consider $\sum_{n=1}^{\infty} \frac{k^n\, n!}{n^n}$. Use the Ratio Test to find the values of k for which the series converges and the values for which it diverges. (Use $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.)

19. Show that $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ converges, and decide whether the convergence is absolute or conditional.

20. Find the least N for which S_N approximates $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ with error less than 0.001, and give S_N as a fraction.

21. Suppose $\sum a_n$ converges absolutely. Explain why $\sum a_n^2$ also converges.

7.5 Power Series and Intervals of Convergence

Find the radius and interval of convergence of a power series with the Ratio Test, check the endpoints, and differentiate and integrate power series term by term.

A **power series** is an infinite polynomial. It defines a function wherever it converges, and calculus on it works one term at a time. The key questions are: for which x does it converge, and how do you differentiate and integrate it?

Power Series, Radius and Interval of Convergence

A power series centered at c has the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

Exactly one of these is true: (1) it converges only at $x = c$ ($R = 0$); (2) it converges for all x ($R = \infty$); (3) there is a number $R > 0$ such that it converges absolutely for $|x - c| < R$ and diverges for $|x - c| > R$. The number R is the **radius of convergence**.

The **interval of convergence** is the set of all x where the series converges. It is centered at c, but each endpoint $c \pm R$ must be tested separately.

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1. **Finding the interval of convergence.**
2. **Ratio Test.** Compute $L(x) = \lim_{n\to\infty} \left|\frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n}\right|$. It will have the form $|x-c|\cdot\rho$.
3. **Solve** $L(x) < 1$. This gives $|x-c| < R$ with $R = 1/\rho$.
4. **Test each endpoint.** Substitute $x = c - R$ and $x = c + R$ and use any convergence test on the resulting number series (the Ratio Test fails here, since $L = 1$).
5. **State the interval** with brackets for included endpoints and parentheses for excluded ones.

EXAMPLE 1

Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n\,3^n}$.

SOLUTION

Ratio: $\left|\frac{(x-2)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n\,3^n}{(x-2)^n}\right| = \frac{n}{n+1}\cdot\frac{|x-2|}{3} \to \frac{|x-2|}{3}$.

Convergence requires $\frac{|x-2|}{3} < 1$, so $|x-2| < 3$. The radius is $R = 3$ and $-1 < x < 5$.

At $x = 5$: the series is $\sum \frac{3^n}{n3^n} = \sum \frac{1}{n}$, which diverges.

At $x = -1$: the series is $\sum \frac{(-3)^n}{n3^n} = \sum \frac{(-1)^n}{n}$, which converges (Alternating Series Test).

The interval of convergence is $[-1, 5)$.

EXAMPLE 2

Find the radius of convergence of (a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and (b) $\displaystyle\sum_{n=0}^{\infty} n!\,x^n$.

SOLUTION

(a) Ratio: $\frac{|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{|x|^n} = \frac{|x|}{n+1} \to 0 < 1$ for every x. So $R = \infty$ and the interval is $(-\infty, \infty)$.

(b) Ratio: $\frac{(n+1)!|x|^{n+1}}{n!\,|x|^n} = (n+1)|x| \to \infty$ for every $x \neq 0$. So the series converges only at $x = 0$, and $R = 0$.

EXAMPLE 3

Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(x+1)^n}{n^2\,4^n}$.

SOLUTION

The ratio is $\left(\frac{n}{n+1}\right)^2 \frac{|x+1|}{4} \to \frac{|x+1|}{4}$, so $|x+1| < 4$: center -1, $R = 4$, and $-5 < x < 3$.

At $x = 3$: $\sum \frac{4^n}{n^2 4^n} = \sum \frac{1}{n^2}$ converges. At $x = -5$: $\sum \frac{(-4)^n}{n^2 4^n} = \sum \frac{(-1)^n}{n^2}$ converges.

The interval of convergence is $[-5, 3]$.

Term-by-Term Differentiation and Integration

If $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ has radius of convergence $R > 0$, then on $|x-c| < R$:

$$f'(x) = \sum_{n=1}^{\infty} n\,a_n(x-c)^{n-1}, \qquad \int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}.$$

Both new series have the **same radius** R. The behavior at the endpoints may change, so test them again.

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EXAMPLE 4

Starting from $\dfrac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ for $|x| < 1$, find a power series for $\dfrac{1}{(1+x)^2}$.

SOLUTION

Differentiate both sides: $-\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^n \, n \, x^{n-1}$.

Multiply by -1: $\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n+1} n \, x^{n-1} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

The radius of convergence is still $R = 1$.

EXAMPLE 5

Use $\dfrac{1}{1-t} = \sum_{n=0}^{\infty} t^n$ for $|t| < 1$ to find a series for $-\ln(1-x)$ and its interval of convergence.

SOLUTION

Integrate from 0 to x: $-\ln(1-x) = \int_0^x \frac{dt}{1-t} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n}$ for $|x| < 1$.

The radius is 1. At $x = 1$ the series is the harmonic series, which diverges. At $x = -1$ it is $\sum \frac{(-1)^n}{n}$, which converges.

The interval of convergence is $[-1, 1)$. The original geometric series converged on $(-1, 1)$, so the endpoint $x = -1$ was gained.

Common mistakes

(1) Skipping the endpoints. The ratio test always gives $L = 1$ at an endpoint, so you must use a different test there. (2) Writing the interval as $|x - c| \le R$ without checking. (3) Forgetting that the index shifts when you differentiate: the sum for f' starts at $n = 1$. (4) Forgetting $+C$ when integrating a series without limits. (5) Assuming the interval of convergence is unchanged after differentiating or integrating: only the radius is guaranteed.

Practice

SET A Skills

Find the requested radius or interval of convergence. [No calculator]

1. Find the radius and interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{5^n}$.
2. Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n \, x^n}{2^n}$.
3. Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{(2n)!}$.
4. Find the radius of convergence of $\displaystyle\sum_{n=1}^{\infty} n^n \, x^n$ and state where it converges.
5. Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2}$.
6. Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n \, 4^n}$.
7. Find the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(3x)^n}{n+1}$.
8. Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{9^n}$.
9. A power series $\sum c_n (x-1)^n$ converges at $x = -2$ and diverges at $x = 4$.
   (a) What are the center and radius of convergence?
   (b) Does the series converge at $x = 3$? At $x = 5$?

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SET B Applications and Mixed

Answer each question. Give decimals to three places. [Calculator]

10. The series $\displaystyle\sum_{n=0}^{\infty}\frac{(x-1)^n}{3^n}$ converges to $\dfrac{3}{4-x}$ on its interval of convergence.

(a) Find the interval of convergence.

(b) Approximate $f(2)$ using the first five terms ($n=0$ to 4), to three decimal places.

(c) Find the exact value of $f(2)$.

11. For $|x|<1$, $\dfrac{1}{1-x^2}=\displaystyle\sum_{n=0}^{\infty}x^{2n}$.

(a) Differentiate term by term to write the first four nonzero terms of the series for $\dfrac{2x}{(1-x^2)^2}$.

(b) Evaluate the sum of those four terms at $x=0.5$, and compare with the exact value of $\dfrac{2x}{(1-x^2)^2}$ at $x=0.5$ to three decimals.

12. Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{n^2(x-4)^n}{5^n}$.

13. For a power series $\sum c_n(x-1)^n$, $\displaystyle\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|=\frac{2}{5}$.

(a) Find the radius of convergence.

(b) For each of $x=0$, $x=3$ and $x=4$, say whether the series converges, diverges or cannot be determined from this information.

14. Let $f(x)=\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n\,2^n}$.

(a) Find the interval of convergence of f.

(b) Write $f'(x)$ as a power series and as a closed-form expression for $|x|<2$.

(c) Find $f'(1)$ and use it to check your closed form.

15. A signal model is $S(t)=\displaystyle\sum_{n=0}^{\infty}\left(-\frac{t}{4}\right)^n$ volts.

(a) For which t does the series converge, and what is its sum?

(b) Find $S(1.5)$ to three decimals, and find the error when the first four terms ($n=0$ to 3) are used.

16. Integrate the series $\dfrac{1}{1-t}=\displaystyle\sum_{n=0}^{\infty}t^n$ term by term from 0 to $\frac{1}{2}$ and use the first five terms to approximate $\ln 2$. Compare with the calculator value of $\ln 2$.

SET C Challenge

Justify your answers. [No calculator]

17. Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\,x^n$.

18. Find the radius of convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n^n}$.

19. Let $f(x)=\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$. Differentiate term by term to show that $f'(x)=f(x)$.

20. A function $y=\displaystyle\sum_{n=0}^{\infty}a_nx^n$ satisfies $y'=2y$ and $y(0)=3$.

(a) Equate coefficients to show $n\,a_n=2a_{n-1}$ for $n\ge 1$.

(b) Find a_0, a_1, a_2, a_3 and a formula for a_n.

21. For $|x|<1$, $\dfrac{1}{1-x}=\displaystyle\sum_{n=0}^{\infty}x^n$.

(a) Show that $\displaystyle\sum_{n=1}^{\infty}n\,x^n=\frac{x}{(1-x)^2}$.

(b) Use part (a) to find $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2^n}$.

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7.6 Maclaurin Series and Taylor Polynomials

Memorize the six standard Maclaurin series, build new series by substitution, differentiation and integration, and use Taylor polynomials to approximate function values.

A Taylor polynomial replaces a complicated function by a polynomial you can evaluate with arithmetic alone. A Maclaurin series is the full infinite version. Memorize the six standard series below: nearly every series problem is built from them.

Taylor Polynomials and Maclaurin Series

If f has n derivatives at 0, its **Maclaurin polynomial of degree** n is

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k.$$

If f has derivatives of all orders at 0, the **Maclaurin series** is $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$. The polynomial P_n matches f and its first n derivatives at $x = 0$.

To read a derivative from a series, remember $f^{(k)}(0) = k! \cdot (\text{coefficient of } x^k)$.

Six Standard Maclaurin Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (\text{all } x)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad (\text{all } x)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad (\text{all } x)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \quad (|x| < 1)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad (-1 < x \le 1)$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \quad (-1 \le x \le 1)$$

EXAMPLE 1

Find the Maclaurin polynomial of degree 3 for $f(x) = \sqrt{1+x}$.

SOLUTION

$f(x) = (1+x)^{1/2}$, $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, $f''(x) = -\frac{1}{4}(1+x)^{-3/2}$, $f'''(x) = \frac{3}{8}(1+x)^{-5/2}$.

At 0: $f(0) = 1$, $f'(0) = \frac{1}{2}$, $f''(0) = -\frac{1}{4}$, $f'''(0) = \frac{3}{8}$.

$P_3(x) = 1 + \frac{1}{2}x - \frac{1/4}{2!}x^2 + \frac{3/8}{3!}x^3 = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$.

EXAMPLE 2

Find the first four nonzero terms of the Maclaurin series for e^{-x^2}.

SOLUTION

Substitute $u = -x^2$ into $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$.

$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$. It converges for all x, since e^u does.

EXAMPLE 3

Find the first three nonzero terms of the Maclaurin series for $x^2 \sin(3x)$.

SOLUTION

$\sin(3x) = 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \cdots = 3x - \frac{9x^3}{2} + \frac{81x^5}{40} - \cdots$

Multiply by x^2: $x^2 \sin(3x) = 3x^3 - \frac{9}{2}x^5 + \frac{81}{40}x^7 - \cdots$

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EXAMPLE 4

Use a Maclaurin series to approximate $\int_0^{0.5} e^{-t^2}\,dt$ with an error less than 0.0001.

SOLUTION

Integrate the series term by term: $\int_0^x e^{-t^2}dt = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \cdots$

At $x = 0.5$: $0.5 - 0.041667 + 0.003125 - 0.000186 + \cdots$. This alternating series has decreasing terms, so the error after four terms is at most the next term, $\frac{0.5^9}{216} \approx 0.000009 < 0.0001$.

The sum of four terms is 0.461272, so the integral is about 0.4613.

EXAMPLE 5

Evaluate $\lim_{x\to 0} \dfrac{x - \sin x}{x^3}$ using a series.

SOLUTION

$x - \sin x = x - \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) = \frac{x^3}{6} - \frac{x^5}{120} + \cdots$

Divide by x^3: $\frac{1}{6} - \frac{x^2}{120} + \cdots \to \frac{1}{6}$ as $x \to 0$.

Common mistakes

(1) Forgetting the factorial: the coefficient of x^k is $f^{(k)}(0)/k!$, not $f^{(k)}(0)$. (2) Substituting without squaring or cubing the constant: for $\sin(3x)$ the cube is $(3x)^3 = 27x^3$. (3) Using the wrong signs: $\sin x$ and $\cos x$ alternate, e^x does not. (4) Writing the series for $\ln(1+x)$ starting at $n = 0$; there is no constant term and no x^0 term. (5) Forgetting that a substituted series has a changed interval of convergence: $\frac{1}{1-3x}$ needs $|3x| < 1$.

Practice

SET A ### Skills

Find the requested polynomials, series or limits. [No calculator]

1. Use derivatives to find the Maclaurin polynomial of degree 3 for $f(x) = e^{3x}$.
2. Write the first four nonzero terms of the Maclaurin series for $\cos(2x)$.
3. Write the first three nonzero terms of the Maclaurin series for $x\,e^{x^2}$.
4. Write the Maclaurin series for $\dfrac{1}{1+3x}$ in summation notation and state its radius of convergence.
5. Write the first three nonzero terms of the Maclaurin series for $\ln(1+x^2)$.
6. Write the first three nonzero terms of the Maclaurin series for $\arctan(x^2)$.
7. Find the Maclaurin polynomial of degree 4 for $f(x) = \sin x + \cos x$.
8. A function f has $f(0) = 2$, $f'(0) = -1$, $f''(0) = 6$, $f'''(0) = -12$. Write its Maclaurin polynomial of degree 3.
9. Let $f(x) = x^2 e^x$. Use the Maclaurin series for e^x to find $f^{(5)}(0)$.

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SET B Applications and Mixed

Give decimal answers to three decimal places unless told otherwise. [Calculator]

10. (a) Use the Maclaurin polynomial $P_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$ to approximate $\sin(0.4)$ to five decimal places.
(b) Use the Alternating Series Error Bound to bound the error.

11. Approximate $e^{0.5}$ using the Maclaurin polynomial of degree 4. Compare with the calculator value and give the actual error to four decimal places.

12. A function f has the derivative values shown at $x = 0$.
(a) Write the Maclaurin polynomial of degree 4 for f.
(b) Use it to approximate $f(0.2)$ to three decimals.

n	0	1	2	3	4
$f^{(n)}(0)$	3	2	-4	6	24

13. Use the first four terms of the Maclaurin series for $\ln(1+x)$ to approximate $\ln(1.2)$ to three decimals. Give an error bound.

14. The series for $\sin(x^2)$ integrates term by term to $\displaystyle\int_0^1 \sin(x^2)\,dx = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \cdots$.
(a) Approximate the integral with the first three terms (three decimals).
(b) Compare with the calculator value of the integral.

15. For small angles a pendulum model uses $\sin\theta \approx \theta - \frac{\theta^3}{6}$. For $\theta = 0.3$ radian, find the approximation, the calculator value of $\sin 0.3$, and the error to six decimal places.

16. (a) Write the Maclaurin series for $\dfrac{1}{2-x}$ (hint: factor out 2) and give its interval of convergence.
(b) Use the degree 3 polynomial to approximate the value at $x = 0.5$, and compare with the exact value.

SET C Challenge

Justify your answers. [No calculator]

17. Let $f(x) = \dfrac{\sin x}{x}$ for $x \neq 0$ and $f(0) = 1$.
(a) Write the first four nonzero terms of its Maclaurin series.
(b) Find $f^{(4)}(0)$.

18. (a) Integrate $\dfrac{1}{1+t^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n t^{2n}$ from 0 to x to obtain the Maclaurin series for $\arctan x$.
(b) The series converges at $x = 1$. Use this to write a series for $\frac{\pi}{4}$.

19. Find the exact sum of each series.
(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\, n!}$
(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}$
(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\, 3^n}$

20. Use Maclaurin series to evaluate $\displaystyle\lim_{x\to 0} \frac{x - \arctan x}{x^3}$.

21. (a) Use the Maclaurin series for $\ln(1+x)$ to write the series for $\ln(1-x)$.
(b) Find the first three nonzero terms of $\ln\dfrac{1+x}{1-x}$ and its radius of convergence.

7.7 Taylor Series and the Lagrange Error Bound

Build Taylor polynomials and series about any center, approximate values, bound the error with the Lagrange bound and the alternating series bound, and read derivatives from a Taylor polynomial.

Maclaurin series are centered at 0, but you often need a polynomial that is accurate near some other point. Taylor series are centered at any $x = a$. Equally important is knowing **how accurate** an approximation is: the Lagrange error bound answers that.

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Taylor Polynomials and Taylor Series

If f has n derivatives at $x = a$, its **Taylor polynomial of degree n about $x = a$** is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If f has derivatives of all orders at a, the **Taylor series** is $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$. Maclaurin series is the case $a = 0$.

The polynomial P_n has the same value and the same first n derivatives as f at $x = a$. Conversely, $f^{(k)}(a) = k! \cdot (\text{coefficient of } (x-a)^k)$.

The Lagrange Error Bound

Suppose f has $n+1$ continuous derivatives on an interval I containing a and x. Let P_n be the degree n Taylor polynomial about a, and let $R_n(x) = f(x) - P_n(x)$. If $|f^{(n+1)}(t)| \le M$ for all t between a and x, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Alternating series error bound (special case). If the Taylor series evaluated at x is an alternating series whose terms decrease to 0 in size, the error after the term of degree n is at most the size of the first omitted term.

EXAMPLE 1

Find the Taylor polynomial of degree 3 for $f(x) = \ln x$ about $x = 1$.

SOLUTION

$f(x) = \ln x$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$.

At $x = 1$: $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$.

$P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{2}{6}(x-1)^3 = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$.

EXAMPLE 2

Find the Taylor polynomial of degree 3 for $f(x) = \sqrt{x}$ about $x = 4$. Then use P_2 to approximate $\sqrt{4.4}$ and bound the error.

SOLUTION

$f = x^{1/2}$, $f' = \frac{1}{2}x^{-1/2}$, $f'' = -\frac{1}{4}x^{-3/2}$, $f''' = \frac{3}{8}x^{-5/2}$. At 4: 2, $\frac{1}{4}$, $-\frac{1}{32}$, $\frac{3}{256}$.

$P_3(x) = 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512}$.

$P_2(4.4) = 2 + \frac{0.4}{4} - \frac{0.16}{64} = 2 + 0.1 - 0.0025 = 2.0975$.

Lagrange bound with $n = 2$: $f'''(t) = \frac{3}{8}t^{-5/2}$ is positive and decreasing, so on $[4, 4.4]$ its maximum is $f'''(4) = \frac{3}{256} = M$. $|R_2(4.4)| \le \frac{3/256}{3!}(0.4)^3 = 0.000125$.

So $\sqrt{4.4}$ is within 0.000125 of 2.0975.

EXAMPLE 3

How many terms of the Maclaurin series for e^x (that is, what degree n) guarantee that $P_n(0.3)$ approximates $e^{0.3}$ with error less than 0.0001?

SOLUTION

On $[0, 0.3]$ every derivative of e^x is $e^t \le e^{0.3} < 2$, so use $M = 2$.

Need $\frac{2(0.3)^{n+1}}{(n+1)!} < 0.0001$. For $n = 3$: $\frac{2(0.0081)}{24} = 0.000675$, too big. For $n = 4$: $\frac{2(0.00243)}{120} = 0.0000405 < 0.0001$.

The degree $n = 4$ polynomial is the smallest that works.

EXAMPLE 4

Approximate $\sin(0.5)$ with $P_3(x) = x - \frac{x^3}{6}$ and bound the error in two ways.

SOLUTION

$P_3(0.5) = 0.5 - \frac{0.125}{6} \approx 0.47917$.

Alternating series: $\sin 0.5 = 0.5 - \frac{0.5^3}{3!} + \frac{0.5^5}{5!} - \cdots$ with decreasing terms, so the error is at most $\frac{0.5^5}{120} \approx 0.00026$.

Lagrange (using $n = 4$, since $P_4 = P_3$): all derivatives of $\sin$ are bounded by 1, so $M = 1$ and the error is at most $\frac{0.5^5}{5!} \approx 0.00026$. Both bounds agree.

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EXAMPLE 5

A function f has Taylor polynomial $P_3(x) = 5 - 3(x-2) + 4(x-2)^2 - 6(x-2)^3$ about $x = 2$. Find $f(2)$, $f'(2)$, $f''(2)$ and $f'''(2)$.

SOLUTION

The coefficient of $(x-2)^k$ is $\frac{f^{(k)}(2)}{k!}$, so $f^{(k)}(2) = k! \cdot \text{coefficient}$.

$f(2) = 5$, $f'(2) = -3$, $f''(2) = 2! \cdot 4 = 8$, $f'''(2) = 3! \cdot (-6) = -36$.

Common mistakes

(1) In the Lagrange bound, M bounds the $(n+1)$st derivative, not the nth. (2) M must work for every t between a and x, and you must justify it (for example, by noting that the derivative is increasing or that $|\sin t| \le 1$). (3) Use $|x-a|$, not $|x|$, and the exponent $n+1$. (4) The Lagrange bound needs f to have $n+1$ derivatives on the interval. (5) Do not use the alternating series bound unless the terms really alternate and decrease.

Practice

SET A Skills

Answer each question. [No calculator]

1. Find the Taylor polynomial of degree 3 for $f(x) = \dfrac{1}{x}$ about $x = 1$.
2. Write the Taylor series for $f(x) = e^x$ about $x = 3$, and its first four terms.
3. Find the Taylor polynomial of degree 4 for $f(x) = \sin x$ about $x = \frac{\pi}{2}$.
4. Find the Taylor series for $f(x) = \dfrac{1}{2+x}$ about $x = -1$ and its interval of convergence.
5. Find the Taylor polynomial of degree 3 for $g(x) = x^3$ about $x = 2$. What do you notice about the result?
6. The Taylor polynomial of degree 3 for f about $x = 1$ is $4 + 2(x-1) - 3(x-1)^2 + 5(x-1)^3$. Find $f(1)$, $f'(1)$, $f''(1)$ and $f'''(1)$.
7. Suppose $|f^{(4)}(t)| \le 6$ for all t in $[1, 3]$ and P_3 is the Taylor polynomial of f about $x = 1$. Use the Lagrange error bound to bound $|f(3) - P_3(3)|$.
8. Use the Taylor polynomial of degree 2 for $\ln x$ about $x = 1$ to approximate $\ln(1.1)$. Bound the error, using the fact that $|f'''(t)| \le 2$ on $[1, 1.1]$.
9. Approximate $\cos(1)$ by the Maclaurin polynomial of degree 4 and use the Alternating Series Error Bound to bound the error. Leave answers as fractions.

SET B Applications and Mixed

Give decimal answers to three decimal places unless told otherwise. [Calculator]

10. (a) Find the Taylor polynomial of degree 2 for $f(x) = x^{1/3}$ about $x = 8$.
    (b) Use it to approximate $\sqrt[3]{8.4}$, and use $|f'''(t)| \le \frac{10}{27 \cdot 256}$ on $[8, 8.4]$ to bound the error.
11. (a) Approximate $e^{0.2}$ using the Maclaurin polynomial of degree 3.
    (b) Use the Lagrange error bound with $M = e^{0.2}$ to bound the error, and compare with the actual error.
12. What is the least degree n for which the Lagrange error bound guarantees that the Maclaurin polynomial P_n for $\sin x$ approximates $\sin(1)$ with error less than 0.0001? (Use $M = 1$.)
13. A function f has $f(2) = 1$, $f'(2) = 3$, $f''(2) = -2$, $f'''(2) = 4$, and $|f^{(4)}(t)| \le 12$ on $[2, 2.1]$.
    (a) Write P_3 about $x = 2$.
    (b) Approximate $f(2.1)$ and give a bound for the error.
14. A swing of length $L = 2$ meters is pulled to an angle $\theta = 0.5$ radian, so its height above the lowest point is $h = 2(1 - \cos\theta)$ meters.
    (a) Use $\cos\theta \approx 1 - \frac{\theta^2}{2}$ to approximate h.
    (b) Find the actual error to three decimals.
15. Use the first five nonzero terms of the series for $\ln(1+x)$ to approximate $\ln(1.5)$. Give the alternating series error bound and the actual error.
16. (a) Write the Taylor polynomial of degree 2 for $\cos x$ about $x = \frac{\pi}{4}$.
    (b) Approximate $\cos(0.9)$ and bound the error using $|f'''(t)| \le 1$.

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SET C Challenge

Justify your answers fully. [No calculator]

17. Show that for $|x| \le 1$, the approximation $\sin x \approx x - \frac{x^3}{6}$ has error at most $\frac{1}{120}$. Justify using the Lagrange error bound.
18. Using $e < 3$ and the Lagrange error bound for e^x about 0 on $[0, 1]$, find the least n for which $P_n(1)$ is guaranteed to approximate e with error less than 0.001.
19. A function f satisfies $f^{(n)}(2) = \dfrac{(-1)^n n!}{3^n}$ for all $n \ge 0$.
    (a) Write the Taylor series for f about $x = 2$ in summation notation.
    (b) Find its radius of convergence and its sum as a function of x.
20. The table gives derivative values of f at $x = 1$. Also $|f^{(4)}(t)| \le 12$ for t in $[1, 1.5]$.
    (a) Write the Taylor polynomial P_3 about $x = 1$.
    (b) Use the Lagrange error bound to bound $|f(1.5) - P_3(1.5)|$.
    (c) Is the graph of f concave up or concave down at $x = 1$? Justify.

n	0	1	2	3
$f^{(n)}(1)$	4	-2	6	-18

21. Let $f(x) = x^2 \cos x$. Use the Maclaurin series of $\cos x$ to find $f^{(6)}(0)$ and $f^{(7)}(0)$.

7.8 Parametric Equations

Work with parametric curves: eliminate the parameter, find slope and concavity, locate tangents, and compute speed and arc length for motion defined by x(t) and y(t).

In **parametric equations**, both coordinates depend on a third variable, the parameter t (often time). This lets a curve cross itself, move back and forth, and carry direction and speed information that $y = f(x)$ cannot.

Parametric Curves

A curve is given by $x = f(t)$, $y = g(t)$ for t in an interval. As t increases, the point (x, y) traces the curve in a definite **direction** (orientation).

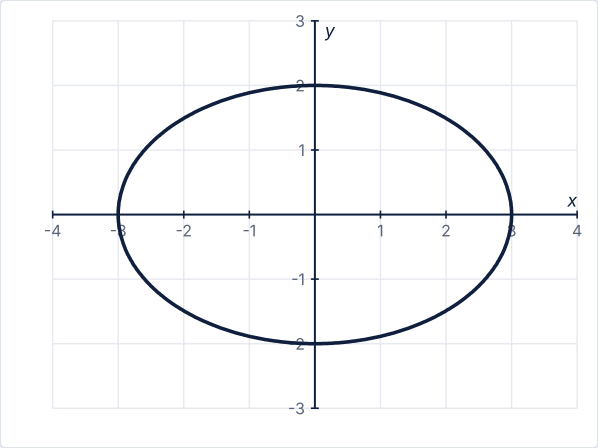
To **eliminate the parameter**, solve one equation for t and substitute, or use an identity such as $\cos^2 t + \sin^2 t = 1$. Always note any restriction on x or y caused by the range of t, and the direction of motion.

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EXAMPLE 1

Eliminate the parameter for $x = 3\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$, and describe the motion.



The curve x = 3 cos t, y = 2 sin t.

SOLUTION

Solve: $\cos t = \frac{x}{3}$ and $\sin t = \frac{y}{2}$.

Use $\cos^2 t + \sin^2 t = 1$: $\frac{x^2}{9} + \frac{y^2}{4} = 1$, an ellipse.

At $t = 0$ the point is $(3, 0)$; at $t = \frac{\pi}{2}$ it is $(0, 2)$. The ellipse is traced once, counterclockwise.

Slope, Concavity, Speed and Arc Length

If $x = f(t)$, $y = g(t)$ are differentiable and $\frac{dx}{dt} \neq 0$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}.$$

The curve has a **horizontal tangent** where $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$, and a **vertical tangent** where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

Speed is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. If the derivatives are continuous, the **arc length** from $t = a$ to $t = b$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

EXAMPLE 2

For $x = t^2 - 4$, $y = t^3 - 3t$, find the points of horizontal and vertical tangency.

SOLUTION

$\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2 - 3 = 3(t-1)(t+1)$.

Horizontal: $\frac{dy}{dt} = 0$ at $t = \pm 1$, where $\frac{dx}{dt} = \pm 2 \neq 0$. Points: $t = 1$ gives $(-3, -2)$; $t = -1$ gives $(-3, 2)$.

Vertical: $\frac{dx}{dt} = 0$ at $t = 0$, where $\frac{dy}{dt} = -3 \neq 0$. Point: $(-4, 0)$.

EXAMPLE 3

For $x = t^2 + 1$, $y = t^3$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $t = 2$, and decide the concavity.

SOLUTION

$\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$ for $t \neq 0$. At $t = 2$: 3.

$\frac{d}{dt}\left(\frac{3t}{2}\right) = \frac{3}{2}$, so $\frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}$. At $t = 2$: $\frac{3}{8}$.

Since $\frac{d^2y}{dx^2} > 0$, the curve is concave up at $t = 2$.

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EXAMPLE 4

Find the length of the curve $x = t^3$, $y = \frac{3}{2}t^2$ for $0 \le t \le 2$.

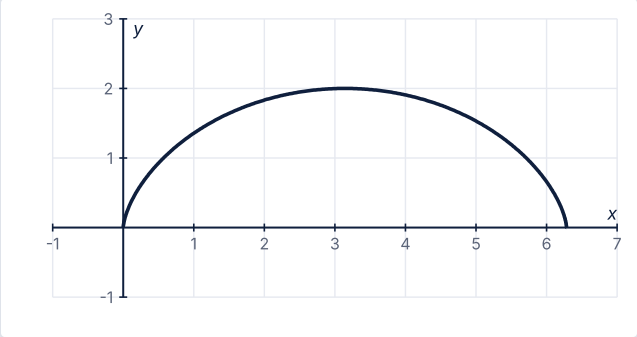
SOLUTION

$\frac{dx}{dt} = 3t^2$, $\frac{dy}{dt} = 3t$. Speed $= \sqrt{9t^4 + 9t^2} = 3t\sqrt{t^2+1}$ (since $t \ge 0$).

$L = \int_0^2 3t\sqrt{t^2+1}\,dt = \left[(t^2+1)^{3/2}\right]_0^2 = 5\sqrt{5} - 1 \approx 10.180.$

EXAMPLE 5

A point moves with $x = t - \sin t$, $y = 1 - \cos t$. Find its speed and the length of one arch, $0 \le t \le 2\pi$.



One arch of x = t - sin t, y = 1 - cos t.

SOLUTION

$\frac{dx}{dt} = 1 - \cos t$, $\frac{dy}{dt} = \sin t$.

Speed $= \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t} = \sqrt{4\sin^2\frac{t}{2}} = 2\sin\frac{t}{2}$ for $0 \le t \le 2\pi$.

$L = \int_0^{2\pi} 2\sin\frac{t}{2}\,dt = \left[-4\cos\frac{t}{2}\right]_0^{2\pi} = 4 + 4 = 8.$

Common mistakes

(1) $\frac{d^2y}{dx^2}$ is **not** $\frac{d^2y/dt^2}{d^2x/dt^2}$. Differentiate $\frac{dy}{dx}$ with respect to t, then divide by $\frac{dx}{dt}$. (2) Do not forget the divisor $\frac{dx}{dt}$ in the second derivative. (3) If both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are 0, you cannot conclude horizontal or vertical. (4) Arc length uses the sum of squares inside one square root, then integrates over t. (5) After eliminating the parameter, check the allowed range of x and y and the direction.

Practice

SET A Skills

Answer each question. [No calculator]

1. Eliminate the parameter: $x = 2t - 1$, $y = t^2 + 3$. Write y as a function of x.
2. Eliminate the parameter: $x = 4\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$. Identify the curve.
3. Eliminate the parameter: $x = e^t$, $y = e^{2t} + 1$. State the domain of x on the resulting curve.
4. For $x = 2\cos t$, $y = 3\sin t$, find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$.
5. For $x = t^2 + 3t$, $y = t^3 - 6t$, find $\frac{dy}{dx}$ at $t = 1$.
6. For $x = t^2 + 2$, $y = t^3 - 12t$, find all points of horizontal tangency and all points of vertical tangency.
7. For $x = 2t$, $y = t^3 - t$, find $\frac{d^2y}{dx^2}$ at $t = 2$.
8. Find the length of the curve $x = t$, $y = \frac{2}{3}t^{3/2}$ for $0 \le t \le 3$.
9. A particle has position $x = t^3$, $y = t^2$. Find its speed at $t = 1$.

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SET B Applications and Mixed

Give decimal answers to three decimal places. [Calculator]

10. A ball is launched so that $x = 4t$ and $y = 20t - 5t^2$ (x, y in meters, t in seconds, $y \geq 0$).
(a) When does the ball land?
(b) Find the maximum height.
(c) At $t = 3$, find $\frac{dy}{dx}$ and the speed (to three decimals).

11. Find the length of the curve $x = t^2$, $y = 2t$ for $0 \leq t \leq 3$ to three decimal places.

12. An ellipse is given by $x = 5\cos t$, $y = 3\sin t$, $0 \leq t \leq 2\pi$.
(a) Eliminate the parameter.
(b) Find $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$ to three decimals.
(c) Write, and evaluate to three decimals, an integral for its perimeter.

13. The table gives information about a particle moving in the plane.
(a) Find $\frac{dy}{dx}$ and the speed at $t = 2$ (three decimals).
(b) Is the particle moving left or right, and up or down, at $t = 2$?
(c) Write an equation of the tangent line at $t = 2$.

t	0	2	4
x	5	-2	-6
y	1	9	26
dx/dt	-4	-3	-1
dy/dt	2	6	11

14. A point on a rolling wheel has position $x = 2(t - \sin t)$, $y = 2(1 - \cos t)$.
(a) Find $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$.
(b) Find the speed at $t = \frac{\pi}{3}$.
(c) Find the distance traveled during one full arch, $0 \leq t \leq 2\pi$.

15. For $x = t^2 + t$, $y = e^t$, find $\dfrac{d^2y}{dx^2}$ at $t = 1$ to three decimals and state whether the curve is concave up or down there.

16. A rider on a Ferris wheel has position $x = 8\cos(0.4t)$, $y = 10 + 8\sin(0.4t)$ meters, t in seconds.
(a) Find the speed of the rider.
(b) How long does one revolution take (three decimals)?
(c) How far does the rider travel in 20 seconds?

SET C Challenge

Justify your answers. [No calculator]

17. Eliminate the parameter: $x = \sin t$, $y = \cos 2t$, $0 \leq t \leq 2\pi$. Describe the curve and the motion.

18. The curve $x = t^2 - 1$, $y = t^3 - t$ crosses itself at one point. Find that point and write equations of the two tangent lines there.

19. Find the exact length of $x = t^2$, $y = \frac{t^3}{3} - t$ for $0 \leq t \leq 3$. (Hint: the speed simplifies to a polynomial.)

20. For $x = \ln t$, $y = t^2$ ($t > 0$), find $\dfrac{d^2y}{dx^2}$ and describe the concavity of the curve.

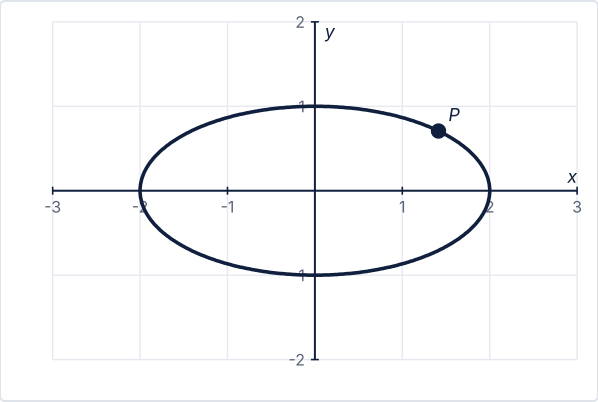
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21. A particle moves on $x = 2\cos t$, $y = \sin t$. The graph is shown below.

(a) Find the value of t in $(0, \frac{\pi}{2})$ at which the particle is at $P\left(\sqrt{2}, \frac{\sqrt{2}}{2}\right)$.

(b) Find the speed at P and $\frac{dy}{dx}$ at P.



The path of the particle.

7.9 Vector-Valued Functions and Motion in the Plane

Describe motion in the plane with vector-valued functions: find velocity, acceleration, speed, position, displacement and distance traveled, and decide the direction of motion.

Motion in the plane is described by a **vector-valued function** $\vec{r}(t) = \langle x(t), y(t)\rangle$ that gives position at time t. Calculus acts on each component, so everything you know about particle motion on a line carries over.

Velocity, Acceleration and Speed

For $\vec{r}(t) = \langle x(t), y(t)\rangle$ with differentiable components:

$$\vec{v}(t) = \vec{r}\,'(t) = \langle x'(t),\, y'(t)\rangle, \qquad \vec{a}(t) = \vec{v}\,'(t) = \langle x''(t),\, y''(t)\rangle.$$

The **speed** is the magnitude of velocity: $|\vec{v}(t)| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}$. Speed is a number (never negative); velocity is a vector and points along the path in the direction of motion.

The slope of the path is $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$ when $x'(t) \neq 0$. The particle is at rest only when **both** $x'(t) = 0$ and $y'(t) = 0$.

Position, Displacement and Distance

If $\vec{v}(t)$ is continuous, integrate each component:

$$\vec{r}(b) = \vec{r}(a) + \int_a^b \vec{v}(t)\,dt = \left\langle x(a) + \int_a^b x'(t)\,dt,\; y(a) + \int_a^b y'(t)\,dt \right\rangle.$$

The integral $\int_a^b \vec{v}\,dt$ is the **displacement** vector $\vec{r}(b) - \vec{r}(a)$. The **total distance traveled** is the integral of speed:

$$\int_a^b |\vec{v}(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

Displacement can be $\vec{0}$ after a round trip, but distance traveled is never negative.

EXAMPLE 1

A particle has position $\vec{r}(t) = \langle t^3 - 3t,\; t^2\rangle$. Find its velocity, acceleration and speed at $t = 2$.

SOLUTION

$\vec{v}(t) = \langle 3t^2 - 3,\; 2t\rangle$ and $\vec{a}(t) = \langle 6t,\; 2\rangle$.

At $t = 2$: $\vec{v}(2) = \langle 9, 4\rangle$, $\vec{a}(2) = \langle 12, 2\rangle$.

Speed $= \sqrt{81 + 16} = \sqrt{97} \approx 9.849$.

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EXAMPLE 2

A particle has acceleration $\vec{a}(t) = \langle 6t,\ -2\rangle$, velocity $\vec{v}(0) = \langle 1, 3\rangle$ and position $\vec{r}(0) = \langle 0, 5\rangle$. Find $\vec{r}(t)$ and the position at $t = 2$.

SOLUTION

Integrate acceleration: $\vec{v}(t) = \langle 3t^2 + C_1,\ -2t + C_2\rangle$. Using $\vec{v}(0) = \langle 1, 3\rangle$: $\vec{v}(t) = \langle 3t^2 + 1,\ -2t + 3\rangle$.

Integrate velocity: $\vec{r}(t) = \langle t^3 + t + C_3,\ -t^2 + 3t + C_4\rangle$. Using $\vec{r}(0) = \langle 0, 5\rangle$: $\vec{r}(t) = \langle t^3 + t,\ -t^2 + 3t + 5\rangle$.

$\vec{r}(2) = \langle 10,\ 7\rangle$.

EXAMPLE 3

A particle has velocity $\vec{v}(t) = \langle 4t,\ 2t^2 - 2\rangle$ for $0 \le t \le 3$. Find the displacement and total distance traveled.

SOLUTION

Displacement $= \int_0^3 \langle 4t, 2t^2 - 2\rangle dt = \left\langle 2t^2,\ \frac{2t^3}{3} - 2t\right\rangle \Big|_0^3 = \langle 18,\ 12\rangle$.

Speed: $\sqrt{16t^2 + 4(t^2 - 1)^2} = 2\sqrt{(t^2 + 1)^2} = 2(t^2 + 1)$.

Distance $= \int_0^3 2(t^2 + 1)\, dt = 2(9 + 3) = 24$. The straight-line distance from start to finish is only $\sqrt{324 + 144} = 6\sqrt{13} \approx 21.633$, which is less than the distance traveled.

EXAMPLE 4

For $0 \le t \le 5$, a particle has $x'(t) = t^2 - 4$ and $y'(t) = 2 - t$. Describe its direction of motion at $t = 1$ and $t = 3$, and find when it is at rest.

SOLUTION

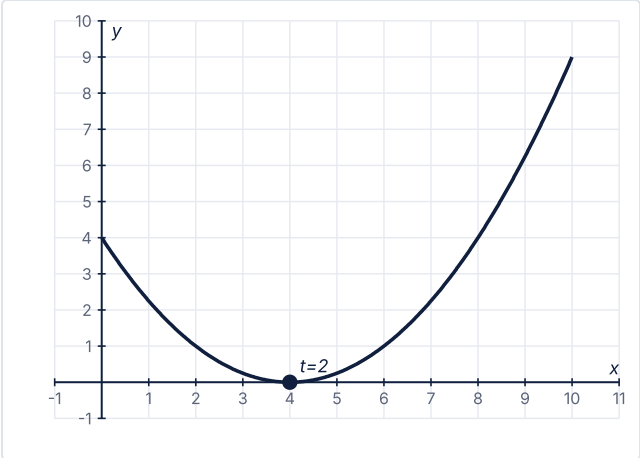
At $t = 1$: $x' = -3 < 0$ (moving left) and $y' = 1 > 0$ (moving up).

At $t = 3$: $x' = 5 > 0$ (moving right) and $y' = -1 < 0$ (moving down).

At rest requires $x' = 0$ and $y' = 0$. $x' = 0$ for $t = 2$ (in the interval) and $y' = 0$ for $t = 2$. So the particle is at rest at $t = 2$ only.

EXAMPLE 5

A particle has position $\vec{r}(t) = \langle 2t,\ t^2 - 4t + 4\rangle$. When is its speed increasing? Justify.



The path r(t) = <2t, t^2 - 4t + 4> for 0 <= t <= 5.

SOLUTION

$\vec{v}(t) = \langle 2,\ 2t - 4\rangle$ and $\vec{a}(t) = \langle 0,\ 2\rangle$.

Speed increases when $\vec{v} \cdot \vec{a} > 0$ (velocity and acceleration make an acute angle): $\vec{v} \cdot \vec{a} = 2(2t - 4) = 4t - 8$.

$4t - 8 > 0$ for $t > 2$, so the speed is increasing for $t > 2$ and decreasing for $0 \le t < 2$. The minimum speed is $|\vec{v}(2)| = |\langle 2, 0\rangle| = 2$, at the vertex of the parabola.

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Common mistakes

(1) Confusing displacement and distance: distance integrates **speed**, not velocity. (2) Reporting velocity when the question asks for speed (or the reverse). (3) Saying a particle is at rest when only one component of velocity is 0. (4) Forgetting the initial position when integrating velocity to get position. (5) Writing speed as $x' + y'$ instead of $\sqrt{x'^2 + y'^2}$.

Practice

SET A Skills

Answer each question. [No calculator]

1. A particle has position $\vec{r}(t) = \langle t^2 + 1,\ 3t^3 \rangle$. Find $\vec{v}(t)$ and $\vec{a}(t)$.
2. A particle has position $\vec{r}(t) = \langle \cos 2t,\ \sin 2t \rangle$. Find $\vec{v}(t)$ and show that its speed is constant.
3. A particle has position $\vec{r}(t) = \langle e^{2t},\ 3t^2 - t \rangle$. Find $\vec{v}(0)$ and $\vec{a}(0)$.
4. A particle has position $\vec{r}(t) = \langle 3t^2,\ 4t^2 \rangle$. Find its velocity and speed at $t = 1$.
5. A particle has $\vec{a}(t) = \langle 2,\ 6t \rangle$, $\vec{v}(0) = \langle 0, 1 \rangle$, $\vec{r}(0) = \langle 1, 0 \rangle$. Find $\vec{r}(t)$.
6. A particle has velocity $\vec{v}(t) = \langle 2t,\ 3 \rangle$ and position $\vec{r}(0) = \langle 1, 1 \rangle$. Find its position at $t = 4$.
7. A particle has velocity $\vec{v}(t) = \langle 6t,\ 8t \rangle$. Find the total distance traveled from $t = 0$ to $t = 3$.
8. For $t \geq 0$, a particle has $x'(t) = t - 3$ and $y'(t) = t^2 - 1$. Find the values of t for which the particle moves to the left and downward.
9. At $t = 1$, a particle has $\vec{v}(1) = \langle -2, 3 \rangle$ and $\vec{a}(1) = \langle 4, 1 \rangle$. Is its speed increasing or decreasing at $t = 1$? Justify.

SET B Applications and Mixed

Give decimal answers to three decimal places. [Calculator]

10. A particle has position $\vec{r}(t) = \langle t^2,\ \sin t \rangle$. Find its velocity, acceleration and speed at $t = 2$ (three decimals).
11. A particle has velocity $\vec{v}(t) = \langle t,\ e^{t/2} \rangle$. Find the total distance traveled from $t = 0$ to $t = 3$ (three decimals).
12. A particle has velocity $\vec{v}(t) = \langle 2\cos t,\ 3\sin t \rangle$ and position $\vec{r}(0) = \langle 1, 0 \rangle$.
    (a) Find $\vec{r}(t)$.
    (b) Find its position and speed at $t = 2$ (three decimals).
13. The table gives $x'(t)$ and $y'(t)$ for a particle with $\vec{r}(0) = \langle 2, 1 \rangle$. Use trapezoidal sums with four subintervals to estimate the particle's position at $t = 4$.

t	0	1	2	3	4
$x'(t)$	3	2	0	-2	-1
$y'(t)$	1	3	4	2	0

14. A projectile has position $\vec{r}(t) = \langle 12t,\ 15t - 4.9t^2 \rangle$ meters, t in seconds.
    (a) Find $\vec{v}(1)$.
    (b) When does it reach its greatest height (three decimals)?
    (c) Find its speed on landing (three decimals).
15. A particle moves with $\vec{r}(t) = \langle 5\cos 2t,\ 5\sin 2t \rangle$.
    (a) Find its speed.
    (b) Find $|\vec{a}(t)|$.
    (c) Show that $\vec{v} \cdot \vec{a} = 0$ and explain what this means about the speed.
16. A particle has $\vec{a}(t) = \langle 2t,\ -1 \rangle$, $\vec{v}(0) = \langle 1, 2 \rangle$ and $\vec{r}(0) = \langle 0, 0 \rangle$.
    (a) Find $\vec{v}(t)$ and $\vec{r}(t)$.
    (b) Is the particle ever at rest?
    (c) Find its speed at $t = 3$ (three decimals).

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SET C Challenge

Justify your answers. [No calculator]

17. A particle starts at the origin with velocity $\vec{v}(t) = \langle 3t^2 - 6t,\ 2t - 2\rangle$.
    (a) Find $\vec{r}(t)$.
    (b) Does the particle ever return to the origin for $t > 0$? Justify.
18. For $\vec{r}(t) = \langle 3\cos t,\ 3\sin t\rangle$, show that the speed is constant even though the acceleration is never $\vec{0}$. Find $|\vec{a}|$.
19. A particle has velocity $\vec{v}(t) = \langle 4t,\ 4t^2 - 1\rangle$ for $0 \le t \le 2$.
    (a) Show that the speed is $4t^2 + 1$.
    (b) Find the total distance traveled.
    (c) Find the displacement vector.
20. A particle has position $x(t) = t^3 - 3t$, $y(t) = t^2 - 2t$ for $t \ge 0$.
    (a) Is the particle at rest at any time?
    (b) When does it move to the right and upward? To the left and downward?
21. A particle has velocity $\vec{v}(t) = \langle 2t - 4,\ t^2 - 4t + 3\rangle$ for $t \ge 0$. Show that the speed is increasing exactly when $t > 2$, and find the minimum speed.

7.10 Polar Coordinates and Graphs

You will learn to locate points with a distance and an angle, convert between polar and rectangular forms, and recognize circles, limacons and roses. Polar graphs are the setting for the calculus in the next section.

A point in the plane can be located by its distance from a fixed point and a direction, instead of by horizontal and vertical distances. The fixed point O is the **pole** and the ray starting at O pointing to the right is the **polar axis**. A point P has polar coordinates (r, θ), where θ is the angle (in radians, measured counterclockwise from the polar axis) to the ray through P, and r is the directed distance from the pole along that ray. If $r < 0$, the point is plotted $|r|$ units in the direction opposite to θ.

Converting between polar and rectangular coordinates

Polar to rectangular:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Rectangular to polar:

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}\ (x \neq 0).$$

For θ, choose the quadrant that matches the signs of x and y. The equation $\tan\theta = y/x$ alone does not tell you the quadrant.

Unlike rectangular coordinates, a point has infinitely many polar names. The point (r, θ) is also named $(r, \theta + 2\pi k)$ and $(-r, \theta + \pi + 2\pi k)$ for every integer k. The pole itself is $(0, \theta)$ for any θ. This is why two polar curves can cross at a point without the two equations sharing a common solution (r, θ).

EXAMPLE 1

(a) Convert $(6, \frac{2\pi}{3})$ to rectangular coordinates. (b) Convert $(-3,\ 3)$ to polar coordinates with $r > 0$ and $0 \le \theta < 2\pi$.

SOLUTION

(a) $x = 6\cos\frac{2\pi}{3} = 6(-\frac{1}{2}) = -3$ and $y = 6\sin\frac{2\pi}{3} = 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$. The point is $(-3, 3\sqrt{3})$.

(b) $r = \sqrt{(-3)^2 + 3^2} = 3\sqrt{2}$. The point is in Quadrant II and $\tan\theta = -1$, so $\theta = \frac{3\pi}{4}$. The point is $(3\sqrt{2}, \frac{3\pi}{4})$.

To convert an equation, substitute $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$, and use algebra. A common trick is to multiply both sides of a polar equation by r so that $r\cos\theta$, $r\sin\theta$ and r^2 appear.

EXAMPLE 2

Convert $r = 4\sin\theta$ to rectangular form and identify the graph. Then convert $y = x + 2$ to polar form.

SOLUTION

Multiply by r: $r^2 = 4r\sin\theta$, so $x^2 + y^2 = 4y$.

Complete the square: $x^2 + (y - 2)^2 = 4$. This is a circle with center $(0, 2)$ and radius 2.

For the line, substitute: $r\sin\theta = r\cos\theta + 2$, so $r(\sin\theta - \cos\theta) = 2$ and $r = \dfrac{2}{\sin\theta - \cos\theta}$.

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Basic polar curves (with $a > 0$)

Circles: $r = a\cos\theta$ and $r = a\sin\theta$ have diameter a and pass through the pole. $r = a$ is a circle of radius a centered at the pole.

Limacons: $r = b \pm a\cos\theta$ or $r = b \pm a\sin\theta$. If $b = a$ it is a **cardioid** (a cusp at the pole). If $b < a$ it has an **inner loop**. If $a < b < 2a$ it is **dimpled**, and if $b \geq 2a$ it is convex.

Roses: $r = a\cos n\theta$ or $r = a\sin n\theta$ has petals of length a. It has n petals if n is odd and $2n$ petals if n is even.

Symmetry tests

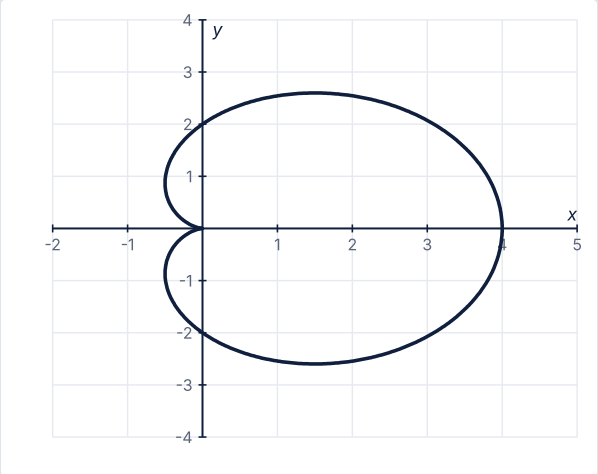
The graph of $r = f(\theta)$ is symmetric about the polar axis (the x-axis) if replacing θ by $-\theta$ gives the same equation. Cosine curves such as $r = b + a\cos\theta$ have this symmetry.

It is symmetric about the line $\theta = \frac{\pi}{2}$ (the y-axis) if replacing θ by $\pi - \theta$ gives the same equation. Sine curves such as $r = b + a\sin\theta$ have this symmetry.

It is symmetric about the pole if replacing r by $-r$ gives the same equation. These tests are sufficient but not necessary, because of the many names for one point.

EXAMPLE 3

Sketch $r = 2 + 2\cos\theta$ and state its symmetry and the maximum value of r.



Graph of $r = 2 + 2\cos\theta$.

SOLUTION

The form $b + a\cos\theta$ with $b = a = 2$ is a cardioid. Since $\cos(-\theta) = \cos\theta$, it is symmetric about the polar axis.

Plot a few points: $\theta = 0$ gives $r = 4$; $\theta = \frac{\pi}{2}$ gives $r = 2$; $\theta = \pi$ gives $r = 0$ (the cusp at the pole). Reflect across the x-axis for $\pi < \theta < 2\pi$.

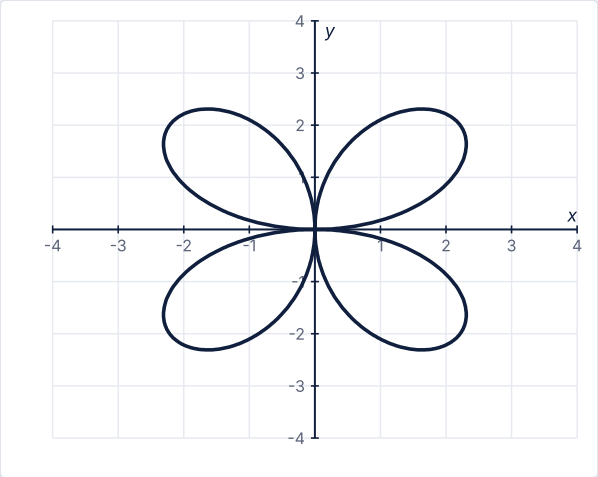
The maximum is $r = 4$ at $\theta = 0$.

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EXAMPLE 4

Describe the graph of $r = 3\sin 2\theta$: the number of petals, their length, and the directions of the petal tips.



Graph of $r = 3\sin 2\theta$.

SOLUTION

Since $n = 2$ is even there are $2n = 4$ petals, each of length 3.

Tips occur where $|\sin 2\theta| = 1$, that is $2\theta = \frac{\pi}{2} + k\pi$, so $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

At $\theta = \frac{3\pi}{4}$ we get $r = 3\sin\frac{3\pi}{2} = -3$, so that tip is plotted in the direction $\frac{7\pi}{4}$. The four tips lie along the lines $y = x$ and $y = -x$, one in each quadrant.

EXAMPLE 5

Find all intersection points of $r = 1$ and $r = 2\sin\theta$.

SOLUTION

Set the radii equal: $2\sin\theta = 1$, so $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, both with $r = 1$.

The points are $(1, \frac{\pi}{6})$ and $(1, \frac{5\pi}{6})$. The circle $r = 2\sin\theta$ passes through the pole but $r = 1$ does not, so there are no other intersections.

Check in rectangular form: $x^2 + y^2 = 1$ and $x^2 + (y-1)^2 = 1$ give $y = \frac{1}{2}$, $x = \pm\frac{\sqrt{3}}{2}$.

Common mistakes

(1) Using $\theta = \arctan(y/x)$ without checking the quadrant. (2) Forgetting that (r, θ) and $(-r, \theta + \pi)$ name the same point, so a curve can pass through a point at a different θ. (3) Finding intersections only by setting the two r expressions equal, which can miss the pole and points where one curve reaches the point with a different name. (4) Counting petals incorrectly: $r = \cos 3\theta$ has 3 petals, not 6.

Practice

SET A **Skills**

Answer each question. [No calculator]

1. Convert the polar point $(4, \frac{\pi}{3})$ to rectangular coordinates.
2. Convert the rectangular point $(-2, 2)$ to polar coordinates (r, θ) with $r > 0$ and $0 \le \theta < 2\pi$.
3. Convert $r = 6\cos\theta$ to rectangular form and identify the graph.
4. Write the line $x = 5$ in polar form $r = f(\theta)$.
5. Convert $r^2 = 9\sin 2\theta$ to rectangular form.
6. Give another polar name for $(5, \frac{\pi}{4})$ that uses a negative value of r and $0 \le \theta < 2\pi$.
7. Identify the curve $r = 3 + 3\cos\theta$, state its symmetry, and give its maximum value of r.

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8. For the rose $r = 4\sin 3\theta$, state the number of petals, the length of each petal, and the first positive θ at which r reaches a maximum.
9. Does the limacon $r = 2 + 3\cos\theta$ have an inner loop? Explain using the constants in the equation.

SET B Applications and Mixed Practice

Give decimal answers rounded to three places. Angles are in radians unless degrees are shown. [Calculator]

10. Convert the polar point $(7.5,\ 2.4)$ (with θ in radians) to rectangular coordinates.
11. Convert $(-3.2,\ -4.7)$ to polar coordinates with $r > 0$ and $0 \le \theta < 2\pi$.
12. A radar display locates an aircraft 9 km from the station in a direction 40° counterclockwise from due east. Find the eastward and northward components of its position.
13. For $r = 1 + 2\sin\theta$, find r at $\theta = 0.8$, $\theta = 2.5$ and $\theta = 4.1$. At which of these is the point plotted on the opposite side of the pole from the ray θ?
14. Find the distance between the polar points $(6.2,\ 0.7)$ and $(4.5,\ 2.9)$ (angles in radians).
15. For the rose $r = 5\cos 3\theta$, find r when $\theta = 0.35$ and the rectangular coordinates of that point.
16. For the limacon $r = 2 + 3\cos\theta$, find the two values of θ in $[0, 2\pi]$ at which the curve passes through the pole. Between them the curve traces its inner loop.

SET C Challenge and AP-Style

Give exact answers and justify. [No calculator]

17. (a) How many petals do $r = 3\sin 4\theta$ and $r = 3\cos 5\theta$ have? (b) Explain why the two counts follow different rules.
18. For $r = 1 + 2\cos\theta$: (a) find all θ in $[0, 2\pi]$ where $r = 0$; (b) for which θ is the inner loop traced; (c) find the rectangular coordinates of the point of the inner loop at $\theta = \pi$.
19. Show that the graph of $r^2 = 4\cos 2\theta$ is symmetric about the polar axis, the line $\theta = \frac{\pi}{2}$, and the pole.
20. Convert $r = \dfrac{2}{1 - \cos\theta}$ to rectangular form and identify the curve.
21. Find all intersection points of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$, including any at the pole.

7.11 Calculus in Polar Coordinates

You will find slopes of tangent lines to polar curves, tangent lines at the pole, areas of regions bounded by polar curves, and areas between two polar curves.

A polar curve $r = f(\theta)$ is also a parametric curve: $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$, with parameter θ. That observation gives the slope formula. Areas are different: because θ sweeps out thin wedges (sectors) instead of vertical strips, the area integrals use r^2.

Slope of a polar curve

If $r = f(\theta)$ is differentiable, then

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta},$$

provided $dx/d\theta \neq 0$.

The tangent is horizontal where $dy/d\theta = 0$ and $dx/d\theta \neq 0$. It is vertical where $dx/d\theta = 0$ and $dy/d\theta \neq 0$.

EXAMPLE 1

Find the slope of the tangent line to $r = 1 + 2\cos\theta$ at $\theta = \frac{\pi}{3}$.

SOLUTION

$r\left(\frac{\pi}{3}\right) = 1 + 2\cdot\frac{1}{2} = 2$ and $r' = -2\sin\theta$, so $r'\left(\frac{\pi}{3}\right) = -\sqrt{3}$.

Numerator: $r'\sin\theta + r\cos\theta = -\sqrt{3}\cdot\frac{\sqrt{3}}{2} + 2\cdot\frac{1}{2} = -\frac{3}{2} + 1 = -\frac{1}{2}$.

Denominator: $r'\cos\theta - r\sin\theta = -\sqrt{3}\cdot\frac{1}{2} - 2\cdot\frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2}$.

$$\frac{dy}{dx} = \frac{-1/2}{-3\sqrt{3}/2} = \frac{1}{3\sqrt{3}} = \frac{\sqrt{3}}{9}.$$

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Tangent lines at the pole

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the curve $r = f(\theta)$ passes through the pole at $\theta = \alpha$ and its tangent line there is the line $\theta = \alpha$ (the line through the pole making angle α with the polar axis).

Reason: at the pole the slope formula reduces to $\frac{f' \sin\alpha}{f' \cos\alpha} = \tan\alpha$.

EXAMPLE 2

Find the tangent lines at the pole to $r = 2\cos 3\theta$ for $0 \leq \theta < \pi$.

SOLUTION

Solve $r = 0$: $\cos 3\theta = 0$, so $3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ and $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.

$r' = -6\sin 3\theta = \mp 6 \neq 0$ at each of these, so each gives a tangent line.

The lines are $\theta = \frac{\pi}{6}$ (slope $\frac{\sqrt{3}}{3}$), $\theta = \frac{\pi}{2}$ (the y-axis) and $\theta = \frac{5\pi}{6}$ (slope $-\frac{\sqrt{3}}{3}$). These are the lines along which the three petals meet at the pole.

Area in polar coordinates

If f is continuous and nonnegative on $[\alpha, \beta]$ with $0 \leq \beta - \alpha \leq 2\pi$, the area of the region swept out by $r = f(\theta)$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\, d\theta.$$

For the region between two curves with $0 \leq g(\theta) \leq f(\theta)$ (outer f, inner g):

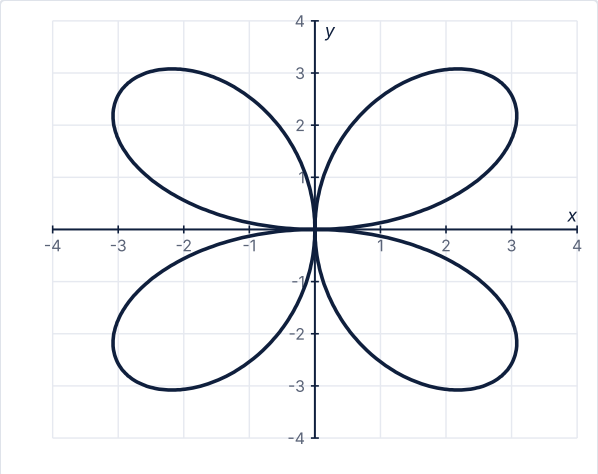
$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left([f(\theta)]^2 - [g(\theta)]^2\right) d\theta.$$

Never subtract first and then square.

Pick limits by finding where the curve starts and stops, usually at the pole ($r = 0$) or at intersections of the two curves.

EXAMPLE 3

Find the area of one petal of $r = 4\sin 2\theta$.



Graph of $r = 4\sin 2\theta$.

SOLUTION

One petal is traced from $r = 0$ at $\theta = 0$ to $r = 0$ at $\theta = \frac{\pi}{2}$, with $r \geq 0$ on that interval.

$A = \frac{1}{2}\int_0^{\pi/2} 16\sin^2 2\theta\, d\theta = 8\int_0^{\pi/2} \frac{1-\cos 4\theta}{2}\, d\theta = 4\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$.

$A = 4 \cdot \frac{\pi}{2} = 2\pi$.

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EXAMPLE 4

Find the area of the region inside $r = 3\cos\theta$ and outside $r = 1 + \cos\theta$.

SOLUTION

Intersections: $3\cos\theta = 1 + \cos\theta$ gives $\cos\theta = \frac{1}{2}$, so $\theta = \pm\frac{\pi}{3}$. On $-\frac{\pi}{3} < \theta < \frac{\pi}{3}$, $3\cos\theta > 1 + \cos\theta > 0$, so the circle is the outer curve.

$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(8\cos^2\theta - 1 - 2\cos\theta\right) d\theta = \int_0^{\pi/3} \left(8\cos^2\theta - 1 - 2\cos\theta\right) d\theta$ by symmetry.

Use $8\cos^2\theta = 4 + 4\cos 2\theta$: the integrand is $3 + 4\cos 2\theta - 2\cos\theta$.

$A = \left[3\theta + 2\sin 2\theta - 2\sin\theta\right]_0^{\pi/3} = \pi + \sqrt{3} - \sqrt{3} = \pi$.

Arc length (BC)

For $r = f(\theta)$ with f' continuous, the length from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Common mistakes

(1) Using $dr/d\theta$ as the slope. It is not: the slope is dy/dx from the quotient above. (2) Using $\frac{1}{2}\int r\, d\theta$ or forgetting the factor $\frac{1}{2}$. (3) Using limits for the whole curve when you want one petal or one loop. (4) In an area between curves, writing $(f - g)^2$ instead of $f^2 - g^2$. (5) Missing the pole as an intersection of two curves, or using an interval on which r is negative without checking what is traced.

Practice

SET A Skills

Give exact answers. [No calculator]

1. Find $\frac{dy}{dx}$ for $r = 4\cos\theta$ at $\theta = \frac{\pi}{6}$.
2. Find the slope of the tangent line to $r = 1 + \cos\theta$ at $\theta = \frac{\pi}{2}$.
3. Find the values of θ in $(0, \pi)$ at which the curve $r = 3\sin\theta$ has a horizontal tangent. (Use $y = r\sin\theta$.)
4. Find the equations of the tangent lines at the pole to $r = 1 + 2\sin\theta$.
5. Find the area enclosed by the circle $r = 3\cos\theta$.
6. Find the area enclosed by the cardioid $r = 1 + \cos\theta$.
7. Find the area of one petal of $r = 2\cos 3\theta$, and then the total area of the three-petal rose.
8. Find the area of the inner loop of $r = 1 + 2\cos\theta$. (The loop is traced for $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$.)
9. Find the area of the region inside $r = 4$ and outside $r = 2$.

SET B Applications and Mixed Practice

Give decimal answers rounded to three places. [Calculator]

10. Find $\frac{dy}{dx}$ for $r = 2 + 3\sin\theta$ at $\theta = 1$.
11. Find the area enclosed by $r = e^{\sin\theta}$ for $0 \le \theta \le 2\pi$.
12. Find the area of the region inside $r = 2 + 2\cos\theta$ and outside $r = 3$.
13. Find the area of the region that lies inside both $r = 3\cos\theta$ and $r = 1 + \cos\theta$.
14. A flower bed has boundary $r = 3 + \sin 3\theta$ (in meters), $0 \le \theta \le 2\pi$, with the pole at its center. Find its area in square meters.
15. Find the length of the spiral $r = \theta$ from $\theta = 0$ to $\theta = 2$.
16. Find the equation of the tangent line (in rectangular form $y = mx + b$) to $r = 2\theta$ at $\theta = \frac{\pi}{3}$.

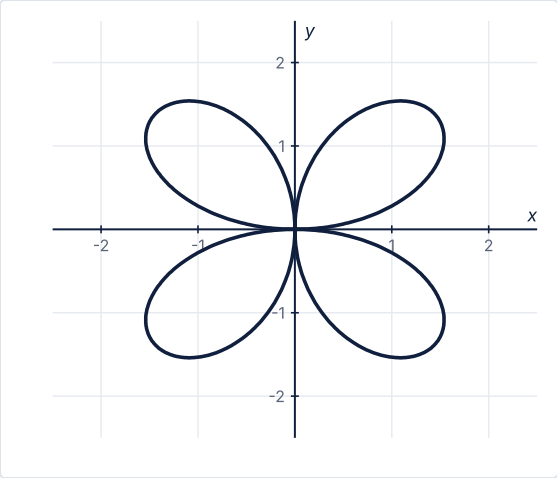
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SET C Challenge and AP-Style

Show justification. [No calculator]

17. Find the area of the region inside $r = 2\cos\theta$ and outside $r = 1$.
18. The graph shows the rose $r = 2\sin 2\theta$. Find the area of the part of the first-quadrant petal that lies outside the circle $r = 1$.



Graph of $r = 2\sin 2\theta$.

19. (a) Find the tangent lines at the pole to $r = 2\cos 2\theta$ for $0 \le \theta < \pi$. (b) Find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{6}$.
20. Find the area of the region inside the cardioid $r = 1 + \sin\theta$ and outside the circle $r = 1$.
21. The area enclosed by the cardioid $r = a(1 + \cos\theta)$, $a > 0$, is 24π. Find a.

7.12 Multiple-Choice Practice on Series, Parametrics, Vectors, and Polar

Thirty multiple-choice problems covering infinite series, Taylor polynomials, parametric curves, vector motion and polar curves.

Work Part A without a calculator and Part B with a calculator. Decimal answers in Part B are rounded to three places. Check each answer against the explanations at the end of the chapter.

SET A Part A [No calculator]

Choose the best answer. [No calculator]

1. What is the sum of the series $\sum_{n=0}^{\infty} 3\left(-\frac{2}{5}\right)^n$?
   (A) 5 (B) $\frac{21}{5}$ (C) $\frac{6}{7}$ (D) 3 (E) $\frac{15}{7}$
2. Which of the following series converges?
   (A) $\sum_{n=1}^{\infty} \frac{n^3}{n+1}$ (B) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (C) $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$
   (D) $\sum_{n=1}^{\infty} \frac{1}{3n-1}$ (E) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$
3. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is
   (A) conditionally convergent
   (B) absolutely convergent
   (C) divergent because the terms do not approach 0
   (D) divergent because it is a p -series with $p = \frac{1}{2}$
   (E) a convergent geometric series
4. What is the interval of convergence of $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n\,3^n}$?
   (A) $-1 \le x < 5$ (B) $-1 < x < 5$ (C) $-1 \le x \le 5$ (D) $-1 < x \le 5$ (E) $-3 \le x < 3$
5. If $f(x) = \sin(x^2)$, what is $f^{(6)}(0)$?
   (A) $-\frac{1}{6}$ (B) -120 (C) 120 (D) -720 (E) 0

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6. The table gives values of a function f and its derivatives at $x = 2$. Use the second-degree Taylor polynomial for f about $x = 2$ to approximate $f(2.1)$.

x	$f(2)$	$f'(2)$	$f''(2)$	$f'''(2)$
value	3	-1	4	6

(A) 2.94 **(B)** 2.90 **(C)** 2.88 **(D)** 2.92 **(E)** 3.12

7. The function $\cos x$ is approximated at $x = \frac{1}{2}$ by its second-degree Maclaurin polynomial. Using the third derivative of $\cos x$, the Lagrange error bound for this approximation is

(A) $\frac{1}{24}$ **(B)** $\frac{1}{16}$ **(C)** $\frac{1}{12}$ **(D)** $\frac{1}{48}$ **(E)** $\frac{1}{8}$

8. A curve is given by $x = t^2 + 1$, $y = t^3 - 3t$. What is $\frac{dy}{dx}$ at $t = 2$?

(A) $\frac{3}{4}$ **(B)** $\frac{9}{4}$ **(C)** $\frac{4}{9}$ **(D)** 9 **(E)** $\frac{9}{2}$

9. For the curve $x = t^2$, $y = t^3$ with $t > 0$, what is $\frac{d^2y}{dx^2}$ at $t = 2$?

(A) 3 **(B)** $\frac{3}{8}$ **(C)** $\frac{3}{2}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{3}{16}$

10. What is the length of the curve $x = 3t^2$, $y = 2t^3$ for $0 \le t \le 1$?

(A) $2\sqrt{2} - 1$ **(B)** $4\sqrt{2}$ **(C)** $\frac{2}{3}\left(2\sqrt{2} - 1\right)$ **(D)** 6 **(E)** $4\sqrt{2} - 2$

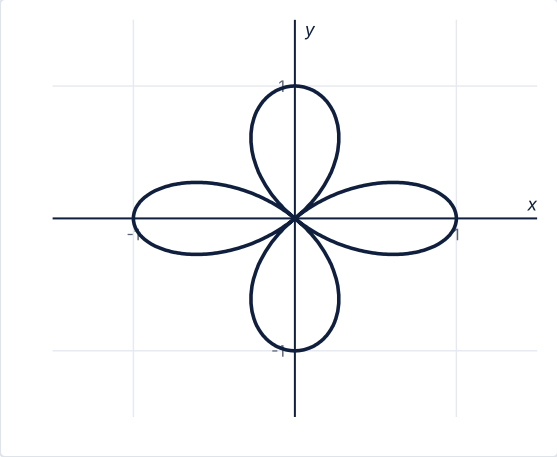
11. A particle has position $\langle 2t^3, 8t \rangle$ for $t \ge 0$. What is its speed at $t = 1$?

(A) 10 **(B)** 14 **(C)** $\sqrt{68}$ **(D)** 6 **(E)** 8

12. A particle moves in the plane with velocity $\langle 2t, 3t^2 \rangle$ and position $\langle 1, -1 \rangle$ at $t = 0$. What is its position at $t = 2$?

(A) $\langle 4, 8 \rangle$ **(B)** $\langle 5, 8 \rangle$ **(C)** $\langle 5, 7 \rangle$ **(D)** $\langle 4, 7 \rangle$ **(E)** $\langle 6, 6 \rangle$

13. The graph of $r = \cos 2\theta$ is shown. What is the area of the petal that lies along the positive x-axis?



Graph of $r = \cos 2\theta$.

(A) $\frac{\pi}{4}$ **(B)** $\frac{\pi}{2}$ **(C)** $\frac{\pi}{16}$ **(D)** $\frac{\pi}{8}$ **(E)** $\frac{1}{8}$

14. Which integral gives the area of the region inside the circle $r = 2\cos\theta$ and outside the circle $r = 1$?

(A) $\frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta - 1)^2 \, d\theta$ **(B)** $\frac{1}{2}\int_{0}^{\pi/3} \left(4\cos^2\theta - 1\right) d\theta$ **(C)** $\frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(4\cos^2\theta - 1\right) d\theta$ **(D)** $\frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(4\cos^2\theta - 1\right) d\theta$ **(E)** $\int_{-\pi/3}^{\pi/3} (2\cos\theta - 1) \, d\theta$

15. Which line is tangent to the curve $r = 2\sin\theta - 1$ at the pole?

(A) $\theta = \frac{\pi}{3}$ **(B)** $\theta = \frac{\pi}{2}$ **(C)** $\theta = \frac{\pi}{6}$ **(D)** $\theta = \frac{2\pi}{3}$ **(E)** $\theta = \frac{\pi}{4}$

16. What is the value of $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^{2n+1}}$?

(A) $\ln\frac{4}{3}$ **(B)** $\arctan\frac{1}{3}$ **(C)** $\frac{3}{10}$ **(D)** $\frac{1}{3}$ **(E)** $\tan\frac{1}{3}$

17. For which values of p does $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converge?

(A) $p > 0$ **(B)** $p \ge 1$ **(C)** $p < 1$ **(D)** $p > 1$ **(E)** $p > 2$

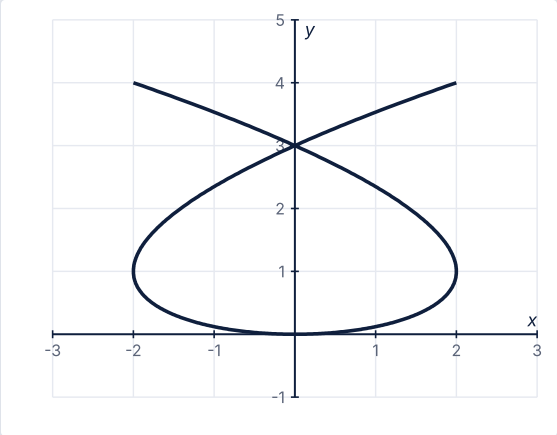
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18. What is the sum of the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$?

(A) 1 (B) $\frac{3}{2}$ (C) 4 (D) The series diverges. (E) 2

19. The curve $x = t^3 - 3t$, $y = t^2$ is shown for $-2 \le t \le 2$. At which values of t is the tangent line vertical?



The curve $x = t^3 - 3t$, $y = t^2$.

(A) $t = 0$ (B) $t = \pm\sqrt{3}$ (C) $t = \pm 1$ (D) $t = 1$ only (E) $t = 0, \pm 1$

20. What is the area of the region enclosed by the spiral $r = \theta$ for $0 \le \theta \le \pi$ and the polar axis?

(A) $\frac{\pi^3}{3}$ (B) $\frac{\pi^2}{2}$ (C) $\frac{\pi^2}{4}$ (D) $\frac{\pi^3}{12}$ (E) $\frac{\pi^3}{6}$

SET B Part B [Calculator]

Choose the best answer. Decimal answers are rounded to three places. [Calculator]

21. The alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ is approximated by its fourth partial sum S_4. The alternating series error bound for $|S - S_4|$ is

(A) 0.016 (B) 0.037 (C) 0.005 (D) 0.008 (E) 0.200

22. What is the length of the curve $x = 2t$, $y = t^2$ for $0 \le t \le 2$?

(A) 5.916 (B) 2.958 (C) 4.472 (D) 8.000 (E) 6.325

23. The Taylor polynomial of degree 3 for $\ln x$ about $x = 1$ is used to approximate $\ln(1.2)$. The Lagrange error bound for this approximation is

(A) 0.0016 (B) 0.0096 (C) 0.0004 (D) 0.0040 (E) 0.0003

24. A particle moves in the plane with velocity $\langle 3\sin t, 2t\rangle$. Its speed at $t = 2$ is

(A) 2.728 (B) 4.842 (C) 4.000 (D) 6.728 (E) 23.441

25. A particle moves in the plane with velocity $\langle e^t, 2t\rangle$. The total distance it travels from $t = 0$ to $t = 2$ is

(A) 7.538 (B) 10.389 (C) 6.389 (D) 4.000 (E) 7.566

26. What is the area of the region bounded by the polar curve $r = e^{\theta/3}$, the polar axis, and the ray $\theta = \pi$?

(A) 10.681 (B) 7.121 (C) 2.670 (D) 5.340 (E) 3.560

27. Let P_4 be the fourth-degree Maclaurin polynomial for $\cos x$. What is $P_4(0.8)$?

(A) 0.697 (B) 0.680 (C) 0.663 (D) 0.360 (E) 1.337

28. A particle moving in the plane has velocity $\langle 2\cos t, e^{-t}\rangle$ and position $\langle 1, 2\rangle$ at $t = 0$. At $t = 3$, its distance from the origin is

(A) 1.282 (B) 2.950 (C) 3.217 (D) 4.232 (E) 3.000

29. A curve is given by $x = t^3 + t$, $y = e^{-t}$. What is $\frac{dy}{dx}$ at $t = 1$?

(A) -0.368 (B) -0.092 (C) -0.184 (D) 0.092 (E) -0.046

30. What is the slope of the tangent line to the polar curve $r = \theta$ at $\theta = 2$?

(A) 0.034 (B) -29.021 (C) 0.077 (D) 0.500 (E) -0.034

7.13 Free-Response Practice on Series, Parametrics, Vectors, and Polar

Four multi-part free-response questions in the style of the exam: a series question, a parametric motion question, a polar question and a Taylor error question.

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Each question is worth 9 points. Show your setup, state theorems and conditions you use, and give units where they apply. Full solutions with scoring are in the chapter answers.

QUESTION 1 Series

No calculator is allowed on this question. Show all work. [No calculator]

1. Let f be the function given by the series

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n\,3^n}$$

on the interval where it converges.

(a) Find the interval of convergence of the series. Show the work that leads to your answer, including the test of each endpoint.

(b) Write the first four nonzero terms and the general term of the series for $f'(x)$.

(c) Find the sum of the series for $f'(x)$ as an expression in closed form, for $|x-1| < 3$.

(d) The series for $f(2)$ is an alternating series. Use the first three terms to approximate $f(2)$, and use the alternating series error bound to show that the approximation differs from $f(2)$ by less than $\frac{1}{300}$.

QUESTION 2 Parametric Motion

A calculator is allowed. Give decimal answers to three places. Show all work. [Calculator]

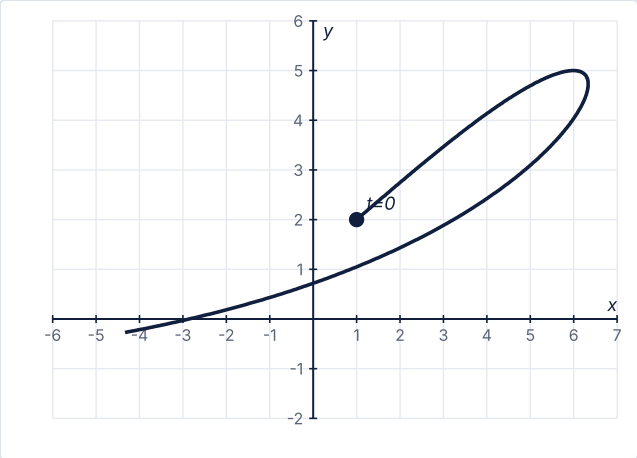
2. A small robot moves in a horizontal plane. For $0 \le t \le 4$ seconds its velocity is $\langle x'(t), y'(t) \rangle = \langle 4 - t^2,\ 3\cos t \rangle$ in meters per second. At $t = 0$ the robot is at $(1, 2)$. The path is shown.

(a) Find the acceleration vector and the speed of the robot at $t = 1.5$.

(b) Is the robot ever at rest for $0 \le t \le 4$? Justify your answer.

(c) Find the position of the robot at $t = 3$.

(d) Find the total distance the robot travels from $t = 0$ to $t = 3$.



Path of the robot for 0 <= t <= 4.

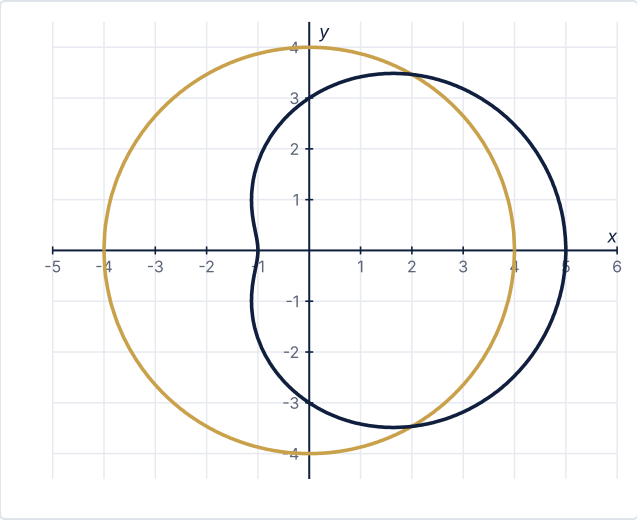
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QUESTION 3 **Polar Curves**

A calculator is allowed. Give decimal answers to three places. Show all work. [Calculator]

3. The graph shows the limacon $r = 3 + 2\cos\theta$ and the circle $r = 4$. The curves intersect at the points where $\theta = \pm\frac{\pi}{3}$. Let R be the region inside the limacon and outside the circle.
   (a) Write an expression involving one or more integrals for the area of R, and find the area.
   (b) Find the slope of the line tangent to the limacon at $\theta = \frac{\pi}{3}$.
   (c) For $0 \le \theta \le \pi$, find the greatest y-coordinate of a point on the limacon.
   (d) A point moves along the limacon with $\frac{d\theta}{dt} = 0.5$ radian per second. Find $\frac{dx}{dt}$ when $\theta = \frac{\pi}{3}$, and say whether the point is moving left or right.



The limacon $r = 3 + 2\cos\theta$ and the circle $r = 4$.

QUESTION 4 **Taylor Error**

No calculator is allowed on this question. Show all work. [No calculator]

4. Let $f(x) = e^{2x}$. [No calculator]
   (a) Find the third-degree Maclaurin polynomial $P_3(x)$ for f.
   (b) Use P_3 to approximate $f(0.1) = e^{0.2}$. Write your answer as a decimal.
   (c) Use the Lagrange error bound to show that $|f(0.1) - P_3(0.1)| < 0.0002$. You may use the fact that $e^{0.2} < 2$.
   (d) Use the ratio test to find the interval of convergence of the Maclaurin series for f.

Answers Chapter 7: Answers and Explanations

Answers for every practice problem, numbered to match. Read the explanation even when your answer is correct.

7.1 Sequences

SET A · SKILLS

1. $-\frac{1}{2},\ \frac{2}{3},\ -\frac{3}{4},\ \frac{4}{5}$. Substitute $n = 1, 2, 3, 4$.
2. $3,\ 5,\ 9,\ 17,\ 33$. Each term is twice the previous term minus 1.
3. $\frac{5}{2}$. Divide by n^2; the lower-order terms vanish.
4. 4. Divide top and bottom by n: $\frac{4+1/n}{\sqrt{1+9/n^2}} \to 4$.
5. 0. L'Hopital on $\frac{\ln x}{x}$ gives $\frac{1/x}{1} \to 0$.
6. e^2. With $\ln y = \frac{\ln(1+2/x)}{1/x} \to 2$, the limit is e^2.
7. Explicit: $a_n = 3n + 1$. Recursive: $a_1 = 4$, $a_{n+1} = a_n + 3$ (common difference 3).
8. $a_n = 2 \cdot 3^{n-1}$. It diverges: the common ratio is $3 > 1$, so the terms grow without bound.
9. It diverges. Since $a_n = (-1)^n$, the terms alternate between -1 and 1 and never settle near one number.

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SET B · APPLICATIONS AND MIXED PRACTICE

10. $d_2 = 90$, $d_3 = 122$, $d_4 = 147.6$, $d_5 = 168.08$, so $d_5 = 168.080$ mg. If the limit L exists, $L = 0.8L + 50$, so $L = 250$ mg. (The sequence is increasing and stays below 250, so the limit does exist.)
11. $n = 10$. Solve $0.6^n < 0.01$: $0.6^9 \approx 0.01008$ is too big, and $0.6^{10} \approx 0.00605 < 0.01$. So $h_{10} \approx 0.060$ m.
12. $a_1 = 1060.000$, $a_4 \approx 1061.364$, $a_{12} \approx 1061.678$. The limit is $1000e^{0.06} \approx 1061.837$ (continuous compounding).
13. $P_2 = 108.000$, $P_3 = 116.467$. A limit L must satisfy $L = L + 0.1L\left(1 - \frac{L}{500}\right)$, so $L = 0$ or $L = 500$. The terms increase from 100, so $L = 500$.
14. The values approach 1. Confirmation: let $u = \frac{1}{n} \to 0$, so $a_n = \frac{\sin u}{u} \to 1$.
15. $a_2 = 2.667$, $a_3 = 2.646$. If $a_n \to L > 0$, then $L = \frac{1}{2}\left(L + \frac{7}{L}\right)$, so $L^2 = 7$ and $L = \sqrt{7} \approx 2.646$.
16. $N = 4998$. Since $a_n - 2 = \frac{-5}{n+3}$, we need $\frac{5}{n+3} < 0.001$, so $n + 3 > 5000$, so $n > 4997$.

SET C · CHALLENGE AND AP-STYLE PROBLEMS

17. Let $f(x) = \frac{x}{x^2+1}$. Then $f'(x) = \frac{1-x^2}{(x^2+1)^2} \le 0$ for $x \ge 1$, so a_n is decreasing. Also $0 < a_n \le \frac{1}{2}$. A decreasing sequence bounded below converges; $\lim \frac{n}{n^2+1} = 0$.
18. If $0 < a_n < 3$ then $a_{n+1} < \sqrt{9} = 3$, so all terms lie in $(0, 3)$. Also $a_{n+1} > a_n \iff 6 + a_n > a_n^2 \iff (3 - a_n)(a_n + 2) > 0$, true since $a_n < 3$. Increasing and bounded, so it converges to L with $L^2 = L + 6$; $L = 3$ (reject -2).
19. $e^{-3/2}$. $\ln y = 3x \ln\left(1 - \frac{1}{2x}\right) = \frac{3\ln(1-1/(2x))}{1/x} \to 3 \cdot \left(-\frac{1}{2}\right) = -\frac{3}{2}$.
20. $\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdots \frac{n}{n}$, and every factor after the first is at most 1, so $0 < \frac{n!}{n^n} \le \frac{1}{n}$. Since $\frac{1}{n} \to 0$, the limit is 0.
21. $a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0$, but $\sqrt{n} \to \infty$, so $\{a_n\}$ diverges.

7.2 Infinite Series: Geometric and Telescoping

SET A · SKILLS

1. $S_4 = 1 + 3 + 5 + 7 = 16$. The pattern $1, 4, 9, 16$ suggests $S_N = N^2$. (The series diverges since $S_N \to \infty$.)
2. $\frac{28}{3}$. $\frac{7}{1-1/4} = \frac{7}{3/4} = \frac{28}{3}$.
3. $-\frac{2}{5}$. First term $-\frac{2}{3}$, $r = -\frac{2}{3}$: $\frac{-2/3}{1+2/3} = \frac{-2/3}{5/3} = -\frac{2}{5}$.
4. $\frac{1}{5}$. The first term is $4 \cdot \frac{1}{25} = \frac{4}{25}$, $r = \frac{1}{5}$: $\frac{4/25}{4/5} = \frac{1}{5}$.
5. 1. Rewrite as $\frac{1}{3}\left(\frac{2}{3}\right)^n$: $\frac{1/3}{1-2/3} = 1$.
6. 1. $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, so $S_N = 1 - \frac{1}{N+1} \to 1$.
7. $\frac{4}{11}$. $0.36 + 0.0036 + \cdots = \frac{0.36}{1-0.01} = \frac{36}{99} = \frac{4}{11}$.
8. No, it diverges. $\lim \frac{n+1}{2n+5} = \frac{1}{2} \neq 0$, so the nth-term test applies.
9. No. It is geometric with $r = \frac{3}{2}$ and $|r| \ge 1$, so it diverges.

SET B · APPLICATIONS AND MIXED PRACTICE

10. After 5 doses: $20 \cdot \frac{1-0.3^5}{0.7} \approx 28.502$ mg. Long term: $\frac{20}{1-0.3} \approx 28.571$ mg.
11. ≈ 45.333 m. Down 8 m, then each rebound contributes up and down: $8 + 2(5.6 + 5.6(0.7) + \cdots) = 8 + \frac{2(5.6)}{0.3} = 8 + 37.333 = 45.333$.
12. $\frac{2}{1-0.85} \approx 13.333$ million dollars (geometric, $a = 2$, $r = 0.85$).
13. $N = 12$. Need $\left(\frac{2}{3}\right)^N < 0.01$. $\left(\frac{2}{3}\right)^{11} \approx 0.01156$ is too large, while $\left(\frac{2}{3}\right)^{12} \approx 0.00771 < 0.01$.
14. Total: $\frac{50}{1-0.92} = 625.000$ cm. The nth swing is $50(0.92)^{n-1} < 1$ when $0.92^{n-1} < 0.02$. $0.92^{46} \approx 0.0216$ and $0.92^{47} \approx 0.0199$, so $n - 1 = 47$ and $n = 48$.
15. Since $\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$, $S_N = \frac{1}{2} - \frac{1}{N+2}$. So $S_{10} = \frac{5}{12} \approx 0.417$ and the sum is $\frac{1}{2}$.
16. $\frac{227}{111}$. $0.\overline{045} = \frac{45}{999} = \frac{5}{111}$, so the number is $2 + \frac{5}{111} = \frac{227}{111}$ (≈ 2.045).

SET C · CHALLENGE AND AP-STYLE PROBLEMS

17. $\frac{3}{2}$. Split: $\sum \left(\frac{1}{3}\right)^n = \frac{1/3}{2/3} = \frac{1}{2}$ and $\sum \left(\frac{1}{2}\right)^n = 1$. Total $\frac{3}{2}$.
18. $1 - \frac{1}{n^2} = \frac{(n-1)(n+1)}{n^2}$, so the partial sum is $\ln \prod_{n=2}^{N} \frac{n-1}{n} \cdot \frac{n+1}{n} = \ln\left(\frac{1}{N} \cdot \frac{N+1}{2}\right) = \ln \frac{N+1}{2N} \to \ln \frac{1}{2} = -\ln 2$.
19. 1. $\frac{2}{(2n-1)(2n+1)} = \frac{1}{2n-1} - \frac{1}{2n+1}$, so $S_N = 1 - \frac{1}{2N+1} \to 1$.
20. Need $\left|\frac{x}{x+2}\right| < 1$, i.e. $x^2 < (x+2)^2$, i.e. $4x + 4 > 0$, so $x > -1$. The sum is $\frac{1}{1-\frac{x}{x+2}} = \frac{x+2}{2}$.
21. Group terms: $\frac{1}{3} + \frac{1}{4} > \frac{2}{4} = \frac{1}{2}$, $\frac{1}{5} + \cdots + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$, and in general each block from $\frac{1}{2^j+1}$ to $\frac{1}{2^{j+1}}$ has 2^j terms each at least $\frac{1}{2^{j+1}}$, so it sums to at least $\frac{1}{2}$. Then $S_{2^k} \ge 1 + \frac{k}{2} \to \infty$, so S_N is unbounded and the series diverges.

7.3 Convergence Tests I

SET A · SKILLS

1. Diverges. $\lim \frac{n}{n+4} = 1 \neq 0$, so the nth-term test applies.
2. Converges. It is $7\sum n^{-4/3}$, a constant times a p-series with $p = \frac{4}{3} > 1$.
3. Diverges. It is $2\sum n^{-1/3}$, a constant times a p-series with $p = \frac{1}{3} \le 1$.
4. Let $f(x) = \frac{x}{x^2+1}$: positive, continuous, and $f'(x) = \frac{1-x^2}{(x^2+1)^2} \le 0$ for $x \ge 1$ (decreasing). $\int_1^\infty f = \lim \left[\frac{1}{2}\ln(x^2+1)\right]_1^b = \infty$. The series diverges.
5. Converges. $0 < \frac{1}{n^2+4} < \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ is a convergent p-series ($p = 2$).
6. $\lim \frac{1/(3^n-2)}{1/3^n} = \lim \frac{3^n}{3^n-2} = 1$, and $\sum \left(\frac{1}{3}\right)^n$ converges (geometric, $r = \frac{1}{3}$). So the series converges.
7. Converges. Compare with $\frac{1}{n^2}$: $\lim \frac{(n+2)/(n^3+5)}{1/n^2} = \lim \frac{n^3+2n^2}{n^3+5} = 1$, and $\sum \frac{1}{n^2}$ converges.
8. Diverges. Compare with $\frac{1}{\sqrt{n}}$: $\lim \frac{\sqrt{n}}{\sqrt{n}+3} = 1$, and $\sum \frac{1}{\sqrt{n}}$ diverges ($p = \frac{1}{2}$).

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9. Diverges. $\cos\frac{1}{n} \to \cos 0 = 1 \neq 0$, so the nth-term test applies.

SET B · APPLICATIONS AND MIXED PRACTICE

10. $S_{10} \approx 1.54977$. Using $\int_{11}^{\infty} x^{-2}dx = \frac{1}{11} \approx 0.0909$ and $\int_{10}^{\infty} x^{-2}dx = 0.1$: $1.640 \le \sum \frac{1}{n^2} \le 1.650$.
11. $R_N \le \int_N^{\infty} x^{-3}dx = \frac{1}{2N^2}$. Need $\frac{1}{2N^2} < 0.001$, so $N^2 > 500$, $N > 22.36$. $N = 23$ terms (check: $N = 22$ gives 0.001033, too big).
12. Need $H_N > 4$. $H_{30} \approx 3.995$ is not enough and $H_{31} \approx 4.027$ is, so $N = 31$. Because $\sum \frac{1}{n}$ diverges, the overhang can be made as large as you like with enough blocks.
13. $S_5 \approx 0.65573$. The tail satisfies $\sum_{n>5} \frac{1}{n^3+n} < \int_5^{\infty} x^{-3}dx = 0.02$. So the sum lies between 0.655 and 0.676.
14. By the integral test comparison, $\int_1^{1001} \frac{dx}{x} \le H_{1000} \le 1 + \int_1^{1000} \frac{dx}{x}$. The bounds are $6.909 \le H_{1000} \le 7.908$ (the actual value is about 7.485).
15. Compare with $\frac{1}{n^{3/2}}$: $\lim \frac{(n+1)n^{3/2}}{n^{5/2}+3} = \lim \frac{n^{5/2}+n^{3/2}}{n^{5/2}+3} = 1$. The p-series with $p = \frac{3}{2}$ converges, so the series converges.
16. $f(x) = xe^{-x/2}$ is positive and continuous, and $f'(x) = e^{-x/2}\left(1 - \frac{x}{2}\right) < 0$ for $x > 2$, so f is decreasing on $[2, \infty)$. $\int_2^{\infty} xe^{-x/2}dx = \left[-2xe^{-x/2} - 4e^{-x/2}\right]_2^{\infty} = 8e^{-1} \approx 2.943$ is finite. So the series converges.

SET C · CHALLENGE AND AP-STYLE PROBLEMS

17. For $p > 0$ the function $\frac{1}{x(\ln x)^p}$ is positive, continuous and decreasing on $[2, \infty)$. With $u = \ln x$, $\int_2^{\infty} \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^{\infty} u^{-p}du$, which converges exactly when $p > 1$. For $p \le 0$ the terms are at least $\frac{1}{n}$ (for $n \ge 3$) so the series diverges.
18. Since $n \le 2^n$, $\frac{2^n+n}{3^n} \le \frac{2 \cdot 2^n}{3^n} = 2\left(\frac{2}{3}\right)^n$. The geometric series $\sum 2\left(\frac{2}{3}\right)^n$ converges, so the given series converges.
19. No. The limit comparison test requires a limit c with $0 < c < \infty$. Here $c = 0$, so the test is inconclusive; in fact $\sum \frac{1}{n^2}$ converges (p-series with $p = 2$).
20. Since $0 \le \sin^2 n \le 1$, $0 \le \frac{\sin^2 n}{n^2} \le \frac{1}{n^2}$. The p-series $\sum \frac{1}{n^2}$ converges, so by direct comparison the series converges.
21. Compare with $\frac{1}{n}$: $\frac{1/n^{1+1/n}}{1/n} = n^{-1/n}$. Since $\ln\left(n^{-1/n}\right) = -\frac{\ln n}{n} \to 0$, the ratio tends to $e^0 = 1$. By limit comparison with the divergent harmonic series, the series diverges.

7.4 Convergence Tests II

SET A · SKILLS

1. $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{5n} \to \frac{1}{5} < 1$. Converges.
2. Ratio $= \frac{10}{n+1} \to 0 < 1$. Converges.
3. Ratio $= \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} = \frac{n+1}{3} \to \infty$. Diverges.
4. Ratio $= \frac{1}{2}\left(\frac{n+1}{n}\right)^3 \to \frac{1}{2} < 1$. Converges.
5. Ratio $= 3\left(\frac{n}{n+1}\right)^2 \to 3 > 1$. Diverges.
6. (a) $\frac{n^2}{(n+1)^2} \to 1$, so $L = 1$ and the test is inconclusive.
   (b) It is a p-series with $p = 2 > 1$, so it converges.
7. Converges by the Alternating Series Test ($\frac{1}{n+4}$ decreases to 0). The absolute series $\sum \frac{1}{n+4}$ diverges (limit comparison with $\sum \frac{1}{n}$), so the convergence is conditional.
8. $|a_n| = \frac{n}{n+1} \to 1$, so a_n does not approach 0 (it oscillates near ± 1). The series diverges by the nth-term test.
9. The absolute series is $\sum \frac{1}{n^3+1}$. Since $0 < \frac{1}{n^3+1} < \frac{1}{n^3}$ and $\sum \frac{1}{n^3}$ is a convergent p-series ($p = 3$), the absolute series converges by direct comparison. So the original converges absolutely.

SET B · APPLICATIONS AND MIXED

10. (a) $S_5 = 1 - \frac{1}{16} + \frac{1}{81} - \frac{1}{256} + \frac{1}{625} \approx 0.948$.
    (b) $|S - S_5| \le b_6 = \frac{1}{1296} \approx 0.001$ (about 0.00077).
11. Need $b_{N+1} = \frac{1}{(N+1)^2+1} < 0.001$, so $(N+1)^2 > 999$ and $N + 1 \ge 32$. Check: $N = 30$ gives $\frac{1}{962} > 0.001$; $N = 31$ gives $\frac{1}{1025} < 0.001$. Answer: $N = 31$.
12. (a) $\frac{a_{11}}{a_{10}} = \frac{121}{100}(0.9) = 1.089$.
    (b) $\frac{(n+1)^2}{n^2}(0.9) \to 0.9 < 1$, so the series converges.
13. $b_n = \frac{\ln n}{n}$ has derivative $\frac{1-\ln n}{n^2} < 0$ for $n \ge 3$, and $\frac{\ln n}{n} \to 0$, so the series converges. For $n \ge 3$, $\frac{\ln n}{n} > \frac{1}{n}$, so the absolute series diverges by comparison with the harmonic series. Conditionally convergent.
14. Take $a_n = \frac{1}{n}$: every ratio $\frac{n}{n+1}$ is less than 1, yet $\sum \frac{1}{n}$ diverges. The Ratio Test requires the **limit** of the ratios to be less than 1; here the limit is 1.
15. $S_4 = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} = \frac{77}{192} \approx 0.401$. The error is at most $b_5 = \frac{1}{5 \cdot 32} = \frac{1}{160} = 0.00625$, so about 0.006.
16. Ratio: $\frac{c^{n+1}}{(n+1)^2} \cdot \frac{n^2}{c^n} \to c$. Converges for $c < 1$, diverges for $c > 1$. At $c = 1$ the series is $\sum \frac{1}{n^2}$, which converges. So $0 < c \le 1$.

SET C · CHALLENGE

17. $\lim \frac{a_{n+1}}{a_n} > 1$ means a_n eventually grows, so $a_n \not\to 0$. Both $\sum a_n$ and $\sum (-1)^n a_n$ diverge (nth-term test; the Ratio Test also applies to $\sum |(-1)^n a_n|$).
18. Ratio $= \frac{k^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{k^n\, n!} = k\left(\frac{n}{n+1}\right)^n = \frac{k}{(1+1/n)^n} \to \frac{k}{e}$. Converges (absolutely) for $0 < k < e$, diverges for $k > e$, and the test is inconclusive at $k = e$. (At $k = e$ the ratio equals $\frac{e}{(1+1/n)^n} > 1$ for every n, so the terms increase and the series diverges by the nth-term test.)
19. $b_n = \frac{n}{n^2+1} \to 0$; for $f(x) = \frac{x}{x^2+1}$, $f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0$ for $x > 1$, so b_n decreases and the series converges. Since $\frac{n}{n^2+1} \ge \frac{1}{2n}$ and $\sum \frac{1}{2n}$ diverges, the absolute series diverges. Conditionally convergent.
20. Need $\frac{1}{(N+1)!} < 0.001$, so $(N+1)! > 1000$. $6! = 720$ is too small, $7! = 5040$ works, so $N = 6$. $S_6 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} = \frac{91}{144}$.
21. Since $\sum |a_n|$ converges, $a_n \to 0$, so $|a_n| \le 1$ for large n. Then $a_n^2 = |a_n|^2 \le |a_n|$. By direct comparison, $\sum a_n^2$ converges.

7.5 Power Series and Intervals of Convergence

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1. Ratio $\to \frac{|x|}{5}$, so $R = 5$. At $x = \pm 5$ the terms are 1 and $(-1)^n$, which do not approach 0. Interval: $(-5, 5)$.
2. Ratio $= \frac{n+1}{n} \cdot \frac{|x|}{2} \to \frac{|x|}{2}$, so $R = 2$. At $x = \pm 2$ the terms n and $(-1)^n n$ do not approach 0. Interval: $(-2, 2)$.
3. Ratio $= \frac{|x-1|}{(2n+2)(2n+1)} \to 0$ for every x. $R = \infty$.
4. Ratio $= \frac{(n+1)^{n+1}}{n^n}|x| = (n+1)\left(1+\frac{1}{n}\right)^n |x| \to \infty$ unless $x = 0$. $R = 0$; the series converges only at $x = 0$.
5. Ratio $\to |x-3|$, so $R = 1$ and $2 < x < 4$. Both endpoints give $\sum \frac{1}{n^2}$ or $\sum \frac{(-1)^n}{n^2}$, which converge. Interval: $[2, 4]$.
6. Ratio $\to \frac{|x+2|}{4}$, so $R = 4$ and $-6 < x < 2$. At $x = 2$: $\sum \frac{1}{n}$ diverges. At $x = -6$: $\sum \frac{(-1)^n}{n}$ converges. Interval: $[-6, 2)$.
7. Ratio $\to |3x|$, so $|x| < \frac{1}{3}$. At $x = \frac{1}{3}$: $\sum \frac{1}{n+1}$ diverges. At $x = -\frac{1}{3}$: $\sum \frac{(-1)^n}{n+1}$ converges. Interval: $\left[-\frac{1}{3}, \frac{1}{3}\right)$.
8. Ratio $\to \frac{x^2}{9} < 1$ gives $|x| < 3$, so $R = 3$.
9. (a) The series is centered at $c = 1$. Convergence at -2 (distance 3) means $R \geq 3$, and divergence at 4 (distance 3) means $R \leq 3$. So $R = 3$.
   (b) $x = 3$ is within 3 of the center ($|3 - 1| = 2 < 3$), so it converges. $x = 5$ is at distance $4 > 3$, so it diverges.

SET B · APPLICATIONS AND MIXED

10. (a) Geometric with ratio $\frac{x-1}{3}$: need $|x - 1| < 3$, so $-2 < x < 4$. At the endpoints the terms are ± 1 or $(-1)^n$, which do not approach 0. Interval: $(-2, 4)$.
    (b) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \approx 1.494$.
    (c) $f(2) = \frac{3}{4-2} = 1.5$.
11. (a) $\frac{d}{dx}\sum x^{2n} = \sum_{n=1}^{\infty} 2n\, x^{2n-1} = 2x + 4x^3 + 6x^5 + 8x^7 + \cdots$
    (b) At $x = 0.5$ the four terms give $1 + 0.5 + 0.1875 + 0.0625 = 1.75$. Exact: $\frac{1}{(0.75)^2} = \frac{16}{9} \approx 1.778$.
12. Ratio $= \left(\frac{n+1}{n}\right)^2 \frac{|x-4|}{5} \to \frac{|x-4|}{5}$, so $R = 5$ and $-1 < x < 9$. At both endpoints the terms n^2 and $(-1)^n n^2$ do not approach 0, so they diverge. Interval: $(-1, 9)$.
13. (a) The ratio test gives $|x - 1| \cdot \frac{2}{5} < 1$, so $|x - 1| < \frac{5}{2}$ and $R = \frac{5}{2}$.
    (b) $x = 0$: distance 1 from the center, converges. $x = 3$: distance 2, converges. $x = 4$: distance $3 > \frac{5}{2}$, diverges.
14. (a) $R = 2$. At $x = 2$: $\sum \frac{1}{n}$ diverges. At $x = -2$: $\sum \frac{(-1)^n}{n}$ converges. Interval: $[-2, 2)$.
    (b) $f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{2^n} = \frac{1}{2} \cdot \frac{1}{1 - x/2} = \frac{1}{2-x}$.
    (c) $f'(1) = \frac{1}{2-1} = 1$. Series: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$. They agree.
15. (a) Geometric with ratio $-\frac{t}{4}$: converges for $|t| < 4$, with sum $\frac{1}{1 + t/4} = \frac{4}{4+t}$.
    (b) $S(1.5) = \frac{4}{5.5} \approx 0.727$. Four-term sum: $1 - 0.375 + 0.1406 - 0.0527 \approx 0.7129$. Error ≈ 0.0144 (about 0.014 volts).
16. $\ln 2 = \int_0^{1/2} \frac{dt}{1-t} = \sum_{n=0}^{\infty} \frac{(1/2)^{n+1}}{n+1}$. Five terms: $0.5 + 0.125 + 0.04167 + 0.015625 + 0.00625 \approx 0.689$. Calculator: $\ln 2 \approx 0.693$. The approximation is slightly low.

SET C · CHALLENGE

17. Ratio $= \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2}|x| = \frac{(n+1)^2}{(2n+2)(2n+1)}|x| \to \frac{|x|}{4}$. So $R = 4$.
18. Ratio $= \frac{|x|^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{|x|^n} = \frac{|x|}{n+1}\left(\frac{n}{n+1}\right)^n \to 0$ for every x. $R = \infty$.
19. $f'(x) = \sum_{n=1}^{\infty} \frac{n\, x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$. Let $m = n - 1$: $\sum_{m=0}^{\infty} \frac{x^m}{m!} = f(x)$. (Valid on the whole interval of convergence $(-\infty, \infty)$.)
20. (a) $y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$ and $2y = \sum 2a_n x^n$. Matching the coefficient of x^{n-1} gives $n\, a_n = 2a_{n-1}$.
    (b) $a_0 = 3$, $a_1 = 6$, $a_2 = 6$, $a_3 = 4$, and $a_n = \frac{3 \cdot 2^n}{n!}$.
21. (a) Differentiate $\sum x^n = \frac{1}{1-x}$: $\sum_{n=1}^{\infty} n\, x^{n-1} = \frac{1}{(1-x)^2}$. Multiply by x: $\sum n\, x^n = \frac{x}{(1-x)^2}$.
    (b) Put $x = \frac{1}{2}$ (inside $|x| < 1$): $\frac{1/2}{(1/2)^2} = 2$.

7.6 Maclaurin Series and Taylor Polynomials

SET A · SKILLS

1. $f^{(k)}(0) = 3^k$, so $P_3(x) = 1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{6} = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3$.
2. $\cos u = 1 - \frac{u^2}{2} + \frac{u^4}{24} - \frac{u^6}{720}$ with $u = 2x$: $1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6$.
3. $e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \cdots$. Multiply by x: $x + x^3 + \frac{x^5}{2} + \cdots$
4. $\frac{1}{1+3x} = \sum_{n=0}^{\infty} (-3x)^n = \sum_{n=0}^{\infty} (-1)^n 3^n x^n$. Converges for $|3x| < 1$, so $R = \frac{1}{3}$.
5. Substitute $u = x^2$ in $u - \frac{u^2}{2} + \frac{u^3}{3}$: $x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots$
6. Substitute $u = x^2$ in $u - \frac{u^3}{3} + \frac{u^5}{5}$: $x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \cdots$
7. Add $\left(x - \frac{x^3}{6}\right)$ and $\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)$: $P_4(x) = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$.
8. $P_3(x) = 2 + (-1)x + \frac{6}{2}x^2 + \frac{-12}{6}x^3 = 2 - x + 3x^2 - 2x^3$.
9. $x^2 e^x = x^2 + x^3 + \frac{x^4}{2} + \frac{x^5}{6} + \cdots$. The coefficient of x^5 is $\frac{1}{6}$, so $f^{(5)}(0) = 5! \cdot \frac{1}{6} = 20$.

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SET B · APPLICATIONS AND MIXED

10. (a) $0.4 - \frac{0.064}{6} + \frac{0.01024}{120} = 0.4 - 0.010667 + 0.000085 = 0.38942$.
    (b) The next term is $\frac{0.4^7}{7!} \approx 0.0000003$, so the error is less than 3.3×10^{-7} (the calculator gives 0.389418).
11. $P_4(0.5) = 1 + 0.5 + 0.125 + 0.020833 + 0.002604 = 1.648438$. Calculator: $e^{0.5} \approx 1.648721$. Error ≈ 0.0003.
12. (a) $P_4(x) = 3 + 2x + \frac{-4}{2}x^2 + \frac{6}{6}x^3 + \frac{24}{24}x^4 = 3 + 2x - 2x^2 + x^3 + x^4$.
    (b) $P_4(0.2) = 3 + 0.4 - 0.08 + 0.008 + 0.0016 = 3.3296 \approx 3.330$.
13. $0.2 - \frac{0.2^2}{2} + \frac{0.2^3}{3} - \frac{0.2^4}{4} = 0.2 - 0.02 + 0.002667 - 0.0004 \approx 0.182$. The error is at most the next term $\frac{0.2^5}{5} = 0.000064$. (Calculator: 0.1823.)
14. (a) $0.333333 - 0.023810 + 0.000758 \approx 0.310$.
    (b) The calculator gives 0.310 (more exactly 0.310268); the next term is about 0.000013.
15. Approximation: $0.3 - \frac{0.027}{6} = 0.2955$. Calculator: $\sin 0.3 = 0.295520$. Error ≈ 0.000020.
16. (a) $\frac{1}{2-x} = \frac{1}{2} \cdot \frac{1}{1-x/2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \cdots$ on $(-2, 2)$ (the endpoints give terms that do not approach 0).
    (b) $P_3(0.5) = 0.5 + 0.125 + 0.03125 + 0.0078125 = 0.664$. Exact: $\frac{1}{1.5} = 0.667$.

SET C · CHALLENGE

17. (a) $\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \cdots$
    (b) The coefficient of x^4 is $\frac{1}{120}$, so $f^{(4)}(0) = 24 \cdot \frac{1}{120} = \frac{1}{5}$.
18. (a) $\arctan x = \int_0^x \sum (-1)^n t^{2n} dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ for $|x| < 1$ (and at $x = \pm 1$ by the Alternating Series Test).
    (b) $\arctan 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.
19. (a) e^x at $x = -\frac{1}{2}$: $e^{-1/2} = \frac{1}{\sqrt{e}}$.
    (b) $\cos x$ at $x = \frac{\pi}{2}$: $\cos \frac{\pi}{2} = 0$.
    (c) This is $\ln(1+x) = \sum \frac{(-1)^{n+1}x^n}{n}$ at $x = \frac{1}{3}$, so the sum is $\ln \frac{4}{3}$.
20. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$, so $x - \arctan x = \frac{x^3}{3} - \frac{x^5}{5} + \cdots$. Divide by x^3: $\frac{1}{3} - \frac{x^2}{5} + \cdots \to \frac{1}{3}$.
21. (a) Replace x by $-x$: $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = -\sum_{n=1}^{\infty} \frac{x^n}{n}$.
    (b) $\ln(1+x) - \ln(1-x) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots$ (even powers cancel). Both series have $R = 1$, so $R = 1$.

7.7 Taylor Series and the Lagrange Error Bound

SET A · SKILLS

1. $f^{(k)}(1) = (-1)^k k!$, so the coefficients are $(-1)^k$: $P_3(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3$.
2. $f^{(k)}(3) = e^3$, so $e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n = e^3\left[1 + (x-3) + \frac{(x-3)^2}{2} + \frac{(x-3)^3}{6} + \cdots\right]$.
3. Derivatives at $\frac{\pi}{2}$: $\sin = 1$, $\cos = 0$, $-\sin = -1$, $-\cos = 0$, $\sin = 1$. $P_4(x) = 1 - \frac{(x-\pi/2)^2}{2} + \frac{(x-\pi/2)^4}{24}$.
4. $\frac{1}{2+x} = \frac{1}{1+(x+1)} = \sum_{n=0}^{\infty}(-1)^n(x+1)^n$. Converges for $|x+1| < 1$, so the interval is $(-2, 0)$ (endpoints give terms that do not approach 0).
5. $g(2) = 8$, $g'(2) = 12$, $g''(2) = 12$, $g'''(2) = 6$. $P_3(x) = 8 + 12(x-2) + 6(x-2)^2 + (x-2)^3$. This equals x^3 exactly (expand it), because a cubic is its own degree 3 Taylor polynomial.
6. $f(1) = 4$, $f'(1) = 2$, $f''(1) = 2! \cdot (-3) = -6$, $f'''(1) = 3! \cdot 5 = 30$.
7. $|R_3(3)| \le \frac{6}{4!}|3-1|^4 = \frac{6 \cdot 16}{24} = 4$.
8. $P_2(x) = (x-1) - \frac{(x-1)^2}{2}$, so $P_2(1.1) = 0.1 - 0.005 = 0.095$. Error $\le \frac{2}{3!}(0.1)^3 = \frac{1}{3000} \approx 0.00033$.
9. $P_4(1) = 1 - \frac{1}{2} + \frac{1}{24} = \frac{13}{24}$. The series for $\cos 1$ alternates with decreasing terms, so the error is at most the next term $\frac{1}{6!} = \frac{1}{720}$.

SET B · APPLICATIONS AND MIXED

10. (a) $f = 2$, $f' = \frac{1}{12}$, $f'' = -\frac{1}{144}$ at 8. $P_2(x) = 2 + \frac{x-8}{12} - \frac{(x-8)^2}{288}$.
    (b) $P_2(8.4) = 2 + 0.03333 - 0.00056 \approx 2.0328$. Error $\le \frac{10}{27 \cdot 256} \cdot \frac{0.4^3}{6} \approx 0.000015$.
11. (a) $1 + 0.2 + \frac{0.04}{2} + \frac{0.008}{6} = 1.22133$.
    (b) Bound: $\frac{1.2214(0.2)^4}{24} \approx 0.000081$. Actual: $e^{0.2} - 1.22133 \approx 0.00007$. The actual error is below the bound.
12. Use $n+1$ in the bound: $\frac{1}{(n+1)!} < 0.0001$ requires $(n+1)! > 10000$. $7! = 5040$ is too small, $8! = 40320$ works. So $n+1 = 8$, $n = 7$.
13. (a) $P_3(x) = 1 + 3(x-2) - (x-2)^2 + \frac{2}{3}(x-2)^3$.
    (b) $P_3(2.1) = 1 + 0.3 - 0.01 + 0.000667 \approx 1.291$. Error $\le \frac{12}{4!}(0.1)^4 = 0.00005$.
14. (a) $h \approx 2 \cdot \frac{0.25}{2} = 0.25$ meter.
    (b) Exact $h = 2(1 - \cos 0.5) \approx 0.24483$. Error ≈ 0.005 meter.
15. $0.5 - 0.125 + 0.041667 - 0.015625 + 0.00625 \approx 0.407$. Error bound: next term $\frac{0.5^6}{6} \approx 0.0026$. Actual: $\ln 1.5 \approx 0.4055$, so the error ≈ 0.0018, which is less than the bound.
16. (a) $P_2(x) = \frac{\sqrt{2}}{2}\left[1 - \left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2\right]$.
    (b) With $h = 0.9 - \frac{\pi}{4} \approx 0.1146$: $P_2(0.9) \approx 0.62143$ (calculator value 0.62161). Error $\le \frac{1}{6}(0.1146)^3 \approx 0.00025$.

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SET C · CHALLENGE

17. Use $n=4$: $P_4(x)=x-\frac{x^3}{6}$ (the x^4 term is 0). Since $f^{(5)}(t)=\cos t$, $|f^{(5)}(t)|\le 1=M$. So $|R_4(x)|\le \frac{1}{5!}|x|^5\le \frac{1}{120}$ for $|x|\le 1$.
18. $f^{(n+1)}(t)=e^t\le e<3$ on $[0,1]$, so $|R_n(1)|\le \frac{3}{(n+1)!}$. Need $(n+1)!>3000$. $6!=720$ and $7!=5040$, so $n+1=7$ and $n=6$.
19. (a) $\frac{f^{(n)}(2)}{n!}=\frac{(-1)^n}{3^n}$, so $\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}(x-2)^n$.
    (b) Geometric with ratio $-\frac{x-2}{3}$: $R=3$ and the sum is $\frac{1}{1+(x-2)/3}=\frac{3}{x+1}$ for $-1<x<5$.
20. (a) $P_3(x)=4-2(x-1)+\frac{6}{2}(x-1)^2-\frac{18}{6}(x-1)^3=4-2(x-1)+3(x-1)^2-3(x-1)^3$.
    (b) $|R_3(1.5)|\le\frac{12}{4!}(0.5)^4=\frac{1}{32}=0.03125$.
    (c) $f''(1)=6>0$, so the graph is concave up at $x=1$.
21. $x^2\cos x=x^2-\frac{x^4}{2}+\frac{x^6}{24}-\cdots$ contains only even powers. $f^{(6)}(0)=6!\cdot\frac{1}{24}=30$ and $f^{(7)}(0)=0$ (the coefficient of x^7 is 0).

7.8 Parametric Equations

SET A · SKILLS

1. $t=\frac{x+1}{2}$, so $y=\frac{(x+1)^2}{4}+3$.
2. $x^2+y^2=16\cos^2 t+16\sin^2 t=16$: a circle of radius 4 centered at the origin, traced once counterclockwise.
3. $y=(e^t)^2+1=x^2+1$ with $x=e^t>0$. So $y=x^2+1$ for $x>0$ (the right half of the parabola).
4. $\frac{dy}{dx}=\frac{3\cos t}{-2\sin t}=-\frac{3}{2}\cot t$. At $t=\frac{\pi}{4}$, $\cot t=1$, so $\frac{dy}{dx}=-\frac{3}{2}$.
5. $\frac{dx}{dt}=2t+3=5$ and $\frac{dy}{dt}=3t^2-6=-3$ at $t=1$. So $\frac{dy}{dx}=-\frac{3}{5}$.
6. $\frac{dx}{dt}=2t$, $\frac{dy}{dt}=3t^2-12$. Horizontal: $t=\pm 2$ gives $(6,-16)$ and $(6,16)$. Vertical: $t=0$ gives $(2,0)$ (where $\frac{dy}{dt}=-12\neq 0$).
7. $\frac{dy}{dx}=\frac{3t^2-1}{2}$. $\frac{d}{dt}\left(\frac{3t^2-1}{2}\right)=3t$. Divide by $\frac{dx}{dt}=2$: $\frac{3t}{2}$. At $t=2$: 3.
8. $\frac{dx}{dt}=1$, $\frac{dy}{dt}=t^{1/2}$. $L=\int_0^3\sqrt{1+t}\,dt=\frac{2}{3}(1+t)^{3/2}\Big|_0^3=\frac{2}{3}(8-1)=\frac{14}{3}$.
9. $\frac{dx}{dt}=3t^2=3$ and $\frac{dy}{dt}=2t=2$ at $t=1$. Speed $=\sqrt{9+4}=\sqrt{13}$.

SET B · APPLICATIONS AND MIXED

10. (a) $y=0$: $5t(4-t)=0$, so $t=4$ s.
    (b) $y'=20-10t=0$ at $t=2$; $y(2)=40-20=20$ m.
    (c) At $t=3$: $\frac{dx}{dt}=4$, $\frac{dy}{dt}=-10$. $\frac{dy}{dx}=-2.5$; speed $=\sqrt{116}\approx 10.770$ m/s.
11. Speed $=\sqrt{4t^2+4}=2\sqrt{t^2+1}$. $L=\int_0^3 2\sqrt{t^2+1}\,dt=\left[t\sqrt{t^2+1}+\ln\left(t+\sqrt{t^2+1}\right)\right]_0^3\approx 11.305$ (calculator).
12. (a) $\frac{x^2}{25}+\frac{y^2}{9}=1$.
    (b) $\frac{dy}{dx}=\frac{3\cos t}{-5\sin t}=-\frac{3}{5}\cot t$. At $t=\frac{\pi}{3}$: $-\frac{3}{5}\cdot\frac{1}{\sqrt{3}}\approx -0.346$.
    (c) $L=\int_0^{2\pi}\sqrt{25\sin^2 t+9\cos^2 t}\,dt\approx 25.527$.
13. (a) $\frac{dy}{dx}=\frac{6}{-3}=-2.000$. Speed $=\sqrt{9+36}=\sqrt{45}\approx 6.708$.
    (b) $\frac{dx}{dt}=-3<0$: moving left; $\frac{dy}{dt}=6>0$: moving up.
    (c) Point $(-2,9)$, slope -2: $y-9=-2(x+2)$.
14. (a) $\frac{dy}{dx}=\frac{2\sin t}{2(1-\cos t)}=\frac{\sin t}{1-\cos t}$. At $t=\frac{\pi}{3}$: $\frac{\sqrt{3}/2}{1/2}=\sqrt{3}\approx 1.732$.
    (b) Speed $=\sqrt{4(1-\cos t)^2+4\sin^2 t}=2\sqrt{2-2\cos t}=4\sin\frac{t}{2}$. At $t=\frac{\pi}{3}$: $4\sin\frac{\pi}{6}=2$.
    (c) $\int_0^{2\pi}4\sin\frac{t}{2}\,dt=\left[-8\cos\frac{t}{2}\right]_0^{2\pi}=16$.
15. $\frac{dy}{dx}=\frac{e^t}{2t+1}$. $\frac{d}{dt}\left(\frac{e^t}{2t+1}\right)=\frac{e^t(2t+1)-2e^t}{(2t+1)^2}=\frac{e^t(2t-1)}{(2t+1)^2}$. Divide by $2t+1$: $\frac{d^2y}{dx^2}=\frac{e^t(2t-1)}{(2t+1)^3}$. At $t=1$: $\frac{e}{27}\approx 0.101>0$, concave up.
16. (a) $\frac{dx}{dt}=-3.2\sin(0.4t)$, $\frac{dy}{dt}=3.2\cos(0.4t)$. Speed $=3.2$ m/s.
    (b) Period $=\frac{2\pi}{0.4}\approx 15.708$ s.
    (c) Distance $=3.2\cdot 20=64$ m.

SET C · CHALLENGE

17. $y=\cos 2t=1-2\sin^2 t=1-2x^2$ with $-1\le x\le 1$: an arc of a downward parabola. The particle starts at $(0,1)$, moves right and down to $(1,-1)$, then left through $(0,1)$ to $(-1,-1)$, and finally right back to $(0,1)$: one full back-and-forth trip along the arc.
18. Self-crossing needs $t_1\neq t_2$ with the same (x,y). x equal gives $t_2=-t_1$. $y(-t)=-y(t)$, so $y=0$: $t(t^2-1)=0$ gives $t=\pm 1$ (nonzero), point $(0,0)$. $\frac{dy}{dx}=\frac{3t^2-1}{2t}$: at $t=1$ slope 1, at $t=-1$ slope -1. Tangent lines: $y=x$ and $y=-x$.
19. $\frac{dx}{dt}=2t$, $\frac{dy}{dt}=t^2-1$. Speed $=\sqrt{4t^2+(t^2-1)^2}=\sqrt{(t^2+1)^2}=t^2+1$. $L=\int_0^3(t^2+1)\,dt=9+3=12$.
20. $\frac{dy}{dx}=\frac{2t}{1/t}=2t^2$. $\frac{d}{dt}(2t^2)=4t$. Divide by $\frac{dx}{dt}=\frac{1}{t}$: $\frac{d^2y}{dx^2}=4t^2>0$ for all $t>0$, so the curve is concave up everywhere.
21. (a) $2\cos t=\sqrt{2}$ and $\sin t=\frac{\sqrt{2}}{2}$ both give $t=\frac{\pi}{4}$.
    (b) $\frac{dx}{dt}=-2\sin t=-\sqrt{2}$, $\frac{dy}{dt}=\cos t=\frac{\sqrt{2}}{2}$. Speed $=\sqrt{2+\frac{1}{2}}=\frac{\sqrt{10}}{2}$. $\frac{dy}{dx}=\frac{\sqrt{2}/2}{-\sqrt{2}}=-\frac{1}{2}$.

7.9 Vector-Valued Functions and Motion in the Plane

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SET A · SKILLS

1. $\vec{v}(t) = \langle 2t,\ 9t^2\rangle$ and $\vec{a}(t) = \langle 2,\ 18t\rangle$.
2. $\vec{v}(t) = \langle -2\sin 2t,\ 2\cos 2t\rangle$. Speed $= \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2$, a constant.
3. $\vec{v}(t) = \langle 2e^{2t},\ 6t - 1\rangle$, so $\vec{v}(0) = \langle 2, -1\rangle$. $\vec{a}(t) = \langle 4e^{2t},\ 6\rangle$, so $\vec{a}(0) = \langle 4, 6\rangle$.
4. $\vec{v}(t) = \langle 6t, 8t\rangle$, so $\vec{v}(1) = \langle 6, 8\rangle$. Speed $= \sqrt{36 + 64} = 10$.
5. $\vec{v}(t) = \langle 2t,\ 3t^2 + 1\rangle$. $\vec{r}(t) = \langle t^2 + 1,\ t^3 + t\rangle$ (constants: 1 and 0).
6. $\vec{r}(4) = \vec{r}(0) + \int_0^4 \langle 2t, 3\rangle dt = \langle 1, 1\rangle + \langle 16, 12\rangle = \langle 17, 13\rangle$.
7. Speed $= \sqrt{36t^2 + 64t^2} = 10t$. Distance $= \int_0^3 10t\,dt = 5t^2\Big|_0^3 = 45$.
8. Left: $x' < 0$ means $t < 3$. Down: $y' < 0$ means $t^2 < 1$, so $0 \le t < 1$. Both: $0 \le t < 1$.
9. $\vec{v}\cdot\vec{a} = (-2)(4) + (3)(1) = -5 < 0$. Velocity and acceleration make an obtuse angle, so the speed is decreasing.

SET B · APPLICATIONS AND MIXED

10. $\vec{v} = \langle 2t, \cos t\rangle$, so $\vec{v}(2) = \langle 4,\ -0.416\rangle$. $\vec{a} = \langle 2, -\sin t\rangle$, so $\vec{a}(2) = \langle 2,\ -0.909\rangle$. Speed $= \sqrt{16 + 0.173} \approx 4.022$.
11. Speed $= \sqrt{t^2 + e^t}$. Distance $= \int_0^3 \sqrt{t^2 + e^t}\,dt \approx 8.343$.
12. (a) $\vec{r}(t) = \langle 1 + 2\sin t,\ 3 - 3\cos t\rangle$ (constants chosen so $\vec{r}(0) = \langle 1, 0\rangle$).
    (b) $\vec{r}(2) \approx \langle 2.819,\ 4.248\rangle$. Speed $= \sqrt{4\cos^2 2 + 9\sin^2 2} \approx 2.852$.
13. Trapezoids, width 1: $\Delta x \approx \frac{3+2}{2} + \frac{2+0}{2} + \frac{0-2}{2} + \frac{-2-1}{2} = 2.5 + 1 - 1 - 1.5 = 1$. $\Delta y \approx \frac{1+3}{2} + \frac{3+4}{2} + \frac{4+2}{2} + \frac{2+0}{2} = 2 + 3.5 + 3 + 1 = 9.5$. Position $\approx \langle 2 + 1,\ 1 + 9.5\rangle = \langle 3,\ 10.5\rangle$.
14. (a) $\vec{v}(t) = \langle 12,\ 15 - 9.8t\rangle$, so $\vec{v}(1) = \langle 12,\ 5.2\rangle$ m/s.
    (b) $y' = 0$ at $t = \frac{15}{9.8} \approx 1.531$ s.
    (c) It lands at $t = \frac{30}{9.8} \approx 3.061$ s, where $\vec{v} = \langle 12, -15\rangle$. Speed $= \sqrt{369} \approx 19.209$ m/s.
15. $\vec{v} = \langle -10\sin 2t,\ 10\cos 2t\rangle$, $\vec{a} = \langle -20\cos 2t,\ -20\sin 2t\rangle$.
    (a) Speed $= 10$.
    (b) $|\vec{a}| = 20$.
    (c) $\vec{v}\cdot\vec{a} = 200\sin 2t\cos 2t - 200\sin 2t\cos 2t = 0$. Since $\frac{d}{dt}|\vec{v}|^2 = 2\vec{v}\cdot\vec{a} = 0$, the speed is constant.
16. (a) $\vec{v}(t) = \langle t^2 + 1,\ -t + 2\rangle$ and $\vec{r}(t) = \left\langle \frac{t^3}{3} + t,\ -\frac{t^2}{2} + 2t\right\rangle$.
    (b) $x' = t^2 + 1 \ge 1 > 0$, so never at rest.
    (c) $\vec{v}(3) = \langle 10, -1\rangle$, speed $= \sqrt{101} \approx 10.050$.

SET C · CHALLENGE

17. (a) $\vec{r}(t) = \langle t^3 - 3t^2,\ t^2 - 2t\rangle$.
    (b) $x = t^2(t - 3) = 0$ only for $t = 0$ or $t = 3$; $y = t(t - 2) = 0$ only for $t = 0$ or $t = 2$. The sets share only $t = 0$, so for $t > 0$ the particle is never at the origin.
18. $\vec{v} = \langle -3\sin t,\ 3\cos t\rangle$, speed $= 3$. $\vec{a} = \langle -3\cos t, -3\sin t\rangle$, $|\vec{a}| = 3 \ne 0$ always. Speed is constant because $\vec{a} \perp \vec{v}$ (only the direction of velocity changes).
19. (a) $\sqrt{16t^2 + (4t^2 - 1)^2} = \sqrt{16t^4 + 8t^2 + 1} = \sqrt{(4t^2 + 1)^2} = 4t^2 + 1$.
    (b) $\int_0^2 (4t^2 + 1)dt = \frac{32}{3} + 2 = \frac{38}{3}$.
    (c) $\left\langle \int_0^2 4t\,dt,\ \int_0^2 (4t^2 - 1)dt\right\rangle = \langle 8,\ \frac{26}{3}\rangle$.
20. $x' = 3t^2 - 3 = 3(t - 1)(t + 1)$ and $y' = 2t - 2 = 2(t - 1)$.
    (a) Both are 0 only at $t = 1$, so the particle is at rest at $t = 1$ only.
    (b) Right and up: $x' > 0$ and $y' > 0$ for $t > 1$. Left and down: $x' < 0$ and $y' < 0$ for $0 \le t < 1$.
21. $\vec{a} = \langle 2,\ 2t - 4\rangle$. $\vec{v}\cdot\vec{a} = 2(2t - 4) + (t^2 - 4t + 3)(2t - 4) = (2t - 4)(t^2 - 4t + 5) = 2(t - 2)\big((t - 2)^2 + 1\big)$. The second factor is always positive, so the sign is that of $t - 2$: speed increasing for $t > 2$ (decreasing for $t < 2$). Minimum speed at $t = 2$: $|\langle 0, -1\rangle| = 1$.

7.10 Polar Coordinates and Graphs

SET A · SKILLS

1. $x = 4\cos\frac{\pi}{3} = 2$, $y = 4\sin\frac{\pi}{3} = 2\sqrt{3}$. The point is $(2, 2\sqrt{3})$.
2. $r = \sqrt{4 + 4} = 2\sqrt{2}$. The point is in Quadrant II with $\tan\theta = -1$, so $\theta = \frac{3\pi}{4}$. Answer: $\left(2\sqrt{2}, \frac{3\pi}{4}\right)$.
3. Multiply by r: $x^2 + y^2 = 6x$, so $(x - 3)^2 + y^2 = 9$. This is a circle with center $(3, 0)$ and radius 3.
4. $r\cos\theta = 5$, so $r = 5\sec\theta$ (or $r = \frac{5}{\cos\theta}$).
5. $r^2 = 18\sin\theta\cos\theta$. Multiply by r^2: $r^4 = 18(r\sin\theta)(r\cos\theta)$, so $(x^2 + y^2)^2 = 18xy$.
6. Use $(-r, \theta + \pi)$: $\left(-5, \frac{5\pi}{4}\right)$. Check: $x = -5\cos\frac{5\pi}{4} = \frac{5\sqrt{2}}{2}$, the same as $5\cos\frac{\pi}{4}$.
7. It is a cardioid ($b = a = 3$). It is symmetric about the polar axis because $\cos(-\theta) = \cos\theta$. The maximum r is 6, at $\theta = 0$.
8. Since $n = 3$ is odd there are 3 petals. Each has length 4. The first maximum $|r| = 4$ occurs when $3\theta = \frac{\pi}{2}$, so $\theta = \frac{\pi}{6}$.
9. Yes. For $r = b + a\cos\theta$ with $b < a$ there is an inner loop. Here $b = 2 < a = 3$. Equivalently, $r = 0$ when $\cos\theta = -\frac{2}{3}$, so r changes sign.

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SET B · APPLICATIONS AND MIXED PRACTICE

10. $x = 7.5\cos 2.4 \approx -5.530$, $y = 7.5\sin 2.4 \approx 5.066$. The point is about $(-5.530, 5.066)$.
11. $r = \sqrt{3.2^2 + 4.7^2} \approx 5.686$. The point is in Quadrant III, so $\theta = \pi + \arctan\frac{4.7}{3.2} \approx 4.115$. Answer: $(5.686, 4.115)$.
12. East: $9\cos 40^\circ \approx 6.894$ km. North: $9\sin 40^\circ \approx 5.785$ km.
13. $r(0.8) \approx 2.435$, $r(2.5) \approx 2.197$, $r(4.1) \approx -0.637$. Only $\theta = 4.1$ gives $r < 0$, so that point is plotted on the opposite side of the pole.
14. Use the law of cosines with included angle $2.9 - 0.7 = 2.2$: $d^2 = 6.2^2 + 4.5^2 - 2(6.2)(4.5)\cos 2.2$, so $d \approx 9.567$.
15. $r = 5\cos 1.05 \approx 2.488$. Then $x = r\cos 0.35 \approx 2.337$ and $y = r\sin 0.35 \approx 0.853$.
16. $2 + 3\cos\theta = 0$ gives $\cos\theta = -\frac{2}{3}$, so $\theta = \arccos(-\frac{2}{3}) \approx 2.301$ and $\theta = 2\pi - 2.301 \approx 3.983$.

SET C · CHALLENGE AND AP-STYLE

17. (a) $r = 3\sin 4\theta$: $n = 4$ is even, so 8 petals. $r = 3\cos 5\theta$: $n = 5$ is odd, so 5 petals.
    (b) For odd n, $\cos(n(\theta + \pi)) = -\cos n\theta$, so the point $(r, \theta + \pi)$ with $r = -r(\theta)$ is the same as the point at θ: the second half of $[0, 2\pi]$ retraces the first, giving n petals. For even n the values at θ and $\theta + \pi$ are equal, so the point at $\theta + \pi$ is the opposite of the point at θ and new petals appear, giving $2n$.
18. (a) $1 + 2\cos\theta = 0$ gives $\cos\theta = -\frac{1}{2}$, so $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$.
    (b) $r < 0$ for $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$, so the inner loop is traced on that interval.
    (c) At $\theta = \pi$, $r = -1$: $x = -\cos\pi = 1$, $y = -\sin\pi = 0$. The point is $(1, 0)$.
19. Replacing θ by $-\theta$: $\cos(-2\theta) = \cos 2\theta$, so the equation is unchanged (polar axis). Replacing θ by $\pi - \theta$: $\cos(2\pi - 2\theta) = \cos 2\theta$, unchanged (line $\theta = \frac{\pi}{2}$). Replacing r by $-r$: $(-r)^2 = r^2$, unchanged (pole).
20. Write $r - r\cos\theta = 2$, so $r = 2 + x$. Square: $x^2 + y^2 = (x+2)^2 = x^2 + 4x + 4$, so $y^2 = 4x + 4$, that is $x = \frac{y^2}{4} - 1$. This is a parabola opening to the right with vertex $(-1, 0)$.
21. Setting $1 + \cos\theta = 1 - \cos\theta$ gives $\cos\theta = 0$, so $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ with $r = 1$: the points $(0, 1)$ and $(0, -1)$. The first curve reaches the pole at $\theta = \pi$ and the second at $\theta = 0$, so the pole is also a common point. Answer: $(0, 1)$, $(0, -1)$ and the pole.

7.11 Calculus in Polar Coordinates

SET A · SKILLS

1. $r' = -4\sin\theta = -2$, $r = 2\sqrt{3}$. Numerator: $-2 \cdot \frac{1}{2} + 2\sqrt{3} \cdot \frac{\sqrt{3}}{2} = -1 + 3 = 2$. Denominator: $-2 \cdot \frac{\sqrt{3}}{2} - 2\sqrt{3} \cdot \frac{1}{2} = -2\sqrt{3}$. Slope $= -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$.
2. At $\theta = \frac{\pi}{2}$: $r = 1$, $r' = -1$. Numerator: $-1 \cdot 1 + 1 \cdot 0 = -1$. Denominator: $-1 \cdot 0 - 1 \cdot 1 = -1$. Slope $= 1$.
3. $y = 3\sin^2\theta$, so $\frac{dy}{d\theta} = 6\sin\theta\cos\theta = 3\sin 2\theta = 0$ on $(0, \pi)$ only at $\theta = \frac{\pi}{2}$. There $\frac{dx}{d\theta} = 3\cos 2\theta = -3 \neq 0$, so the tangent is horizontal at $\theta = \frac{\pi}{2}$ only.
4. $r = 0$ when $\sin\theta = -\frac{1}{2}$: $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$. Here $r' = 2\cos\theta \neq 0$. Tangent lines: $\theta = \frac{7\pi}{6}$, i.e. $y = \frac{\sqrt{3}}{3}x$, and $\theta = \frac{11\pi}{6}$, i.e. $y = -\frac{\sqrt{3}}{3}x$.
5. The circle is traced once for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. $A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} 9\cos^2\theta\, d\theta = \frac{9}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2} = \frac{9\pi}{4}$. (Check: radius $\frac{3}{2}$.)
6. $A = \frac{1}{2}\int_0^{2\pi}(1 + 2\cos\theta + \cos^2\theta)\, d\theta = \frac{1}{2}(2\pi + 0 + \pi) = \frac{3\pi}{2}$.
7. One petal: $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$. $A = \frac{1}{2}\int 4\cos^2 3\theta\, d\theta = \int_{-\pi/6}^{\pi/6}(1 + \cos 6\theta)\, d\theta = \frac{\pi}{3}$. Three petals: $3 \cdot \frac{\pi}{3} = \pi$.
8. $A = \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(1 + 2\cos\theta)^2 d\theta = \frac{1}{2}\left[3\theta + 4\sin\theta + \sin 2\theta\right]_{2\pi/3}^{4\pi/3} = \frac{1}{2}(2\pi - 4\sqrt{3} + \sqrt{3}) = \pi - \frac{3\sqrt{3}}{2}$.
9. $A = \frac{1}{2}\int_0^{2\pi}(16 - 4)\, d\theta = 6 \cdot 2\pi = 12\pi$. (Check: $16\pi - 4\pi$.)

SET B · APPLICATIONS AND MIXED PRACTICE

10. $r = 2 + 3\sin\theta$, $r' = 3\cos\theta$. $\dfrac{dy}{dx} = \dfrac{3\cos\theta\sin\theta + (2 + 3\sin\theta)\cos\theta}{3\cos^2\theta - (2 + 3\sin\theta)\sin\theta}$. At $\theta = 1$ this is about -1.299.
11. $A = \frac{1}{2}\int_0^{2\pi} e^{2\sin\theta} d\theta \approx 7.162$.
12. The curves meet where $2 + 2\cos\theta = 3$, $\theta = \pm\frac{\pi}{3}$, and the cardioid is outside $r = 3$ for $|\theta| < \frac{\pi}{3}$. $A = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[(2 + 2\cos\theta)^2 - 9\right] d\theta \approx 4.653$.
13. The curves meet at $\theta = \pm\frac{\pi}{3}$. For $0 \le \theta \le \frac{\pi}{3}$ the cardioid is nearer the pole; for $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$ the circle is. By symmetry $A = 2\left[\frac{1}{2}\int_0^{\pi/3}(1 + \cos\theta)^2 d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} 9\cos^2\theta\, d\theta\right] \approx 3.927$ (exactly $\frac{5\pi}{4}$).
14. $A = \frac{1}{2}\int_0^{2\pi}(3 + \sin 3\theta)^2 d\theta \approx 29.845$ square meters.
15. $L = \int_0^2 \sqrt{\theta^2 + 1}\, d\theta \approx 2.958$.
16. $r = \frac{2\pi}{3} \approx 2.094$, $r' = 2$. The point is $(1.047, 1.814)$. The slope is $\frac{2\sin\theta + r\cos\theta}{2\cos\theta - r\sin\theta} \approx -3.415$. Then $b = 1.814 + 3.415(1.047) \approx 5.390$. Line: $y \approx -3.415x + 5.390$.

SET C · CHALLENGE AND AP-STYLE

17. Intersections: $2\cos\theta = 1$, $\theta = \pm\frac{\pi}{3}$. Circle outer. $A = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(4\cos^2\theta - 1)\, d\theta = \frac{1}{2}\int(1 + 2\cos 2\theta)\, d\theta = \frac{1}{2}\left[\theta + \sin 2\theta\right]_{-\pi/3}^{\pi/3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$.
18. The petal is outside $r = 1$ when $2\sin 2\theta > 1$, i.e. $\frac{\pi}{12} < \theta < \frac{5\pi}{12}$. $A = \frac{1}{2}\int_{\pi/12}^{5\pi/12}(4\sin^2 2\theta - 1)\, d\theta = \frac{1}{2}\int(1 - 2\cos 4\theta)\, d\theta = \frac{1}{2}\left[\theta - \frac{\sin 4\theta}{2}\right]_{\pi/12}^{5\pi/12} = \frac{\pi}{6} + \frac{\sqrt{3}}{4}$.
19. (a) $r = 0$ when $\cos 2\theta = 0$: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$. $r' = -4\sin 2\theta \neq 0$ there, so the tangent lines are $y = x$ and $y = -x$.
    (b) $r = 2\cos\frac{\pi}{3} = 1$, $r' = -4\sin\frac{\pi}{3} = -2\sqrt{3}$. Numerator: $-2\sqrt{3} \cdot \frac{1}{2} + 1 \cdot \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$. Denominator: $-2\sqrt{3} \cdot \frac{\sqrt{3}}{2} - 1 \cdot \frac{1}{2} = -\frac{7}{2}$. Slope $= \frac{\sqrt{3}}{7}$.
20. The curves meet where $\sin\theta = 0$. For $0 \le \theta \le \pi$, $1 + \sin\theta \ge 1$, so the cardioid is outside the circle there; for $\pi < \theta < 2\pi$ it is inside. $A = \frac{1}{2}\int_0^{\pi}\left[(1 + \sin\theta)^2 - 1\right] d\theta = \frac{1}{2}\int_0^{\pi}(2\sin\theta + \sin^2\theta)\, d\theta = \frac{1}{2}\left(4 + \frac{\pi}{2}\right) = 2 + \frac{\pi}{4}$.
21. Area $= \frac{1}{2}\int_0^{2\pi} a^2(1 + \cos\theta)^2 d\theta = \frac{a^2}{2}(2\pi + \pi) = \frac{3\pi a^2}{2}$. Setting this equal to 24π gives $a^2 = 16$, so $a = 4$.

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7.12 Multiple-Choice Practice on Series, Parametrics, Vectors, and Polar

SET A · PART A [NO CALCULATOR]

1. E. This is geometric with first term 3 and ratio $-\frac{2}{5}$, and $|r|<1$, so the sum is $\frac{3}{1+2/5}=\frac{15}{7}$.
2. A. Since $\frac{n}{n^3+1}<\frac{1}{n^2}$, the comparison test with a convergent p-series ($p=2$) applies. The others diverge: $p=\frac{1}{2}$; terms tend to 1; limit comparison with $\frac{1}{n}$; terms do not tend to 0.
3. A. The terms $\frac{1}{\sqrt{n}}$ decrease to 0, so the alternating series test gives convergence. The absolute series is a p-series with $p=\frac{1}{2}\le 1$, which diverges, so the convergence is conditional.
4. A. The ratio test gives $|x-2|<3$, so the radius is 3 and the center is 2. At $x=5$ the series is $\sum\frac{1}{n}$ (diverges). At $x=-1$ it is $\sum\frac{(-1)^n}{n}$ (converges).
5. B. $\sin(x^2)=x^2-\frac{x^6}{6}+\cdots$. The coefficient of x^6 equals $\frac{f^{(6)}(0)}{6!}$, so $f^{(6)}(0)=-\frac{720}{6}=-120$.
6. D. $P_2(2.1)=3+(-1)(0.1)+\frac{4}{2}(0.1)^2=3-0.1+0.02=2.92$.
7. D. The third derivative of $\cos x$ is $\sin x$, with $|\sin x|\le 1$. The bound is $\frac{1}{3!}\left(\frac{1}{2}\right)^3=\frac{1}{48}$.
8. B. $\frac{dy}{dt}=3t^2-3=9$ and $\frac{dx}{dt}=2t=4$ at $t=2$, so $\frac{dy}{dx}=\frac{9}{4}$.
9. B. $\frac{dy}{dx}=\frac{3t^2}{2t}=\frac{3t}{2}$. Then $\frac{d^2y}{dx^2}=\frac{d/dt(3t/2)}{dx/dt}=\frac{3/2}{2t}=\frac{3}{4t}=\frac{3}{8}$ at $t=2$.
10. E. The speed is $\sqrt{36t^2+36t^4}=6t\sqrt{1+t^2}$. An antiderivative is $2(1+t^2)^{3/2}$, so $L=2\left(2^{3/2}-1\right)=4\sqrt{2}-2$.
11. A. The velocity is $\langle 6t^2,8\rangle=\langle 6,8\rangle$ at $t=1$, so the speed is $\sqrt{36+64}=10$.
12. C. Integrate: $x=t^2+1$, $y=t^3-1$. At $t=2$ the position is $\langle 5,7\rangle$.
13. D. That petal is traced for $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$. $A=\frac{1}{2}\int\cos^2 2\theta\,d\theta=\frac{1}{4}\left[\theta+\frac{\sin 4\theta}{4}\right]_{-\pi/4}^{\pi/4}=\frac{\pi}{8}$.
14. C. The circles meet where $2\cos\theta=1$, $\theta=\pm\frac{\pi}{3}$. Between those angles the outer radius is $2\cos\theta$ and the inner is 1, so square each separately before subtracting.
15. C. The curve is at the pole when $2\sin\theta-1=0$, i.e. $\theta=\frac{\pi}{6}$ (or $\frac{5\pi}{6}$), and $r'=2\cos\theta\neq 0$ there. For the other choices $r\neq 0$.
16. B. This is the Maclaurin series $\arctan x=\sum\frac{(-1)^n x^{2n+1}}{2n+1}$ evaluated at $x=\frac{1}{3}$, inside its interval of convergence.
17. D. By the integral test, $\int_2^\infty\frac{dx}{x(\ln x)^p}=\int\frac{du}{u^p}$ with $u=\ln x$, which converges exactly when $p>1$.
18. E. Differentiate $\sum x^n=\frac{1}{1-x}$ to get $\sum nx^{n-1}=\frac{1}{(1-x)^2}$, so $\sum nx^n=\frac{x}{(1-x)^2}$. At $x=\frac{1}{2}$ this is $\frac{1/2}{1/4}=2$.
19. C. Vertical tangents need $\frac{dx}{dt}=3t^2-3=0$ with $\frac{dy}{dt}=2t\neq 0$. So $t=\pm 1$. At $t=0$, $\frac{dx}{dt}=-3\neq 0$ and $\frac{dy}{dt}=0$, so the tangent is horizontal.
20. E. $A=\frac{1}{2}\int_0^\pi\theta^2\,d\theta=\frac{1}{2}\cdot\frac{\pi^3}{3}=\frac{\pi^3}{6}$.

SET B · PART B [CALCULATOR]

21. D. The error is bounded by the first omitted term, $a_5=\frac{1}{125}=0.008$.
22. A. $L=\int_0^2\sqrt{4+4t^2}\,dt=2\int_0^2\sqrt{1+t^2}\,dt\approx 5.916$.
23. C. $f^{(4)}(x)=-\frac{6}{x^4}$, so $|f^{(4)}|\le 6$ on $[1,1.2]$. The bound is $\frac{6}{4!}(0.2)^4=0.0004$.
24. B. Speed $=\sqrt{(3\sin 2)^2+4^2}=\sqrt{7.443+16}\approx 4.842$.
25. E. Distance $=\int_0^2\sqrt{e^{2t}+4t^2}\,dt\approx 7.566$. (The straight-line displacement $\sqrt{(e^2-1)^2+4^2}\approx 7.538$ is smaller.)
26. D. $A=\frac{1}{2}\int_0^\pi e^{2\theta/3}d\theta=\frac{3}{4}\left(e^{2\pi/3}-1\right)\approx 5.340$.
27. A. $P_4(x)=1-\frac{x^2}{2}+\frac{x^4}{24}$, so $P_4(0.8)=1-0.32+0.0171\approx 0.697$.
28. C. $x(3)=1+2\sin 3\approx 1.282$ and $y(3)=2+(1-e^{-3})\approx 2.950$. The distance is $\sqrt{1.282^2+2.950^2}\approx 3.217$.
29. B. $\frac{dy}{dx}=\frac{-e^{-t}}{3t^2+1}$. At $t=1$ this is $-\frac{e^{-1}}{4}\approx -0.092$.
30. E. With $r'=1$: $\frac{dy}{dx}=\frac{\sin 2+2\cos 2}{\cos 2-2\sin 2}\approx\frac{0.0770}{-2.2347}\approx -0.034$.

7.13 Free-Response Practice on Series, Parametrics, Vectors, and Polar

QUESTION 1 · SERIES

1. (a) Ratio test: $\lim\left|\frac{(x-1)^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{(x-1)^n}\right|=\frac{|x-1|}{3}<1$, so $|x-1|<3$ and $-2<x<4$. At $x=4$: $\sum\frac{(-1)^{n+1}}{n}$ converges by the alternating series test (terms decrease to 0). At $x=-2$: the terms are $\frac{(-1)^{n+1}(-3)^n}{n3^n}=-\frac{1}{n}$, so the series is $-\sum\frac{1}{n}$, which diverges. The interval of convergence is $(-2,4]$.
   (b) Differentiate term by term: $f'(x)=\frac{1}{3}-\frac{x-1}{9}+\frac{(x-1)^2}{27}-\frac{(x-1)^3}{81}+\cdots$ with general term $\frac{(-1)^{n+1}(x-1)^{n-1}}{3^n}$.
   (c) This is geometric with first term $\frac{1}{3}$ and ratio $-\frac{x-1}{3}$, with $|\text{ratio}|<1$ for $|x-1|<3$. The sum is $\frac{1/3}{1+(x-1)/3}=\frac{1}{x+2}$.
   (d) $f(2)=\frac{1}{3}-\frac{1}{18}+\frac{1}{81}-\cdots$. The sum of the first three terms is $\frac{54-9+2}{162}=\frac{47}{162}$. The series alternates, its terms $\frac{1}{n3^n}$ decrease in magnitude and tend to 0 , so the error is at most the first omitted term $\frac{1}{4\cdot 81}=\frac{1}{324}<\frac{1}{300}$.
   Scoring: 9 points. (a) 3 points: ratio test and radius (1), endpoint $x=4$ (1), endpoint $x=-2$ and interval (1). (b) 2 points: terms (1), general term (1). (c) 2 points: identifies geometric with correct first term and ratio (1), correct sum (1). (d) 2 points: approximation $\frac{47}{162}$ (1), error bound with justification (1).

QUESTION 2 · PARAMETRIC MOTION

2. (a) Acceleration $=\langle -2t,-3\sin t\rangle$, which at $t=1.5$ is $\langle -3.000,-2.992\rangle$. Velocity at $t=1.5$ is $\langle 1.75,0.212\rangle$, so the speed is $\sqrt{1.75^2+0.212^2}\approx 1.763$ m/s.
   (b) The robot is at rest only if both components of velocity are 0. $x'(t)=0$ only at $t=2$ on this interval, and $y'(2)=3\cos 2\approx -1.248\neq 0$. So the robot is never at rest.
   (c) $x(3)=1+\int_0^3(4-t^2)\,dt=1+12-9=4$ and $y(3)=2+\int_0^3 3\cos t\,dt=2+3\sin 3\approx 2.423$. The position is $(4,\,2.423)$.
   (d) Distance $=\int_0^3\sqrt{(4-t^2)^2+9\cos^2 t}\,dt\approx 9.734$ meters.
   Scoring: 9 points. (a) 2 points: acceleration (1), speed (1). (b) 2 points: condition that both components vanish (1), conclusion with $y'(2)\neq 0$ (1). (c) 2 points: integral setup or antiderivatives (1), correct coordinates (1). (d) 3 points: integrand (1), limits (1), answer (1).

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QUESTION 3 · POLAR CURVES

3. (a) For $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$ the limacon is outside the circle, so $A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left[(3+2\cos\theta)^2 - 4^2\right] d\theta$. Evaluating gives $A = \frac{13\sqrt{3}}{2} - \frac{5\pi}{3} \approx 6.022$.

(b) $r\left(\frac{\pi}{3}\right) = 4$ and $r' = -2\sin\theta = -\sqrt{3}$. $\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-\sqrt{3}\cdot\frac{\sqrt{3}}{2} + 2}{-\frac{\sqrt{3}}{2} - 2\sqrt{3}} = \frac{1/2}{-5\sqrt{3}/2} = -\frac{\sqrt{3}}{15} \approx -0.115$.

(c) $y = (3+2\cos\theta)\sin\theta$, $\frac{dy}{d\theta} = 4\cos^2\theta + 3\cos\theta - 2 = 0$ gives $\cos\theta = \frac{-3+\sqrt{41}}{8} \approx 0.425$, $\theta \approx 1.131$. y increases before this value and decreases after (and $y = 0$ at both ends), so the maximum is $y(1.131) \approx 3.485$.

(d) $x = (3+2\cos\theta)\cos\theta$, $\frac{dx}{d\theta} = -\sin\theta(3+4\cos\theta)$. At $\theta = \frac{\pi}{3}$ this is $-\frac{\sqrt{3}}{2}\cdot 5 \approx -4.330$, so $\frac{dx}{dt} = -4.330(0.5) \approx -2.165$. Since $\frac{dx}{dt} < 0$, the point is moving to the left.

Scoring: 9 points. (a) 3 points: integrand with outer and inner radius (1), limits (1), area (1). (b) 2 points: slope formula or derivatives (1), answer (1). (c) 2 points: sets $\frac{dy}{d\theta} = 0$ (1), maximum value with justification (1). (d) 2 points: $\frac{dx}{dt}$ by the chain rule (1), value and direction (1).

QUESTION 4 · TAYLOR ERROR

4. (a) $f^{(k)}(x) = 2^k e^{2x}$, so $f(0) = 1$, $f'(0) = 2$, $f''(0) = 4$, $f'''(0) = 8$. $P_3(x) = 1 + 2x + \frac{4}{2}x^2 + \frac{8}{6}x^3 = 1 + 2x + 2x^2 + \frac{4}{3}x^3$.

(b) $P_3(0.1) = 1 + 0.2 + 0.02 + \frac{4}{3}(0.001) = 1.2213333\ldots \approx 1.22133$.

(c) $f^{(4)}(x) = 16e^{2x}$, which is increasing, so on $[0, 0.1]$ we have $|f^{(4)}(x)| \le 16e^{0.2} < 32$. The error is at most $\frac{32}{4!}(0.1)^4 = \frac{4}{3}\times 10^{-4} \approx 0.000133 < 0.0002$.

(d) The series is $\sum \frac{(2x)^n}{n!}$. $\lim \left|\frac{(2x)^{n+1}/(n+1)!}{(2x)^n/n!}\right| = \lim \frac{2|x|}{n+1} = 0 < 1$ for every x. The series converges for all real x: interval $(-\infty, \infty)$.

Scoring: 9 points. (a) 2 points: derivative values (1), polynomial (1). (b) 1 point: value. (c) 3 points: fourth derivative (1), bound on $|f^{(4)}|$ with justification (1), error bound and comparison to 0.0002 (1). (d) 3 points: ratio setup (1), limit 0 (1), interval (1).

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Chapter 8: After the AP Exam.

3 sections with explanations and worked examples, then 63 practice problems. Answers and full explanations are at the end of the chapter.

8.1 Hyperbolic Functions

You will meet sinh, cosh and tanh, their identities, derivatives and integrals, and the inverse hyperbolic functions. They model hanging cables and motion with air resistance.

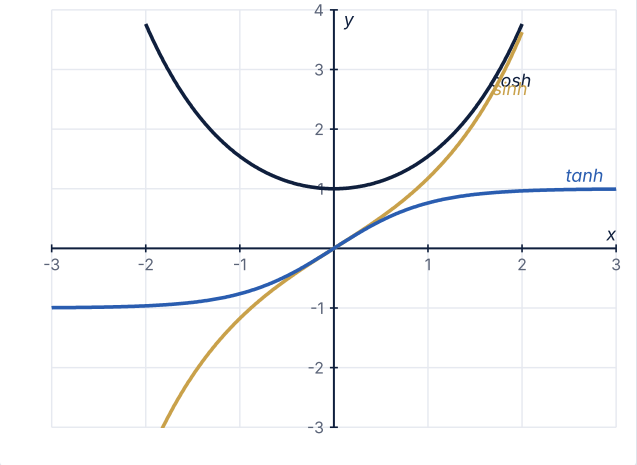
Certain combinations of e^x and e^{-x} appear so often in physics and engineering (hanging cables, wave motion, motion with air resistance) that they have their own names. They behave like the trigonometric functions in many ways, but they are tied to a hyperbola instead of a circle.

Definitions

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}, \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Also $\operatorname{sech} x = \frac{1}{\cosh x}$, $\operatorname{csch} x = \frac{1}{\sinh x}$, $\coth x = \frac{\cosh x}{\sinh x}$.

$\sinh$ is odd, $\cosh$ is even, and $\cosh x \geq 1$ for all x. The graph of $\cosh x$ is the shape of a hanging cable (a catenary), and $\tanh x$ rises from -1 to 1 with horizontal asymptotes $y = \pm 1$.



Graphs of $\cosh x$, $\sinh x$ and $\tanh x$.

Identities

$$\cosh^2 x - \sinh^2 x = 1, \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x,$$
$$\sinh 2x = 2 \sinh x \cosh x, \qquad \cosh 2x = \cosh^2 x + \sinh^2 x,$$
$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x}.$$

The first identity shows that $(x, y) = (\cosh t, \sinh t)$ lies on the hyperbola $x^2 - y^2 = 1$. Note the minus sign: it is $\cosh^2 - \sinh^2 = 1$, not $+$.

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EXAMPLE 1

Evaluate $\sinh(\ln 2)$, $\cosh(\ln 2)$ and $\tanh(\ln 2)$ exactly. Then verify $\cosh^2 x - \sinh^2 x = 1$ at $x = \ln 2$.

SOLUTION

Use $e^{\ln 2} = 2$ and $e^{-\ln 2} = \frac{1}{2}$.

$\sinh(\ln 2) = \frac{2 - 1/2}{2} = \frac{3}{4}$ and $\cosh(\ln 2) = \frac{2 + 1/2}{2} = \frac{5}{4}$, so $\tanh(\ln 2) = \frac{3/4}{5/4} = \frac{3}{5}$.

Check: $\frac{25}{16} - \frac{9}{16} = 1$.

EXAMPLE 2

Prove $\cosh^2 x - \sinh^2 x = 1$.

SOLUTION

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{4} = \frac{4}{4} = 1.$$

Derivatives

$$\frac{d}{dx}\sinh x = \cosh x, \quad \frac{d}{dx}\cosh x = \sinh x, \quad \frac{d}{dx}\tanh x = \operatorname{sech}^2 x.$$

Unlike the trigonometric case there is no minus sign for $\cosh$. Each rule follows from differentiating e^x and e^{-x} directly. Combine with the chain, product and quotient rules as usual.

EXAMPLE 3

Differentiate (a) $y = \sinh(3x^2)$, (b) $y = \ln(\cosh x)$, (c) $y = x \tanh x$.

SOLUTION

(a) Chain rule: $y' = \cosh(3x^2) \cdot 6x = 6x\cosh(3x^2)$.

(b) $y' = \frac{\sinh x}{\cosh x} = \tanh x$.

(c) Product rule: $y' = \tanh x + x\operatorname{sech}^2 x$.

Integrals

$$\int \sinh x\,dx = \cosh x + C, \quad \int \cosh x\,dx = \sinh x + C, \quad \int \operatorname{sech}^2 x\,dx = \tanh x + C,$$
$$\int \tanh x\,dx = \ln(\cosh x) + C.$$

The last one needs no absolute value because $\cosh x > 0$. It follows from $u = \cosh x$, $du = \sinh x\,dx$.

EXAMPLE 4

Evaluate (a) $\int \sinh x \cosh^3 x\,dx$ and (b) $\int_0^{\ln 2} \cosh x\,dx$.

SOLUTION

(a) Let $u = \cosh x$, $du = \sinh x\,dx$: $\int u^3\,du = \frac{u^4}{4} + C = \frac{\cosh^4 x}{4} + C$.

(b) $[\sinh x]_0^{\ln 2} = \sinh(\ln 2) - 0 = \frac{3}{4}$.

Inverse hyperbolic functions

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \text{ (all } x\text{)}, \qquad \cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \ (x \geq 1),$$
$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1 + x}{1 - x} \ (|x| < 1).$$

Derivatives:

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}}, \quad \frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}} \ (x > 1), \quad \frac{d}{dx}\tanh^{-1} x = \frac{1}{1 - x^2} \ (|x| < 1).$$

Antiderivative: $\displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + C = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$ for $a > 0$.

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EXAMPLE 5

(a) Differentiate $y = \sinh^{-1}(x^2)$. (b) Evaluate $\int_0^1 \frac{dx}{\sqrt{x^2+1}}$.

SOLUTION

(a) Chain rule: $y' = \frac{1}{\sqrt{(x^2)^2+1}} \cdot 2x = \frac{2x}{\sqrt{x^4+1}}$.

(b) The integral equals $\sinh^{-1}(1) - \sinh^{-1}(0) = \ln\left(1+\sqrt{2}\right) - \ln 1 = \ln\left(1+\sqrt{2}\right) \approx 0.881$.

Common mistakes

(1) Writing $\cosh^2 x + \sinh^2 x = 1$; the correct identity has a minus sign. (2) Using $\frac{d}{dx}\cosh x = -\sinh x$ by analogy with cosine. (3) Applying $\cosh^{-1} x$ for $x < 1$: its domain is $x \geq 1$. (4) Confusing $\sinh^{-1} x$ with $\frac{1}{\sinh x}$.

Practice

SET A Skills

Give exact answers. [No calculator]

1. Evaluate $\cosh(\ln 3)$ exactly.
2. Evaluate $\tanh(\ln 3)$ exactly.
3. Differentiate $y = x^2 \sinh x$.
4. Differentiate $y = \ln(\sinh x)$ for $x > 0$.
5. Differentiate $y = \tanh(5x)$.
6. Find $\int \cosh(2x)\,dx$.
7. Find $\int \operatorname{sech}^2(3x)\,dx$.
8. Evaluate $\int_0^{\ln 3} \sinh x\,dx$.
9. If $\sinh x = \frac{3}{4}$, find $\cosh x$ and $\tanh x$.

SET B Applications and Mixed Practice

Give decimal answers rounded to three places. [Calculator]

10. Find $\sinh 1.2$, $\cosh 1.2$ and $\tanh 1.2$.
11. Solve $\sinh x = 2$ for x.
12. Solve $\cosh x = 3$ for $x > 0$.
13. Let $f(x) = x\cosh x$. Find $f'(0.5)$.
14. Find the area of the region under $y = \cosh x$ and above the x-axis for $0 \leq x \leq 2$.
15. A hanging cable has the shape $y = 20\cosh\frac{x}{20}$ (in meters) for $-15 \leq x \leq 15$. Find (a) the length of the cable and (b) how much higher the ends are than the lowest point.
16. A falling object has speed $v(t) = 30\tanh\frac{t}{3}$ meters per second after t seconds. (a) What is the limiting speed? (b) At what time does the speed reach 25 meters per second?

SET C Challenge and Proof

Show your reasoning. [No calculator]

17. Prove from the definitions that $\sinh 2x = 2\sinh x\cosh x$.
18. Show that $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$ for $x > 1$ by differentiating $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$.
19. Solve $y = \sinh x$ for x to derive $\sinh^{-1} y = \ln\left(y + \sqrt{y^2+1}\right)$.
20. Evaluate $\int_0^4 \frac{dx}{\sqrt{x^2+9}}$ exactly.
21. Evaluate $\int_0^{\ln 3} \tanh x\,dx$ exactly.

8.2 Surface Area of a Solid of Revolution

Learn to compute the area of the outer skin of a solid of revolution about either axis, and see how it differs from volume.

A **surface of revolution** is the skin of the solid you get by spinning a curve about a line. Think of a vase, a bowl or a light bulb: the volume tells you how much fits inside, and the surface area tells you how much material or paint covers the outside. To measure it, slice the curve into tiny pieces of length ds. Each piece sweeps out a thin band, roughly a strip of width ds wrapped around a circle of radius r, so its area is about $2\pi r\,ds$. Adding the bands gives an integral.

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Surface area about the x-axis

Let f be nonnegative with a continuous derivative f' on $[a, b]$. The area of the surface formed by revolving $y = f(x)$, $a \le x \le b$, about the x-axis is

$$S = \int_a^b 2\pi\, f(x)\sqrt{1 + [f'(x)]^2}\, dx.$$

Here $2\pi f(x)$ is the circumference of the circle swept by the point $(x, f(x))$, and $\sqrt{1 + [f'(x)]^2}\, dx$ is the arc length element ds.

Surface area about the y-axis

Let f' be continuous on $[a, b]$ with $0 \le a < b$. Revolving $y = f(x)$, $a \le x \le b$, about the y-axis gives

$$S = \int_a^b 2\pi\, x\sqrt{1 + [f'(x)]^2}\, dx.$$

Now the radius of each circle is the distance from the point to the y-axis, which is x. If the curve is written as $x = g(y)$ for $c \le y \le d$ and revolved about the y-axis, use $S = \int_c^d 2\pi\, g(y)\sqrt{1 + [g'(y)]^2}\, dy$.

The pattern is always **surface area** $= \int 2\pi\,(\text{radius})\, ds$. The radius is the distance from the curve to the axis of revolution, and $ds = \sqrt{1 + (dy/dx)^2}\, dx$ is arc length. The hypotheses matter: the radius must be nonnegative (otherwise you would subtract area), and f' must be continuous so the arc length integrand is defined and integrable.

1. **Sketch and identify the axis.** Decide which line the curve spins about and which variable measures the distance to it.
2. **Write the radius.** It is $f(x)$ for the x-axis and x for the y-axis.
3. **Compute $1 + [f'(x)]^2$.** Simplify it. For nice problems it is a perfect square or the product with the radius is easy to integrate.
4. **Set up and integrate.** Multiply $2\pi \cdot (\text{radius}) \cdot \sqrt{1 + [f'(x)]^2}$, then use a u-substitution when the radius is a multiple of u'.

EXAMPLE 1: A CONE

Find the surface area generated by revolving $y = x$, $0 \le x \le 3$, about the x-axis.

SOLUTION

Radius $f(x) = x$, and $f'(x) = 1$, so $\sqrt{1 + 1^2} = \sqrt{2}$.

$S = \int_0^3 2\pi x\sqrt{2}\, dx = 2\sqrt{2}\,\pi\left[\frac{x^2}{2}\right]_0^3 = 9\sqrt{2}\,\pi.$

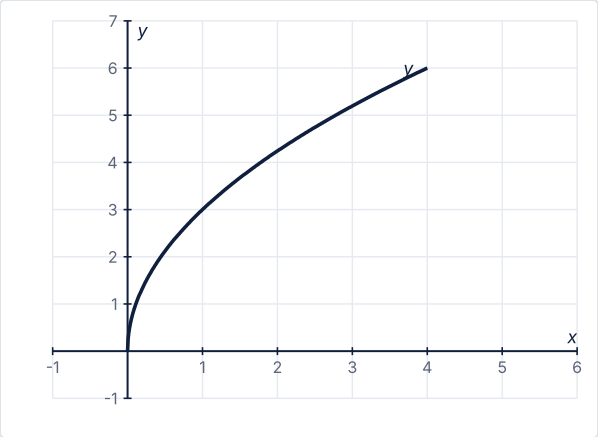
Check with geometry: the cone has radius 3 and slant height $3\sqrt{2}$, so lateral area $\pi r\ell = \pi \cdot 3 \cdot 3\sqrt{2} = 9\sqrt{2}\,\pi$.

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EXAMPLE 2: SUBSTITUTION ABOUT THE X-AXIS

Find the area of the surface formed by revolving $y = 3\sqrt{x}$, $0 \le x \le 4$, about the x-axis.



The curve y = 3 sqrt(x) on [0, 4], revolved about the x-axis.

SOLUTION

$f'(x) = \dfrac{3}{2\sqrt{x}}$, so $1 + [f'(x)]^2 = 1 + \dfrac{9}{4x} = \dfrac{4x+9}{4x}$. (The derivative is not continuous at $x = 0$, but the integrand stays bounded, as shown next, so the integral is fine.)

$f(x)\sqrt{1+[f'(x)]^2} = 3\sqrt{x} \cdot \dfrac{\sqrt{4x+9}}{2\sqrt{x}} = \dfrac{3}{2}\sqrt{4x+9}$.

$S = 2\pi \int_0^4 \frac{3}{2}(4x+9)^{1/2}\,dx$. Let $u = 4x+9$, $du = 4\,dx$: $S = 3\pi \cdot \frac{1}{4}\int_9^{25} u^{1/2}\,du = \frac{3\pi}{4} \cdot \frac{2}{3}\left[u^{3/2}\right]_9^{25}$.

$S = \frac{\pi}{2}(125 - 27) = 49\pi$.

EXAMPLE 3: REVOLVING ABOUT THE Y-AXIS

Find the surface area generated by revolving $y = 2x^2$, $0 \le x \le 1$, about the y-axis.

SOLUTION

The radius is x. $f'(x) = 4x$, so $\sqrt{1+16x^2}$.

$S = \int_0^1 2\pi x\sqrt{1+16x^2}\,dx$. Let $u = 1+16x^2$, $du = 32x\,dx$, so $x\,dx = \frac{1}{32}du$. Limits: $x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 17$.

$S = 2\pi \cdot \frac{1}{32} \cdot \frac{2}{3}\left[u^{3/2}\right]_1^{17} = \dfrac{\pi}{24}\left(17\sqrt{17} - 1\right)$.

Common mistakes

(1) Forgetting the factor $2\pi \cdot (\text{radius})$ and integrating only the arc length. (2) Using y as the radius when the axis is the y-axis (the radius is then x). (3) Writing $\sqrt{1+f'(x)^2}$ as $1 + f'(x)$: you cannot take the square root term by term. (4) Forgetting to change the limits after a u-substitution.

Comparing with volume. Revolving the same curve gives a solid whose volume (disks) is $V = \pi\int_a^b [f(x)]^2\,dx$. Volume uses the area πr^2 of a disk; surface area uses the circumference $2\pi r$ times arc length. Volume has no square root of a derivative, so it is usually easier to integrate, and the two quantities can behave very differently.

Quantity	About the x-axis	About the y-axis (shells, $0 \le a < b$)
Volume	$\pi\int_a^b [f(x)]^2\,dx$	$2\pi\int_a^b x\,f(x)\,dx$
Surface area	$2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$	$2\pi\int_a^b x\sqrt{1+[f'(x)]^2}\,dx$

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EXAMPLE 4: VOLUME AND SURFACE AREA TOGETHER

For the solid in Example 2 (the curve $y = 3\sqrt{x}$ on $[0, 4]$ revolved about the x-axis), find the volume and compare it with the surface area.

SOLUTION

$V = \pi \int_0^4 (3\sqrt{x})^2\,dx = \pi \int_0^4 9x\,dx = \pi \cdot \frac{9}{2} \cdot 16 = 72\pi$.

From Example 2, $S = 49\pi$. Numerically $V \approx 226.19$ and $S \approx 153.94$, and they have different units (cubic versus square), so only their formulas, not their sizes, are comparable.

Doubling every length multiplies S by 4 and V by 8.

Practice

SET A Skills

Evaluate or set up each surface area as directed. [No calculator]

1. Find the surface area generated by revolving $y = 2x$, $0 \le x \le 3$, about the x-axis.
2. Find the surface area generated by revolving $y = \dfrac{x^3}{3}$, $0 \le x \le 1$, about the x-axis.
3. Find the surface area generated by revolving $y = \sqrt{25 - x^2}$, $-3 \le x \le 3$, about the x-axis. (Hint: simplify $y\sqrt{1 + (y')^2}$ first.)
4. Find the surface area generated by revolving $y = \sqrt{2x}$, $0 \le x \le 4$, about the x-axis.
5. Find the surface area generated by revolving $y = x^2$, $0 \le x \le 1$, about the y-axis.
6. Find the surface area generated by revolving $y = 3x$, $1 \le x \le 2$, about the y-axis.
7. The curve $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$, $1 \le x \le 2$, is revolved about the x-axis. Show that $1 + (y')^2 = \left(\dfrac{x^2}{2} + \dfrac{1}{2x^2}\right)^2$ and then find the surface area.
8. Write, but do not evaluate, an integral for the area of the surface formed by revolving $y = e^x$, $0 \le x \le 1$, about the x-axis.
9. Explain in one or two sentences why the surface area integral contains the factor $2\pi f(x)$ and why the arc length factor $\sqrt{1 + [f'(x)]^2}$ also appears.

SET B Applications

Use the surface area formula in context. Give decimals to three decimal places where asked. [Calculator]

10. The curve $y = \sin x$, $0 \le x \le \pi$, is revolved about the x-axis. Find the surface area to three decimal places.
11. A reflecting dish is formed by revolving $y = \dfrac{x^2}{4}$, $0 \le x \le 2$ (meters), about the y-axis. Find the area of the dish, to three decimal places.
12. A glass cap is the part of a sphere of radius 10 cm formed by revolving $y = \sqrt{100 - x^2}$, $6 \le x \le 8$, about the x-axis. Find its surface area in exact form and to three decimal places.
13. A storage tank is formed by revolving $y = \sqrt{x + 1}$, $0 \le x \le 3$ (meters), about the x-axis. Find the area of its curved surface to three decimal places.
14. The curved surface of the tank formed by revolving $y = \sqrt{x + 1}$, $0 \le x \le 3$ (meters), about the x-axis is coated at a cost of 12 dollars per square meter. Find the cost to the nearest cent.
15. The curve $y = \sqrt{x}$, $0 \le x \le 4$, is revolved about the x-axis. Find the volume V and the surface area S to three decimal places, and compute S/V.
16. A cone-shaped funnel is made by revolving $y = \dfrac{x}{2}$, $0 \le x \le 8$ (cm), about the x-axis. Find the lateral surface area using an integral, then check it with $\pi r \ell$.

SET C Challenge

Justify each result clearly. [No calculator]

17. Revolve $y = \sqrt{r^2 - x^2}$, $-r \le x \le r$, about the x-axis ($r > 0$ constant). Show that the surface area is $4\pi r^2$. Why must you be careful at the endpoints?
18. Let $0 < a < b$ and $m > 0$. Revolve $y = mx$, $a \le x \le b$, about the x-axis. Show that $S = \pi(r_1 + r_2)\ell$, where $r_1 = ma$, $r_2 = mb$ and ℓ is the slant length of the frustum.
19. The curve $y = 2\sqrt{x}$, $0 \le x \le b$, is revolved about the x-axis and the surface area is $\dfrac{208\pi}{3}$. Find b. Justify that b is unique.

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20. The curve $x = 2\sqrt{y}$, $0 \le y \le 3$, is revolved about the y-axis. Find the surface area.
21. A curve in the upper half plane has arc length 10 and its centroid is at height $\bar{y} = 3$. Using $S = 2\pi \bar{y}\, L$ for revolution about the x-axis, find S. Then confirm the formula for a semicircle of radius 5 (arc length 5π, centroid height $10/\pi$).

8.3 Linear First-Order Differential Equations

Master the integrating factor method for equations of the form y' + P(x)y = Q(x), then apply it to mixing tanks and simple circuits.

Many changing quantities obey a rule of the form rate of change equals rate in minus rate out: salt in a tank, the current in a circuit, the amount of medicine in the blood. When the unknown appears only to the first power in its own rate equation, the equation is **linear**, and there is a clean method that always works: multiply by a well-chosen function so the left side becomes the derivative of a product.

Linear first-order equation and integrating factor

A first-order linear equation can be written in standard form

$$\frac{dy}{dx} + P(x)\, y = Q(x),$$

where P and Q are continuous on an interval I.

The **integrating factor** is $\mu(x) = e^{\int P(x)\,dx}$ (any one antiderivative of P works; take the constant to be 0). Then $\mu' = P\mu$, so

$$\frac{d}{dx}\big[\mu(x)\, y\big] = \mu(x)\, Q(x).$$

General solution

Integrating both sides of $\frac{d}{dx}[\mu y] = \mu Q$ gives

$$y = \frac{1}{\mu(x)} \left(\int \mu(x)\, Q(x)\, dx + C \right).$$

If P and Q are continuous on I and $x_0 \in I$, the initial value problem $y(x_0) = y_0$ has exactly one solution on all of I.

1. **Put the equation in standard form.** Divide so that the coefficient of y' is 1. For example $x\,y' + 3y = x^2$ becomes $y' + \frac{3}{x}y = x$ (for $x > 0$).
2. **Find μ.** Compute $\int P\,dx$ and exponentiate. Note that $e^{\ln x^3} = x^3$ for $x > 0$.
3. **Multiply and recognize the product rule.** The left side collapses to $(\mu y)'$.
4. **Integrate and divide by μ.** Do not forget $+C$ and do not forget to divide every term, including C, by μ.
5. **Apply any initial condition last.** Use it after you have the general solution.

EXAMPLE 1: CONSTANT COEFFICIENT

Solve $y' + 2y = 6e^x$ with $y(0) = 1$.

SOLUTION

Standard form already; $P = 2$, so $\mu = e^{2x}$.

$(e^{2x}y)' = e^{2x} \cdot 6e^x = 6e^{3x}$, so $e^{2x}y = 2e^{3x} + C$.

$y = 2e^x + Ce^{-2x}$. From $y(0) = 1$: $2 + C = 1$, so $C = -1$.

$y = 2e^x - e^{-2x}$.

EXAMPLE 2: VARIABLE COEFFICIENT

Solve $x\,y' + 3y = x^2$ for $x > 0$.

SOLUTION

Divide by x: $y' + \frac{3}{x}\,y = x$. Here $P = \frac{3}{x}$ is continuous for $x > 0$.

$\mu = e^{3\ln x} = x^3$. Then $(x^3 y)' = x^3 \cdot x = x^4$.

$x^3 y = \frac{x^5}{5} + C$, so $y = \frac{x^2}{5} + \frac{C}{x^3}$.

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Common mistakes

(1) Computing μ before dividing by the coefficient of y'. (2) Writing $\mu = e^{P}$ instead of $e^{\int P}$. (3) Forgetting to divide C by μ, which gives a wrong family of solutions. (4) Using the integrating factor on an equation that is not linear, such as $y' + y^2 = x$.

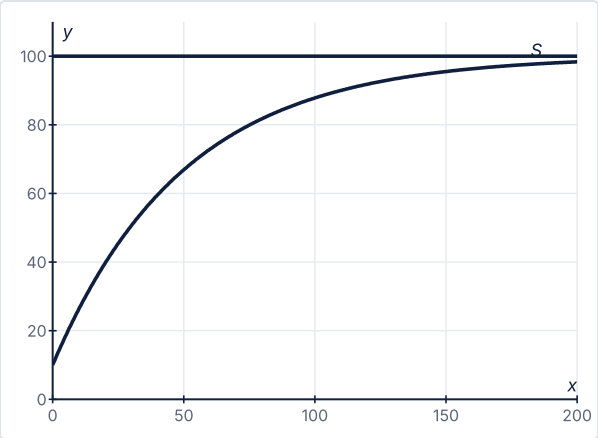
Mixing problems. Let $S(t)$ be the amount of solute in a well-stirred tank. Then

$$\frac{dS}{dt} = (\text{rate in}) - (\text{rate out}),$$

where rate in $=$ (inflow volume rate)(inflow concentration) and rate out $=$ (outflow volume rate)(concentration in the tank) $=$ (outflow volume rate)$\cdot S/V$. If the tank volume V is constant, the equation is linear with constant coefficients.

EXAMPLE 3: SALT IN A TANK

A tank holds 200 L of solution containing 10 kg of salt. Brine with 0.5 kg of salt per liter flows in at 4 L/min, and the well-mixed solution flows out at 4 L/min. Find $S(t)$, the kilograms of salt after t minutes, and the time when $S = 50$.



Salt S (kg) versus time t (min): the level rises toward 100 kg.

SOLUTION

Rate in $= 4(0.5) = 2$ kg/min. Rate out $= 4 \cdot \dfrac{S}{200} = \dfrac{S}{50}$ kg/min. So $S' + \dfrac{1}{50}S = 2$, $S(0) = 10$.

$\mu = e^{t/50}$, so $(e^{t/50}S)' = 2e^{t/50}$ and $e^{t/50}S = 100e^{t/50} + C$, giving $S = 100 + Ce^{-t/50}$.

$S(0) = 10$ gives $C = -90$: $S(t) = 100 - 90e^{-t/50}$. The salt level approaches 100 kg, the amount of salt in 200 L of the incoming brine.

$S = 50$: $90e^{-t/50} = 50$, so $t = 50 \ln \dfrac{9}{5} \approx 29.389$ minutes.

Circuits in plain language. In a simple loop with a battery of voltage E, a resistor of R ohms and a coil (inductor) of L henrys, the coil resists changes in current. The current $i(t)$ obeys

$$L\frac{di}{dt} + Ri = E.$$

Dividing by L gives the standard form $i' + \frac{R}{L}i = \frac{E}{L}$, which has the same shape as the mixing equation. The solution with $i(0) = 0$ is $i(t) = \dfrac{E}{R}\left(1 - e^{-Rt/L}\right)$: the current climbs toward the steady value E/R, and L/R is the **time constant**.

EXAMPLE 4: AN RL CIRCUIT

A coil with $L = 2$ henrys and a resistor with $R = 6$ ohms are connected to a 12 volt battery at $t = 0$ with $i(0) = 0$. Find $i(t)$ and the steady-state current.

SOLUTION

$2i' + 6i = 12$, so $i' + 3i = 6$. Then $\mu = e^{3t}$ and $(e^{3t}i)' = 6e^{3t}$.

$e^{3t}i = 2e^{3t} + C$, so $i = 2 + Ce^{-3t}$. With $i(0) = 0$, $C = -2$.

$i(t) = 2\left(1 - e^{-3t}\right)$ amperes. As $t \to \infty$, $i \to 2 = E/R$. The time constant is $L/R = 1/3$ second.

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Practice

SET A Skills

Solve each differential equation. [No calculator]

1. Solve $y' + 4y = 8$.
2. Solve $y' + y = e^{-x}$.
3. Solve $y' - 2y = x$.
4. Solve $y' + \dfrac{2}{x}y = 3$ for $x > 0$.
5. Solve the initial value problem $y' + y = 2x$, $y(0) = 3$.
6. Solve $x\,y' - y = x^2$ for $x > 0$ with $y(1) = 4$.
7. Solve $y' + (\tan x)\,y = \cos x$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ with $y(0) = 2$.
8. For $y' + 3x^2 y = x^2$, (a) find the integrating factor and (b) find the general solution.
9. Explain why you may choose the constant of integration to be zero when you compute $\mu(x) = e^{\int P(x)\,dx}$.

SET B Applications

Model and solve. Give decimals to three decimal places where asked. [Calculator]

10. A tank holds 500 L of pure water. Brine containing 0.2 kg of salt per liter enters at 5 L/min, and the well-mixed solution leaves at 5 L/min. Find $S(t)$ and the amount of salt after 30 minutes, to three decimal places.
11. For the tank in the previous problem (500 L, 0.2 kg per liter in, 5 L/min in and out, starting with pure water), find the time when the tank contains 60 kg of salt, to three decimal places. Also state the long-term amount of salt.
12. A coil with inductance $L = 0.5$ henry and resistance $R = 20$ ohms is connected to a 10 volt battery at $t = 0$ with $i(0) = 0$. Find $i(t)$, and the time at which the current reaches 90% of its steady value, to three decimal places (seconds).
13. A cup of liquid at 90°C sits in a room at 20°C. Its temperature satisfies $T' = -k(T - 20)$. After 5 minutes the temperature is 60°C. Find k and the time when $T = 30^\circ$C, each to three decimal places.
14. A drug is infused into the blood at 8 mg per hour and is removed at a rate equal to 0.25 times the current amount A (mg), so $A' = 8 - 0.25A$, with $A(0) = 0$. Find $A(4)$ to three decimal places and the long-term amount.
15. A tank initially has 100 L with 20 kg of dissolved salt. Brine at 0.1 kg per liter enters at 3 L/min, and well-mixed solution leaves at 2 L/min, so the volume is $100 + t$ liters. (a) Write the equation for $S(t)$. (b) Solve it. (c) Find $S(50)$ to three decimal places.
16. Solve $y' + 2xy = x$ with $y(0) = 3$ and find $y(1)$ to three decimal places.

SET C Challenge

Justify each step. [No calculator]

17. [No calculator] Verify that $y = \dfrac{1}{\mu}\left(\int \mu Q\,dx + C\right)$ with $\mu = e^{\int P\,dx}$ satisfies $y' + Py = Q$.
18. [No calculator] Solve $y' \cos x + y \sin x = 1$ on $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.
19. [No calculator] A 50 L tank starts with pure water. Brine of 3 kg per liter enters at 2 L/min and the well-mixed solution leaves at 2 L/min. After 25 minutes, the inflow is switched to pure water (same rates). Find the kilograms of salt at $t = 50$ in exact form.
20. [No calculator] Solve $y' - 2y = e^{2x}$ with $y(0) = 1$. Explain why trying $y = ae^{2x}$ for a constant a cannot work.
21. [No calculator] Find all solutions of $y' + y = \cos x$. Which one is periodic, and what happens to the other terms as $x \to \infty$?

Answers Chapter 8: Answers and Explanations

Answers for every practice problem, numbered to match. Read the explanation even when your answer is correct.

8.1 Hyperbolic Functions

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SET A · SKILLS

1. $\cosh(\ln 3) = \frac{3+1/3}{2} = \frac{10/3}{2} = \frac{5}{3}$.
2. $\tanh(\ln 3) = \frac{3-1/3}{3+1/3} = \frac{8/3}{10/3} = \frac{4}{5}$.
3. Product rule: $y' = 2x\sinh x + x^2\cosh x$.
4. $y' = \frac{\cosh x}{\sinh x} = \coth x$.
5. Chain rule: $y' = 5\,\mathrm{sech}^2(5x)$.
6. $u = 2x$: $\frac{1}{2}\sinh(2x) + C$.
7. Since $\frac{d}{dx}\tanh(3x) = 3\,\mathrm{sech}^2(3x)$, the antiderivative is $\frac{1}{3}\tanh(3x) + C$.
8. $[\cosh x]_0^{\ln 3} = \cosh(\ln 3) - 1 = \frac{3+1/3}{2} - 1 = \frac{5}{3} - 1 = \frac{2}{3}$.
9. $\cosh^2 x = 1 + \sinh^2 x = 1 + \frac{9}{16} = \frac{25}{16}$ and $\cosh x > 0$, so $\cosh x = \frac{5}{4}$. Then $\tanh x = \frac{3/4}{5/4} = \frac{3}{5}$.

SET B · APPLICATIONS AND MIXED PRACTICE

10. $\sinh 1.2 \approx 1.509$, $\cosh 1.2 \approx 1.811$, $\tanh 1.2 \approx 0.834$.
11. $x = \sinh^{-1} 2 = \ln(2 + \sqrt{5}) \approx 1.444$.
12. $x = \cosh^{-1} 3 = \ln(3 + \sqrt{8}) \approx 1.763$.
13. $f'(x) = \cosh x + x\sinh x$, so $f'(0.5) \approx 1.388$.
14. $\int_0^2 \cosh x\,dx = \sinh 2 \approx 3.627$.
15. (a) $L = \int_{-15}^{15} \cosh\frac{x}{20}\,dx = 40\sinh 0.75 \approx 32.893$ m (since $\sqrt{1+\sinh^2} = \cosh$). (b) $20\cosh 0.75 - 20 \approx 5.894$ m.
16. (a) As $t \to \infty$, $\tanh\frac{t}{3} \to 1$, so the limit is 30 m/s. (b) $\tanh\frac{t}{3} = \frac{5}{6}$ gives $\frac{t}{3} = \tanh^{-1}\frac{5}{6} = \frac{1}{2}\ln 11$, so $t = \frac{3}{2}\ln 11 \approx 3.597$ s.

SET C · CHALLENGE AND PROOF

17. $2\sinh x\cosh x = 2\cdot\frac{e^x - e^{-x}}{2}\cdot\frac{e^x + e^{-x}}{2} = \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x$.
18. $\frac{d}{dx}\ln\left(x + \sqrt{x^2-1}\right) = \frac{1 + \frac{x}{\sqrt{x^2-1}}}{x + \sqrt{x^2-1}} = \frac{\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}$.
19. $y = \frac{e^x - e^{-x}}{2}$. Let $u = e^x$: $u^2 - 2yu - 1 = 0$, so $u = y \pm \sqrt{y^2+1}$. Since $u > 0$ and $\sqrt{y^2+1} > |y|$, take the plus sign. Then $x = \ln u = \ln\left(y + \sqrt{y^2+1}\right)$.
20. $\left[\sinh^{-1}\frac{x}{3}\right]_0^4 = \sinh^{-1}\frac{4}{3} = \ln\left(\frac{4}{3} + \sqrt{\frac{16}{9}+1}\right) = \ln\left(\frac{4}{3}+\frac{5}{3}\right) = \ln 3$.
21. $\int \tanh x\,dx = \ln\cosh x$. $\cosh(\ln 3) = \frac{3+1/3}{2} = \frac{5}{3}$ and $\cosh 0 = 1$, so the value is $\ln\frac{5}{3}$.

8.2 Surface Area of a Solid of Revolution

SET A · SKILLS

1. $18\sqrt{5}\,\pi$. Radius $2x$, $\sqrt{1+4} = \sqrt{5}$: $S = \int_0^3 4\pi\sqrt{5}\,x\,dx = 2\sqrt{5}\pi\cdot 9 = 18\sqrt{5}\,\pi$. (Matches the cone formula $\pi r\ell$ with $r = 6$, $\ell = 3\sqrt{5}$.)
2. $\frac{\pi}{9}\left(2\sqrt{2} - 1\right)$. $y' = x^2$, so $S = 2\pi\int_0^1 \frac{x^3}{3}\sqrt{1+x^4}\,dx$. With $u = 1 + x^4$, $S = \frac{2\pi}{3}\cdot\frac{1}{4}\cdot\frac{2}{3}\left[u^{3/2}\right]_1^2 = \frac{\pi}{9}(2\sqrt{2} - 1)$.
3. 60π. $y' = -x/y$, so $y\sqrt{1+(y')^2} = \sqrt{y^2+x^2} = 5$. $S = 2\pi\int_{-3}^{3} 5\,dx = 60\pi$ (a spherical zone: $2\pi Rh = 2\pi\cdot 5\cdot 6$).
4. $\frac{52\pi}{3}$. $y' = \frac{1}{\sqrt{2x}}$, so $y\sqrt{1+(y')^2} = \sqrt{2x+1}$. $S = 2\pi\left[\frac{(2x+1)^{3/2}}{3}\right]_0^4 = \frac{2\pi}{3}(27 - 1) = \frac{52\pi}{3}$.
5. $\frac{\pi}{6}\left(5\sqrt{5} - 1\right)$. Radius x, $y' = 2x$: $S = \int_0^1 2\pi x\sqrt{1+4x^2}\,dx$. With $u = 1 + 4x^2$, $S = \frac{2\pi}{8}\cdot\frac{2}{3}\left[u^{3/2}\right]_1^5 = \frac{\pi}{6}(5\sqrt{5} - 1)$.
6. $3\sqrt{10}\,\pi$. Radius x, $\sqrt{1+9} = \sqrt{10}$: $S = 2\pi\sqrt{10}\left[\frac{x^2}{2}\right]_1^2 = 2\pi\sqrt{10}\cdot\frac{3}{2} = 3\sqrt{10}\,\pi$.
7. $\frac{47\pi}{16}$. $y' = \frac{x^2}{2} - \frac{1}{2x^2}$, and $1 + (y')^2 = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$. Then $y\sqrt{\cdot} = \frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}$, and $S = 2\pi\left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}\right]_1^2 = \frac{47\pi}{16}$.
8. $S = \int_0^1 2\pi\,e^x\sqrt{1+e^{2x}}\,dx$. Radius $y = e^x$ and $y' = e^x$, so $(y')^2 = e^{2x}$.
9. Each point of the curve travels around a circle of radius $f(x)$, so a small piece of curve sweeps a band of circumference $2\pi f(x)$. The width of the band is the arc length $ds = \sqrt{1 + [f'(x)]^2}\,dx$ along the curve, not just dx, because the curve is slanted.

SET B · APPLICATIONS

10. $2\pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right) \approx 14.424$. $S = 2\pi\int_0^{\pi}\sin x\sqrt{1+\cos^2 x}\,dx$. With $u = \cos x$ this becomes $2\pi\int_{-1}^{1}\sqrt{1+u^2}\,du$; a calculator integral gives 14.424.
11. $\frac{8\pi}{3}\left(2\sqrt{2} - 1\right) \approx 15.318$ square meters. $y' = x/2$, so $S = \int_0^2 2\pi x\sqrt{1 + x^2/4}\,dx$. With $u = 1 + x^2/4$, $S = 4\pi\cdot\frac{2}{3}\left[u^{3/2}\right]_1^2$.
12. $40\pi \approx 125.664$ cm². $y\sqrt{1+(y')^2} = \sqrt{y^2+x^2} = 10$, so $S = 2\pi\int_6^8 10\,dx = 40\pi$ (zone formula $2\pi Rh$).
13. $\frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \approx 30.846$ m². $y\sqrt{1+(y')^2} = \sqrt{x + \frac{5}{4}}$, so $S = 2\pi\cdot\frac{2}{3}\left[\left(x+\frac{5}{4}\right)^{3/2}\right]_0^3$.
14. About 370.16 dollars. The area is about 30.846 square meters, so cost $= 12 \times 30.846 \approx 370.158$, which is 370.16 dollars to the nearest cent.
15. $V = 8\pi \approx 25.133$, $S = \frac{\pi}{6}\left(17\sqrt{17} - 1\right) \approx 36.177$, so $S/V \approx 1.439$. Here $V = \pi\int_0^4 x\,dx = 8\pi$ and $S = 2\pi\int_0^4\sqrt{x + \frac{1}{4}}\,dx$.
16. $16\sqrt{5}\,\pi \approx 112.397$ cm². $S = \int_0^8 2\pi\cdot\frac{x}{2}\cdot\frac{\sqrt{5}}{2}\,dx = \frac{\sqrt{5}\pi}{2}\cdot 32 = 16\sqrt{5}\pi$. Check: $r = 4$, $\ell = \sqrt{64+16} = 4\sqrt{5}$, so $\pi r\ell = 16\sqrt{5}\pi$.

SET C · CHALLENGE

17. $y' = -x/y$, so $y\sqrt{1+(y')^2} = \sqrt{y^2+x^2} = r$, and $S = 2\pi\int_{-r}^{r} r\,dx = 4\pi r^2$. At $x = \pm r$ the derivative y' is undefined (vertical tangent), but the product $y\sqrt{1+(y')^2} = r$ stays bounded, so the integral is fine after simplifying.
18. $S = \int_a^b 2\pi m x\sqrt{1+m^2}\,dx = \pi m\sqrt{1+m^2}\,(b^2 - a^2)$. The slant length is $\ell = (b-a)\sqrt{1+m^2}$ and $r_1 + r_2 = m(a+b)$, so $\pi(r_1+r_2)\ell = \pi m(a+b)(b-a)\sqrt{1+m^2}$, the same.
19. $b = 8$. $y\sqrt{1+(y')^2} = 2\sqrt{x}\cdot\sqrt{\frac{x+1}{x}} = 2\sqrt{x+1}$, so $S(b) = \frac{8\pi}{3}\left[(b+1)^{3/2} - 1\right]$. Setting this to $\frac{208\pi}{3}$ gives $(b+1)^{3/2} = 27$, so $b = 8$. $S(b)$ is strictly increasing for $b > 0$, so b is unique.

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20. $\frac{56\pi}{3}$. Here $dx/dy = 1/\sqrt{y}$, so $x\sqrt{1+(dx/dy)^2} = 2\sqrt{y}\cdot\sqrt{\frac{y+1}{y}} = 2\sqrt{y+1}$. $S = 4\pi\int_0^3 \sqrt{y+1}\,dy = 4\pi\cdot\frac{2}{3}\cdot 7 = \frac{56\pi}{3}$.
21. $S = 2\pi(3)(10) = 60\pi$. For the semicircle: $S = 2\pi\cdot\frac{10}{\pi}\cdot 5\pi = 100\pi$, which equals the sphere area $4\pi\cdot 5^2 = 100\pi$.

8.3 Linear First-Order Differential Equations

SET A · SKILLS

1. $y = 2 + Ce^{-4x}$. $\mu = e^{4x}$, $(e^{4x}y)' = 8e^{4x}$, so $e^{4x}y = 2e^{4x} + C$.
2. $y = (x+C)e^{-x}$. $\mu = e^{x}$, $(e^{x}y)' = 1$, so $e^{x}y = x + C$.
3. $y = Ce^{2x} - \frac{x}{2} - \frac{1}{4}$. $\mu = e^{-2x}$, $(e^{-2x}y)' = xe^{-2x}$. By parts, $\int xe^{-2x}dx = -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C$.
4. $y = x + \frac{C}{x^2}$. $\mu = x^2$, $(x^2y)' = 3x^2$, so $x^2y = x^3 + C$.
5. $y = 2x - 2 + 5e^{-x}$. $\mu = e^{x}$, $e^{x}y = \int 2xe^{x}dx = 2xe^{x} - 2e^{x} + C$. Then $y(0) = 3$ gives $-2 + C = 3$, so $C = 5$.
6. $y = x^2 + 3x$. Standard form $y' - \frac{1}{x}y = x$, $\mu = \frac{1}{x}$, $(y/x)' = 1$, so $y = x(x + C)$. Then $y(1) = 1 + C = 4$, so $C = 3$.
7. $y = (x+2)\cos x$. $\mu = e^{\int \tan x\,dx} = \sec x$ (valid since $\cos x > 0$). $(y\sec x)' = 1$, so $y = (x + C)\cos x$, and $y(0) = 2$ gives $C = 2$.
8. (a) $\mu = e^{x^3}$. (b) $(e^{x^3}y)' = x^2e^{x^3}$, so $e^{x^3}y = \frac{1}{3}e^{x^3} + C$ and $y = \frac{1}{3} + Ce^{-x^3}$.
9. Any antiderivative works. A different constant k gives $\mu_k = e^{k}\mu$, a constant multiple of μ. Multiplying the whole equation by a constant does not change its solutions, and the constant cancels when you divide by μ.

SET B · APPLICATIONS

10. $S(t) = 100\left(1 - e^{-t/100}\right)$; $S(30) \approx 25.918$ kg. Rate in $= 5(0.2) = 1$, rate out $= 5S/500 = S/100$, so $S' + S/100 = 1$, $S(0) = 0$.
11. $t = 100\ln 2.5 \approx 91.629$ minutes. Set $100(1 - e^{-t/100}) = 60$. The long-term amount is 100 kg (concentration 0.2 kg/L times 500 L).
12. $i(t) = 0.5\left(1 - e^{-40t}\right)$. Steady value 0.5. Need $1 - e^{-40t} = 0.9$, so $t = \dfrac{\ln 10}{40} \approx 0.058$ s.
13. $k = \dfrac{\ln(7/4)}{5} \approx 0.112$ per minute. $T - 20 = 70e^{-kt}$; at 5 min, $40 = 70e^{-5k}$. For $T = 30$: $70e^{-kt} = 10$, $t = \dfrac{\ln 7}{k} \approx 17.386$ minutes.
14. $A(t) = 32\left(1 - e^{-t/4}\right)$, so $A(4) = 32(1 - e^{-1}) \approx 20.228$ mg. Long-term amount $= 8/0.25 = 32$ mg.
15. (a) $S' = 0.3 - \dfrac{2S}{100 + t}$. (b) $\mu = (100+t)^2$, so $\left[(100+t)^2S\right]' = 0.3(100+t)^2$, giving $(100+t)^2S = 0.1(100+t)^3 + C$; $S(0) = 20$ gives $C = 100000$, so $S = 0.1(100+t) + \dfrac{100000}{(100+t)^2}$. (c) $S(50) = 15 + \dfrac{100000}{22500} \approx 19.444$ kg.
16. $y = \frac{1}{2} + \frac{5}{2}e^{-x^2}$; $y(1) = \frac{1}{2} + \frac{5}{2}e^{-1} \approx 1.420$. $\mu = e^{x^2}$, $(e^{x^2}y)' = xe^{x^2}$, so $e^{x^2}y = \frac{1}{2}e^{x^2} + C$.

SET C · CHALLENGE

17. Differentiate $y = \mu^{-1}\left(\int \mu Q\,dx + C\right)$ with the product rule. Since $(\mu^{-1})' = -\mu^{-2}\mu' = -P\mu^{-1}$ and $\left(\int \mu Q\,dx\right)' = \mu Q$, we get $y' = -P\mu^{-1}\left(\int \mu Q\,dx + C\right) + \mu^{-1}\mu Q = -Py + Q$. Therefore $y' + Py = Q$.
18. $y = \sin x + C\cos x$. Divide by $\cos x > 0$: $y' + (\tan x)y = \sec x$. $\mu = \sec x$, $(y\sec x)' = \sec^2 x$, so $y\sec x = \tan x + C$.
19. $150\left(e^{-1} - e^{-2}\right)$ kg. For $0 \le t \le 25$: $S' = 6 - \frac{S}{25}$, $S(0) = 0$ gives $S = 150(1 - e^{-t/25})$, so $S(25) = 150(1 - e^{-1})$. Then $S' = -\frac{S}{25}$, so $S(50) = S(25)e^{-1} = 150(1 - e^{-1})e^{-1} = 150(e^{-1} - e^{-2})$.
20. $y = (x+1)e^{2x}$. $\mu = e^{-2x}$, $(e^{-2x}y)' = 1$, so $y = (x + C)e^{2x}$, and $y(0) = 1$ gives $C = 1$. If $y = ae^{2x}$ then $y' - 2y = 0 \neq e^{2x}$: the forcing term is already a solution of the homogeneous equation $y' - 2y = 0$.
21. $y = \frac{1}{2}(\cos x + \sin x) + Ce^{-x}$. $\mu = e^{x}$ and $\int e^{x}\cos x\,dx = \frac{1}{2}e^{x}(\cos x + \sin x)$. The solution with $C = 0$ is periodic. The term Ce^{-x} is a transient that tends to 0 as $x \to \infty$.

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Chapter 9: Practice and Review.

7 sections with explanations and worked examples, then 189 practice problems. Answers and full explanations are at the end of the chapter.

9.3 Derivatives in Context

Mixed applications of the derivative: tangent lines, optimization, related rates, motion, and the Mean Value Theorem with written justifications.

This set mixes the main applications of the derivative. Before starting, recall the justification habits that earn credit: name the theorem, check its hypotheses, and state the sign information that supports your conclusion.

Justification toolkit

Mean Value Theorem: if f is continuous on $[a, b]$ and differentiable on (a, b), then $f'(c) = \frac{f(b)-f(a)}{b-a}$ for some c in (a, b).
Extrema: a sign change of f' (negative to positive is a minimum, positive to negative is a maximum), or the sign of f'' at a critical number. On a closed interval, compare the endpoints too.
Related rates: write an equation connecting the variables, differentiate with respect to t, and only then substitute the instantaneous values.

Common mistake

Substituting a specific value for a changing quantity before differentiating. A radius of 5 at one instant is not a constant, so differentiate first and plug in last.

SET A Core applications

Solve each problem. Show the key steps. [No calculator]

1. Find an equation of the tangent line to $y = x^2 e^x$ at $x = 1$.
2. Find an equation of the normal line to $y = \sqrt{4x + 1}$ at $x = 2$.
3. At what points does the graph of $f(x) = x^3 - 12x + 5$ have a horizontal tangent line? Give the points.
4. The area of a circle grows at a constant rate of $10\ \text{cm}^2/\text{s}$. How fast is the radius increasing at the instant the radius is 5 cm? [No calculator]
5. A farmer has 60 m of fence for a rectangular pen. One side of the pen is a long barn wall and needs no fence. What dimensions give the largest area, and what is that area?
6. Let $f(x) = x^2 + 4x$ on $[1, 5]$. Verify the hypotheses of the Mean Value Theorem and find all values c that satisfy its conclusion.
7. A particle moves on a line with position $s(t) = t^3 - 9t^2 + 15t$ for $t \geq 0$. Find the times at which the particle is at rest and its acceleration at the first such time.
8. Find all critical numbers of $g(x) = x^4 - 8x^2$ and classify each as a local maximum, local minimum, or neither. Justify.
9. Use the tangent line to $f(x) = \sqrt[3]{x}$ at $x = 27$ to approximate $\sqrt[3]{28}$. Give an exact fraction.

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SET B Applied problems

Solve each problem. Where decimals are requested, give three decimal places. [Calculator]

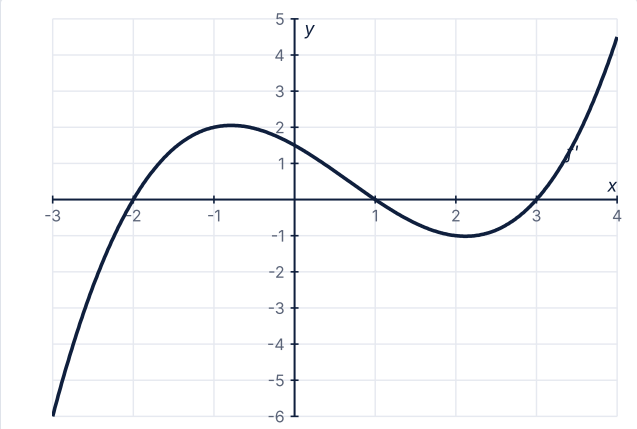
10. A 13 ft ladder leans against a vertical wall. The bottom slides away from the wall at 2 ft/s. How fast is the top sliding down the wall when the bottom is 5 ft from the wall? Give a decimal. [Calculator]
11. A closed cylindrical can must hold $500\ \text{cm}^3$. Find the radius that minimizes the total surface area and the minimum surface area. Give decimals. [Calculator]
12. A particle on a line has position $s(t) = te^{-t/2}$ meters for $t \geq 0$. At what time is the particle farthest to the right, and what is its position then? Give decimals. [Calculator]
13. A function f is differentiable on $[2, 8]$ with $f(2) = 7$ and $f(8) = 25$. Explain why there is a time c in $(2, 8)$ with $f'(c) = 3$, and name the theorem. [Calculator]
14. A bacteria population is modeled by $P(t) = \dfrac{1200}{1 + 5e^{-0.4t}}$, where t is in hours. Find $P'(3)$ and interpret it with units. [Calculator]
15. The tangent line to $f(x) = \sqrt{x}$ at $x = 9$ is used to approximate $\sqrt{9.6}$. Find the approximation and the true value to three decimals, and state whether the approximation is an overestimate or an underestimate, with a reason. [Calculator]
16. Water is poured at $3\ \text{cm}^3/\text{s}$ into an inverted cone with height 10 cm and top radius 4 cm. How fast is the water depth rising when the depth is 5 cm? Give a decimal. [Calculator]
17. A company sells x units at price $p = 40 - 0.03x$ dollars per unit. The cost of producing x units is $C(x) = 500 + 12x + 0.01x^2$ dollars. Find the production level that maximizes profit and the maximum profit. [Calculator]
18. The velocity $v(t)$ of a particle (m/s) is differentiable, and selected values are given. Explain why there must be a time in $(5, 9)$ when the acceleration equals $-0.5\ \text{m/s}^2$. [Calculator]

t (s)	0	5	9
$v(t)$	3	12	10

SET C AP-style justification

Solve each problem and justify your conclusions in complete sentences. [No calculator]

19. The graph shows f', the derivative of a function f that is continuous on $[-3, 4]$, with $f'(x) = \frac{1}{4}(x + 2)(x - 1)(x - 3)$.
    (a) On what intervals is f decreasing? Justify.
    (b) Find the x-coordinate of each local maximum and local minimum of f. Justify.
    (c) Find the x-coordinates of the inflection points of f. Justify.



Graph of f'.

20. The curve $x^2 - xy + y^2 = 3$ passes through $(1, 2)$.
    (a) Show that the tangent line at $(1, 2)$ is horizontal.
    (b) Find $\frac{d^2y}{dx^2}$ at $(1, 2)$ and decide whether the curve has a local maximum or minimum in y there.
21. Find the point on the curve $y = \sqrt{x}$ closest to the point $(4, 0)$, and the minimum distance. Justify that it is a minimum. [No calculator]

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22. A rocket rises vertically from a launch pad. An observer stands 3 km from the pad and watches the rocket through an instrument that tracks the angle of elevation θ. When the rocket is 4 km high it is rising at 0.5 km/s. Find $\frac{d\theta}{dt}$ at that instant. [No calculator]
23. A student claims that a differentiable function f exists with $f(1) = -2$, $f(5) = 10$, and $|f'(x)| \le 2$ for all x in $[1, 5]$. Use the Mean Value Theorem to show this claim is impossible. [No calculator]
24. Let $f(x) = x^3 + x + 1$. Explain why f has an inverse function g, and find $g'(3)$. [No calculator]
25. A particle moves on a line with velocity $v(t) = t^2 - 4t + 3$ for $0 \le t \le 5$. On which open intervals is the speed increasing? Justify. [No calculator]
26. Let $f(x) = \begin{cases} x^2 + ax, & x \le 1 \\ bx^3 + 2, & x > 1 \end{cases}$. Find a and b so that f is differentiable at $x = 1$. [No calculator]
27. Find the absolute maximum and absolute minimum values of $f(x) = x^3 - 3x^2$ on $[-1, 4]$, and where each occurs. [No calculator]

9.4 Integral Skills

Quick drills on antiderivatives and definite integrals: substitution, rewriting, geometry, and the Fundamental Theorem.

Fast, accurate integration comes from choosing the right tool at once. Scan the integrand: can you rewrite it, does one factor look like the derivative of another, or does it describe a familiar shape?

Integral toolkit

Fundamental Theorem, part 1: if f is continuous, $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\,g'(x)$.
Fundamental Theorem, part 2: if $F' = f$ and f is continuous on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.
Substitution: with $u = g(x)$, $\int f(g(x))g'(x)\,dx = \int f(u)\,du$. For definite integrals, change the limits to u-values.

Common mistake

Forgetting to change the limits (or to convert back to x) after a substitution, and dropping the constant of integration on an indefinite integral.

SET A Core skills

Evaluate or find each. [No calculator]

1. Find $\int \left(8x^3 - \frac{5}{x^2} + 3\sqrt{x}\right) dx$.
2. Find $\int x(x^2 + 5)^4\,dx$.
3. Find $\int \left(\sin 2x + e^{-x}\right) dx$.
4. Find $\int \frac{x}{x^2 + 1}\,dx$.
5. Evaluate $\int_0^2 3x^2 e^{x^3}\,dx$.
6. Evaluate $\int_1^4 \frac{x + 1}{\sqrt{x}}\,dx$.
7. Evaluate $\int_0^{\pi/2} \cos x\,\sin^3 x\,dx$.
8. Use geometry to evaluate $\int_{-3}^{3} \left(2 + \sqrt{9 - x^2}\right) dx$.
9. Find $\int \frac{x^2 - 9}{x - 3}\,dx$.

SET B Numerical and applied

Give decimal answers to three places unless an exact value is asked for. [Calculator]

10. Evaluate $\int_0^3 e^{-x^2}\,dx$ to three decimal places. [Calculator]
11. Evaluate $\int_0^2 \sqrt{1 + x^3}\,dx$ to three decimal places. [Calculator]
12. The table gives values of a continuous function f. Use a trapezoid sum with the three subintervals shown to approximate $\int_0^{10} f(x)\,dx$. [Calculator]

x	0	3	7	10
$f(x)$	2	5	9	4

13. Let $f(x) = \sqrt{1 + x}$. Find the left Riemann sum with 4 equal subintervals for $\int_0^2 f(x)\,dx$, the exact value of the integral, and say whether the left sum is an underestimate or overestimate. Give decimals. [Calculator]

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14. Let $G(x) = 3 + \int_1^x \sin(t^2)\,dt$. Find $G(2)$ and $G'(2)$ to three decimal places. [Calculator]

15. Find the average value of $f(x) = e^{\sin x}$ on $[0, \pi]$ to three decimal places. [Calculator]

16. Water flows into a tank at rate $R(t) = 12 + 8\sin\left(\frac{t}{3}\right)$ liters per minute for $0 \le t \le 6$ minutes. The tank holds 50 liters at $t = 0$. Find the amount of water in the tank at $t = 6$ to three decimal places. [Calculator]

17. Find the value of $k > 0$ for which $\int_0^k \frac{1}{1+x^2}\,dx = 1$. Give three decimals. [Calculator]

18. Evaluate $\int_0^3 x^2 \ln(x+1)\,dx$ to three decimal places. [Calculator]

SET C AP-style integrals

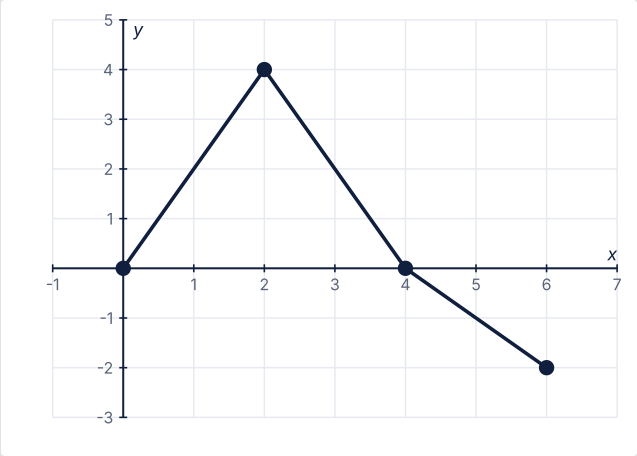
Show your work and justify where asked. [No calculator]

19. The graph shows f, which consists of line segments joining $(0,0)$, $(2,4)$, $(4,0)$, $(6,-2)$. Let $g(x) = \int_0^x f(t)\,dt$.

(a) Find $g(2)$, $g(4)$, $g(6)$.

(b) On $[0,6]$, where does g have its absolute maximum? Justify.

(c) Find the x-coordinate of each inflection point of g on $(0,6)$. Justify.



Graph of f.

20. Let $H(x) = \int_1^{x^2} \ln(1+t)\,dt$. Find $H'(x)$ and $H'(1)$. [No calculator]

21. Evaluate $\int_1^e \frac{(\ln x)^2}{x}\,dx$. [No calculator]

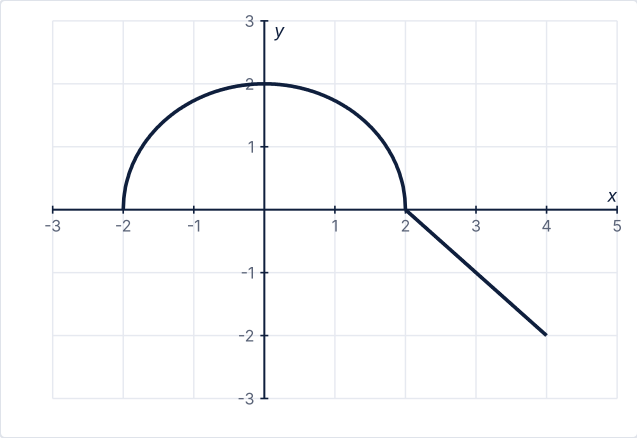
22. Evaluate $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$. [No calculator]

23. Evaluate $\int_0^1 \frac{x^2+3x+3}{x+1}\,dx$. [No calculator]

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24. The graph of f on $[-2, 4]$ consists of a semicircle and a line segment: $f(x) = \sqrt{4 - x^2}$ for $-2 \le x \le 2$ and $f(x) = -(x - 2)$ for $2 < x \le 4$.
    (a) Evaluate $\int_{-2}^{4} f(x)\,dx$.
    (b) Evaluate $\int_{-2}^{4} |f(x)|\,dx$. [No calculator]



Graph of f.

25. The function f satisfies $f'(x) = xe^{-x^2}$ and $f(0) = 2$. Find $f(x)$. [No calculator]
26. Evaluate $\int_{-2}^{2} (x^5 - 4x^3 + 6x^2 + 1)\,dx$ without finding a full antiderivative, and explain the symmetry argument. [No calculator]
27. A differentiable function f has $f(3) = 7$ and $\int_{3}^{8} f'(x)\,dx = -4$. Find $f(8)$ and $\int_{3}^{8} (f'(x) + 2)\,dx$. [No calculator]

9.5 Particle Motion and Accumulation

Particle motion with derivatives and integrals: displacement, total distance, speed, and net change from rate functions.

Motion and accumulation problems share one idea: a rate integrated over time gives a net change. Position is the integral of velocity, velocity is the integral of acceleration, and amounts in tanks, lots and bloodstreams are integrals of net rates.

Motion and accumulation summary

Net change: $\int_a^b f'(t)\,dt = f(b) - f(a)$. Displacement is $\int_a^b v(t)\,dt$ and final position is $x(a) + \int_a^b v(t)\,dt$.
Total distance: $\int_a^b |v(t)|\,dt$. Split at each time where v changes sign.
Speeding up: speed is increasing exactly when v and a have the same sign (both nonzero).

Common mistake

Reporting displacement when the question asks for total distance, or forgetting the initial value. Ask yourself: do I need the net change, the total traveled, or the final amount?

SET A Core skills

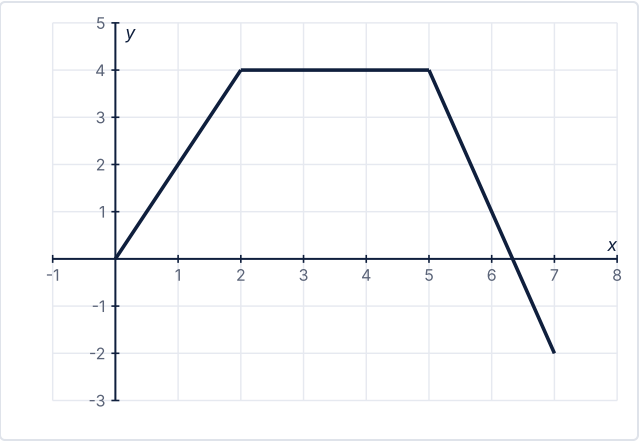
Solve each problem. Show the key steps. [No calculator]

1. A particle moves on a number line with velocity $v(t) = 3t^2 - 12t + 9$ for $0 \le t \le 4$. Find its displacement and the total distance it travels. [No calculator]
2. A particle has acceleration $a(t) = 4 - 2t$, initial velocity $v(0) = 3$, and initial position $s(0) = 1$. Find $v(t)$ and $s(3)$. [No calculator]
3. A particle has velocity $v(t) = t^2 - 5t + 4$. Is its speed increasing or decreasing at $t = 2$? Justify. [No calculator]
4. A particle has velocity $v(t) = \cos 2t$ for $0 \le t \le \pi$. At what times does it change direction? Justify. [No calculator]
5. A particle has position $s(t) = t^2 - 4t$ for $0 \le t \le 5$. Find its average velocity and its average speed on this interval. [No calculator]
6. The volume $W(t)$ of liquid in a tank changes at the rate $W'(t) = 12 - 3t$ liters per minute, and $W(0) = 40$. Find $W(4)$ and explain why W has its maximum on $[0, 6]$ at $t = 4$. [No calculator]

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7. The graph shows the velocity $v(t)$ of a particle for $0 \le t \le 7$ (segments joining $(0,0)$, $(2,4)$, $(5,4)$, $(7,-2)$). Find the displacement, the total distance, and the acceleration at $t = 6$. [No calculator]



Graph of v.

8. A particle has velocity $v(t) = 4\sin 2t$. Find its displacement and total distance for $0 \le t \le \pi$. [No calculator]
9. Medicine enters a patient's bloodstream at the rate $R(t) = 100e^{-0.5t}$ milligrams per hour. Find the exact amount that enters during the first 2 hours. [No calculator]

SET B Calculator practice

Give decimal answers to three places. [Calculator]

10. A particle has velocity $v(t) = t\cos t$ for $0 \le t \le 4$. Find the total distance traveled to three decimal places. [Calculator]
11. A particle has velocity $v(t) = 2\sin\left(\frac{t^2}{3}\right)$ and $s(0) = -1$. Find $s(3)$ and the speed at $t = 3$ to three decimal places. [Calculator]
12. A particle has velocity $v(t) = 4 - te^{t/3}$ for $t \ge 0$ and starts at $x(0) = 1$. Show that it changes direction exactly once, and find that time and the position then. [Calculator]
13. The rate $R(t)$ at which water drains from a tank, in liters per minute, is measured and given in the table. Use a left Riemann sum with the three subintervals shown to estimate the amount drained during the first 10 minutes. [Calculator]

t (min)	0	4	6	10
$R(t)$	12	9	10	15

14. Cars enter a lot at rate $E(t) = 80 + 30\sin\left(\frac{t}{2}\right)$ and leave at the constant rate 90 cars per hour for $0 \le t \le 10$ hours. The lot holds 200 cars at $t = 0$. (a) Find all times when the net rate of change is zero. (b) At what time is the number of cars greatest, and how many cars are there then? Give three decimals. [Calculator]
15. A particle has velocity $v(t) = \sin t - 0.4t$ for $0 \le t \le 4$. Find the average velocity and the average speed on $[0,4]$ to three decimal places. [Calculator]
16. A particle has acceleration $a(t) = 3\ln(t+1)$ and $v(0) = -2$. Find $v(2)$ to three decimals and decide whether the speed is increasing or decreasing at $t = 2$. [Calculator]
17. Oil leaks from a pipe at rate $R(t) = 2 + te^{-t/2}$ liters per hour for $0 \le t \le 6$. Find the average rate of leakage on $[0,6]$ and every time at which $R(t)$ equals it. Give three decimals. [Calculator]
18. The amount $A(t)$ of a drug in a patient (mg) satisfies $A'(t) = 6e^{-0.3t} - 1.5$ for $t \ge 0$ hours, with $A(0) = 0$. Find the time of the maximum amount and the maximum amount. Give decimals. [Calculator]

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SET C AP-style motion and accumulation

Justify your answers with complete sentences. [No calculator]

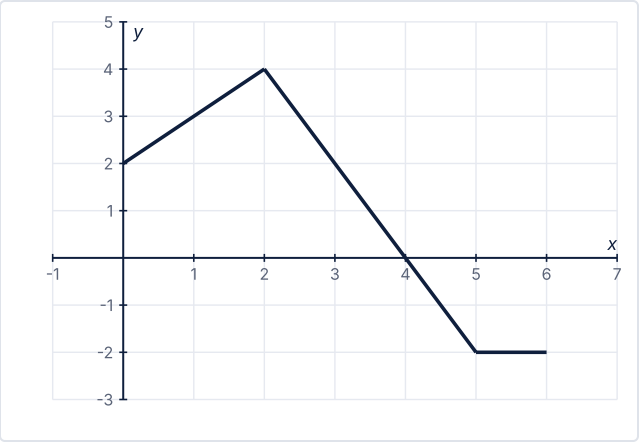
19. Oil in a tank changes at rate $N(t)$ gallons per minute, shown in the graph (segments joining $(0,2)$, $(2,4)$, $(5,-2)$, $(6,-2)$). The tank holds 30 gallons at $t=0$.

(a) How many gallons are in the tank at $t=5$?

(b) At what time on $[0,6]$ is the amount greatest? How much is it?

(c) Find the amount at $t=6$.

(d) On what interval is the amount increasing at a decreasing rate? Justify.



Graph of N.

20. A particle moves on a line with continuous velocity $v(t)$ (m/s). Selected values: $v(0)=4$, $v(2)=6$, $v(5)=1$, $v(8)=-3$, $v(10)=2$, and the position is $x(0)=5$.

(a) Use a trapezoid sum with the four subintervals to estimate $x(10)$.

(b) Explain why v has at least two zeros on $(5,10)$.

(c) Explain why the acceleration equals $-\frac{5}{3}$ at some time in $(2,5)$ if v is differentiable.

t	0	2	5	8	10
$v(t)$	4	6	1	-3	2

21. A particle has velocity $v(t)=(t-2)(t-5)$ for $0\le t\le 6$. (a) On what intervals is the speed increasing? Justify. (b) Find the total distance traveled. [No calculator]

22. A colony of bacteria grows at the rate $r(t)=30e^{0.2t}$ cells per hour, and the colony has 200 cells at $t=0$. (a) Find the exact number of cells added during the first 5 hours. (b) Find the exact average rate of growth on $[0,5]$. (c) Write the number of cells at $t=5$. [No calculator]

23. A particle has velocity $v(t)=3\cos\left(\frac{\pi t}{2}\right)$ for $0\le t\le 3$ and position $x(0)=2$. (a) When does the particle change direction? (b) Find $x(3)$ and the total distance traveled. [No calculator]

24. Water enters a tank at rate $12t-3t^2$ and leaves at rate 9, both in liters per minute, for $0\le t\le 4$. The tank holds 20 liters at $t=0$. Find the absolute maximum and absolute minimum amounts on $[0,4]$, with justification. [No calculator]

25. The position of a particle is $x(t)=4+\displaystyle\int_0^t (u^2-4)\,du$ for $t\ge 0$. (a) Find $v(t)$ and $a(t)$. (b) Show that the particle changes direction at $t=2$ and find $x(2)$. (c) Find the total distance traveled for $0\le t\le 3$. [No calculator]

26. A particle has velocity $v(t)=(t-2)^2(t+1)$ for $t\ge 0$. (a) Show that the particle is at rest at $t=2$ but does not change direction. (b) Find the displacement and total distance on $[0,3]$. [No calculator]

27. A particle moving on a line has acceleration $a(t)=12t-6$, velocity $v(1)=0$, and position $x(0)=2$. (a) Find $v(t)$. (b) When is the particle at rest? (c) Find $x(2)$ and the total distance traveled on $[0,2]$. [No calculator]

9.6 Areas and Volumes

Mixed practice with area between curves, disks, washers, shells, and solids with known cross sections.

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Every area and volume problem starts with a sketch and a decision: which variable do you integrate with, and what shape is the typical slice? Write the slice's size, then add up.

Area and volume formulas

Area between curves: $\int_a^b [f(x) - g(x)]\,dx$ where $f \geq g$ on $[a, b]$; use $\int_c^d [\text{right} - \text{left}]\,dy$ for horizontal slices.

Disks and washers (about a horizontal or vertical line): $V = \pi \int \left[R^2 - r^2\right]$ where R and r are the outer and inner distances to the axis.

Shells: $V = 2\pi \int (\text{radius})(\text{height})$ with the slice parallel to the axis.

Cross sections: $V = \int A(x)\,dx$, where A is the area of the cross section (square: s^2; semicircle: $\frac{\pi}{8}d^2$; equilateral triangle: $\frac{\sqrt{3}}{4}s^2$).

Common mistake

Squaring the difference of radii instead of subtracting the squares: a washer needs $R^2 - r^2$, not $(R - r)^2$. Also measure radii from the axis of revolution, not from the x-axis.

SET A Core skills

Find each exact area or volume. [No calculator]

1. Find the area of the region bounded by $y = x^2$ and $y = 2x$. [No calculator]
2. Find the area of the region bounded by $y = \sqrt{x}$ and $y = \frac{x}{3}$. [No calculator]
3. Find the area of the region bounded by $y = \cos x$, $y = \sin x$, and the y-axis, for $0 \leq x \leq \frac{\pi}{4}$. [No calculator]
4. Find the area of the region bounded by $x = y^2$ and $x = 6 - y$ by integrating with respect to y. [No calculator]
5. The region under $y = \sqrt{x}$ for $0 \leq x \leq 4$ is revolved about the x-axis. Find the volume. [No calculator]
6. The region between $y = x$ and $y = x^2$ is revolved about the x-axis. Find the volume. [No calculator]
7. The region bounded by $y = x^2$, $y = 4$, and $x = 0$ (with $x \geq 0$) is revolved about the y-axis. Find the volume. [No calculator]
8. The region under $y = 3x - x^2$ and above the x-axis is revolved about the y-axis. Use cylindrical shells to find the volume. [No calculator]
9. The base of a solid is the region between $y = 0$ and $y = 4 - x^2$. Cross sections perpendicular to the x-axis are squares. Find the volume. [No calculator]

SET B Calculator practice

Give decimal answers to three places. [Calculator]

10. Find the area of the region enclosed by $y = \cos x$ and $y = x^2$. Give three decimals. [Calculator]
11. The region enclosed by $y = \cos x$ and $y = x^2$ is revolved about the x-axis. Find the volume to three decimals. [Calculator]
12. The region under $y = e^{-x^2}$ for $0 \leq x \leq 2$ is revolved about the y-axis. Find the volume to three decimals. [Calculator]
13. The region between $y = \sqrt{x}$ and $y = x^2$ is revolved about the line $y = -1$. Find the volume to three decimals. [Calculator]
14. A pond has the shape shown in the table: its width $w(x)$ in meters is measured at points x meters along its length. The pond is modeled as a solid whose cross sections perpendicular to its length are squares with side $w(x)$. Use a trapezoid sum with four subintervals to estimate the volume in cubic meters. [Calculator]

x (m)	0	5	10	15	20
$w(x)$ (m)	4	7	9	6	3

15. Find the total area of the region(s) enclosed by $y = x^3 - 4x$ and $y = x^2$. Give three decimals. [Calculator]
16. The base of a solid is the region bounded by $y = \ln x$, the x-axis, and $x = 3$. Cross sections perpendicular to the x-axis are squares. Find the volume to three decimals. [Calculator]
17. The region between $y = 2 + \sin x$ and the x-axis for $0 \leq x \leq 2\pi$ is revolved about the x-axis. Find the volume to three decimals. [Calculator]
18. The line $x = k$ with $0 < k < 3$ divides the region under $y = x^2$ on $[0, 3]$ into two parts of equal area. Find k to three decimals. [Calculator]

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SET C AP-style region problems

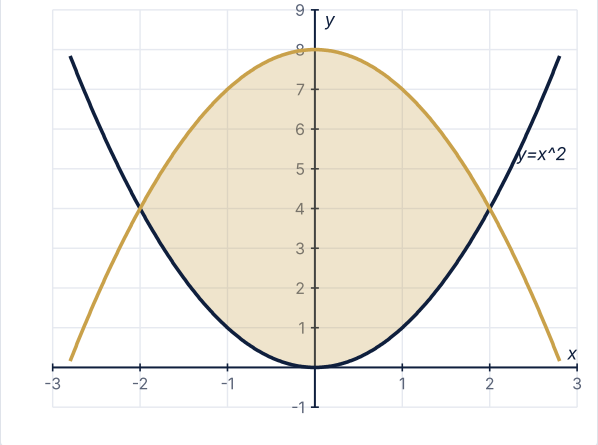
Show setups and give exact answers unless told otherwise. [No calculator]

19. Let R be the region enclosed by $y = x^2$ and $y = 8 - x^2$.

(a) Find the area of R.

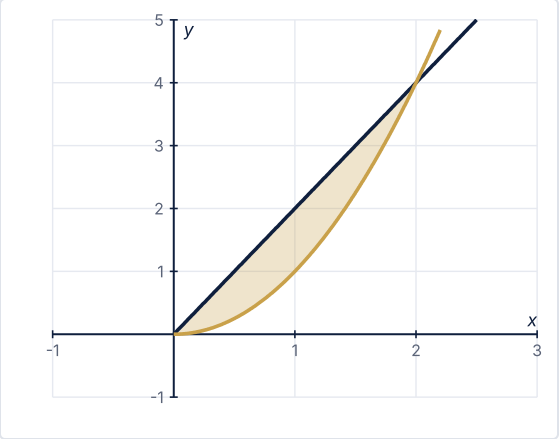
(b) Find the volume when R is revolved about the x-axis.

(c) Write, but do not evaluate, an integral for the volume when R is revolved about the line $y = -1$.



Region R.

20. Let R be the region in the first quadrant bounded by $y = 2x$ and $y = x^2$. A solid has base R, and its cross sections perpendicular to the x-axis are isosceles right triangles with one leg in the base. Find the volume of the solid. [No calculator]



Region R.

21. Let R be the region bounded by $y = x^3$, $y = 8$, and the y-axis. (a) Find the volume when R is revolved about the y-axis using disks. (b) Verify the same result using shells. (c) Find the volume when R is revolved about the x-axis. [No calculator]

22. Let R be the region bounded by $y = \frac{1}{x}$, the x-axis, $x = 1$, and $x = 3$. (a) Find the area of R. (b) Find the volume when R is revolved about the x-axis. (c) Find the volume when R is revolved about the line $y = -1$. [No calculator]

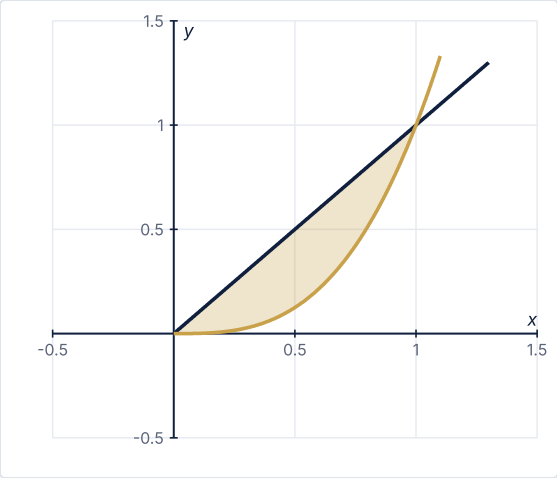
23. For a constant $k > 0$, the region bounded by $y = x^2$ and $y = kx$ has area 36. Find k. [No calculator]

24. The base of a solid is the disk $x^2 + y^2 \le 9$. Cross sections perpendicular to the x-axis are (a) squares, and in a second solid (b) semicircles, in each case with a diameter in the base. Find each volume. [No calculator]

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25. Let R be the region in the first quadrant bounded by $y = x$ and $y = x^3$.
(a) Find the area of R.
(b) Find the volume when R is revolved about the x-axis.
(c) Find the volume when R is revolved about the line $x = 1$.



Region R.

26. Find the area of the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{2}$. [No calculator]

27. A tank has cross-sectional area $A(y) = 9 - y^2$ square meters at height y meters above its base, for $0 \le y \le 3$. (a) Write and evaluate an integral for the volume of the tank. (b) Water is poured in at $2\ \text{m}^3/\text{min}$. At what rate is the depth increasing when the depth is 2 m? [No calculator]

9.7 Differential Equations

Practice with separable equations, slope fields, Euler's method, exponential growth and decay, and logistic models.

Differential equations ask you to find a function from a rule about its rate of change. The main AP skills are separating variables, reading slope fields, using Euler's method, and recognizing exponential and logistic models.

Differential equation toolkit

Separable equations: write $\frac{dy}{dx} = g(x)h(y)$ as $\frac{dy}{h(y)} = g(x)\,dx$, integrate both sides, add one constant C, then use the initial condition.

Euler's method: $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h\left.\frac{dy}{dx}\right|_{(x_n, y_n)}$.

Models: $\frac{dy}{dt} = ky$ gives $y = y_0 e^{kt}$; $\frac{dy}{dt} = -k(y - A)$ gives $y = A + Ce^{-kt}$; $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$ is logistic with carrying capacity L, fastest growth at $P = \frac{L}{2}$.

Common mistake

Forgetting the constant of integration before using the initial condition, or losing the sign when removing absolute values. Always solve for C right after integrating.

SET A Core skills

Solve each problem. [No calculator]

1. Solve $\frac{dy}{dx} = 3x^2 y$ with $y(0) = 2$. [No calculator]

2. Solve $\frac{dy}{dx} = \frac{x}{y}$ with $y(0) = 3$. State the solution as a function of x. [No calculator]

3. Solve $\frac{dy}{dx} = e^{x-y}$ with $y(0) = \ln 3$. [No calculator]

4. Solve $\frac{dy}{dx} = y^2 \cos x$ with $y(0) = 1$, and state the largest interval containing $x = 0$ on which the solution is defined. [No calculator]

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5. For the differential equation $\frac{dy}{dx} = xy$, find the slope of the solution curve at $(1, 2)$, $(-2, 1)$, $(0, 5)$, and $(3, -1)$, and describe where the slope field has horizontal segments. [No calculator]
6. Let $\frac{dy}{dx} = x + y$ with $y(0) = 1$. Use Euler's method with two steps of size $h = 0.5$ to approximate $y(1)$. [No calculator]
7. A population satisfies $\frac{dP}{dt} = 0.04P$ with $P(0) = 500$. Find $P(t)$, then $P(10)$ exactly, and the time for the population to double. [No calculator]
8. A hot object cools according to $\frac{dT}{dt} = -k(T - 20)$ with T in degrees Celsius. If $T(0) = 90$ and $T(5) = 55$, find k and $T(10)$. [No calculator]
9. A population obeys $\frac{dP}{dt} = 0.02P\left(1 - \frac{P}{800}\right)$ with $P(0) = 100$. Find $\lim_{t\to\infty} P(t)$, the value of P at which the population grows fastest, and that maximum growth rate. [No calculator]

SET B Models and numerical work

Give decimal answers to three places. [Calculator]

10. Let $\frac{dy}{dx} = xe^{-y}$ with $y(0) = 1$. Use Euler's method with $h = 0.2$ (three steps) to approximate $y(0.6)$. Give three decimals. [Calculator]
11. Let $\frac{dy}{dx} = y\cos x$ with $y(0) = 2$. (a) Use Euler's method with $h = 0.25$ to approximate $y(1)$. (b) Solve the equation exactly and compare. Give three decimals. [Calculator]
12. A town has population $P(t)$ with $\frac{dP}{dt} = 0.03P$ and $P(0) = 1200$, where t is in years. Find $P(15)$ and the time when the population reaches 3000. Give three decimals. [Calculator]
13. A population satisfies $\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{2000}\right)$ with $P(0) = 100$. The solution is $P(t) = \frac{2000}{1 + 19e^{-0.3t}}$. Find $P(10)$ and the time at which $P = 1000$. Give three decimals. [Calculator]
14. A liquid cools according to $\frac{dT}{dt} = -0.12(T - 22)$ with $T(0) = 95$ (degrees Celsius, t in minutes). When does the temperature reach 40 degrees? Give three decimals. [Calculator]
15. Solve $\frac{dy}{dx} = \frac{x^2}{y}$ with $y(1) = 2$, then find $y(3)$ to three decimals. [Calculator]
16. Salt enters a tank so that the amount $S(t)$ in kilograms satisfies $\frac{dS}{dt} = 4 - 0.05S$ with $S(0) = 10$. Find $S(t)$, then $S(20)$ to three decimals, and $\lim_{t\to\infty} S(t)$. [Calculator]
17. A radioactive sample has half-life 12 days and initial mass 50 g. Find the mass after 30 days and the instantaneous rate of change of mass at that time. Give three decimals. [Calculator]
18. A population satisfies $\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{50}\right)$ with $P(0) = 10$. (a) Find $\frac{dP}{dt}$ at $t = 0$. (b) Use the tangent line at $t = 0$ to estimate $P(0.5)$. (c) Find the maximum growth rate. [Calculator]

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SET C **AP-style differential equations**

Show your work and justify. [No calculator]

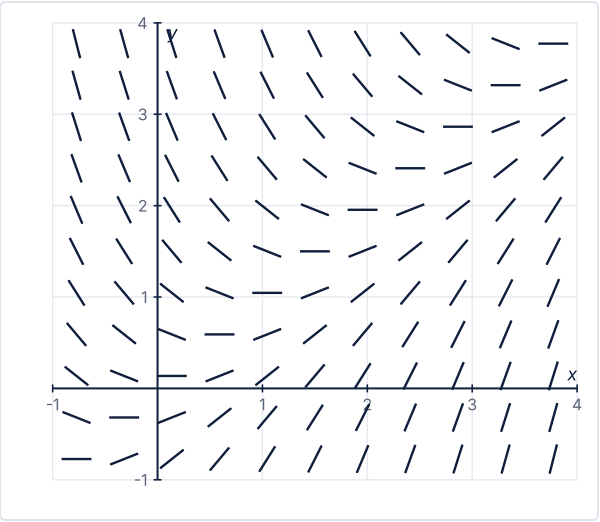
19. The slope field shown is for the differential equation $\frac{dy}{dx} = x - y$. Let $y = f(x)$ be the solution with $f(0) = 2$.

(a) Find the slope at $(0, 2)$, and find all points where the slope is zero.

(b) Show that $y = x - 1$ is a solution.

(c) Find $\frac{d^2y}{dx^2}$ in terms of x and y, and decide whether the graph of f is concave up or down at $x = 0$.

(d) Use Euler's method with two steps of size $h = 1$ to approximate $f(2)$.



Slope field for dy/dx = x - y.

20. Solve $\frac{dy}{dx} = \frac{x}{y^2}$ with $y(2) = 1$, and state the domain of the solution. [No calculator]

21. A population satisfies $\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{500}\right)$. (a) If $P(0) = 100$, find $\lim_{t\to\infty} P(t)$. (b) For what positive values of P is the graph of P concave up? (c) What is the greatest possible value of $\frac{dP}{dt}$? (d) If $P(0) = 700$, describe the behavior of $P(t)$. [No calculator]

22. A culture of bacteria grows at a rate proportional to its size and doubles every 3 hours. (a) Write a differential equation for the size $B(t)$ and find k. (b) By what factor does it grow in 9 hours? (c) How long does it take to triple? [No calculator]

23. An object's temperature satisfies $\frac{dT}{dt} = -\frac{1}{4}(T - 10)$ with $T(0) = 50$. (a) Verify that $T = 10 + 40e^{-t/4}$ is the solution. (b) Find the time at which $T = 30$. (c) Explain why $T(t) > 10$ for all t and find $\lim_{t\to\infty} T$. [No calculator]

24. Solve $\frac{dy}{dx} = y\cos x$ with $y\left(\frac{\pi}{2}\right) = e$. [No calculator]

25. Let $\frac{dy}{dx} = x - 2y$ with $y(0) = 1$. (a) Use Euler's method with $h = 0.5$ (three steps) to approximate $y(1.5)$. (b) Find $\frac{d^2y}{dx^2}$ at $(0, 1)$ and decide whether the first Euler estimate $y(0.5) \approx 0$ is an overestimate or underestimate. [No calculator]

26. Consider $\frac{dy}{dx} = 3(y - 2)$. (a) Show that $y = 2 + Ce^{3x}$ is a solution for every constant C. (b) Find the particular solution with $y(0) = 5$. (c) For this solution, find $\lim_{x\to-\infty} y$. [No calculator]

27. Consider $\frac{dy}{dx} = y(y - 3)(y + 1)$. (a) Find the equilibrium solutions. (b) Classify each as stable or unstable. (c) If $y(0) = 1$, find $\lim_{x\to\infty} y$. [No calculator]

9.8 Mixed Calculus AB Review

A cumulative review of limits, derivatives, applications of derivatives, integrals, and applications of integrals.

This cumulative set pulls together limits and continuity, derivatives and their applications, integrals, and applications of integrals. Pick the right tool, show the key step, and justify conclusions with the theorem you are using.

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Core results to keep ready

Continuity at a: $f(a)$ exists, $\lim_{x\to a} f(x)$ exists, and the two are equal. Differentiable implies continuous, not the other way around.

Extreme Value and Mean Value Theorems: both need continuity on the closed interval $[a, b]$; the Mean Value Theorem also needs differentiability on (a, b).

Fundamental Theorem: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ and $\int_a^b f'(x)\,dx = f(b) - f(a)$ for continuous f and f'.

Common mistake

Concluding that a critical number is a maximum or minimum without a sign change in f' or a sign check of f''. State the evidence, not just the answer.

SET A Mixed skills

Solve each problem. [No calculator]

1. Evaluate $\lim_{x\to 3} \dfrac{x^2 - 9}{x^2 - 5x + 6}$. [No calculator]
2. Evaluate $\lim_{x\to\infty} \dfrac{4x^3 - x}{2x^3 + 7}$ and interpret the result for the graph. [No calculator]
3. Evaluate $\lim_{x\to 0} \dfrac{\sin 5x}{2x}$. [No calculator]
4. Let $f(x) = \dfrac{x^2 - 4}{x - 2}$ for $x \neq 2$ and $f(2) = k$. For what value of k is f continuous at $x = 2$? Justify using the definition of continuity. [No calculator]
5. Differentiate $f(x) = e^{2x}\sin 3x$. [No calculator]
6. Differentiate $y = \dfrac{2x}{x^2 + 1}$ and find where the graph has a horizontal tangent. [No calculator]
7. The curve $x^3 + y^3 = 6xy$ passes through $(3, 3)$. Find $\dfrac{dy}{dx}$ there. [No calculator]
8. The table gives values of differentiable functions f and g. Let $h(x) = f(g(x))$ and $k(x) = f(x)\,g(x)$. Find $h'(2)$ and $k'(2)$. [No calculator]

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	1	3	-2
3	7	5	2	6

9. Find the absolute maximum and minimum values of $f(x) = x^3 - 6x^2 + 9x + 1$ on $[0, 5]$. [No calculator]

SET B Mixed applications

Give decimal answers to three places. [Calculator]

10. A spherical balloon is inflated so that its volume increases at $20\ \text{cm}^3/\text{s}$. How fast is the radius increasing when the radius is 6 cm? Give three decimals. [Calculator]
11. A particle moves on a line with velocity $v(t) = \ln(t + 1) - 1$ for $0 \le t \le 4$. Find the time it changes direction, its displacement, and the total distance traveled. Give three decimals. [Calculator]
12. Let $f(x) = xe^{-x}$ for $x \ge 0$. Find the absolute maximum value of f and the x-coordinate and y-coordinate of the inflection point. Give three decimals. [Calculator]
13. A box with no top is made from a 24 cm by 18 cm sheet by cutting equal squares of side x from the corners and folding up the sides. Find the value of x that maximizes the volume and the maximum volume. Give three decimals. [Calculator]
14. Find the area of the region enclosed by $y = e^x$ and $y = x + 3$. Give three decimals. [Calculator]
15. The region bounded by $y = xe^{-x}$, the x-axis, and $x = 2$ is revolved about the x-axis. Find the volume to three decimals. [Calculator]
16. The velocity $v(t)$ of a particle in m/s is recorded in the table. Use a right Riemann sum with the three subintervals shown to approximate the displacement from $t = 0$ to $t = 9$, and give the units. [Calculator]

t (s)	0	2	5	9
$v(t)$	5	3	6	2

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17. Find the average value of $g(x) = x\sqrt{1+x}$ on $[0, 3]$ to three decimals. [Calculator]

18. Find $f'(2)$ for $f(x) = x^x$ with $x > 0$, to three decimals. [Calculator]

SET C AP-style mixed problems

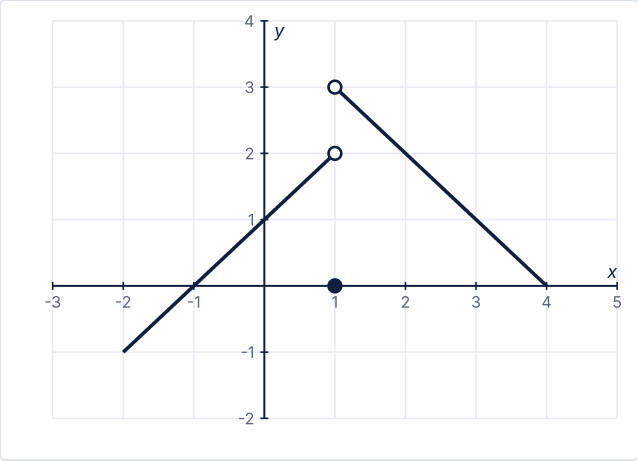
Justify your answers in complete sentences. [No calculator]

19. The graph of g on $[-2, 4]$ consists of the line $y = x + 1$ for $-2 \le x < 1$ (open circle at $(1, 2)$), the single point $(1, 0)$, and the line $y = 4 - x$ for $1 < x \le 4$ (open circle at $(1, 3)$).

    (a) Find $\lim_{x \to 1^-} g(x)$, $\lim_{x \to 1^+} g(x)$, and $\lim_{x \to 1} g(x)$.

    (b) Find $g(1)$. Is g continuous at $x = 1$? Justify, and name the type of discontinuity.

    (c) Find $\lim_{x \to 3} g(x)$ and $g'(3)$.



Graph of g.

20. Let $f(x) = xe^{-x^2}$. (a) Find the critical numbers of f and classify each. (b) Find the x-coordinates of the inflection points. (c) Find $\lim_{x \to \infty} f(x)$ and use it to explain why f has an absolute maximum value, and state it. [No calculator]

21. Sand falls at $10\ \text{m}^3/\text{min}$ onto a pile that is a cone whose height always equals its radius. How fast is the height increasing when the height is 5 m? [No calculator]

22. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. Find the dimensions of the rectangle of greatest area and justify that it is a maximum. [No calculator]

23. Let $F(x) = \displaystyle\int_1^x (t^2 - t - 6)\,dt$. (a) Find $F'(x)$ and the critical numbers of F. (b) Classify each critical number. (c) Find $F(3)$. (d) Is F concave up or down at $x = 0$? Justify. [No calculator]

24. A water tank leaks at a rate $R(t)$ liters per hour, which is decreasing and differentiable. Selected values are in the table.

    (a) Find the left Riemann sum and the right Riemann sum for $\int_0^6 R(t)\,dt$ with three equal subintervals.

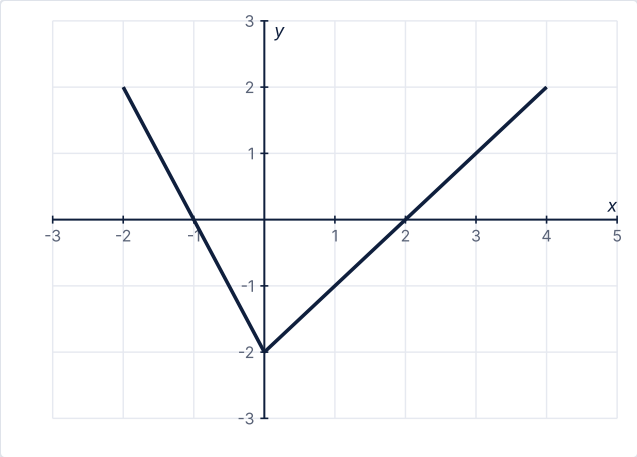
    (b) Which sum is an overestimate of the true amount that leaked? Justify. (c) What does the trapezoid sum give?

t (h)	0	2	4	6
$R(t)$	8	6	5	3

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25. The graph shows f', the derivative of a continuous function f on $[-2, 4]$. It consists of segments joining $(-2, 2)$, $(0, -2)$, $(2, 0)$, and $(4, 2)$. Also $f(0) = 1$.
    (a) Find the x-coordinate of each local maximum and local minimum of f. Justify.
    (b) Find the x-coordinate of each inflection point of f. Justify.
    (c) Find $f(2)$.



Graph of f'.

26. Solve $\frac{dy}{dx} = -2xy$ with $y(0) = 3$. Then find $\lim_{x \to \infty} y$ and the x-coordinates where the graph of y has inflection points. [No calculator]
27. Let R be the region bounded by $y = \sqrt{x}$, $y = x - 2$, and the y-axis. (a) Find the area of R. (b) Write an integral expression for the volume when R is revolved about the line $y = -2$, and evaluate it. [No calculator]

9.9 Mixed Calculus BC Review

A cumulative set that revisits series, parametric and vector motion, polar curves, and integration techniques. Use it to find gaps before the exam.

Work Set A without a calculator, Set B with one, and Set C without. Give decimals to three places in Set B. In Set C, write full justifications as you would on a free-response question.

How to use this review

Mark every problem you miss, then return to the matching section. Series and polar problems reward a clean setup: write the formula first, then substitute.

SET A Skills

Solve each problem. [No calculator]

1. Evaluate $\int xe^{2x}\,dx$.
2. Evaluate $\int \frac{3x + 5}{(x + 1)(x + 3)}\,dx$.
3. Evaluate $\int_2^{\infty} \frac{4}{x^3}\,dx$.
4. Find the radius and the interval of convergence of $\sum_{n=1}^{\infty} \frac{(x - 2)^n}{n\,3^n}$.
5. Write the first three nonzero terms of the Maclaurin series for $f(x) = xe^{-x^2}$.
6. A curve is given by $x = t^2 + 1$, $y = t^3 - 3t$.
   (a) Find $\frac{dy}{dx}$ at $t = 2$.
   (b) Find $\frac{d^2y}{dx^2}$ at $t = 2$.
7. A particle has position $\mathbf{r}(t) = \langle t^3, 3t^2 \rangle$.
   (a) Find its speed at $t = 1$.
   (b) Find its acceleration vector at $t = 1$.

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8. Find the area of one petal of the polar curve $r = 4\cos(2\theta)$, the petal that is centered on the positive x-axis.
9. Let $\frac{dy}{dx} = x + y$ with $y(0) = 1$. Use Euler's method with two steps of size $h = 0.5$ to approximate $y(1)$.

SET B Applications and Mixed Practice

Solve each problem. Give decimal answers to three decimal places. [Calculator]

10. Find the length of the parametric curve $x = t^2$, $y = t^3$ for $0 \le t \le 2$.
11. Find the length of the polar curve $r = \theta$ for $0 \le \theta \le \pi$.
12. Use the third-degree Maclaurin polynomial for $\sin x$ to approximate $\sin(0.3)$, and use the Lagrange error bound to bound the error.
13. Let S_4 be the sum of the first four terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$. Compute S_4 and use the alternating series error bound to give a bound on $|S - S_4|$.
14. A particle moves with velocity $\mathbf{v}(t) = \langle \cos(t^2), \sin t \rangle$ and starts at $(1, 2)$ when $t = 0$.
    (a) Find the position at $t = 2$.
    (b) Is the particle moving farther from the line $y = 2$ or closer to it at $t = 1$? Explain.
15. Find the area of the region inside $r = 3\cos\theta$ and outside $r = 1 + \cos\theta$.
16. Use the first three nonzero terms of the Maclaurin series for $\ln(1 + x)$ to approximate $\ln(1.2)$, and give a bound for the error.
17. Evaluate $\int_1^{\infty} \frac{1}{\sqrt{x^3 + 1}}\,dx$ numerically, after first explaining why it converges.
18. Let $\frac{dy}{dx} = x^2 + y$ with $y(0) = 1$. Use Euler's method with step size 0.2 to approximate $y(0.6)$. The exact solution is $y = 3e^x - x^2 - 2x - 2$; find the error of the approximation.

SET C AP-Style Challenges

Show all work and justify. [No calculator]

19. Let $f(x) = \int_0^x e^{-t^2}\,dt$.
    (a) Write the first four nonzero terms of the Maclaurin series for f.
    (b) Use the series to find $f'''(0)$.
20. A particle has acceleration $\mathbf{a}(t) = \langle 6t, 2 \rangle$, initial velocity $\mathbf{v}(0) = \langle 0, 1 \rangle$, and initial position $\mathbf{r}(0) = \langle 2, 0 \rangle$.
    (a) Find $\mathbf{r}(t)$.
    (b) Find the speed at $t = 1$.
21. For the polar curve $r = 1 + \sin\theta$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$.
22. Evaluate $\int \frac{x^2 + 1}{x(x-1)^2}\,dx$.
23. Evaluate $\int_0^3 (x-1)^{-2/3}\,dx$, or show that it diverges.
24. Find the length of one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$ for $0 \le t \le 2\pi$.
25. Classify each series as absolutely convergent, conditionally convergent, or divergent.
    (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$
    (b) $\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{2n+1}$
    (c) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$
26. A function f has $f(2) = 3$, $f'(2) = -1$, $f''(2) = 4$, and $|f'''(x)| \le 6$ for $2 \le x \le 2.5$. Use the second-degree Taylor polynomial about $x = 2$ to approximate $f(2.5)$ and give an error bound.
27. A population satisfies $\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{800}\right)$ with $P(0) = 100$.
    (a) Find $\lim_{t \to \infty} P(t)$.
    (b) For what value of P is the population growing fastest, and what is that rate?

Answers Chapter 9: Answers and Explanations

Answers for every practice problem, numbered to match. Read the explanation even when your answer is correct.

9.3 Derivatives in Context

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SET A · CORE APPLICATIONS

1. $y = 3ex - 2e$. Since $y' = (2x + x^2)e^x$, the slope at $x = 1$ is $3e$ and $y(1) = e$. Then $y = e + 3e(x - 1) = 3ex - 2e$.
2. $y - 3 = -\frac{3}{2}(x - 2)$. Here $y' = \frac{2}{\sqrt{4x+1}}$, so the tangent slope at $x = 2$ is $\frac{2}{3}$ and the normal slope is $-\frac{3}{2}$ through $(2, 3)$.
3. $(-2, 21)$ and $(2, -11)$. Solve $f'(x) = 3x^2 - 12 = 0$ to get $x = \pm 2$, then $f(-2) = -8 + 24 + 5 = 21$ and $f(2) = 8 - 24 + 5 = -11$.
4. $\frac{1}{\pi}$ cm/s. From $A = \pi r^2$, $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$, so $10 = 10\pi \frac{dr}{dt}$ and $\frac{dr}{dt} = \frac{1}{\pi}$.
5. 15 m (perpendicular to the wall) by 30 m (parallel), area 450 m^2. With width w, length $60 - 2w$ and $A = 60w - 2w^2$, $A' = 60 - 4w = 0$ gives $w = 15$. Since $A'' = -4 < 0$, this is a maximum.
6. $c = 3$. f is a polynomial, so it is continuous on $[1, 5]$ and differentiable on $(1, 5)$. The average rate is $\frac{45-5}{4} = 10$, and $f'(c) = 2c + 4 = 10$ gives $c = 3$, which lies in $(1, 5)$.
7. At rest at $t = 1$ and $t = 5$; $a(1) = -12$. We have $v = 3t^2 - 18t + 15 = 3(t - 1)(t - 5)$, and $a = 6t - 18$, so $a(1) = -12$.
8. $x = 0$ is a local maximum; $x = \pm 2$ are local minima. $g'(x) = 4x(x - 2)(x + 2)$ changes sign from $-$ to $+$ at -2, from $+$ to $-$ at 0, and from $-$ to $+$ at 2.
9. $\frac{82}{27}$. $f'(x) = \frac{1}{3}x^{-2/3}$, so $f'(27) = \frac{1}{27}$ and $L(x) = 3 + \frac{1}{27}(x - 27)$. Then $L(28) = 3 + \frac{1}{27} = \frac{82}{27}$.

SET B · APPLIED PROBLEMS

10. The top moves down at about 0.833 ft/s ($\frac{dy}{dt} \approx -0.833$). From $x^2 + y^2 = 169$, $x\,x' + y\,y' = 0$; with $x = 5$, $y = 12$: $y' = -\frac{5}{12}(2) = -\frac{5}{6}$.
11. $r \approx 4.301$ cm and minimum area ≈ 348.734 cm^2. With $h = \frac{500}{\pi r^2}$, $S = 2\pi r^2 + \frac{1000}{r}$, and $S' = 4\pi r - \frac{1000}{r^2} = 0$ gives $r^3 = \frac{250}{\pi}$. $S'' > 0$ confirms a minimum (the height equals the diameter, about 8.603 cm).
12. At $t = 2$ s, position $\frac{2}{e} \approx 0.736$ m. $v(t) = \left(1 - \frac{t}{2}\right)e^{-t/2}$ is positive for $t < 2$ and negative for $t > 2$, so s has its absolute maximum at $t = 2$ (and $s \to 0$ as $t \to \infty$).
13. By the Mean Value Theorem. f is differentiable, hence continuous, on $[2, 8]$, so some c in $(2, 8)$ has $f'(c) = \frac{25-7}{8-2} = 3$.
14. $P'(3) \approx 115.108$ bacteria per hour. At $t = 3$ hours the population is increasing at about this instantaneous rate. (Use $P'(t) = \frac{2400e^{-0.4t}}{(1+5e^{-0.4t})^2}$.)
15. Approximation 3.100, true value ≈ 3.098; overestimate. $L(x) = 3 + \frac{1}{6}(x - 9)$ gives $L(9.6) = 3.1$. Because $f''(x) = -\frac{1}{4}x^{-3/2} < 0$, the graph is concave down and lies below its tangent line.
16. About 0.239 cm/s. By similar triangles $r = 0.4h$, so $V = \frac{0.16\pi}{3}h^3$ and $\frac{dV}{dt} = 0.16\pi h^2 \frac{dh}{dt}$. Then $3 = 0.16\pi(25)\frac{dh}{dt}$ gives $\frac{dh}{dt} = \frac{3}{4\pi}$.
17. $x = 350$ units, maximum profit 4400 dollars. Profit is $P(x) = x(40 - 0.03x) - C(x) = 28x - 0.04x^2 - 500$. $P'(x) = 28 - 0.08x = 0$ gives $x = 350$, and $P'' = -0.08 < 0$, so this is the maximum. $P(350) = 4400$.
18. Because v is differentiable on $[5, 9]$, the Mean Value Theorem gives a time c in $(5, 9)$ with $v'(c) = \frac{v(9)-v(5)}{9-5} = \frac{10-12}{4} = -0.5$, so $a(c) = -0.5$ m/s^2.

SET C · AP-STYLE JUSTIFICATION

19. (a) f is decreasing on $(-3, -2)$ and $(1, 3)$, where $f' < 0$.
    (b) Local minima at $x = -2$ and $x = 3$ (f' changes from negative to positive); local maximum at $x = 1$ (f' changes from positive to negative).
    (c) $f''(x) = \frac{1}{4}(3x^2 - 4x - 5) = 0$ at $x = \frac{2 \pm \sqrt{19}}{3}$, and f'' changes sign at each (it is a quadratic with two simple roots), so both are inflection points.
20. (a) Differentiating, $2x - y - xy' + 2yy' = 0$, so $y' = \frac{y - 2x}{2y - x}$. At $(1, 2)$, $y' = \frac{0}{3} = 0$.
    (b) Differentiating again and using $y' = 0$: $2 - 2y' - xy'' + 2(y')^2 + 2yy'' = 0$ becomes $2 + (2y - x)y'' = 0$, so $y'' = -\frac{2}{3}$. The curve is concave down, so $(1, 2)$ is a local maximum.
21. Point $\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$, distance $\frac{\sqrt{15}}{2}$. Minimize $D = (x - 4)^2 + x$ (squared distance): $D' = 2x - 7 = 0$ gives $x = \frac{7}{2}$, and $D'' = 2 > 0$. Then $D = \frac{1}{4} + \frac{7}{2} = \frac{15}{4}$.
22. $\frac{3}{50} = 0.06$ rad/s. We have $\tan\theta = \frac{h}{3}$, so $\sec^2\theta\,\theta' = \frac{1}{3}h'$. At $h = 4$ the hypotenuse is 5 and $\sec^2\theta = \frac{25}{9}$. Then $\theta' = \frac{9}{25} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{3}{50}$.
23. By the Mean Value Theorem there is c in $(1, 5)$ with $f'(c) = \frac{10-(-2)}{5-1} = 3$. But $|f'(c)| \le 2$ and $3 > 2$, a contradiction. No such function exists.
24. $g'(3) = \frac{1}{4}$. Since $f'(x) = 3x^2 + 1 > 0$, f is strictly increasing, hence one-to-one. As $f(1) = 3$, $g(3) = 1$, and $g'(3) = \frac{1}{f'(1)} = \frac{1}{4}$.
25. $(1, 2)$ and $(3, 5)$. Here $v = (t - 1)(t - 3)$ and $a = 2t - 4$. Speed increases when v and a have the same sign: on $(1, 2)$ both are negative; on $(3, 5)$ both are positive. On $(0, 1)$ and $(2, 3)$ the signs differ.
26. $a = \frac{5}{2}$, $b = \frac{3}{2}$. Continuity at 1: $1 + a = b + 2$. Matching derivatives: $2x + a = 3bx^2$ at $x = 1$ gives $2 + a = 3b$. Substituting $a = b + 1$ gives $b + 3 = 3b$, so $b = \frac{3}{2}$ and $a = \frac{5}{2}$.
27. Absolute maximum 16 at $x = 4$; absolute minimum -4 at $x = -1$ and at $x = 2$. $f'(x) = 3x(x - 2)$ gives critical numbers 0 and 2. Compare $f(-1) = -4$, $f(0) = 0$, $f(2) = -4$, $f(4) = 16$.

9.4 Integral Skills

SET A · CORE SKILLS

1. $2x^4 + \frac{5}{x} + 2x^{3/2} + C$. Rewrite as $8x^3 - 5x^{-2} + 3x^{1/2}$ and apply the power rule term by term.
2. $\frac{(x^2+5)^5}{10} + C$. Let $u = x^2 + 5$, $du = 2x\,dx$, so the integral is $\frac{1}{2}\int u^4\,du = \frac{u^5}{10} + C$.
3. $-\frac{1}{2}\cos 2x - e^{-x} + C$. Each term needs a factor from the inside function: $\int \sin 2x\,dx = -\frac{1}{2}\cos 2x$ and $\int e^{-x}dx = -e^{-x}$.
4. $\frac{1}{2}\ln(x^2 + 1) + C$. With $u = x^2 + 1$, $du = 2x\,dx$, the integral is $\frac{1}{2}\int \frac{du}{u}$. No absolute value is needed since $x^2 + 1 > 0$.
5. $e^8 - 1$. Let $u = x^3$, $du = 3x^2dx$; the limits become 0 and 8, so the integral is $\int_0^8 e^u\,du = e^8 - 1$.
6. $\frac{20}{3}$. Rewrite the integrand as $x^{1/2} + x^{-1/2}$. An antiderivative is $\frac{2}{3}x^{3/2} + 2x^{1/2}$, which equals $\frac{28}{3}$ at 4 and $\frac{8}{3}$ at 1.
7. $\frac{1}{4}$. Let $u = \sin x$, $du = \cos x\,dx$; the limits go from 0 to 1, giving $\int_0^1 u^3 du = \frac{1}{4}$.

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8. $12 + \frac{9\pi}{2}$. The integral of the constant 2 is a rectangle of area $2 \cdot 6 = 12$. The graph of $y = \sqrt{9 - x^2}$ is the upper half of a circle of radius 3, with area $\frac{1}{2}\pi(3)^2 = \frac{9\pi}{2}$.
9. $\frac{x^2}{2} + 3x + C$. Factor: $\frac{x^2-9}{x-3} = x + 3$ for $x \neq 3$, then integrate term by term.

SET B · NUMERICAL AND APPLIED

10. 0.886. No elementary antiderivative exists, so use numerical integration of the continuous integrand on $[0, 3]$.
11. 3.241. Evaluate the definite integral numerically.
12. 58. Trapezoid areas: $\frac{1}{2}(3)(2+5) = 10.5$, $\frac{1}{2}(4)(5+9) = 28$, $\frac{1}{2}(3)(9+4) = 19.5$. The sum is 58.
13. Left sum ≈ 2.610; exact value $= \frac{2}{3}(3^{3/2} - 1) \approx 2.797$; underestimate. f is increasing, so left endpoints give heights below the curve.
14. $G(2) \approx 3.495$ and $G'(2) = \sin 4 \approx -0.757$. $G(2) = 3 + \int_1^2 \sin(t^2)dt$ by calculator, and $G'(x) = \sin(x^2)$ by the Fundamental Theorem.
15. 1.976. The average value is $\frac{1}{\pi - 0}\int_0^{\pi} e^{\sin x}dx$, evaluated numerically.
16. ≈ 155.988 liters. The amount is $50 + \int_0^6 R(t)\,dt$, and $\int_0^6 R\,dt = 105.988$ by calculator (exactly $72 + 24(1 - \cos 2)$).
17. $k = \tan 1 \approx 1.557$. An antiderivative is $\arctan x$, so $\arctan k = 1$ and $k = \tan 1$.
18. 10.439. The integrand is continuous on $[0, 3]$; evaluate numerically.

SET C · AP-STYLE INTEGRALS

19. (a) $g(2) = \frac{1}{2}(2)(4) = 4$, $g(4) = 4 + 4 = 8$, $g(6) = 8 - \frac{1}{2}(2)(2) = 6$.
(b) $g' = f$ is positive on $(0, 4)$ and negative on $(4, 6)$, so g increases then decreases. The candidates are $g(0) = 0$, $g(4) = 8$, $g(6) = 6$, so the absolute maximum is 8 at $x = 4$.
(c) $g'' = f'$, which is 2 on $(0, 2)$ and -2 on $(2, 4)$ and -1 on $(4, 6)$. It changes sign only at $x = 2$, so the only inflection point is $x = 2$.
20. $H'(x) = 2x\ln(1 + x^2)$ and $H'(1) = 2\ln 2$. Substitute $t = x^2$ into the integrand and multiply by $\frac{d}{dx}(x^2) = 2x$.
21. $\frac{1}{3}$. Let $u = \ln x$, $du = \frac{dx}{x}$; the limits 1 and e become 0 and 1. Then $\int_0^1 u^2 du = \frac{1}{3}$.
22. $\frac{8}{3}$. Let $u = x + 1$, so $x = u - 1$ and the limits are 1 to 4. The integral is $\int_1^4 (u^{1/2} - u^{-1/2})du = \left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^4 = \frac{4}{3} - \left(-\frac{4}{3}\right) = \frac{8}{3}$.
23. $\frac{5}{2} + \ln 2$. Divide: $x^2 + 3x + 3 = (x + 1)(x + 2) + 1$, so the integrand is $x + 2 + \frac{1}{x+1}$. Then $\left[\frac{x^2}{2} + 2x + \ln(x + 1)\right]_0^1 = \frac{5}{2} + \ln 2$.
24. (a) $2\pi - 2$. The semicircle of radius 2 has area $\frac{1}{2}\pi(4) = 2\pi$. The triangle below the axis on $[2, 4]$ has legs 2 and 2, area 2, counted negatively.
(b) $2\pi + 2$. Absolute value makes the triangle's contribution positive.
25. $f(x) = \frac{5}{2} - \frac{1}{2}e^{-x^2}$. Let $u = -x^2$: $\int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2} + C$. Then $f(0) = 2$ gives $-\frac{1}{2} + C = 2$, so $C = \frac{5}{2}$.
26. 36. The terms x^5 and $-4x^3$ are odd functions, so their integrals over $[-2, 2]$ are 0. The even part gives $2\int_0^2 (6x^2 + 1)\,dx = 2(16 + 2) = 36$.
27. $f(8) = 3$ and the second integral is 6. By the Fundamental Theorem, $f(8) - f(3) = -4$, so $f(8) = 3$. Then $\int_3^8 (f' + 2)\,dx = -4 + 2(5) = 6$.

9.5 Particle Motion and Accumulation

SET A · CORE SKILLS

1. Displacement 4; total distance 12. Since $v = 3(t - 1)(t - 3)$, the particle moves right on $(0, 1)$, left on $(1, 3)$, right on $(3, 4)$. With $s(t) = t^3 - 6t^2 + 9t$ (taking $s(0) = 0$): $s(1) = 4$, $s(3) = 0$, $s(4) = 4$. Distance $= 4 + 4 + 4 = 12$.
2. $v(t) = 3 + 4t - t^2$ and $s(3) = 19$. Integrate a: $v = 4t - t^2 + 3$. Integrate again: $s = 2t^2 - \frac{t^3}{3} + 3t + 1$, so $s(3) = 18 - 9 + 9 + 1 = 19$.
3. Increasing. $v(2) = 4 - 10 + 4 = -2$ and $a(t) = 2t - 5$, so $a(2) = -1$. The velocity and acceleration are both negative, so the speed is increasing.
4. $t = \frac{\pi}{4}$ and $t = \frac{3\pi}{4}$. Zeros of $\cos 2t$ occur when $2t = \frac{\pi}{2}, \frac{3\pi}{2}$. The velocity is positive on $\left(0, \frac{\pi}{4}\right)$, negative on $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$, and positive on $\left(\frac{3\pi}{4}, \pi\right)$, so it changes sign at both.
5. Average velocity 1; average speed $\frac{13}{5}$. $s(5) - s(0) = 5$, so the average velocity is 1. Since $v = 2t - 4$ changes sign at $t = 2$, distance $= |s(2) - s(0)| + |s(5) - s(2)| = 4 + 9 = 13$, and the average speed is $\frac{13}{5}$.
6. $W(4) = 64$ liters. $W(4) = 40 + \int_0^4 (12 - 3t)\,dt = 40 + [12t - 1.5t^2]_0^4 = 40 + 24 = 64$. Since $W' > 0$ for $t < 4$ and $W' < 0$ for $t > 4$, W is greatest at $t = 4$.
7. Displacement 18, total distance $\frac{58}{3}$, $a(6) = -3$. Areas: triangle $\frac{1}{2}(2)(4) = 4$; rectangle $3 \cdot 4 = 12$. The velocity crosses zero at $t = 5 + \frac{4}{3} = \frac{19}{3}$: positive triangle $\frac{1}{2} \cdot \frac{4}{3} \cdot 4 = \frac{8}{3}$, negative triangle $\frac{1}{2} \cdot \frac{2}{3} \cdot 2 = \frac{2}{3}$. Displacement $= 4 + 12 + \frac{8}{3} - \frac{2}{3} = 18$; distance $= 16 + \frac{10}{3} = \frac{58}{3}$. The slope on $(5, 7)$ is -3.
8. Displacement 0; total distance 8. An antiderivative is $-2\cos 2t$. On $\left[0, \frac{\pi}{2}\right]$ the velocity is nonnegative and the change in position is 4; on $\left[\frac{\pi}{2}, \pi\right]$ it is nonpositive and the change is -4.
9. $200(1 - e^{-1})$ mg. The amount is $\int_0^2 100e^{-0.5t}dt = \left[-200e^{-0.5t}\right]_0^2 = 200 - 200e^{-1}$.

SET B · CALCULATOR PRACTICE

10. 5.822. The velocity changes sign only at $t = \frac{\pi}{2}$ on this interval (the next zero is $\frac{3\pi}{2} \approx 4.712$), so distance $= \int_0^{\pi/2} t\cos t\,dt - \int_{\pi/2}^4 t\cos t\,dt$.
11. $s(3) \approx 2.090$ and speed ≈ 0.282. $s(3) = -1 + \int_0^3 v(t)\,dt$ by calculator, and $|v(3)| = 2\sin 3$.
12. At $t \approx 2.032$, where $x \approx 5.845$. The function $te^{t/3}$ is increasing for $t \geq 0$ and starts at 0, so v decreases from 4 through 0 exactly once. Solve $te^{t/3} = 4$, then compute $x = 1 + \int_0^t v$.
13. About 106 liters. Left sum $= 4(12) + 2(9) + 4(10) = 48 + 18 + 40 = 106$. The result has units liters because rate times time gives volume.
14. (a) $t \approx 0.680$ and $t \approx 5.604$. (b) At $t \approx 5.604$, with about 260.533 cars. The net rate is $30\sin(t/2) - 10$, positive between the two zeros and negative elsewhere. Candidates: $N(0) = 200$, $N(10) \approx 142.980$, $N(5.604) \approx 260.533$, so the maximum is at $t \approx 5.604$.
15. Average velocity ≈ -0.387; average speed ≈ 0.698. The velocity is positive until $t \approx 2.125$ and negative afterward, so distance $= \int_0^{2.125} v - \int_{2.125}^4 v \approx 2.793$ while displacement is $\int_0^4 v \approx -1.546$.

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16. $v(2) = 9\ln 3 - 8 \approx 1.888$; speed is increasing. Since $v(2) = -2 + \int_0^2 3\ln(t+1)\,dt > 0$ and $a(2) = 3\ln 3 > 0$, velocity and acceleration have the same sign.
17. Average rate ≈ 2.534 L/h, attained at $t \approx 0.794$ and $t \approx 4.055$. The average is $\frac{1}{6}\int_0^6 R(t)\,dt$. Since R is continuous, such a time exists; R rises to a maximum at $t = 2$ and then falls, so there are two solutions.
18. At $t = \frac{\ln 4}{0.3} \approx 4.621$ hours, $A \approx 8.069$ mg. $A' = 0$ when $e^{-0.3t} = \frac{1}{4}$; A' changes from positive to negative there. $A = \int_0^t A' = 20(1 - e^{-0.3t}) - 1.5t = 15 - 5\ln 4$.

SET C · AP-STYLE MOTION AND ACCUMULATION

19. (a) 39 gallons. $\int_0^2 N = \frac{2+4}{2}(2) = 6$; on $[2,5]$ the graph crosses zero at $t = 4$, area above $\frac{1}{2}(2)(4) = 4$, below $\frac{1}{2}(1)(2) = 1$. So $\int_0^5 N = 9$ and the amount is $30 + 9 = 39$.
    (b) At $t = 4$: N changes from positive to negative there, and the amount is $30 + 6 + 4 = 40$ gallons; the endpoint $t = 6$ gives 37.
    (c) 37 gallons, since $\int_5^6 N = -2$.
    (d) $(2,4)$: the amount increases when $N > 0$ and its rate N decreases when $N' < 0$, which holds on $(2,4)$.
20. (a) $x(10) \approx 5 + 16.5 = 21.5$ m. The trapezoids give $10 + 10.5 - 3 - 1 = 16.5$.
    (b) v is continuous with $v(5) = 1 > 0$ and $v(8) = -3 < 0$, so by the Intermediate Value Theorem there is a zero in $(5,8)$. Also $v(8) = -3 < 0 < 2 = v(10)$ gives a zero in $(8,10)$. These intervals do not overlap, so there are at least two.
    (c) By the Mean Value Theorem, some c in $(2,5)$ has $a(c) = v'(c) = \frac{v(5)-v(2)}{5-2} = \frac{1-6}{3} = -\frac{5}{3}$ m/s².
21. (a) $\left(2, \frac{7}{2}\right)$ and $(5,6)$. Here $a(t) = 2t - 7$. Speed increases when v and a share a sign: on $\left(2, \frac{7}{2}\right)$ both are negative; on $(5,6)$ both are positive. On $(0,2)$ and $\left(\frac{7}{2}, 5\right)$ they differ.
    (b) 15. An antiderivative is $\frac{t^3}{3} - \frac{7t^2}{2} + 10t$, which takes the values $0, \frac{26}{3}, \frac{25}{6}, 6$ at $t = 0, 2, 5, 6$. Distance $= \frac{26}{3} + \frac{9}{2} + \frac{11}{6} = 15$.
22. (a) $150(e-1)$. $\int_0^5 30e^{0.2t}dt = \left[150e^{0.2t}\right]_0^5 = 150e - 150$.
    (b) $30(e-1)$ cells per hour: divide the net change by 5.
    (c) $200 + 150(e-1) = 50 + 150e$ cells.
23. (a) At $t = 1$: $v > 0$ on $[0,1)$ and $v < 0$ on $(1,3)$ (the next zero is at the endpoint $t = 3$).
    (b) $x(3) = 2 - \frac{6}{\pi}$ and distance $\frac{18}{\pi}$. An antiderivative of v is $\frac{6}{\pi}\sin\frac{\pi t}{2}$. The change on $[0,1]$ is $\frac{6}{\pi}$; on $[1,3]$ it is $\frac{6}{\pi}(-1-1) = -\frac{12}{\pi}$. Distance $= \frac{6}{\pi} + \frac{12}{\pi}$.
24. Maximum 20 liters (at $t = 0$ and $t = 3$); minimum 16 liters (at $t = 1$ and $t = 4$). The net rate is $-3(t-1)(t-3)$, negative on $(0,1)$, positive on $(1,3)$, negative on $(3,4)$. The amount is $20 + 6t^2 - t^3 - 9t$, so $A(0) = 20$, $A(1) = 16$, $A(3) = 20$, $A(4) = 16$. Compare these candidates.
25. (a) $v(t) = t^2 - 4$ and $a(t) = 2t$ (Fundamental Theorem).
    (b) $v < 0$ on $(0,2)$ and $v > 0$ for $t > 2$, so it changes direction at $t = 2$. $x(2) = 4 + \left(\frac{8}{3} - 8\right) = -\frac{4}{3}$.
    (c) $\frac{23}{3}$. Distance $= \int_0^2 (4 - t^2)\,dt + \int_2^3 (t^2 - 4)\,dt = \frac{16}{3} + \frac{7}{3}$.
26. (a) $v(2) = 0$, but $(t-2)^2 \geq 0$ and $t + 1 > 0$, so $v \geq 0$ everywhere; v does not change sign at $t = 2$.
    (b) Both equal $\frac{21}{4}$. Since $v \geq 0$, distance equals displacement. Expanding, $v = t^3 - 3t^2 + 4$, so $\int_0^3 v\,dt = \left[\frac{t^4}{4} - t^3 + 4t\right]_0^3 = \frac{81}{4} - 27 + 12 = \frac{21}{4}$.
27. (a) $v(t) = 6t^2 - 6t = 6t(t-1)$ (since $v = 6t^2 - 6t + C$ and $v(1) = 0$ gives $C = 0$).
    (b) At $t = 0$ and $t = 1$.
    (c) $x(t) = 2t^3 - 3t^2 + 2$, so $x(2) = 16 - 12 + 2 = 6$. The velocity is negative on $(0,1)$ and positive on $(1,2)$: $x(0) = 2$, $x(1) = 1$, $x(2) = 6$. Distance $= 1 + 5 = 6$.

9.6 Areas and Volumes

SET A · CORE SKILLS

1. $\frac{4}{3}$. The curves meet at $x = 0$ and $x = 2$, with $y = 2x$ above. Area $= \int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$.
2. $\frac{9}{2}$. Setting $\sqrt{x} = \frac{x}{3}$ gives $x = 0$ and $x = 9$, and $\sqrt{x}$ is on top between them. Area $= \left[\frac{2}{3}x^{3/2} - \frac{x^2}{6}\right]_0^9 = 18 - \frac{27}{2} = \frac{9}{2}$.
3. $\sqrt{2} - 1$. On $\left[0, \frac{\pi}{4}\right]$, $\cos x \geq \sin x$. Area $= [\sin x + \cos x]_0^{\pi/4} = \sqrt{2} - 1$.
4. $\frac{125}{6}$. Solve $y^2 = 6 - y$: $y = -3$ and $y = 2$. The line $x = 6 - y$ is to the right, so area $= \int_{-3}^2 (6 - y - y^2)\,dy = \left[6y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-3}^2 = \frac{125}{6}$.
5. 8π. Disks of radius $\sqrt{x}$: $V = \pi\int_0^4 x\,dx = \pi\left[\frac{x^2}{2}\right]_0^4 = 8\pi$.
6. $\frac{2\pi}{15}$. On $[0,1]$ the line $y = x$ is above the parabola, so washers have outer radius x and inner radius x^2. $V = \pi\int_0^1 (x^2 - x^4)\,dx = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$.
7. 8π. Using disks perpendicular to the y-axis, the radius is $x = \sqrt{y}$: $V = \pi\int_0^4 y\,dy = 8\pi$. (Shells give the same: $2\pi\int_0^2 x(4 - x^2)\,dx = 2\pi(8 - 4) = 8\pi$.)
8. $\frac{27\pi}{2}$. The region lies on $0 \leq x \leq 3$. $V = 2\pi\int_0^3 x(3x - x^2)\,dx = 2\pi\left[x^3 - \frac{x^4}{4}\right]_0^3 = 2\pi\left(27 - \frac{81}{4}\right) = \frac{27\pi}{2}$.
9. $\frac{512}{15}$. The side of the square at x is $4 - x^2$, so the area is $(4 - x^2)^2$. $V = \int_{-2}^2 (16 - 8x^2 + x^4)\,dx = 2\left(32 - \frac{64}{3} + \frac{32}{5}\right) = \frac{512}{15}$.

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SET B · CALCULATOR PRACTICE

10. 1.095. The curves intersect at $x = \pm 0.824$ (solve $\cos x = x^2$), with $\cos x$ on top. By symmetry, area $= 2\int_0^{0.824} (\cos x - x^2)\,dx$.
11. 3.677. Washers have outer radius $\cos x$ and inner radius x^2 on $[-0.824, 0.824]$. By symmetry $V = 2\pi \int_0^{0.824} (\cos^2 x - x^4)\,dx$.
12. 3.084. By shells, $V = 2\pi \int_0^2 x e^{-x^2} dx = \pi(1 - e^{-4})$.
13. 3.037. On $[0, 1]$, $\sqrt{x} \geq x^2$. Distances from $y = -1$ give outer radius $\sqrt{x} + 1$ and inner radius $x^2 + 1$: $V = \pi \int_0^1 \left[(\sqrt{x} + 1)^2 - (x^2 + 1)^2\right] dx$.
14. About $892.5\ \text{m}^3$. The area is w^2, taking the values $16, 49, 81, 36, 9$. The trapezoid sum is $\frac{5}{2}\left[(16 + 49) + (49 + 81) + (81 + 36) + (36 + 9)\right] = \frac{5}{2}(357) = 892.5$.
15. 10.083. Solve $x^3 - x^2 - 4x = 0$: $x = 0$ and $x = \frac{1 \pm \sqrt{17}}{2} \approx -1.562,\ 2.562$. The graphs cross at $x = 0$, so there are two regions: the cubic is above the parabola on $(-1.562, 0)$ and below it on $(0, 2.562)$. Add the two areas $\int |y_1 - y_2|\,dx$.
16. 1.029. The region runs from $x = 1$ to $x = 3$ with side $\ln x$. $V = \int_1^3 (\ln x)^2 dx = \left[x \ln^2 x - 2x \ln x + 2x\right]_1^3 = 3\ln^2 3 - 6\ln 3 + 4$.
17. 88.826. $V = \pi \int_0^{2\pi} (2 + \sin x)^2 dx = \pi \int_0^{2\pi} \left(4 + 4\sin x + \sin^2 x\right) dx = \pi(8\pi + 0 + \pi) = 9\pi^2$.
18. $k = 2.381$. The total area is $\int_0^3 x^2 dx = 9$, so we need $\int_0^k x^2 dx = \frac{k^3}{3} = \frac{9}{2}$. Thus $k^3 = \frac{27}{2}$ and $k = \left(\frac{27}{2}\right)^{1/3}$.

SET C · AP-STYLE REGION PROBLEMS

19. (a) $\frac{64}{3}$. The curves meet where $x^2 = 8 - x^2$, so $x = \pm 2$. Area $= \int_{-2}^2 (8 - 2x^2)\,dx = \left[8x - \frac{2x^3}{3}\right]_{-2}^2 = \frac{64}{3}$.
    (b) $\frac{512\pi}{3}$. $V = \pi \int_{-2}^2 \left[(8 - x^2)^2 - x^4\right] dx = \pi \int_{-2}^2 (64 - 16x^2)\,dx = \pi\left(256 - \frac{256}{3}\right) = \frac{512\pi}{3}$.
    (c) $V = \pi \int_{-2}^2 \left[(9 - x^2)^2 - (x^2 + 1)^2\right] dx$. Distances from $y = -1$: outer radius $8 - x^2 + 1$, inner radius $x^2 + 1$.
20. $\frac{8}{15}$. The curves meet at $x = 0$ and $x = 2$. A leg has length $2x - x^2$, so the area of a triangle is $\frac{1}{2}(2x - x^2)^2$. $V = \frac{1}{2}\int_0^2 (4x^2 - 4x^3 + x^4)\,dx = \frac{1}{2}\left(\frac{32}{3} - 16 + \frac{32}{5}\right) = \frac{1}{2} \cdot \frac{16}{15} = \frac{8}{15}$.
21. (a) $\frac{96\pi}{5}$. Solving $x = y^{1/3}$: $V = \pi \int_0^8 y^{2/3} dy = \pi \cdot \frac{3}{5} \cdot 32 = \frac{96\pi}{5}$.
    (b) $2\pi \int_0^2 x(8 - x^3)\,dx = 2\pi\left(16 - \frac{32}{5}\right) = \frac{96\pi}{5}$.
    (c) $\frac{768\pi}{7}$. Washers: outer radius 8, inner radius x^3, so $V = \pi \int_0^2 (64 - x^6)\,dx = \pi\left(128 - \frac{128}{7}\right)$.
22. (a) $\ln 3$. $\int_1^3 \frac{1}{x}\,dx = \ln 3 - \ln 1$.
    (b) $\frac{2\pi}{3}$. $V = \pi \int_1^3 x^{-2} dx = \pi\left[-\frac{1}{x}\right]_1^3 = \pi\left(1 - \frac{1}{3}\right)$.
    (c) $\pi\left(\frac{2}{3} + 2\ln 3\right)$. Outer radius $\frac{1}{x} + 1$, inner radius 1: $V = \pi \int_1^3 \left[\left(\frac{1}{x} + 1\right)^2 - 1\right] dx = \pi \int_1^3 \left(x^{-2} + \frac{2}{x}\right) dx$.
23. $k = 6$. The curves meet at $x = 0$ and $x = k$ and the line is above. Area $= \int_0^k (kx - x^2)\,dx = \frac{k^3}{2} - \frac{k^3}{3} = \frac{k^3}{6}$. Setting $\frac{k^3}{6} = 36$ gives $k^3 = 216$, so $k = 6$.
24. (a) 144. The side is the chord length $2\sqrt{9 - x^2}$, so the area is $4(9 - x^2)$ and $V = 4\int_{-3}^3 (9 - x^2)\,dx = 4(36) = 144$.
    (b) 18π. The radius is $\sqrt{9 - x^2}$, so the area is $\frac{\pi}{2}(9 - x^2)$ and $V = \frac{\pi}{2}(36) = 18\pi$.
25. (a) $\frac{1}{4}$. The curves meet at $x = 0$ and $x = 1$ with $y = x$ above: $\int_0^1 (x - x^3)\,dx = \frac{1}{2} - \frac{1}{4}$.
    (b) $\frac{4\pi}{21}$. $V = \pi \int_0^1 (x^2 - x^6)\,dx = \pi\left(\frac{1}{3} - \frac{1}{7}\right)$.
    (c) $\frac{7\pi}{30}$. Shell radius is $1 - x$ and height $x - x^3$: $V = 2\pi \int_0^1 (1 - x)(x - x^3)\,dx = 2\pi \int_0^1 (x - x^2 - x^3 + x^4)\,dx = 2\pi \cdot \frac{7}{60}$.
26. $2\sqrt{2} - 2$. The curves cross at $x = \frac{\pi}{4}$, where $\cos x$ is on top before and $\sin x$ after. Area $= \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx = (\sqrt{2} - 1) + (\sqrt{2} - 1)$.
27. (a) $V = \int_0^3 (9 - y^2)\,dy = 27 - 9 = 18\ \text{m}^3$.
    (b) $\frac{2}{5}$ m/min. The volume up to depth h is $V(h) = \int_0^h A(y)\,dy$, so $\frac{dV}{dt} = A(h)\frac{dh}{dt}$. At $h = 2$: $2 = (9 - 4)\frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{2}{5}$.

9.7 Differential Equations

SET A · CORE SKILLS

1. $y = 2e^{x^3}$. Separate: $\frac{dy}{y} = 3x^2 dx$, so $\ln|y| = x^3 + C$ and $y = Ae^{x^3}$. The condition $y(0) = 2$ gives $A = 2$.
2. $y = \sqrt{x^2 + 9}$. Separate: $y\,dy = x\,dx$, so $\frac{y^2}{2} = \frac{x^2}{2} + C$, giving $y^2 = x^2 + 9$ using $y(0) = 3$. The positive root is chosen because $y(0) > 0$.
3. $y = \ln(e^x + 2)$. Since $e^{x-y} = e^x e^{-y}$, separate to $e^y\,dy = e^x\,dx$. Then $e^y = e^x + C$. At $x = 0$: $3 = 1 + C$, so $C = 2$.
4. $y = \frac{1}{1 - \sin x}$ on $-\frac{3\pi}{2} < x < \frac{\pi}{2}$. Separate: $\frac{dy}{y^2} = \cos x\,dx$, so $-\frac{1}{y} = \sin x + C$. At $(0, 1)$, $C = -1$, so $y = \frac{1}{1 - \sin x}$, which is undefined where $\sin x = 1$ ($x = \frac{\pi}{2}$ and $x = -\frac{3\pi}{2}$).
5. Slopes 2, -2, 0, -3. Substitute into xy. The slope is zero exactly when $x = 0$ or $y = 0$, so horizontal segments lie along the two coordinate axes.
6. $y(1) \approx 2.5$. Step 1: $y_1 = 1 + 0.5(0 + 1) = 1.5$ at $x = 0.5$. Step 2: $y_2 = 1.5 + 0.5(0.5 + 1.5) = 2.5$ at $x = 1$.
7. $P(t) = 500e^{0.04t}$, $P(10) = 500e^{0.4}$, doubling time $25\ln 2$. Exponential growth has the form $P = P_0 e^{kt}$. Doubling: $e^{0.04t} = 2$ gives $t = \frac{\ln 2}{0.04} = 25\ln 2$.
8. $k = \frac{\ln 2}{5}$ and $T(10) = 37.5$. Then $T - 20 = 70e^{-kt}$. At $t = 5$: $35 = 70e^{-5k}$, so $e^{-5k} = \frac{1}{2}$. At $t = 10$: $T - 20 = 70 \cdot \frac{1}{4} = 17.5$, so $T = 37.5$.
9. Limit 800; fastest growth at $P = 400$; maximum rate 4. The carrying capacity is 800. The rate $0.02P\left(1 - \frac{P}{800}\right)$ is a downward parabola in P with vertex halfway between its zeros, $P = 400$, where it equals $0.02(400)(0.5) = 4$.

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SET B · MODELS AND NUMERICAL WORK

10. $y(0.6) \approx 1.044$. Steps: $y(0.2) \approx 1.000$, $y(0.4) \approx 1.015$, $y(0.6) \approx 1.044$, each using $y_{n+1} = y_n + 0.2\,x_n e^{-y_n}$.
11. (a) $y(1) \approx 4.480$. (b) Separating gives $\ln|y| = \sin x + C$, so $y = 2e^{\sin x}$ and $y(1) = 2e^{\sin 1} \approx 4.640$. Euler's estimate is lower than the true value by about 0.160.
12. $P(15) = 1200e^{0.45} \approx 1881.975$ and $t = \frac{\ln 2.5}{0.03} \approx 30.543$ years. $P = 1200e^{0.03t}$; set $1200e^{0.03t} = 3000$.
13. $P(10) \approx 1027.773$ and $t = \frac{\ln 19}{0.3} \approx 9.815$. Setting $P = 1000$: $1 + 19e^{-0.3t} = 2$, so $e^{-0.3t} = \frac{1}{19}$. (Check: $P(0) = \frac{2000}{20} = 100$.)
14. $t \approx 11.667$ minutes. $T - 22 = 73e^{-0.12t}$. Setting $T = 40$: $18 = 73e^{-0.12t}$, so $t = \frac{1}{0.12}\ln\frac{73}{18}$.
15. $y = \sqrt{\frac{2x^3+10}{3}}$ and $y(3) = \sqrt{\frac{64}{3}} \approx 4.619$. Separate: $y\,dy = x^2dx$, so $\frac{y^2}{2} = \frac{x^3}{3} + C$. With $(1,2)$: $C = \frac{5}{3}$, so $y^2 = \frac{2x^3+10}{3}$.
16. $S(t) = 80 - 70e^{-0.05t}$, $S(20) = 80 - 70e^{-1} \approx 54.248$, limit 80. Rewrite as $\frac{dS}{dt} = -0.05(S - 80)$, so $S - 80 = Ce^{-0.05t}$ with $C = -70$.
17. Mass ≈ 8.839 g and rate ≈ -0.511 g/day. The model is $m = 50\left(\frac{1}{2}\right)^{t/12}$ with $\frac{dm}{dt} = -\frac{\ln 2}{12}m$.
18. (a) 4. $0.5(10)(1 - 0.2) = 4$. (b) 12. $P(0.5) \approx 10 + 4(0.5) = 12$. (c) 6.25. The rate is largest at $P = 25$, half the carrying capacity 50: $0.5(25)(0.5) = 6.25$.

SET C · AP-STYLE DIFFERENTIAL EQUATIONS

19. (a) The slope at $(0,2)$ is $0 - 2 = -2$. The slope is zero where $x - y = 0$, that is, along the line $y = x$.
    (b) If $y = x - 1$ then $y' = 1$ and $x - y = x - (x - 1) = 1$. The equation holds for all x.
    (c) $\frac{d^2y}{dx^2} = 1 - \frac{dy}{dx} = 1 - (x - y) = 1 - x + y$. At $(0,2)$ this is $3 > 0$, so f is concave up at $x = 0$.
    (d) $f(1) \approx 2 + 1(-2) = 0$, then $f(2) \approx 0 + 1(1 - 0) = 1$.
20. $y = \sqrt[3]{\frac{3x^2}{2} - 5}$. Separate: $y^2dy = x\,dx$, so $\frac{y^3}{3} = \frac{x^2}{2} + C$. At $(2,1)$: $\frac{1}{3} = 2 + C$, so $C = -\frac{5}{3}$ and $y^3 = \frac{3x^2}{2} - 5$. The differential equation requires $y \neq 0$, which happens at $x = \pm\sqrt{10/3}$, so the solution through $(2,1)$ is valid on $x > \sqrt{\frac{10}{3}}$.
21. (a) 500, the carrying capacity. (b) $0 < P < 250$ and $P > 500$. $\frac{d^2P}{dt^2} = 0.1\left(1 - \frac{2P}{500}\right)\frac{dP}{dt}$. For $0 < P < 250$ both factors are positive; for $250 < P < 500$ the first is negative and $\frac{dP}{dt}$ is positive, giving concave down; for $P > 500$ both factors are negative, giving concave up. (c) 12.5, at $P = 250$: $0.1(250)(0.5)$. (d) P decreases and approaches 500 from above, since $\frac{dP}{dt} < 0$ when $P > 500$.
22. (a) $\frac{dB}{dt} = kB$ with $k = \frac{\ln 2}{3}$, so $B = B_0e^{kt} = B_0 2^{t/3}$. (b) $2^{9/3} = 8$. (c) $e^{kt} = 3$ gives $t = \frac{\ln 3}{k} = \frac{3\ln 3}{\ln 2}$ hours.
23. (a) $T' = -10e^{-t/4} = -\frac{1}{4}(40e^{-t/4}) = -\frac{1}{4}(T - 10)$, and $T(0) = 10 + 40 = 50$. (b) $20 = 40e^{-t/4}$, so $e^{-t/4} = \frac{1}{2}$ and $t = 4\ln 2$. (c) $40e^{-t/4} > 0$ for all t, so $T > 10$; as $t \to \infty$, $e^{-t/4} \to 0$ and $T \to 10$.
24. $y = e^{\sin x}$. Separate: $\frac{dy}{y} = \cos x\,dx$, so $\ln|y| = \sin x + C$. At $x = \frac{\pi}{2}$: $1 = 1 + C$, so $C = 0$. Since $y > 0$ near the initial point, $y = e^{\sin x}$.
25. (a) $y(0.5) \approx 1 + 0.5(-2) = 0$; $y(1) \approx 0 + 0.5(0.5 - 0) = 0.25$; $y(1.5) \approx 0.25 + 0.5(1 - 0.5) = 0.5$. So $y(1.5) \approx 0.5$.
    (b) $y'' = 1 - 2y' = 1 - 2(x - 2y)$. At $(0,1)$: $y' = -2$, so $y'' = 1 + 4 = 5 > 0$. The solution is concave up, so the tangent line lies below it: the Euler estimate is an underestimate.
26. (a) $y' = 3Ce^{3x} = 3(y - 2)$ since $y - 2 = Ce^{3x}$. (b) $5 = 2 + C$, so $C = 3$ and $y = 2 + 3e^{3x}$. (c) As $x \to -\infty$, $e^{3x} \to 0$, so $y \to 2$.
27. (a) $y = -1$, $y = 0$, $y = 3$. (b) Signs of $y(y - 3)(y + 1)$: negative for $y < -1$, positive on $(-1, 0)$, negative on $(0, 3)$, positive for $y > 3$. Solutions move toward $y = 0$ from both sides, so it is stable; they move away from -1 and from 3, so those are unstable. (c) 0, since for $0 < y < 3$ the derivative is negative and y decreases toward the equilibrium at 0.

9.8 Mixed Calculus AB Review

SET A · MIXED SKILLS

1. 6. Factor: $\frac{(x-3)(x+3)}{(x-3)(x-2)} = \frac{x+3}{x-2}$ for $x \neq 3$, which tends to $\frac{6}{1} = 6$.
2. 2. Divide numerator and denominator by x^3: $\frac{4 - x^{-2}}{2 + 7x^{-3}} \to \frac{4}{2}$. The line $y = 2$ is a horizontal asymptote.
3. $\frac{5}{2}$. Write $\frac{\sin 5x}{2x} = \frac{5}{2} \cdot \frac{\sin 5x}{5x}$ and use $\lim_{u \to 0}\frac{\sin u}{u} = 1$.
4. $k = 4$. Continuity requires $\lim_{x \to 2} f(x) = f(2)$. For $x \neq 2$, $f(x) = x + 2$, so the limit is 4. Thus $f(2) = k = 4$.
5. $f'(x) = e^{2x}(2\sin 3x + 3\cos 3x)$. By the product rule: $2e^{2x}\sin 3x + e^{2x} \cdot 3\cos 3x$.
6. $y' = \frac{2 - 2x^2}{(x^2+1)^2}$; horizontal tangents at $x = \pm 1$. Quotient rule: $\frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}$. The numerator $2(1 - x^2)$ is zero at $x = \pm 1$.
7. -1. Differentiate implicitly: $3x^2 + 3y^2y' = 6y + 6xy'$. At $(3,3)$: $27 + 27y' = 18 + 18y'$, so $9y' = -9$.
8. $h'(2) = -10$ and $k'(2) = -5$. Chain rule: $h'(2) = f'(g(2))\,g'(2) = f'(3)(-2) = 5(-2) = -10$. Product rule: $k'(2) = f'(2)g(2) + f(2)g'(2) = 1(3) + 4(-2) = -5$.
9. Absolute maximum 21 at $x = 5$; absolute minimum 1 at $x = 0$ and $x = 3$. $f'(x) = 3(x - 1)(x - 3)$, so the critical numbers are 1 and 3. Values: $f(0) = 1$, $f(1) = 5$, $f(3) = 1$, $f(5) = 21$.

SET B · MIXED APPLICATIONS

10. $\frac{dr}{dt} = 0.044$ cm/s. From $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$, so $20 = 4\pi(36)\frac{dr}{dt}$.
11. Direction change at $t = e - 1 \approx 1.718$. Displacement ≈ 0.047 (exactly $5\ln 5 - 8$); total distance ≈ 1.484. The velocity is negative before $t = e - 1$ and positive after. Distance is $-\int_0^{e-1} v\,dt + \int_{e-1}^{4} v\,dt$.
12. Maximum $\frac{1}{e} \approx 0.368$ at $x = 1$; inflection point at $\left(2, \frac{2}{e^2}\right) \approx (2, 0.271)$. $f'(x) = (1 - x)e^{-x}$ changes from positive to negative at $x = 1$. $f''(x) = (x - 2)e^{-x}$ changes sign at $x = 2$.
13. $x \approx 3.394$ cm and $V \approx 654.977$ cm^3. $V(x) = x(24 - 2x)(18 - 2x) = 4x^3 - 84x^2 + 432x$ for $0 < x < 9$. $V'(x) = 12x^2 - 168x + 432 = 0$ gives $x = 7 \pm \sqrt{13}$, and only $x = 7 - \sqrt{13} \approx 3.394$ is in the domain; $V'' < 0$ there.

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14. 5.694. The curves meet at $x \approx -2.948$ and $x \approx 1.505$, with the line above. Area $= \int_{-2.948}^{1.505} (x + 3 - e^x)\,dx$.
15. 0.598. $V = \pi \int_0^2 x^2 e^{-2x} dx$, evaluated numerically.
16. 32 meters. Right sum $= 2(3) + 3(6) + 4(2) = 6 + 18 + 8 = 32$. The units are meters because velocity (m/s) times time (s) is distance.
17. 2.578. Average value $= \frac{1}{3}\int_0^3 x\sqrt{1+x}\,dx$. (Exactly, with $u = 1 + x$, the integral is $\frac{116}{15}$.)
18. $f'(2) = 4(1 + \ln 2) \approx 6.773$. Write $f(x) = e^{x \ln x}$, so $f'(x) = x^x(\ln x + 1)$.

SET C · AP-STYLE MIXED PROBLEMS

19. (a) The left limit is 2 and the right limit is $4 - 1 = 3$. Because they differ, $\lim_{x\to 1} g(x)$ does not exist.
(b) $g(1) = 0$. The function is not continuous at $x = 1$ because $\lim_{x\to 1} g(x)$ does not exist (a jump discontinuity).
(c) $\lim_{x\to 3} g(x) = 4 - 3 = 1$ and $g'(3) = -1$, the slope of the line $y = 4 - x$.

20. (a) $f'(x) = (1 - 2x^2)e^{-x^2}$, zero at $x = \pm\frac{1}{\sqrt{2}}$. $f' > 0$ on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and negative outside, so there is a local minimum at $-\frac{1}{\sqrt{2}}$ and a local maximum at $\frac{1}{\sqrt{2}}$.
(b) $f''(x) = (4x^3 - 6x)e^{-x^2} = 2x(2x^2 - 3)e^{-x^2}$, which changes sign at $x = 0$ and $x = \pm\sqrt{\frac{3}{2}}$, so these are the inflection points.
(c) The limit is 0 (the exponential dominates). Since $f > 0$ for $x > 0$ and $f \to 0$, and f is odd, the local maximum is the absolute maximum: $f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{-1/2}$.

21. $\frac{2}{5\pi}$ m/min. With $r = h$, $V = \frac{1}{3}\pi h^2 \cdot h = \frac{\pi h^3}{3}$. Then $\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}$, so $10 = 25\pi \frac{dh}{dt}$ and $\frac{dh}{dt} = \frac{2}{5\pi}$.
22. Width 4, height 8, area 32. With vertices at $(\pm x, 12 - x^2)$, $A(x) = 2x(12 - x^2) = 24x - 2x^3$ for $0 < x < \sqrt{12}$. $A'(x) = 24 - 6x^2 = 0$ gives $x = 2$. $A''(x) = -12x < 0$ at $x = 2$, so this is a maximum.
23. (a) $F'(x) = x^2 - x - 6 = (x - 3)(x + 2)$, so the critical numbers are $x = 3$ and $x = -2$.
(b) $F' > 0$ for $x < -2$, < 0 on $(-2, 3)$, > 0 for $x > 3$: local maximum at $x = -2$, local minimum at $x = 3$.
(c) $F(3) = \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^3 = -\frac{27}{2} - \left(-\frac{37}{6}\right) = -\frac{22}{3}$.
(d) $F''(x) = 2x - 1$, and $F''(0) = -1 < 0$, so F is concave down at $x = 0$.

24. (a) Left sum $= 2(8 + 6 + 5) = 38$ liters; right sum $= 2(6 + 5 + 3) = 28$ liters.
(b) The left sum is an overestimate: since R is decreasing, the value at each left endpoint is the largest value on its subinterval (and the right sum is an underestimate).
(c) The trapezoid sum is the average of the two, $\frac{38+28}{2} = 33$ liters.

25. (a) The segment from $(-2, 2)$ to $(0, -2)$ crosses zero at $x = -1$. Thus $f' > 0$ on $(-2, -1)$, $f' < 0$ on $(-1, 2)$, $f' > 0$ on $(2, 4)$. So f has a local maximum at $x = -1$ and a local minimum at $x = 2$.
(b) f'' is the slope of f': -2 on $(-2, 0)$ and $+1$ on $(0, 4)$. It changes sign only at $x = 0$, so the only inflection point is $x = 0$.
(c) $f(2) = f(0) + \int_0^2 f'(x)\,dx = 1 + \left(-\frac{1}{2} \cdot 2 \cdot 2\right) = -1$.

26. $y = 3e^{-x^2}$; the limit is 0; inflection points at $x = \pm\frac{1}{\sqrt{2}}$. Separate: $\frac{dy}{y} = -2x\,dx$, so $\ln|y| = -x^2 + C$ and $y = Ae^{-x^2}$ with $A = 3$. Then $y'' = 3(4x^2 - 2)e^{-x^2}$, which changes sign at $x^2 = \frac{1}{2}$.

27. (a) $\frac{16}{3}$. The curves meet where $\sqrt{x} = x - 2$, at $x = 4$ (check: $2 = 2$). On $[0, 4]$, $\sqrt{x} \geq x - 2$. Area $= \int_0^4 (\sqrt{x} - x + 2)\,dx = \frac{16}{3} - 8 + 8$.
(b) $V = \pi \int_0^4 \left[(\sqrt{x} + 2)^2 - x^2\right] dx = 24\pi$. Distances from $y = -2$: outer radius $\sqrt{x} + 2$, inner radius $(x - 2) + 2 = x$. Expanding, $V = \pi \int_0^4 (x + 4\sqrt{x} + 4 - x^2)\,dx$.

9.9 Mixed Calculus BC Review

SET A · SKILLS

1. $\left(\frac{x}{2} - \frac{1}{4}\right)e^{2x} + C$. Parts with $u = x$, $dv = e^{2x}dx$: $\frac{x}{2}e^{2x} - \int \frac{1}{2}e^{2x}dx$.
2. $\ln|x + 1| + 2\ln|x + 3| + C$. Write $\frac{3x+5}{(x+1)(x+3)} = \frac{1}{x+1} + \frac{2}{x+3}$ (set $x = -1$ and $x = -3$).
3. $\frac{1}{2}$. $\int_2^b 4x^{-3}dx = \left[-2x^{-2}\right]_2^b = \frac{1}{2} - \frac{2}{b^2} \to \frac{1}{2}$.
4. $R = 3$, interval $[-1, 5)$. Ratio test gives $|x - 2| < 3$. At $x = 5$ the series is $\sum 1/n$, which diverges. At $x = -1$ it is $\sum (-1)^n/n$, which converges by the alternating series test.
5. $x - x^3 + \frac{x^5}{2}$. Replace u by $-x^2$ in $e^u = 1 + u + u^2/2 + \cdots$ and multiply by x.
6. (a) $\frac{dy}{dx} = \frac{3t^2 - 3}{2t} = \frac{9}{4}$ at $t = 2$.
(b) $\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{3}{2}\left(1 + \frac{1}{t^2}\right) = \frac{15}{8}$; divide by $dx/dt = 4$ to get $\frac{15}{32}$.
7. (a) $\mathbf{v} = \langle 3t^2, 6t\rangle$, so speed $= \sqrt{9 + 36} = 3\sqrt{5}$.
(b) $\mathbf{a} = \langle 6t, 6\rangle = \langle 6, 6\rangle$.
8. 2π. The petal runs from $\theta = -\pi/4$ to $\pi/4$. Area $= \frac{1}{2}\int_{-\pi/4}^{\pi/4} 16\cos^2(2\theta)\,d\theta = 8 \cdot \frac{\pi}{4} = 2\pi$.
9. 2.5. Step 1: slope 1, $y(0.5) = 1.5$. Step 2: slope $0.5 + 1.5 = 2$, $y(1) = 1.5 + 0.5 \cdot 2 = 2.5$.

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SET B · APPLICATIONS AND MIXED PRACTICE

10. 9.073. $L=\int_0^2\sqrt{4t^2+9t^4}\,dt=\int_0^2 t\sqrt{4+9t^2}\,dt=\frac{1}{27}\left(40^{3/2}-8\right)\approx 9.073$.
11. 6.110. $L=\int_0^{\pi}\sqrt{\theta^2+1}\,d\theta\approx 6.110$.
12. $P_3(0.3)=0.3-\frac{0.027}{6}=0.2955$. Error $\le\frac{(0.3)^4}{4!}=0.000338$ because every derivative of $\sin x$ is at most 1 in size.
13. $S_4=1-\frac{1}{8}+\frac{1}{27}-\frac{1}{64}\approx 0.896$. The error is less than the first omitted term, $\frac{1}{125}=0.008$.
14. (a) $x(2)=1+\int_0^2\cos(t^2)\,dt\approx 1.461$ and $y(2)=2+\int_0^2\sin t\,dt=3-\cos 2\approx 3.416$. Position $\approx(1.461,\,3.416)$.
    (b) The vertical distance from $y=2$ is $|y-2|$ and $y'=\sin t>0$ at $t=1$ with $y-2>0$, so the particle is moving farther from the line.
15. $\pi\approx 3.142$. The curves meet where $\cos\theta=\frac{1}{2}$, $\theta=\pm\frac{\pi}{3}$. Area $=\int_0^{\pi/3}\left(9\cos^2\theta-(1+\cos\theta)^2\right)d\theta=\pi$.
16. $\ln(1.2)\approx 0.2-\frac{0.04}{2}+\frac{0.008}{3}\approx 0.183$. The series alternates with decreasing terms, so the error is less than the next term $\frac{0.2^4}{4}=0.0004$.
17. For $x\ge 1$, $\frac{1}{\sqrt{x^3+1}}<x^{-3/2}$ and $\int_1^{\infty}x^{-3/2}dx$ converges, so the integral converges by comparison. Its value is ≈ 1.895.
18. Steps: $y(0.2)=1.2$, $y(0.4)=1.448$, $y(0.6)=1.7696\approx 1.770$. Exact $y(0.6)=3e^{0.6}-0.36-1.2-2\approx 1.906$. The approximation is low by about 0.137.

SET C · AP-STYLE CHALLENGES

19. (a) $e^{-t^2}=1-t^2+\frac{t^4}{2}-\frac{t^6}{6}+\cdots$. Integrate: $f(x)=x-\frac{x^3}{3}+\frac{x^5}{10}-\frac{x^7}{42}+\cdots$.
    (b) The x^3 coefficient is $\frac{f'''(0)}{3!}=-\frac{1}{3}$, so $f'''(0)=-2$.
20. (a) $\mathbf{v}=\langle 3t^2,\,2t+1\rangle$, so $\mathbf{r}=\langle t^3+2,\;t^2+t\rangle$.
    (b) $\mathbf{v}(1)=\langle 3,3\rangle$, speed $=3\sqrt{2}$.
21. -1. $\frac{dy}{d\theta}=\cos\theta(1+2\sin\theta)=\frac{1+\sqrt{3}}{2}$ and $\frac{dx}{d\theta}=\cos^2\theta-(1+\sin\theta)\sin\theta=-\frac{1+\sqrt{3}}{2}$ at $\theta=\frac{\pi}{3}$. The ratio is -1.
22. $\ln|x|-\dfrac{2}{x-1}+C$. Write $\frac{x^2+1}{x(x-1)^2}=\frac{1}{x}+\frac{0}{x-1}+\frac{2}{(x-1)^2}$ (set $x=0$, $x=1$, then compare x^2 terms).
23. $3+3\sqrt[3]{2}$. The integrand is unbounded at $x=1$, so split there. $\int_0^1=\left[3(x-1)^{1/3}\right]_0^1=3$ and $\int_1^3=3\cdot 2^{1/3}$. Both converge.
24. 8. Speed $=\sqrt{(1-\cos t)^2+\sin^2 t}=\sqrt{2-2\cos t}=2\sin\frac{t}{2}$ on $[0,2\pi]$. Then $L=\int_0^{2\pi}2\sin\frac{t}{2}\,dt=\left[-4\cos\frac{t}{2}\right]_0^{2\pi}=8$.
25. (a) Conditionally convergent: terms alternate, decrease to 0, but $\sum\frac{1}{\sqrt{n+1}}$ diverges ($p=\frac{1}{2}$).
    (b) Divergent: $|a_n|\to\frac{1}{2}\neq 0$, so the terms do not tend to 0.
    (c) Absolutely convergent: ratio $\frac{2}{n+1}\to 0<1$.
26. $P_2(2.5)=3+(-1)(0.5)+\frac{4}{2}(0.5)^2=3$. Error $\le\frac{6}{3!}(0.5)^3=0.125$ by the Lagrange error bound.
27. (a) The carrying capacity is 800, so the limit is 800.
    (b) The rate $0.1P(1-P/800)$ is largest at $P=400$, where it equals $0.1\cdot 400\cdot\frac{1}{2}=20$.

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Chapter 10: Challenge Problems.

3 sections with explanations and worked examples, then 60 practice problems. Answers and full explanations are at the end of the chapter.

10.1 Challenge Set A

Hard problems on limits, derivatives, and function analysis. Several ask for short proofs, so write complete sentences with every hypothesis checked.

These problems combine several ideas and are meant to take time. Set A is [No calculator]. Take notes on which theorem each solution uses: most hard problems become routine once the right theorem is named.

Strategy

For a proof, state what is given, name the theorem you will use, check its hypotheses (continuity on a closed interval, differentiability on the open interval), and then conclude in a sentence.

SET A Limits and Parameters

Solve each problem. Give exact answers. [No calculator]

1. Evaluate $\lim_{x\to\infty} x\left(\sqrt{x^2+4x}-x-2\right)$.
2. Let $f(x)=\begin{cases} x^2+ax, & x\le 4 \\ b\sqrt{x}, & x>4 \end{cases}$
   Find the values of a and b that make f differentiable at $x=4$. Justify each equation you use.
3. Show that $x^3+3x-5=0$ has exactly one real solution, and find an interval of length 1 that contains it.
4. Let $f(x)=x^{1/x}$ for $x>0$.
   (a) Show that f has an absolute maximum at $x=e$.
   (b) Use part (a) to decide which is larger, e^π or π^e.
5. Find equations of all lines through the point $(1,-3)$ that are tangent to the parabola $y=x^2$.
6. Let $f(x)=x^3+2x+1$.
   (a) Show that f has an inverse function.
   (b) Find $(f^{-1})'(4)$ and $(f^{-1})''(4)$.
7. For which values of k does $x^4-4x^3+k=0$ have exactly two distinct real solutions? Justify your answer.

SET B Derivatives and Proofs

Solve each problem. Give exact answers. [No calculator]

8. Use the Mean Value Theorem to prove that $|\sin a-\sin b|\le|a-b|$ for all real a and b.
9. Suppose f is differentiable for all real x, $f'(x)=f(x)$, and $f(0)=0$. Prove that $f(x)=0$ for all x.
10. Let $f(x)=x|x|$.
    (a) Use the limit definition to show that $f'(0)$ exists and find it.
    (b) Find $f'(x)$ for $x\neq 0$, and show that $f''(0)$ does not exist.
11. Find the point on the curve $y=\sqrt{x}$ that is closest to the point $(4,0)$, and find the minimum distance.
12. Let $f(x)=x^2e^{-x}$.
    (a) Find the x-coordinates of all inflection points and justify.
    (b) Find $\lim_{x\to\infty} f(x)$ and the absolute maximum value of f on $[0,\infty)$.

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13. The curve $x^3 + y^3 = 6xy$ passes through $(3, 3)$.
    (a) Find $\frac{dy}{dx}$ at $(3, 3)$.
    (b) Find the x-coordinate of the point in the first quadrant, other than the origin, where the tangent line is horizontal.
14. Find constants a and b so that $\lim_{x \to 0} \frac{\sin 3x + ax + bx^3}{x^5}$ is finite, and find the limit.

SET C Proofs and Deep Analysis

Solve each problem. Justify every step. [No calculator]

15. Let $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
    (a) Use the definition to show that $f'(0) = 0$.
    (b) Show that f' is not continuous at 0.
16. Let f be continuous on $[0, 1]$ with $f(0) = f(1)$. Prove that there is a number c in $[0, \frac{1}{2}]$ with $f(c) = f\left(c + \frac{1}{2}\right)$.
17. Suppose f is differentiable for all x, $f(0) = 2$, and $f'(x) \leq 3$ for all x. Prove that $f(4) \leq 14$, and give a function showing that 14 can be attained.
18. Let $f(x) = x^{\ln x}$ for $x > 0$. Find $f'(e)$.
19. Let $f(x) = x^3 - 3cx$, where c is a constant.
    (a) For which values of c does f have no local extrema?
    (b) For $c > 0$, find the local maximum value in terms of c, and find c so that it equals 16.
20. Prove that $\lim_{x \to \infty} \frac{\ln x}{x} = 0$ by first showing that $\ln t \leq t - 1$ for all $t > 0$.

10.2 Challenge Set B

Hard problems on integrals, area, volume, differential equations, and accumulation. Expect multi-step setups where the picture decides the method.

All three sets are [No calculator]; answers are exact. Sketch every region before choosing between slicing, washers, and shells, and decide which variable gives the simpler integrand.

> **Strategy**
>
> For accumulation problems, say in words what the integral measures and include units. For differential equations, separate variables first, then use the initial condition to fix the constant before simplifying.

SET A Integrals and Regions

Solve each problem. Give exact answers. [No calculator]

1. Evaluate $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx$. (Hint: let I be the integral and consider the same integral with $\cos x$ in the numerator.)
2. Let $F(x) = \int_x^{x^2} \sqrt{1 + t^4}\, dt$.
   Find $F'(x)$ and evaluate $F'(1)$.
3. Find the area of the region bounded by the parabola $y = x^2$, its tangent line at $(1, 1)$, and the x-axis.
4. The region bounded by $y = x^2$, the x-axis, and $x = 2$ is revolved about the line $x = 3$. Find the volume of the solid.
5. Solve $\frac{dy}{dx} = y^2 e^x$ with $y(0) = \frac{1}{2}$.
   State the largest interval containing $x = 0$ on which your solution is valid, and describe what happens at the right endpoint.
6. Evaluate $\int_0^{\infty} x^3 e^{-x}\, dx$.
7. A solid has as its base the region in the first quadrant between $y = x$ and $y = x^2$. Cross sections perpendicular to the x-axis are squares. Find the volume.

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SET B Accumulation and Applications

Solve each problem. Give exact answers. [No calculator]

8. Let $F(x) = \int_0^x (t^2 - 4t + 3)\,dt$.
   Find the absolute maximum and absolute minimum values of F on $[0, 4]$, and where they occur.
9. A tank holds 100 liters of water in which no salt is dissolved. Brine containing 0.5 kilogram of salt per liter enters at 2 liters per minute, and the well-mixed solution leaves at 2 liters per minute. Let $S(t)$ be the kilograms of salt at time t minutes.
   (a) Write and solve a differential equation for S.
   (b) Find the time at which $S = 25$ and $\lim_{t\to\infty} S(t)$.
10. Find the area of the region bounded by the parabola $x = y^2 - 2$ and the line $y = x$.
11. The region bounded by $y = \sqrt{x}$ and $y = \dfrac{x}{2}$ is revolved about the line $y = -1$.
    Find the volume of the solid.
12. Consider $\dfrac{dy}{dx} = y(2 - y)$ with $y(0) = 1$.
    (a) Find y as a function of x.
    (b) Find $y(\ln 3)$ and $\lim_{x\to\infty} y$.
    (c) Find the value of y at which the solution has an inflection point.
13. A particle moves on a line with velocity $v(t) = t^2 - 6t + 8$ for $0 \le t \le 5$.
    (a) Find the displacement.
    (b) Find the total distance traveled.
14. Evaluate $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \dfrac{n}{n^2 + k^2}$ by recognizing a Riemann sum.

SET C Extended Reasoning

Solve each problem. Justify every step. [No calculator]

15. The base of a solid is the disk $x^2 + y^2 \le 4$. Cross sections perpendicular to the x-axis are isosceles right triangles with the hypotenuse on the base. Find the volume.
16. A cup of liquid at 90° is placed in a room at 20°. By Newton's law of cooling, its temperature T satisfies $\dfrac{dT}{dt} = -k(T - 20)$. After 5 minutes the temperature is 60°.
    Find k and the time (in minutes) at which the temperature reaches 30°. Give exact answers.
17. Evaluate $\int_0^1 \dfrac{\ln x}{\sqrt{x}}\,dx$, or show that it diverges.
18. A continuous function f satisfies $f(x) = 1 + 2\int_0^x t\,f(t)\,dt$ for all x.
    Find $f(x)$.
19. The region under $y = \cos x$ for $0 \le x \le \frac{\pi}{2}$ (in the first quadrant) is revolved about the y-axis. Find the volume.
20. The region R is bounded by $y = x^2$ and $y = 4$. Find the value of k so that the line $y = k$ divides R into two regions of equal area.

10.3 Challenge Set C

Hard problems on series, parametric and polar curves, vector motion, and integration techniques. These reward careful setup and complete justification.

All three sets are [No calculator] and ask for exact answers. For series, name each convergence test and check its hypotheses. For parametric and polar curves, write the derivative formula before substituting.

Strategy

Many series problems become easy once you recognize a known Maclaurin series or differentiate or integrate a geometric series term by term inside its interval of convergence.

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SET A Series

Solve each problem. Give exact answers. [No calculator]

1. The geometric series $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$ holds for $|x| < 1$.
   (a) Use differentiation to find $\sum_{n=1}^{\infty} \dfrac{n}{2^n}$.
   (b) Find $\sum_{n=1}^{\infty} \dfrac{n^2}{2^n}$.
2. Let $a_n = \dfrac{(2n)!}{(n!)^2\, 4^n}$ and consider $\sum_{n=1}^{\infty} \dfrac{(2n)!}{(n!)^2} x^n$.
   (a) Find the radius of convergence.
   (b) Show that $a_n \geq \dfrac{1}{2n}$ for all $n \geq 1$ and use this to decide convergence at $x = \frac{1}{4}$.
   (c) Decide convergence at $x = -\frac{1}{4}$ and state the interval of convergence.
3. Find the exact sum of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} \left(\frac{\pi}{3}\right)^{2n-1}$.
4. Let $f(x) = x^2 e^{x^2}$. Use a Maclaurin series to find $f^{(10)}(0)$.
5. Evaluate $\displaystyle\lim_{x \to 0} \frac{x\cos x - \sin x}{x^3}$ using Maclaurin series.
6. Using the Maclaurin series for e^x and the Lagrange error bound with $e^{0.5} < 2$, find the smallest degree n for which the error in approximating $e^{0.5}$ is guaranteed to be less than 0.001.
7. The function $y(x)$ satisfies $y' = x^2 - y$ and $y(0) = 1$.
   (a) Find the first four nonzero terms of its Maclaurin series.
   (b) Verify that your terms agree with the exact solution $y = x^2 - 2x + 2 - e^{-x}$.

SET B Parametric, Polar, and Vector Motion

Solve each problem. Give exact answers. [No calculator]

8. A curve is given by $x = t^3 - 3t$, $y = t^2 - 4$.
   Find all points at which the curve has a horizontal tangent line or a vertical tangent line, and say which is which.
9. Find the length of the curve $x = e^t \cos t$, $y = e^t \sin t$ for $0 \leq t \leq \ln 3$.
10. Find the area of the region that lies inside the circle $r = 4\cos\theta$ and outside the circle $r = 2$.
11. Find the area of the inner loop of the limacon $r = 1 + 2\cos\theta$.
12. A particle moves with position $x = t^2$, $y = t^3 - 3t$ for $t \geq 0$.
    Find the time at which its speed is least, and find that least speed.
13. For the cardioid $r = 1 + \cos\theta$, $0 \leq \theta \leq 2\pi$, find the rectangular coordinates of every point, other than the pole, at which the tangent line is vertical.
    Justify that the tangent really is vertical at each.
14. A particle has position $\mathbf{r}(t) = \langle t - \sin t,\ 1 - \cos t \rangle$.
    (a) Show that its acceleration vector always has magnitude 1.
    (b) For $0 < t < 2\pi$, find when the velocity vector is horizontal and the speed there.
    (c) Compute $\mathbf{v} \cdot \mathbf{a}$ at $t = \frac{\pi}{2}$ and say whether the particle is speeding up or slowing down.

SET C Techniques and Convergence

Solve each problem. Justify every step. [No calculator]

15. Evaluate $\displaystyle\int_0^{\pi} e^x \sin x \, dx$.
16. Evaluate $\displaystyle\int_0^{1} \ln(1 + x^2)\, dx$.
17. Evaluate $\displaystyle\int \frac{2x^2 + x}{(x-1)(x^2+2)}\, dx$.
18. Use the integral test to decide whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.
    Check all hypotheses and state the value of the improper integral you use.

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19. The curve $x = t^2 - 1$, $y = t^3 - t$ forms a loop for $-1 \le t \le 1$ (the points $t = -1$ and $t = 1$ coincide at the origin). Find the area enclosed by the loop.

20. Find the exact sum of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

Answers Chapter 10: Answers and Explanations

Answers for every practice problem, numbered to match. Read the explanation even when your answer is correct.

10.1 Challenge Set A

SET A · LIMITS AND PARAMETERS

1. -2. Multiply and divide by the conjugate: $\sqrt{x^2+4x} - (x+2) = \dfrac{(x^2+4x) - (x+2)^2}{\sqrt{x^2+4x} + x + 2} = \dfrac{-4}{\sqrt{x^2+4x} + x + 2}$.
   Then $x \cdot \dfrac{-4}{\sqrt{x^2+4x} + x + 2} = \dfrac{-4}{\sqrt{1+4/x} + 1 + 2/x} \to \dfrac{-4}{2} = -2$.

2. $a = -12$, $b = -16$.
   Continuity at 4: $16 + 4a = b\sqrt{4} = 2b$.
   Matching derivatives: $2x + a = 8 + a$ from the left and $\dfrac{b}{2\sqrt{x}} = \dfrac{b}{4}$ from the right, so $8 + a = \dfrac{b}{4}$.
   Substituting $b = 32 + 4a$ into $16 + 4a = 2b$ gives $16 + 4a = 64 + 8a$, so $a = -12$ and $b = -16$.

3. Let $f(x) = x^3 + 3x - 5$. f is continuous, $f(1) = -1 < 0$ and $f(2) = 9 > 0$, so by the Intermediate Value Theorem there is a root in $(1, 2)$.
   Since $f'(x) = 3x^2 + 3 > 0$ for all x, f is strictly increasing, so it crosses zero only once. If there were two roots, Rolle's Theorem would give a point with $f' = 0$, which is impossible.

4. (a) $\ln f = \dfrac{\ln x}{x}$, so $\dfrac{f'}{f} = \dfrac{1 - \ln x}{x^2}$ and $f'(x) = x^{1/x}\dfrac{1 - \ln x}{x^2}$. Since $x^{1/x} > 0$, $f' > 0$ for $0 < x < e$ and $f' < 0$ for $x > e$. So f has its absolute maximum at $x = e$.
   (b) Since $f(e) > f(\pi)$, $e^{1/e} > \pi^{1/\pi}$. Raise both sides to the power $e\pi$: $e^{\pi} > \pi^{e}$.

5. $y = 6x - 9$ and $y = -2x - 1$.
   The tangent at (a, a^2) is $y = 2ax - a^2$. Passing through $(1, -3)$ requires $-3 = 2a - a^2$, so $a^2 - 2a - 3 = 0$ and $a = 3$ or $a = -1$.
   For $a = 3$: $y = 6x - 9$. For $a = -1$: $y = -2x - 1$.

6. (a) $f'(x) = 3x^2 + 2 > 0$ for all x, so f is strictly increasing, hence one-to-one, hence invertible.
   (b) $f(1) = 4$, so $f^{-1}(4) = 1$. Then $(f^{-1})'(4) = \dfrac{1}{f'(1)} = \dfrac{1}{5}$.
   Differentiate $f'(f^{-1}(x))\,(f^{-1})'(x) = 1$ again: $f''(f^{-1})\,[(f^{-1})']^2 + f'(f^{-1})\,(f^{-1})'' = 0$. With $f''(1) = 6$: $6 \cdot \frac{1}{25} + 5\,(f^{-1})''(4) = 0$, so $(f^{-1})''(4) = -\dfrac{6}{125}$.

7. $k < 27$.
   Let $g(x) = x^4 - 4x^3$, so we want $g(x) = -k$. $g'(x) = 4x^2(x - 3)$, which is negative for $x < 3$ (except at 0, where it is 0 without changing sign) and positive for $x > 3$. So g decreases on $(-\infty, 3]$ and increases on $[3, \infty)$, with absolute minimum $g(3) = -27$, and $g \to \infty$ at both ends.
   The equation $g(x) = -k$ has exactly two solutions when $-k > -27$, one when $k = 27$, and none when $k > 27$.

SET B · DERIVATIVES AND PROOFS

8. If $a = b$ both sides are 0. Otherwise suppose $a < b$. $\sin x$ is continuous on $[a, b]$ and differentiable on (a, b), so by the Mean Value Theorem there is c in (a, b) with $\dfrac{\sin b - \sin a}{b - a} = \cos c$.
   Since $|\cos c| \le 1$, $|\sin b - \sin a| \le |b - a|$. The case $a > b$ is symmetric.

9. Let $g(x) = f(x)e^{-x}$. Then $g'(x) = f'(x)e^{-x} - f(x)e^{-x} = 0$ for all x.
   So g is constant. Since $g(0) = f(0) = 0$, $g(x) = 0$ for all x. Because $e^{-x} \neq 0$, $f(x) = 0$ for all x.

10. (a) $\dfrac{f(h) - f(0)}{h} = \dfrac{h|h|}{h} = |h| \to 0$ as $h \to 0$. So $f'(0) = 0$.
    (b) For $x > 0$, $f = x^2$; for $x < 0$, $f = -x^2$. So $f'(x) = 2|x|$ for all x (including 0). Then $\dfrac{f'(h) - f'(0)}{h} = \dfrac{2|h|}{h}$, which is 2 from the right and -2 from the left, so $f''(0)$ does not exist.

11. Point $\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$, distance $\dfrac{\sqrt{15}}{2}$.
    Minimize $D = (x - 4)^2 + x$ for $x \ge 0$. $D' = 2(x - 4) + 1 = 0$ gives $x = \frac{7}{2}$. D is an upward parabola in x, so this is the absolute minimum.
    $D\left(\frac{7}{2}\right) = \frac{1}{4} + \frac{7}{2} = \frac{15}{4}$, so the distance is $\frac{\sqrt{15}}{2}$.

12. (a) $f'(x) = (2x - x^2)e^{-x}$ and $f''(x) = (x^2 - 4x + 2)e^{-x}$. $f'' = 0$ at $x = 2 \pm \sqrt{2}$. f'' changes sign at each (positive, negative, positive across them), so both are inflection points.
    (b) $\lim_{x\to\infty} x^2 e^{-x} = 0$. The critical points are $x = 0$ and $x = 2$; $f(0) = 0$, $f(2) = 4e^{-2} \approx 0.54$. So the absolute maximum on $[0, \infty)$ is $4e^{-2}$.

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13. (a) -1. Differentiating, $3x^2+3y^2y'=6y+6xy'$, so $y'=\dfrac{2y-x^2}{y^2-2x}$. At $(3,3)$: $\dfrac{6-9}{9-6}=-1$.
(b) $x=2\sqrt[3]{2}$. Horizontal means $2y=x^2$ with $y^2\neq 2x$. Substituting $y=x^2/2$: $x^3+\frac{x^6}{8}=3x^3$, so $x^3=16$ and $x=16^{1/3}=2\sqrt[3]{2}$.

14. $a=-3$, $b=\frac{9}{2}$, limit $\dfrac{81}{40}$.
Use $\sin 3x=3x-\dfrac{27x^3}{6}+\dfrac{243x^5}{120}-\cdots=3x-\dfrac{9}{2}x^3+\dfrac{81}{40}x^5-\cdots$. For the quotient to have a finite limit the x and x^3 terms must cancel, so $a=-3$ and $b=\frac{9}{2}$. Then the limit is $\frac{81}{40}$.

SET C · PROOFS AND DEEP ANALYSIS

15. (a) $\dfrac{f(h)-f(0)}{h}=h\sin\dfrac{1}{h}$. Since $-|h|\le h\sin\frac{1}{h}\le|h|$, the Squeeze Theorem gives limit 0. So $f'(0)=0$.
(b) For $x\neq 0$, $f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}$. As $x\to 0$, $2x\sin\frac{1}{x}\to 0$ but $\cos\frac{1}{x}$ oscillates between -1 and 1, so $\lim_{x\to 0}f'(x)$ does not exist. Hence f' is not continuous at 0 even though $f'(0)$ exists.

16. Let $g(x)=f\left(x+\frac{1}{2}\right)-f(x)$ on $\left[0,\frac{1}{2}\right]$. g is continuous as a difference of continuous functions.
$g(0)=f\left(\frac{1}{2}\right)-f(0)$ and $g\left(\frac{1}{2}\right)=f(1)-f\left(\frac{1}{2}\right)=f(0)-f\left(\frac{1}{2}\right)=-g(0)$.
So $g(0)$ and $g\left(\frac{1}{2}\right)$ are opposites. If $g(0)=0$ take $c=0$. Otherwise they have opposite signs, and by the Intermediate Value Theorem $g(c)=0$ for some c in $\left(0,\frac{1}{2}\right)$. Then $f(c)=f\left(c+\frac{1}{2}\right)$.

17. f is continuous on $[0,4]$ and differentiable on $(0,4)$. By the Mean Value Theorem there is c with $f'(c)=\dfrac{f(4)-f(0)}{4}$.
Since $f'(c)\le 3$, $f(4)-2\le 12$, so $f(4)\le 14$. The function $f(x)=3x+2$ has $f'=3$, $f(0)=2$ and $f(4)=14$.

18. 2. Take logs: $\ln f=(\ln x)^2$. Differentiate: $\dfrac{f'}{f}=\dfrac{2\ln x}{x}$.
At $x=e$: $f(e)=e^1=e$ and $\dfrac{f'}{f}=\dfrac{2}{e}$, so $f'(e)=e\cdot\dfrac{2}{e}=2$.

19. (a) $f'(x)=3x^2-3c=3(x^2-c)$. If $c\le 0$, $f'\ge 0$ with no sign change (at most a single zero at $x=0$ when $c=0$), so f has no local extrema for $c\le 0$.
(b) For $c>0$, f' changes from positive to negative at $x=-\sqrt{c}$, a local maximum. Its value is $f(-\sqrt{c})=-c^{3/2}+3c^{3/2}=2c^{3/2}$. Setting $2c^{3/2}=16$ gives $c^{3/2}=8$ and $c=4$.

20. Let $h(t)=t-1-\ln t$ for $t>0$. $h'(t)=1-\frac{1}{t}$ is negative for $t<1$ and positive for $t>1$, so h has its absolute minimum $h(1)=0$. Hence $\ln t\le t-1$.
For $x>1$ apply this with $t=\sqrt{x}$: $\frac{1}{2}\ln x\le\sqrt{x}-1<\sqrt{x}$, so $0<\ln x<2\sqrt{x}$. Divide by x: $0<\dfrac{\ln x}{x}<\dfrac{2}{\sqrt{x}}\to 0$. By the Squeeze Theorem, $\lim_{x\to\infty}\dfrac{\ln x}{x}=0$.

10.2 Challenge Set B

SET A · INTEGRALS AND REGIONS

1. $\dfrac{\pi}{4}$. Let $I=\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}dx$. The substitution $u=\frac{\pi}{2}-x$ turns it into $\int_0^{\pi/2}\frac{\cos u}{\cos u+\sin u}du=J$.
Then $I+J=\int_0^{\pi/2}1\,dx=\frac{\pi}{2}$ and $I=J$, so $I=\frac{\pi}{4}$.

2. $F'(x)=2x\sqrt{1+x^8}-\sqrt{1+x^4}$ by the Fundamental Theorem with the chain rule applied to the upper limit.
$F'(1)=2\sqrt{2}-\sqrt{2}=\sqrt{2}$.

3. $\dfrac{1}{12}$. The tangent at $(1,1)$ is $y=2x-1$, crossing the x-axis at $x=\frac{1}{2}$.
Area under the parabola from 0 to 1 is $\frac{1}{3}$. Remove the triangle under the tangent, from $x=\frac{1}{2}$ to 1: $\int_{1/2}^1(2x-1)dx=\frac{1}{4}$.
Area $=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$.

4. 8π. Use shells: radius $3-x$, height x^2.
$V=2\pi\int_0^2(3-x)x^2\,dx=2\pi\left[x^3-\frac{x^4}{4}\right]_0^2=2\pi(8-4)=8\pi$.

5. $y=\dfrac{1}{3-e^x}$.
Separate: $\int y^{-2}dy=\int e^x dx$, so $-\frac{1}{y}=e^x+C$. With $y(0)=\frac{1}{2}$: $-2=1+C$, so $C=-3$ and $y=\frac{1}{3-e^x}$.
The solution is valid for $x<\ln 3$. As $x\to\ln 3^-$, $y\to+\infty$ (a vertical asymptote).

6. 6. Repeated integration by parts: $\int x^n e^{-x}dx=-x^ne^{-x}+n\int x^{n-1}e^{-x}dx$, and the boundary terms vanish at 0 and at ∞.
So $\int_0^\infty x^3e^{-x}=3\cdot 2\cdot 1\cdot\int_0^\infty e^{-x}dx=6$.

7. $\dfrac{1}{30}$. The curves meet at $x=0$ and $x=1$, with $y=x$ above. Each cross section has side $x-x^2$.
$V=\int_0^1(x-x^2)^2dx=\int_0^1(x^2-2x^3+x^4)dx=\frac{1}{3}-\frac{1}{2}+\frac{1}{5}=\frac{1}{30}$.

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SET B · ACCUMULATION AND APPLICATIONS

8. Maximum $\frac{4}{3}$ at $x = 1$ and $x = 4$; minimum 0 at $x = 0$ and $x = 3$.
$F'(x) = (x-1)(x-3)$, and $F(x) = \frac{x^3}{3} - 2x^2 + 3x$. Critical numbers 1 and 3 lie in $[0, 4]$.
Candidates: $F(0) = 0$, $F(1) = \frac{4}{3}$, $F(3) = 0$, $F(4) = \frac{64}{3} - 32 + 12 = \frac{4}{3}$. Compare them (Candidates Test).

9. (a) Salt in: $2 \cdot 0.5 = 1$ kg/min. Salt out: $2 \cdot \frac{S}{100}$. So $\frac{dS}{dt} = 1 - \frac{S}{50}$ with $S(0) = 0$, giving $S = 50\left(1 - e^{-t/50}\right)$.
(b) $50(1 - e^{-t/50}) = 25$ gives $e^{-t/50} = \frac{1}{2}$, so $t = 50 \ln 2$ minutes. As $t \to \infty$, $S \to 50$ kg.

10. $\frac{9}{2}$. The curves meet where $y^2 - 2 = y$, so $y = -1$ and $y = 2$. Integrate with respect to y; the line $x = y$ is to the right of the parabola.
Area $= \int_{-1}^{2} \left(y - (y^2 - 2)\right) dy = \left[\frac{y^2}{2} - \frac{y^3}{3} + 2y\right]_{-1}^{2} = \frac{10}{3} - \left(-\frac{7}{6}\right) = \frac{9}{2}$.

11. $\frac{16\pi}{3}$. The curves meet at $x = 0$ and $x = 4$ with $\sqrt{x}$ on top. Washers about $y = -1$: outer radius $\sqrt{x} + 1$, inner radius $\frac{x}{2} + 1$.
$V = \pi \int_0^4 \left[(\sqrt{x} + 1)^2 - \left(\frac{x}{2} + 1\right)^2\right] dx = \pi \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \pi\left(\frac{32}{3} - \frac{16}{3}\right) = \frac{16\pi}{3}$.

12. (a) Separate: $\int \frac{dy}{y(2-y)} = \int dx$. Since $\frac{1}{y(2-y)} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{2-y}\right)$, we get $\frac{1}{2}\ln\left|\frac{y}{2-y}\right| = x + C$. With $y(0) = 1$, $C = 0$, so $\frac{y}{2-y} = e^{2x}$ and $y = \frac{2}{1 + e^{-2x}}$.
(b) $y(\ln 3) = \frac{2}{1 + \frac{1}{9}} = \frac{9}{5}$. As $x \to \infty$, $y \to 2$.
(c) $y' = 2y - y^2$, so $y'' = (2 - 2y)y'$, which changes sign at $y = 1$ (and $y' > 0$ there). Inflection at $y = 1$, which is $x = 0$.

13. (a) $\int_0^5 (t^2 - 6t + 8)dt = \left[\frac{t^3}{3} - 3t^2 + 8t\right]_0^5 = \frac{20}{3}$.
(b) $v = (t-2)(t-4)$ is positive on $(0, 2)$, negative on $(2, 4)$, positive on $(4, 5)$. Net changes: $\frac{20}{3}$, $-\frac{4}{3}$, $\frac{4}{3}$. Distance $= \frac{20}{3} + \frac{4}{3} + \frac{4}{3} = \frac{28}{3}$.

14. $\frac{\pi}{4}$. Write $\sum_{k=1}^{n} \frac{n}{n^2 + k^2} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1 + (k/n)^2}$, a right Riemann sum for $\int_0^1 \frac{dx}{1+x^2}$.
$\int_0^1 \frac{dx}{1+x^2} = \arctan 1 = \frac{\pi}{4}$.

SET C · EXTENDED REASONING

15. $\frac{32}{3}$. At x the hypotenuse is the chord of length $h = 2\sqrt{4 - x^2}$. An isosceles right triangle with hypotenuse h has area $\frac{h^2}{4} = 4 - x^2$.
$V = \int_{-2}^{2} (4 - x^2)\, dx = 16 - \frac{16}{3} = \frac{32}{3}$.

16. $k = \frac{1}{5}\ln\frac{7}{4}$, and $t = \frac{5\ln 7}{\ln(7/4)}$ minutes.
$T - 20 = 70e^{-kt}$ from $T(0) = 90$. At $t = 5$: $40 = 70e^{-5k}$, so $e^{-5k} = \frac{4}{7}$ and $k = \frac{1}{5}\ln\frac{7}{4}$.
Set $T = 30$: $10 = 70e^{-kt}$, so $e^{-kt} = \frac{1}{7}$ and $t = \frac{\ln 7}{k} = \frac{5\ln 7}{\ln(7/4)}$.

17. -4. The integrand is unbounded at 0, so write $\lim_{a \to 0^+} \int_a^1 x^{-1/2} \ln x\, dx$. By parts ($u = \ln x$, $dv = x^{-1/2}dx$): antiderivative $2\sqrt{x}\ln x - 4\sqrt{x}$.
At 1 it is -4. At $a \to 0^+$, $2\sqrt{a}\ln a \to 0$ and $-4\sqrt{a} \to 0$. So the integral converges to -4.

18. $f(x) = e^{x^2}$. Differentiate both sides (the integrand is continuous): $f'(x) = 2xf(x)$. Also $f(0) = 1$.
Separating, $\frac{f'}{f} = 2x$, so $\ln f = x^2 + C$, and $f(0) = 1$ gives $C = 0$. Check: $1 + 2\int_0^x te^{t^2} dt = 1 + (e^{x^2} - 1) = e^{x^2}$.

19. $\pi^2 - 2\pi$. Shells: radius x, height $\cos x$.
$V = 2\pi \int_0^{\pi/2} x\cos x\, dx = 2\pi\left[x\sin x + \cos x\right]_0^{\pi/2} = 2\pi\left(\frac{\pi}{2} - 1\right) = \pi^2 - 2\pi$.

20. $k = 2\sqrt[3]{2}$ (that is, $4^{2/3}$). Total area $= \int_{-2}^{2}(4 - x^2)dx = \frac{32}{3}$, so each part must have area $\frac{16}{3}$.
The part below $y = k$ has width $2\sqrt{y}$ at height y: area $= \int_0^k 2\sqrt{y}\, dy = \frac{4}{3}k^{3/2}$.
Setting $\frac{4}{3}k^{3/2} = \frac{16}{3}$ gives $k^{3/2} = 4$, so $k = 4^{2/3} = 2\sqrt[3]{2}$, which lies in $(0, 4)$.

10.3 Challenge Set C

SET A · SERIES

1. (a) 2. For $|x| < 1$, differentiate to get $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$, so $\sum nx^n = \frac{x}{(1-x)^2}$. At $x = \frac{1}{2}$: $\frac{1/2}{1/4} = 2$.
(b) 6. Apply $x\frac{d}{dx}$ again: $\sum n^2x^n = x \cdot \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{x(1+x)}{(1-x)^3}$. At $x = \frac{1}{2}$: $\frac{(1/2)(3/2)}{1/8} = 6$.

2. (a) $R = \frac{1}{4}$. The ratio of coefficients is $\frac{(2n+2)(2n+1)}{(n+1)^2} = \frac{2(2n+1)}{n+1} \to 4$, so $|x| < \frac{1}{4}$.
(b) At $x = \frac{1}{4}$ the terms are a_n. $a_1 = \frac{1}{2} \geq \frac{1}{2}$. If $a_n \geq \frac{1}{2n}$, then $a_{n+1} = a_n \cdot \frac{2n+1}{2n+2} \geq \frac{2n+1}{2n(2n+2)} \geq \frac{1}{2n+2}$ since $\frac{2n+1}{2n} \geq 1$. By induction $a_n \geq \frac{1}{2n}$. Since $\sum \frac{1}{2n}$ diverges, the series diverges by direct comparison.
(c) At $x = -\frac{1}{4}$ the series is $\sum(-1)^n a_n$. Here $\frac{a_{n+1}}{a_n} = \frac{2n+1}{2n+2} < 1$, so a_n decreases, and $a_n \to 0$ (indeed $a_n^2 \leq \frac{1}{2n+1}$ by induction, since $(2n+1)(2n+3) \leq (2n+2)^2$). By the alternating series test it converges. Interval: $\left[-\frac{1}{4}, \frac{1}{4}\right)$.

3. $\frac{\sqrt{3}}{2}$. The series is $\sin x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!}$ at $x = \frac{\pi}{3}$.
So the sum is $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

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4. 151200. $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$, so $x^2 e^{x^2} = \sum \frac{x^{2n+2}}{n!}$.
The x^{10} term has $2n+2=10$, so $n=4$ and the coefficient is $\frac{1}{24}$. Then $\frac{f^{(10)}(0)}{10!} = \frac{1}{24}$, so $f^{(10)}(0) = \frac{3628800}{24} = 151200$.

5. $-\frac{1}{3}$. $x\cos x = x - \frac{x^3}{2} + \frac{x^5}{24} - \cdots$ and $\sin x = x - \frac{x^3}{6} + \cdots$.
The difference is $-\frac{x^3}{2} + \frac{x^3}{6} + \cdots = -\frac{x^3}{3} + \cdots$. Divide by x^3 and let $x \to 0$.

6. $n=4$. For $0 \le c \le 0.5$, $|f^{(n+1)}(c)| = e^c < 2$, so the error is at most $\dfrac{2(0.5)^{n+1}}{(n+1)!}$.
$n=3$: $\frac{2/16}{24} = \frac{1}{192} \approx 0.0052$ (not small enough). $n=4$: $\frac{2/32}{120} = \frac{1}{1920} \approx 0.00052 < 0.001$.

7. (a) $1 - x + \dfrac{x^2}{2} + \dfrac{x^3}{6} - \cdots$ (the next term is $-\frac{x^4}{24}$). From the equation: $y'(0) = 0 - 1 = -1$; $y'' = 2x - y'$, so $y''(0) = 1$; $y''' = 2 - y''$, so $y'''(0) = 1$; $y^{(4)} = -y'''$, so $y^{(4)}(0) = -1$. Divide by factorials: $1, -1, \frac{1}{2}, \frac{1}{6}, -\frac{1}{24}$.
(b) $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots$, so $x^2 - 2x + 2 - e^{-x} = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \cdots$, the same.

SET B · PARAMETRIC, POLAR, AND VECTOR MOTION

8. Horizontal: $(0, -4)$. Vertical: $(-2, -3)$ and $(2, -3)$.
$\frac{dx}{dt} = 3t^2 - 3 = 0$ at $t = \pm 1$ and $\frac{dy}{dt} = 2t = 0$ at $t = 0$.
At $t=0$: $\frac{dx}{dt} = -3 \neq 0$, so the tangent is horizontal at $(0,-4)$. At $t=1$: $\frac{dy}{dt} = 2 \neq 0$, vertical at $(-2,-3)$. At $t=-1$: $\frac{dy}{dt} = -2 \neq 0$, vertical at $(2,-3)$.

9. $2\sqrt{2}$. $x' = e^t(\cos t - \sin t)$ and $y' = e^t(\sin t + \cos t)$, so $x'^2 + y'^2 = 2e^{2t}$ and the speed is $\sqrt{2}\,e^t$.
$L = \int_0^{\ln 3} \sqrt{2}\,e^t\,dt = \sqrt{2}(3-1) = 2\sqrt{2}$.

10. $\dfrac{4\pi}{3} + 2\sqrt{3}$. The circles meet where $4\cos\theta = 2$, so $\theta = \pm\frac{\pi}{3}$. Inside the large circle and outside $r=2$ for $|\theta| \le \frac{\pi}{3}$.
Area $= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(16\cos^2\theta - 4\right) d\theta = \int_0^{\pi/3} (8 + 8\cos 2\theta - 4)\,d\theta = [4\theta + 4\sin 2\theta]_0^{\pi/3} = \frac{4\pi}{3} + 2\sqrt{3}$.

11. $\pi - \dfrac{3\sqrt{3}}{2}$. The inner loop is traced for $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$, where $r \le 0$ (it starts and ends at the pole).
Area $= \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1 + 2\cos\theta)^2 d\theta = \frac{1}{2}\left[3\theta + 4\sin\theta + \sin 2\theta\right]_{2\pi/3}^{4\pi/3} = \frac{1}{2}\left(2\pi - 3\sqrt{3}\right) = \pi - \frac{3\sqrt{3}}{2}$.

12. Least speed $\dfrac{4\sqrt{2}}{3}$ at $t = \dfrac{\sqrt{7}}{3}$.
Speed squared: $S = x'^2 + y'^2 = 4t^2 + (3t^2 - 3)^2$. Then $S' = 8t + 36t(t^2 - 1) = 4t(9t^2 - 7)$, which is negative for $0 < t < \frac{\sqrt{7}}{3}$ and positive after, so S (and the speed) is least at $t = \frac{\sqrt{7}}{3}$.
$S\left(\frac{\sqrt{7}}{3}\right) = \frac{28}{9} + \frac{4}{9} = \frac{32}{9}$, so the speed is $\frac{4\sqrt{2}}{3}$. (This is less than $S(0) = 9$.)

13. $(2, 0)$, $\left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$, $\left(-\frac{1}{4}, -\frac{\sqrt{3}}{4}\right)$.
$x = (1 + \cos\theta)\cos\theta$, so $\frac{dx}{d\theta} = -\sin\theta(1 + 2\cos\theta)$, zero at $\theta = 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$. $y = (1 + \cos\theta)\sin\theta$, so $\frac{dy}{d\theta} = \cos\theta + \cos 2\theta$.
At $\theta = 0$: $\frac{dy}{d\theta} = 2 \neq 0$, point $(2, 0)$. At $\theta = \frac{2\pi}{3}$: $\frac{dy}{d\theta} = -1 \neq 0$, $r = \frac{1}{2}$, point $\left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$. At $\theta = \frac{4\pi}{3}$: $\frac{dy}{d\theta} = -1 \neq 0$, point $\left(-\frac{1}{4}, -\frac{\sqrt{3}}{4}\right)$. The case $\theta = \pi$ is the pole, excluded.

14. (a) $\mathbf{v} = \langle 1 - \cos t, \sin t\rangle$ and $\mathbf{a} = \langle \sin t, \cos t\rangle$, so $|\mathbf{a}| = \sqrt{\sin^2 t + \cos^2 t} = 1$.
(b) Horizontal means $\sin t = 0$, so $t = \pi$ in $(0, 2\pi)$. There $\mathbf{v} = \langle 2, 0\rangle$ and the speed is 2.
(c) At $t = \frac{\pi}{2}$: $\mathbf{v} = \langle 1, 1\rangle$, $\mathbf{a} = \langle 1, 0\rangle$, and $\mathbf{v} \cdot \mathbf{a} = 1 > 0$, so the speed is increasing.

SET C · TECHNIQUES AND CONVERGENCE

15. $\dfrac{1 + e^{\pi}}{2}$. Integrate by parts twice with $I = \int e^x \sin x\,dx$: $I = e^x \sin x - \int e^x \cos x\,dx = e^x \sin x - e^x \cos x - I$.
So $I = \frac{e^x(\sin x - \cos x)}{2}$. At π: $\frac{e^{\pi}}{2}$. At 0: $-\frac{1}{2}$. The definite integral is $\frac{e^{\pi}+1}{2}$.

16. $\ln 2 - 2 + \dfrac{\pi}{2}$. By parts with $u = \ln(1 + x^2)$, $dv = dx$: $\int = x\ln(1 + x^2) - \int \dfrac{2x^2}{1 + x^2} dx$.
$\dfrac{2x^2}{1 + x^2} = 2 - \dfrac{2}{1 + x^2}$, so $\int_0^1 = \left[x\ln(1 + x^2) - 2x + 2\arctan x\right]_0^1 = \ln 2 - 2 + \frac{\pi}{2}$.

17. $\ln|x - 1| + \dfrac{1}{2}\ln(x^2 + 2) + \sqrt{2}\arctan\dfrac{x}{\sqrt{2}} + C$.
Write $\dfrac{2x^2 + x}{(x - 1)(x^2 + 2)} = \dfrac{A}{x - 1} + \dfrac{Bx + C}{x^2 + 2}$, so $2x^2 + x = A(x^2 + 2) + (Bx + C)(x - 1)$. At $x = 1$: $3 = 3A$, so $A = 1$. The x^2 terms give $2 = A + B$, so $B = 1$.
The constant terms give $0 = 2A - C$, so $C = 2$.
Then $\int \frac{x+2}{x^2+2} dx = \frac{1}{2}\ln(x^2 + 2) + \sqrt{2}\arctan\frac{x}{\sqrt{2}}$.

18. Converges. Let $f(x) = \dfrac{1}{x(\ln x)^2}$ for $x \ge 2$: it is positive and continuous. The denominator $x(\ln x)^2$ is increasing for $x \ge 2$ (its derivative $(\ln x)^2 + 2\ln x > 0$), so f is decreasing.
$\int_2^{\infty} \dfrac{dx}{x(\ln x)^2} = \lim_{b\to\infty}\left[-\dfrac{1}{\ln x}\right]_2^b = \dfrac{1}{\ln 2}$, which is finite. By the integral test the series converges.

19. $\dfrac{8}{15}$. The loop is symmetric about the x-axis: for $0 \le t \le 1$, $y = t^3 - t \le 0$ (lower half) and $t \mapsto -t$ gives the upper half. Here x runs from -1 (at $t = 0$) to 0.
Height of the loop at a given x is $2(t - t^3)$ and $dx = 2t\,dt$, so Area $= \int_0^1 2(t - t^3)(2t)\,dt = 4\int_0^1 (t^2 - t^4) dt = 4\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{8}{15}$.

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20. $\frac{3}{4}$. Partial fractions: $\frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$.
The partial sum telescopes: $S_N = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right) \to \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$.

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Appendix: Formulas and Reference.

Quick reference for formulas, identities, tables of derivatives and integrals, and the theorems you will use all year.

A.1 Formulas from Geometry and Algebra

Area, volume, surface area, and perimeter formulas, algebra identities, and the laws of exponents and logarithms that calculus uses constantly.

Plane figures: area and perimeter

In every table below, letters stand for positive lengths and angles are measured in radians.

Figure	Area	Perimeter or circumference
Rectangle (length l, width w)	lw	$2l + 2w$
Square (side s)	s^2	$4s$
Triangle (base b, height h)	$\frac{1}{2}bh$	sum of the three sides
Triangle with sides a, b and included angle C	$\frac{1}{2}ab\sin C$	sum of the three sides
Equilateral triangle (side s)	$\frac{\sqrt{3}}{4}s^2$	$3s$
Parallelogram (base b, height h)	bh	sum of the four sides
Trapezoid (parallel sides a, b, height h)	$\frac{1}{2}(a+b)h$	sum of the four sides
Rhombus (diagonals d_1, d_2)	$\frac{1}{2}d_1d_2$	$4\times$ side
Circle (radius r)	πr^2	$2\pi r$
Sector (radius r, central angle θ)	$\frac{1}{2}r^2\theta$	$2r + r\theta$ (arc length $s = r\theta$)
Ellipse (semi-axes a, b)	πab	no elementary formula
Regular polygon, n sides of length s	$\dfrac{ns^2}{4\tan(\pi/n)}$	ns

Right triangles and coordinate geometry

Pythagorean Theorem: $a^2 + b^2 = c^2$ for legs a, b and hypotenuse c.

Special right triangles: $45°$-$45°$-$90°$ has sides $1, 1, \sqrt{2}$. $30°$-$60°$-$90°$ has sides $1, \sqrt{3}, 2$.

Distance between (x_1, y_1) and (x_2, y_2): $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$. Midpoint: $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$.

Slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Point-slope form: $y - y_1 = m(x - x_1)$. Slope-intercept form: $y = mx + b$.

Parallel lines have equal slopes. Perpendicular lines (neither vertical) have slopes with product -1.

Solid	Volume	Surface area
Rectangular box (l, w, h)	lwh	$2(lw + lh + wh)$
Cube (edge s)	s^3	$6s^2$
Prism or cylinder-like solid (base area B, height h)	Bh	lateral area plus two bases
Right circular cylinder (radius r, height h)	$\pi r^2 h$	$2\pi rh + 2\pi r^2$ (lateral part $2\pi rh$)

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Solid	Volume	Surface area
Pyramid (base area B, height h)	$\frac{1}{3}Bh$	lateral faces plus base
Right circular cone (radius r, height h, slant height $\ell = \sqrt{r^2+h^2}$)	$\frac{1}{3}\pi r^2 h$	$\pi r\ell + \pi r^2$ (lateral part $\pi r\ell$)
Sphere (radius r)	$\frac{4}{3}\pi r^3$	$4\pi r^2$
Frustum of a cone (radii R, r, height h, slant height ℓ)	$\frac{1}{3}\pi h\left(R^2+Rr+r^2\right)$	lateral part $\pi(R+r)\ell$
Spherical cap (sphere radius r, cap height h)	$\frac{1}{3}\pi h^2(3r-h)$	curved part $2\pi rh$
Torus (tube radius r, center distance R)	$2\pi^2 Rr^2$	$4\pi^2 Rr$

Reading the volume table

Many volumes have the form (area of a cross section) times (height) for prisms and cylinders, and one third of that for pyramids and cones. Calculus recovers every one of these by integrating cross-sectional areas.

Factoring and expanding

$a^2 - b^2 = (a-b)(a+b)$

$a^2 \pm 2ab + b^2 = (a \pm b)^2$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$ and $a^3 + b^3 = (a+b)(a^2-ab+b^2)$

$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$

$x^2 + (p+q)x + pq = (x+p)(x+q)$

Quadratics

For $ax^2+bx+c=0$ with $a \neq 0$: $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$.

The discriminant b^2-4ac is positive for two real roots, zero for one repeated real root, and negative for no real roots.

Completing the square: $ax^2+bx+c = a\left(x+\dfrac{b}{2a}\right)^2 + c - \dfrac{b^2}{4a}$. The vertex is at $x = -\dfrac{b}{2a}$.

For roots r_1, r_2: $r_1 + r_2 = -\dfrac{b}{a}$ and $r_1 r_2 = \dfrac{c}{a}$.

Binomial theorem, sequences, and sums

$(a+b)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$, where $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$.

Arithmetic sequence $a_n = a_1 + (n-1)d$ has sum $S_n = \dfrac{n}{2}(a_1 + a_n)$.

Geometric sequence $a_n = a_1 r^{n-1}$ has sum $S_n = a_1\dfrac{1-r^n}{1-r}$ for $r \neq 1$, and for $|r|<1$ the infinite sum is $\dfrac{a_1}{1-r}$.

$\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, $\displaystyle\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Row of Pascal's triangle	Entries
$n=0$	1
$n=1$	1, 1
$n=2$	1, 2, 1
$n=3$	1, 3, 3, 1
$n=4$	1, 4, 6, 4, 1
$n=5$	1, 5, 10, 10, 5, 1
$n=6$	1, 6, 15, 20, 15, 6, 1

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Laws of exponents

$a^m a^n = a^{m+n}$, $\dfrac{a^m}{a^n} = a^{m-n}$, $(a^m)^n = a^{mn}$, $(ab)^n = a^n b^n$, $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.

$a^0 = 1$ for $a \neq 0$, $a^{-n} = \dfrac{1}{a^n}$, $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$.

$\sqrt{a^2} = |a|$ for every real a.

Laws of logarithms

For $b > 0$, $b \neq 1$, and positive x, y: $\log_b(xy) = \log_b x + \log_b y$, $\log_b \dfrac{x}{y} = \log_b x - \log_b y$, $\log_b(x^r) = r \log_b x$.

Inverse relationship: $b^{\log_b x} = x$ for $x > 0$ and $\log_b(b^x) = x$ for all x. Also $\log_b 1 = 0$ and $\log_b b = 1$.

Change of base: $\log_b x = \dfrac{\ln x}{\ln b}$.

Natural logarithm: $\ln x = \log_e x$, so $e^{\ln x} = x$ for $x > 0$, $\ln(e^x) = x$, $\ln e = 1$, $\ln 1 = 0$.

Exponential form: $a^x = e^{x \ln a}$ for $a > 0$.

Absolute value, inequalities, and rational expressions

$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$. $|x| < a$ means $-a < x < a$, and $|x| > a$ means $x < -a$ or $x > a$ (for $a > 0$).

Triangle inequality: $|a + b| \leq |a| + |b|$.

Multiplying or dividing an inequality by a negative number reverses its direction.

Rationalizing: $\dfrac{1}{\sqrt{a} - \sqrt{b}} = \dfrac{\sqrt{a} + \sqrt{b}}{a - b}$. Complex fractions are simplified by multiplying numerator and denominator by the common denominator.

Equations of conic sections

Circle: $(x - h)^2 + (y - k)^2 = r^2$.

Ellipse: $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$.

Hyperbola: $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$, with asymptotes $y - k = \pm\dfrac{b}{a}(x - h)$.

Parabola: $y = a(x - h)^2 + k$ has vertex (h, k) and opens upward when $a > 0$.

A.2 Trigonometry Reference

The unit circle, identities, inverse trigonometric functions, and a summary of the six trigonometric graphs, all in radians unless stated.

Definitions

On the unit circle, the point at angle θ is $(\cos\theta, \sin\theta)$. For a right triangle with acute angle θ: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$, $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$, $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$.

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\csc\theta = \dfrac{1}{\sin\theta}$.

Radians and degrees: $180^\circ = \pi$ radians, so x degrees equals $\dfrac{\pi x}{180}$ radians. Arc length $s = r\theta$ and sector area $\frac{1}{2}r^2\theta$ require θ in radians.

Angle θ (radians)	Degrees	$\cos\theta$	$\sin\theta$	$\tan\theta$
0	0 degrees	1	0	0
$\frac{\pi}{6}$	30 degrees	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	45 degrees	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	60 degrees	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	90 degrees	0	1	undefined

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Angle θ (radians)	Degrees	$\cos\theta$	$\sin\theta$	$\tan\theta$
$\frac{2\pi}{3}$	120 degrees	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
$\frac{3\pi}{4}$	135 degrees	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
$\frac{5\pi}{6}$	150 degrees	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{3}$
π	180 degrees	-1	0	0
$\frac{7\pi}{6}$	210 degrees	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{5\pi}{4}$	225 degrees	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
$\frac{4\pi}{3}$	240 degrees	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{3\pi}{2}$	270 degrees	0	-1	undefined
$\frac{5\pi}{3}$	300 degrees	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
$\frac{7\pi}{4}$	315 degrees	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
$\frac{11\pi}{6}$	330 degrees	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{3}$
2π	360 degrees	1	0	0

Memory aid

In the first quadrant, sine takes the values $\frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}$ at 30, 45, 60 degrees, and cosine takes them in the reverse order. Signs by quadrant: all functions are positive in Quadrant I, only sine in II, only tangent in III, only cosine in IV.

Fundamental identities

Pythagorean: $\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$, $1 + \cot^2\theta = \csc^2\theta$.

Even and odd: $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$, $\tan(-\theta) = -\tan\theta$.

Cofunction: $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

Periodicity: $\sin(\theta + 2\pi) = \sin\theta$, $\cos(\theta + 2\pi) = \cos\theta$, $\tan(\theta + \pi) = \tan\theta$.

Supplementary: $\sin(\pi - \theta) = \sin\theta$ and $\cos(\pi - \theta) = -\cos\theta$.

Sum and difference identities

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

Double-angle, half-angle, and power-reducing identities

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Power reducing (used constantly in integration): $\sin^2\theta = \frac{1 - \cos 2\theta}{2}$ and $\cos^2\theta = \frac{1 + \cos 2\theta}{2}$.

Half angle: $\tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta} = \frac{1 - \cos\theta}{\sin\theta}$.

Triple angle: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ and $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

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Product-to-sum and sum-to-product

$$\sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

$$\cos A\cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$$

$$\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$\sin A+\sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \text{ and } \cos A+\cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$$

Triangles that are not right

For a triangle with sides a, b, c opposite angles A, B, C:

Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$.

Area: $\frac{1}{2}ab\sin C$.

Function	Domain	Range	Range restriction used for the inverse
$\sin x$	all reals	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\cos x$	all reals	$[-1,1]$	$[0,\pi]$
$\tan x$	$x \neq \frac{\pi}{2}+k\pi$	all reals	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$\csc x$	$x \neq k\pi$	$(-\infty,-1]\cup[1,\infty)$	$\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$
$\sec x$	$x \neq \frac{\pi}{2}+k\pi$	$(-\infty,-1]\cup[1,\infty)$	$\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$
$\cot x$	$x \neq k\pi$	all reals	$(0,\pi)$

Inverse function	Domain	Range	Sample values
$\arcsin x = \sin^{-1} x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$\arcsin\frac{1}{2}=\frac{\pi}{6}$, $\arcsin(-1)=-\frac{\pi}{2}$
$\arccos x = \cos^{-1} x$	$[-1,1]$	$[0,\pi]$	$\arccos\frac{1}{2}=\frac{\pi}{3}$, $\arccos(-1)=\pi$
$\arctan x = \tan^{-1} x$	all reals	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	$\arctan 1=\frac{\pi}{4}$, $\arctan(-\sqrt{3})=-\frac{\pi}{3}$
$\operatorname{arcsec} x$	$\lvert x\rvert \geq 1$	$\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$	$\operatorname{arcsec} 2=\frac{\pi}{3}$
$\operatorname{arccsc} x$	$\lvert x\rvert \geq 1$	$\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$	$\operatorname{arccsc} 2=\frac{\pi}{6}$
$\operatorname{arccot} x$	all reals	$(0,\pi)$	$\operatorname{arccot} 1=\frac{\pi}{4}$

Common mistake

$\sin^{-1} x$ means the inverse function, not $\frac{1}{\sin x}$. Also, $\arcsin(\sin x) = x$ only when x lies in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$; for example $\arcsin(\sin\pi) = 0$, not π. Likewise $\sin(\arcsin x) = x$ holds for $-1 \leq x \leq 1$.

Useful composite values

$\cos(\arcsin x) = \sqrt{1-x^2}$, $\sin(\arccos x) = \sqrt{1-x^2}$, $\tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$ (for $|x|<1$), $\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$, $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$.

$\arcsin x + \arccos x = \frac{\pi}{2}$ for $-1 \leq x \leq 1$.

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Graph	Period	Zeros (in one period)	Key features
$y = \sin x$	2π	$x = 0, \pi$	Amplitude 1. Maximum 1 at $\frac{\pi}{2}$, minimum -1 at $\frac{3\pi}{2}$. Odd.
$y = \cos x$	2π	$x = \frac{\pi}{2}, \frac{3\pi}{2}$	Amplitude 1. Maximum 1 at 0, minimum -1 at π. Even.
$y = \tan x$	π	$x = 0$	Vertical asymptotes at $x = \frac{\pi}{2} + k\pi$. Increasing on each branch. Odd.
$y = \csc x$	2π	none	Vertical asymptotes at $x = k\pi$. Never takes a value in $(-1, 1)$.
$y = \sec x$	2π	none	Vertical asymptotes at $x = \frac{\pi}{2} + k\pi$. Never takes a value in $(-1, 1)$.
$y = \cot x$	π	$x = \frac{\pi}{2}$	Vertical asymptotes at $x = k\pi$. Decreasing on each branch.

Transformations of sinusoids

$y = A\sin\big(B(x - C)\big) + D$ has amplitude $|A|$, period $\dfrac{2\pi}{|B|}$, phase shift C to the right, and midline $y = D$. The same holds for cosine.

Maximum value $D + |A|$ and minimum value $D - |A|$.

For $y = \tan(B(x - C)) + D$ the period is $\dfrac{\pi}{|B|}$.

Limits and derivatives that start from trigonometry

$\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ (radians) and $\lim\limits_{x\to 0} \dfrac{1 - \cos x}{x} = 0$.

Squeeze setup: for $0 < x < \frac{\pi}{2}$, $\sin x < x < \tan x$.

Polar and rectangular conversion: $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, $\tan\theta = \dfrac{y}{x}$ (choose the quadrant from the signs of x and y).

A.3 Calculus Reference

Derivative and integral tables, the major theorems with their hypotheses, standard series, and parametric, vector, and polar formulas, collected for quick lookup.

Limits and continuity

$\lim_{x\to a} f(x) = L$ means $f(x)$ gets arbitrarily close to L as x approaches a from both sides. It exists exactly when both one-sided limits exist and are equal.

f is continuous at a when $f(a)$ is defined, $\lim_{x\to a} f(x)$ exists, and the two are equal.

Standard limits: $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, $\lim\limits_{x\to 0} \dfrac{1 - \cos x}{x} = 0$, $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$, $\lim\limits_{x\to \infty} \left(1 + \dfrac{1}{x}\right)^x = e$.

Growth comparison as $x \to \infty$: $\ln x$ grows slower than any positive power of x, which grows slower than e^x.

Definition of the derivative: $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{t\to x} \dfrac{f(t) - f(x)}{t - x}$.

Differentiation rules

Constant multiple and sum: $(cf)' = cf'$, $(f \pm g)' = f' \pm g'$.

Product rule: $(fg)' = f'g + fg'$.

Quotient rule: $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$.

Chain rule: $\dfrac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$.

Inverse functions: if f is differentiable and one-to-one with $f'(f^{-1}(a)) \neq 0$, then $\big(f^{-1}\big)'(a) = \dfrac{1}{f'\left(f^{-1}(a)\right)}$.

Implicit differentiation: differentiate both sides with respect to x, treating y as a function of x, then solve for $\dfrac{dy}{dx}$.

Logarithmic differentiation: for $y = f(x)^{g(x)}$, write $\ln y = g(x)\ln f(x)$ and differentiate both sides.

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$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	$\sin x$	$\cos x$
e^x	e^x	$\cos x$	$-\sin x$
a^x $(a>0)$	$a^x \ln a$	$\tan x$	$\sec^2 x$
$\ln x$ $(x>0)$	$\dfrac{1}{x}$	$\cot x$	$-\csc^2 x$
$\ln \lvert x\rvert$ $(x\neq 0)$	$\dfrac{1}{x}$	$\sec x$	$\sec x \tan x$
$\log_a x$	$\dfrac{1}{x \ln a}$	$\csc x$	$-\csc x \cot x$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$	$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$	$\operatorname{arccot} x$	$-\dfrac{1}{1+x^2}$
$\operatorname{arcsec} x$	$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\operatorname{arccsc} x$	$-\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$

Common mistake

The power rule $\frac{d}{dx}x^n = nx^{n-1}$ does not apply to a^x or to x^x. Use $\frac{d}{dx}a^x = a^x \ln a$ and logarithmic differentiation for x^x: $\frac{d}{dx}x^x = x^x(\ln x + 1)$.

Antiderivative table: $f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
x^n $(n\neq -1)$	$\dfrac{x^{n+1}}{n+1}+C$	$\sec^2 x$	$\tan x + C$
$\dfrac{1}{x}$	$\ln \lvert x\rvert + C$	$\csc^2 x$	$-\cot x + C$
e^x	$e^x + C$	$\sec x \tan x$	$\sec x + C$
a^x $(a>0,\, a\neq 1)$	$\dfrac{a^x}{\ln a}+C$	$\csc x \cot x$	$-\csc x + C$
$\sin x$	$-\cos x + C$	$\tan x$	$-\ln \lvert\cos x\rvert + C = \ln \lvert\sec x\rvert + C$
$\cos x$	$\sin x + C$	$\cot x$	$\ln \lvert\sin x\rvert + C$
$\sec x$	$\ln \lvert\sec x + \tan x\rvert + C$	$\csc x$	$-\ln \lvert\csc x + \cot x\rvert + C$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\arcsin \dfrac{x}{a} + C$	$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\arctan \dfrac{x}{a} + C$
$\dfrac{1}{x\sqrt{x^2-a^2}}$	$\dfrac{1}{a}\operatorname{arcsec} \dfrac{\lvert x\rvert}{a} + C$	$\dfrac{1}{a^2-x^2}$ $(\lvert x\rvert < a)$	$\dfrac{1}{2a}\ln \left\lvert \dfrac{a+x}{a-x}\right\rvert + C$
$\ln x$	$x\ln x - x + C$	$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln \left\lvert \dfrac{x-a}{x+a}\right\rvert + C$
$\sin^2 x$	$\dfrac{x}{2} - \dfrac{\sin 2x}{4} + C$	$\cos^2 x$	$\dfrac{x}{2} + \dfrac{\sin 2x}{4} + C$
$\arcsin x$	$x\arcsin x + \sqrt{1-x^2} + C$	$\arctan x$	$x\arctan x - \frac{1}{2}\ln(1+x^2) + C$
xe^{ax} $(a\neq 0)$	$\dfrac{(ax-1)e^{ax}}{a^2}+C$	$\sec^3 x$	$\frac{1}{2}(\sec x\tan x + \ln \lvert\sec x + \tan x\rvert) + C$
$e^{ax}\sin bx$	$\dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}+C$	$e^{ax}\cos bx$	$\dfrac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}+C$

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Techniques of integration

Substitution: $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$ with $u = g(x)$. For definite integrals change the limits to u-values.

Integration by parts: $\int u\,dv = uv - \int v\,du$. Choose u in the order logarithmic, inverse trigonometric, algebraic, trigonometric, exponential (as a guide).

Partial fractions: for a proper rational function, a factor $(x-r)$ gives $\dfrac{A}{x-r}$; a repeated factor $(x-r)^2$ also gives $\dfrac{B}{(x-r)^2}$; an irreducible quadratic factor gives $\dfrac{Bx+C}{x^2+px+q}$. If the numerator's degree is at least the denominator's, divide first.

Powers of sine and cosine: if one power is odd, save one factor and convert the rest with $\sin^2 x + \cos^2 x = 1$. If both are even, use $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$.

Improper integrals: $\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$. If f is unbounded at c, replace the integral by a limit on each side of c. The integral converges only if the limit is finite.

$\int_1^\infty \dfrac{dx}{x^p}$ converges if $p > 1$ and diverges if $p \le 1$. $\int_0^1 \dfrac{dx}{x^p}$ converges if $p < 1$ and diverges if $p \ge 1$.

Definite integrals and accumulation

Riemann sum: $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\,\Delta x$ with $\Delta x = \dfrac{b-a}{n}$.

Trapezoidal rule with equal subintervals: $T_n = \dfrac{\Delta x}{2}\left[y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n\right]$. Left, right, and midpoint sums use the left endpoint, right endpoint, and midpoint of each subinterval.

Properties: $\int_a^b f = -\int_b^a f$, $\int_a^c f + \int_c^b f = \int_a^b f$, $\left|\int_a^b f\right| \le \int_a^b |f|$. If f is even, $\int_{-a}^{a} f = 2\int_0^a f$; if f is odd, $\int_{-a}^{a} f = 0$.

Average value of f on $[a,b]$: $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

Accumulation: $F(x) = F(a) + \displaystyle\int_a^x F'(t)\,dt$. Net change of a quantity equals the integral of its rate of change.

Applications of integrals

Area between curves: $\displaystyle\int_a^b \left[f(x) - g(x)\right]dx$ where $f \ge g$ on $[a,b]$.

Disks: $V = \pi\displaystyle\int_a^b [R(x)]^2\,dx$. Washers: $V = \pi\displaystyle\int_a^b \left([R(x)]^2 - [r(x)]^2\right)dx$, with R and r measured from the axis of revolution.

Shells (about a vertical axis): $V = 2\pi\displaystyle\int_a^b (\text{radius})(\text{height})\,dx$.

Known cross sections with area $A(x)$: $V = \displaystyle\int_a^b A(x)\,dx$. A square with side s has area s^2, an equilateral triangle with side s has area $\frac{\sqrt{3}}{4}s^2$, a semicircle on diameter d has area $\frac{\pi}{8}d^2$.

Arc length of $y = f(x)$: $L = \displaystyle\int_a^b \sqrt{1 + \left[f'(x)\right]^2}\,dx$.

Motion on a line: velocity $v = s'$, acceleration $a = v'$, speed $|v|$. Displacement $= \int_a^b v\,dt$; total distance $= \int_a^b |v|\,dt$.

Differential equations

Separable: $\dfrac{dy}{dx} = g(x)h(y)$ becomes $\displaystyle\int \frac{dy}{h(y)} = \int g(x)\,dx$. Include the constant of integration before using the initial condition.

Exponential growth or decay: $\dfrac{dy}{dt} = ky$ has solution $y = y_0 e^{kt}$.

Logistic: $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{L}\right)$ has solution $P = \dfrac{L}{1 + Ae^{-kt}}$ with $A = \dfrac{L - P_0}{P_0}$. Growth is fastest at $P = \dfrac{L}{2}$, and $\lim_{t\to\infty} P = L$.

Euler's method with step h: $x_{n+1} = x_n + h$, $y_{n+1} = y_n + h\,f(x_n, y_n)$.

Slope fields: the segment at (x,y) has slope $f(x,y)$; equilibrium solutions are the horizontal lines where $f(x,y) = 0$ for all x.

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Theorem	Statement	Hypotheses
Intermediate Value Theorem	For any k strictly between $f(a)$ and $f(b)$, there is c in (a,b) with $f(c)=k$.	f continuous on $[a,b]$.
Extreme Value Theorem	f attains an absolute maximum and an absolute minimum on $[a,b]$.	f continuous on the closed interval $[a,b]$.
Rolle's Theorem	There is c in (a,b) with $f'(c)=0$.	f continuous on $[a,b]$, differentiable on (a,b), and $f(a)=f(b)$.
Mean Value Theorem	There is c in (a,b) with $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.	f continuous on $[a,b]$ and differentiable on (a,b).
Fundamental Theorem, Part 1	If $F(x)=\int_a^x f(t)\,dt$, then $F'(x)=f(x)$. With a variable upper limit $g(x)$: $F'(x)=f(g(x))\,g'(x)$.	f continuous on an interval containing a and x.
Fundamental Theorem, Part 2	$\int_a^b f(x)\,dx=F(b)-F(a)$ for any antiderivative F of f.	f continuous on $[a,b]$.
Mean Value Theorem for Integrals	There is c in $[a,b]$ with $f(c)=\dfrac{1}{b-a}\int_a^b f$.	f continuous on $[a,b]$.
Candidates Test (absolute extrema)	Compare f at critical numbers and endpoints.	f continuous on $[a,b]$.
First Derivative Test	If f' changes from positive to negative at c, then $f(c)$ is a local maximum. If it changes from negative to positive, a local minimum.	f continuous at c and c a critical number.
Second Derivative Test	If $f'(c)=0$ and $f''(c)>0$, local minimum. If $f''(c)<0$, local maximum. If $f''(c)=0$, no conclusion.	f'' exists near c.
L'Hospital's Rule	$\lim\dfrac{f(x)}{g(x)}=\lim\dfrac{f'(x)}{g'(x)}$.	The limit of f/g has form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, f,g differentiable near the point with $g'\neq 0$, and $\lim f'/g'$ exists or is infinite.
Squeeze Theorem	If $g\le f\le h$ near a and $\lim g=\lim h=L$, then $\lim f=L$.	Inequality holds on an open interval around a (except possibly at a).
Differentiable implies continuous	If $f'(a)$ exists, then f is continuous at a.	The converse is false: $\lvert x\rvert$ at 0.

Series tests

nth-term test: if $\lim a_n\neq 0$ (or does not exist), $\sum a_n$ diverges. If $\lim a_n=0$ no conclusion.

Geometric: $\sum ar^n$ converges to $\dfrac{a}{1-r}$ if $|r|<1$ and diverges if $|r|\ge 1$.

p-series: $\sum\dfrac{1}{n^p}$ converges if $p>1$, diverges if $p\le 1$ (the harmonic series is $p=1$).

Integral test: if f is positive, continuous, and decreasing on $[N,\infty)$ with $f(n)=a_n$, then $\sum a_n$ and $\int_N^\infty f$ converge or diverge together.

Direct comparison: if $0\le a_n\le b_n$, then $\sum b_n$ converging forces $\sum a_n$ to converge, and $\sum a_n$ diverging forces $\sum b_n$ to diverge.

Limit comparison: if $a_n,b_n>0$ and $\lim\dfrac{a_n}{b_n}=c$ with $0<c<\infty$, the two series behave alike.

Alternating series test: if $b_n>0$, b_n is decreasing, and $b_n\to 0$, then $\sum(-1)^n b_n$ converges. The error after N terms satisfies $|S-S_N|\le b_{N+1}$.

Ratio test: with $L=\lim\left|\dfrac{a_{n+1}}{a_n}\right|$: converges absolutely if $L<1$, diverges if $L>1$, inconclusive if $L=1$.

Absolute and conditional convergence: if $\sum|a_n|$ converges, so does $\sum a_n$. A series that converges but not absolutely is conditionally convergent.

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Taylor and Maclaurin series

Taylor polynomial of degree n about $x = a$: $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$. A Maclaurin series is the case $a = 0$.

Lagrange error bound: if $|f^{(n+1)}(t)| \le M$ for all t between a and x, then $|f(x) - P_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$.

A power series $\sum c_n(x-a)^n$ has a radius of convergence R: it converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$. Check the endpoints separately. Within the interval, it may be differentiated and integrated term by term.

Function	Maclaurin series	Interval of convergence
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(-\infty, \infty)$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$	$(-\infty, \infty)$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$	$(-\infty, \infty)$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$\frac{1}{1+x}$	$\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$	$(-1, 1)$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$	$(-1, 1]$
$\arctan x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$	$[-1, 1]$
$(1+x)^k$	$1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$(-1, 1)$ in general

Building new series

Substitute into a known series (for example e^{-x^2} from e^x), multiply by a power of x, differentiate, or integrate term by term. The new series converges at least where the substituted expression lies inside the original interval.

Parametric curves and vector-valued functions

For $x = x(t)$, $y = y(t)$: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ when $\frac{dx}{dt} \neq 0$, and $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$.

Horizontal tangent: $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$. Vertical tangent: $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

Arc length for $a \le t \le b$: $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

Position $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, velocity $\mathbf{v} = \mathbf{r}' = \langle x', y' \rangle$, acceleration $\mathbf{a} = \mathbf{v}'$, speed $|\mathbf{v}| = \sqrt{(x')^2 + (y')^2}$.

Position from velocity: $\mathbf{r}(t) = \mathbf{r}(a) + \int_a^t \mathbf{v}(s)\, ds$ (integrate each component). Total distance traveled $= \int_a^b |\mathbf{v}(t)|\, dt$.

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Polar coordinates

Conversions: $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, $\tan\theta = \dfrac{y}{x}$.

Slope of the tangent line: $\dfrac{dy}{dx} = \dfrac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$, where $r' = \dfrac{dr}{d\theta}$.

Area enclosed by $r = f(\theta)$ for $\alpha \le \theta \le \beta$: $A = \dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2\,d\theta$. Region between two curves with $R \ge r \ge 0$: $A = \dfrac{1}{2}\displaystyle\int_\alpha^\beta \left(R^2 - r^2\right)d\theta$.

Arc length: $L = \displaystyle\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$.

Common curves: circle $r = a$; circle $r = 2a\cos\theta$ (center on the x-axis); limacon $r = a + b\cos\theta$ (cardioid if $a = b$); rose $r = a\cos n\theta$ with n petals if n is odd and $2n$ petals if n is even; spiral $r = a\theta$.

Name	Lowercase	Uppercase	Typical use in calculus
alpha	α	A	angles, constants
beta	β	B	angles, constants
gamma	γ	Γ	constants, the gamma function
delta	δ	Δ	small distance in a limit definition (δ); change in a quantity (Δx)
epsilon	ε	E	small tolerance in a limit definition
zeta	ζ	Z	special functions
eta	η	H	constants, efficiency
theta	θ	Θ	angles, polar angle
lambda	λ	Λ	wavelength, decay constant
mu	μ	M	mean, coefficient of friction
nu	ν	N	frequency
xi	ξ	Ξ	a number in an interval
pi	π	Π	the constant $3.14159\ldots$; Π for a product
rho	ρ	P	density, polar radius in some texts
sigma	σ	Σ	Σ for a sum; σ for standard deviation
tau	τ	T	time constant, torque
phi	φ	Φ	angles, golden ratio
omega	ω	Ω	angular speed

Notation reminder

The letter e is Euler's number, $e \approx 2.71828$. The symbol ∞ is not a number: writing $\lim_{x\to a} f(x) = \infty$ describes unbounded growth, and the limit does not exist as a real number. Use $\sum$ for sums, and write $f^{(n)}$ for the nth derivative.

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Full-Length Practice Exams.

Three full-length practice exams in the AP format: multiple choice and free response, with full solutions and scoring breakdowns after each exam.

Exam 1 Practice Exam 1 (Calculus AB Style)

A full-length practice exam in the Calculus AB format: 30 multiple-choice questions and 6 free-response questions. Take it under timed conditions, then check your work against the full solutions.

Exam Instructions

This practice exam follows the Calculus AB format. Section I has 30 multiple-choice questions: Part A (20 questions, no calculator) and Part B (10 questions, graphing calculator allowed). The real exam has 45 multiple-choice questions in 105 minutes; this shortened version suggests about 70 minutes for Section I. Section II has 6 free-response questions worth 9 points each: Part A (2 questions, graphing calculator allowed) and Part B (4 questions, no calculator). Allow about 90 minutes for Section II. Use a calculator only in Section I Part B and Section II Part A. Do not use a calculator in Section I Part A or in Section II Part B. For free-response questions, show your work: answers without supporting work or justification may earn little credit. Unless a problem says otherwise, give decimal answers rounded to three places. Exact answers are expected when no calculator is allowed.

SECTION I, PART A Multiple Choice [No calculator]

Choose the best answer for each question. Do not use a calculator. [No calculator]

1. $\lim_{x \to 3} \frac{x^2 - 9}{x^2 - 2x - 3}$ is

   (A) 0 (B) $\frac{3}{2}$ (C) 3 (D) 6 (E) nonexistent

2. $\lim_{x \to \infty} \frac{5x^2 + 3}{2x^3 - x}$ is

   (A) 0 (B) 1 (C) $\frac{5}{2}$ (D) 5 (E) ∞

3. If $f(x) = \tan(3x)$, then $f'\left(\frac{\pi}{12}\right) =$

   (A) 2 (B) 3 (C) 4 (D) $3\sqrt{2}$ (E) 6

4. If $x^2 + xy + y^2 = 7$, then at the point $(1, 2)$ the value of $\frac{dy}{dx}$ is

   (A) -8 (B) $-\frac{5}{4}$ (C) -1 (D) $-\frac{4}{5}$ (E) $\frac{4}{5}$

5. The table gives values of the differentiable functions f and g and their derivatives. If $h(x) = f(g(x))$, then $h'(2) =$

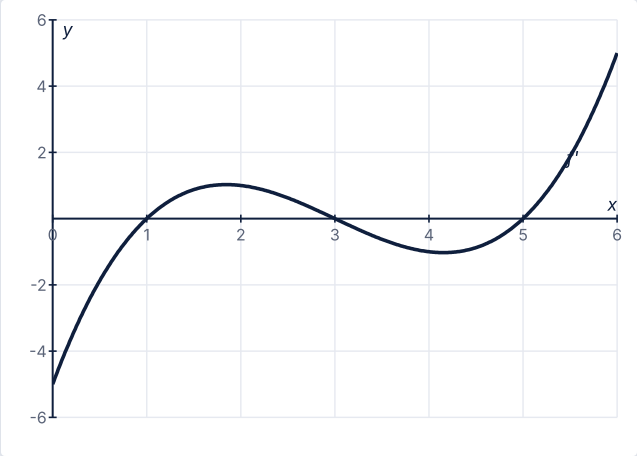
x	1	2	3
$f(x)$	3	4	7
$f'(x)$	1	-3	5
$g(x)$	2	3	1
$g'(x)$	4	-2	6

   (A) -18 (B) -10 (C) 6 (D) 10 (E) 30

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6. The graph of f', the derivative of the twice-differentiable function f, is shown for $0 < x < 6$. How many points of inflection does the graph of f have on $0 < x < 6$?



Graph of f'.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

7. For $f(x) = x^3$ on the interval $[1, 3]$, the value of c guaranteed by the Mean Value Theorem is

(A) $\frac{\sqrt{13}}{3}$ **(B)** $\frac{3}{2}$ **(C)** 2 **(D)** $\sqrt{\frac{13}{3}}$ **(E)** $\sqrt{13}$

8. A 13-foot ladder leans against a vertical wall. The bottom of the ladder slides away from the wall at 2 feet per second. At the instant the bottom is 5 feet from the wall, how fast, in feet per second, is the top of the ladder sliding down the wall?

(A) $\frac{5}{6}$ **(B)** $\frac{6}{5}$ **(C)** $\frac{5}{3}$ **(D)** 2 **(E)** $\frac{12}{5}$

9. What is the absolute maximum value of $f(x) = x^3 - 6x^2 + 9x + 1$ on the closed interval $[0, 5]$?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 21 **(E)** 25

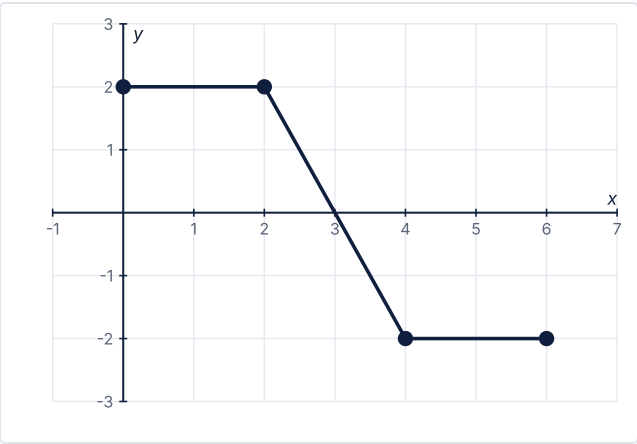
10. $\int_0^1 xe^{-x^2}\,dx =$

(A) $\frac{1}{2}\left(1 - e^{-1}\right)$ **(B)** e^{-1} **(C)** $1 - e^{-1}$
(D) $\frac{1}{2}(e - 1)$ **(E)** $e - 1$

11. Let $F(x) = \int_0^{x^2} \sqrt{1 + t^2}\,dt$. Then $F'(1) =$

(A) 0 **(B)** $\sqrt{2}$ **(C)** 2 **(D)** $2\sqrt{2}$ **(E)** 4

12. The graph of the continuous function f is shown on $[0, 6]$, and it consists of line segments. If $g(x) = \int_0^x f(t)\,dt$, then $g(5) =$



Graph of f.

(A) -2 **(B)** 0 **(C)** 2 **(D)** 4 **(E)** 5

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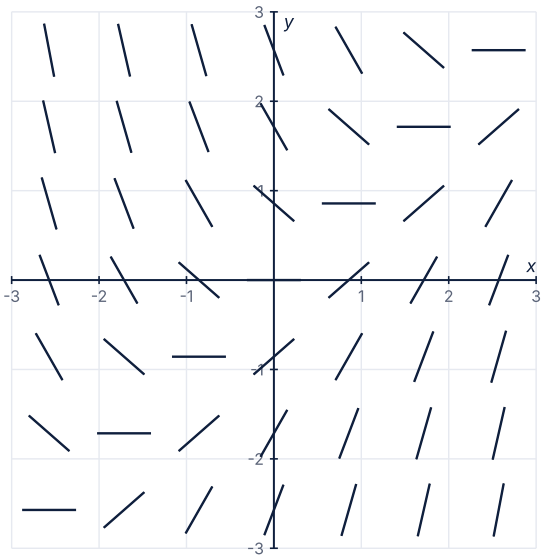
13. What is the area of the region bounded by the graphs of $y = 8 - x^2$ and $y = x^2$?

(A) $\frac{64}{3}$ **(B)** $\frac{80}{3}$ **(C)** 32 **(D)** $\frac{128}{3}$ **(E)** 64

14. The region in the first quadrant bounded by $y = x$ and $y = x^2$ is revolved about the x-axis. What is the volume of the solid generated?

(A) $\frac{\pi}{30}$ **(B)** $\frac{2}{15}$ **(C)** $\frac{\pi}{15}$ **(D)** $\frac{\pi}{10}$ **(E)** $\frac{2\pi}{15}$

15. The slope field shown could be a graph of solutions to which differential equation?



A slope field.

(A) $\frac{dy}{dx} = xy$ **(B)** $\frac{dy}{dx} = x + y$ **(C)** $\frac{dy}{dx} = y - x$ **(D)** $\frac{dy}{dx} = \frac{x}{y}$ **(E)** $\frac{dy}{dx} = x - y$

16. If $\frac{dy}{dx} = \frac{x}{y}$ with $y(1) = 2$ and $y > 0$, then $y(3) =$

(A) 3 **(B)** $2\sqrt{3}$ **(C)** $\sqrt{13}$ **(D)** 4 **(E)** $2\sqrt{6}$

17. Let $f(x) = \begin{cases} ax^2 + 1, & x \le 1 \\ 4x + b, & x > 1 \end{cases}$. If f is differentiable at $x = 1$, then $a + b =$

(A) -2 **(B)** -1 **(C)** 0 **(D)** $\frac{1}{2}$ **(E)** 1

18. The rate $r(t)$, in gallons per minute, at which a tank is filled is given in the table. Using a trapezoidal sum with the three subintervals shown, what is the approximation of the number of gallons added from $t = 0$ to $t = 9$?

t (min)	0	2	5	9
$r(t)$	3	5	4	8

(A) 22.75 **(B)** 37 **(C)** 45.5 **(D)** 54 **(E)** 91

19. A particle moves along a line with velocity $v(t) = t^2 - 4t + 3$ for $0 \le t \le 4$. What is the total distance traveled by the particle?

(A) $\frac{4}{3}$ **(B)** $\frac{8}{3}$ **(C)** 4 **(D)** $\frac{16}{3}$ **(E)** $\frac{20}{3}$

20. The line tangent to the graph of $y = \sqrt[3]{x}$ at $x = 27$ is used to approximate $\sqrt[3]{28}$. What value does this approximation give?

(A) $3 + \frac{1}{81}$ **(B)** $3 + \frac{1}{27}$ **(C)** $3 + \frac{1}{9}$ **(D)** $3 + \frac{1}{3}$ **(E)** 6

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SECTION I, PART B Multiple Choice [Calculator]

Choose the best answer for each question. A graphing calculator may be used. [Calculator]

21. The average value of $f(x) = \sqrt{1 + x^3}$ on the interval $[0, 2]$ is

(A) 1.621 **(B)** 2.000 **(C)** 3.000 **(D)** 3.241 **(E)** 6.483

22. A particle moves along a line with velocity $v(t) = \sin\left(t^2\right)$ for $0 \le t \le 2$. The total distance the particle travels is

(A) 0.805 **(B)** 0.985 **(C)** 1.610 **(D)** 1.772 **(E)** 2.000

23. What is the area of the region bounded by the graphs of $y = 3 - x^2$ and $y = e^x$?

(A) 1.006 **(B)** 1.826 **(C)** 2.646 **(D)** 3.000 **(E)** 3.652

24. The region bounded by the graph of $y = \ln x$, the x-axis, and the line $x = 4$ is revolved about the x-axis. What is the volume of the solid generated?

(A) 2.597 **(B)** 7.996 **(C)** 8.158 **(D)** 16.317 **(E)** 46.121

25. Water flows into a tank at the rate $r(t) = 9 + 4\sin\left(\frac{t}{2}\right)$ liters per minute and is pumped out at a constant rate of 10 liters per minute. The tank holds 40 liters at time $t = 0$. How many liters are in the tank at $t = 8$ minutes?

(A) 5.229 **(B)** 32.000 **(C)** 40.000 **(D)** 45.229 **(E)** 53.229

26. A closed cylindrical can must hold exactly 500 cubic centimeters. What is the minimum possible total surface area, in square centimeters?

(A) 4.301 **(B)** 8.602 **(C)** 232.489 **(D)** 348.734 **(E)** 357.080

27. The function f is continuous on the interval $[0, 9]$, and some of its values are given in the table. What is the minimum number of solutions of $f(x) = 8$ in the interval $(0, 9)$ that the Intermediate Value Theorem guarantees?

x	0	2	5	9
$f(x)$	3	9	6	14

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

28. Euler's method with two steps of equal size $h = 0.25$, starting at $(1, 2)$, is used to approximate $y(1.5)$ for the solution of $\frac{dy}{dx} = x^2 + y$ with $y(1) = 2$. The approximation is

(A) 2.750 **(B)** 3.500 **(C)** 3.641 **(D)** 3.688 **(E)** 3.828

29. Let $f(x) = xe^{-x}$. The Mean Value Theorem guarantees a number c in $(0, 3)$ with $f'(c) = \frac{f(3) - f(0)}{3 - 0}$. What is c?

(A) 0.880 **(B)** 1.000 **(C)** 1.500 **(D)** 2.000 **(E)** 3.000

30. On the interval $[0, 5]$, at what value of x does $f(x) = e^{x/2} - 2x$ attain its minimum value?

(A) 1.386 **(B)** 2.773 **(C)** 4.000 **(D)** 5.000 **(E)** 5.545

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SECTION II, PART A **Free Response [Calculator]**

Each question is worth 9 points. A graphing calculator may be used; show the setup of any integral or equation you evaluate. [Calculator]

31. Water flows into a tank at a rate $R(t)$, in gallons per minute, where t is measured in minutes. R is a differentiable function, and selected values are given in the table. Water also drains from the tank at the rate $D(t) = 4 + 6\sin\left(\frac{\pi t}{60}\right)$ gallons per minute. The tank contains 50 gallons at time $t = 0$.

(a) Use the data in the table to approximate $R'(16)$. Show the computation that leads to your answer, and indicate units of measure.

(b) Use a trapezoidal sum with the four subintervals indicated by the table to approximate $\int_0^{30} R(t)\,dt$. Explain the meaning of this integral in the context of the problem.

(c) Using the approximation from part (b), approximate the number of gallons of water in the tank at time $t = 30$.

(d) Is the amount of water in the tank increasing or decreasing at time $t = 20$? Give a reason for your answer.

t (min)	0	5	12	20	30
$R(t)$ (gal/min)	10	14	13	8	4

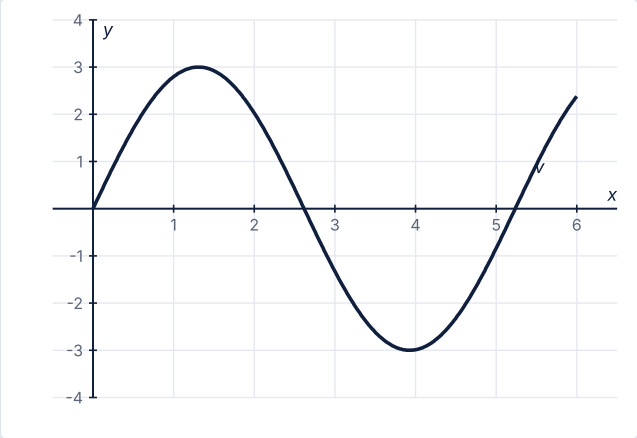
32. A particle moves along the x-axis with velocity $v(t) = 3\sin(1.2t)$ for $0 \le t \le 6$, where t is measured in seconds and v in meters per second. The graph of v is shown. At time $t = 0$ the particle is at $x = 2$.

(a) Find the acceleration of the particle at $t = 2$. Is the speed of the particle increasing or decreasing at $t = 2$? Give a reason.

(b) Find the position of the particle at time $t = 4$.

(c) Find the total distance traveled by the particle from $t = 0$ to $t = 6$.

(d) At what time t in $[0, 6]$ is the particle farthest to the right? Justify your answer.



Graph of v(t).

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SECTION II, PART B **Free Response [No calculator]**

Each question is worth 9 points. Do not use a calculator. Show all work and justify your answers. [No calculator]

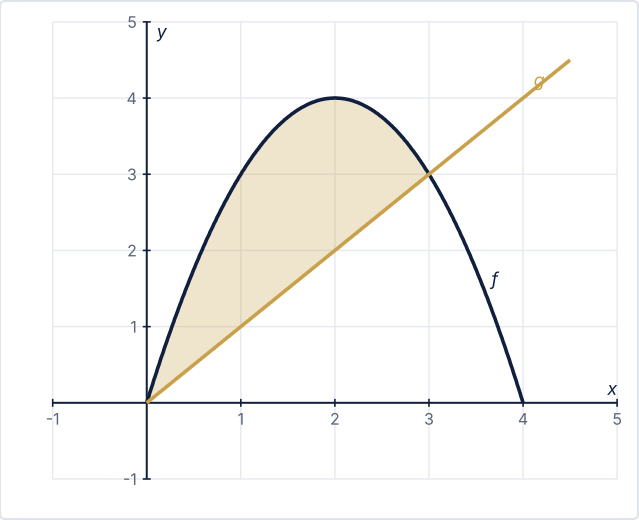
33. Let R be the region bounded by the graphs of $f(x) = 4x - x^2$ and $g(x) = x$, as shown in the figure.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) The region R is the base of a solid. For each x, the cross section of the solid perpendicular to the x-axis is a square. Find the volume of the solid.

(d) Write, but do not evaluate, an integral expression for the volume of the solid generated when R is revolved about the horizontal line $y = -1$.



The region R between f and g.

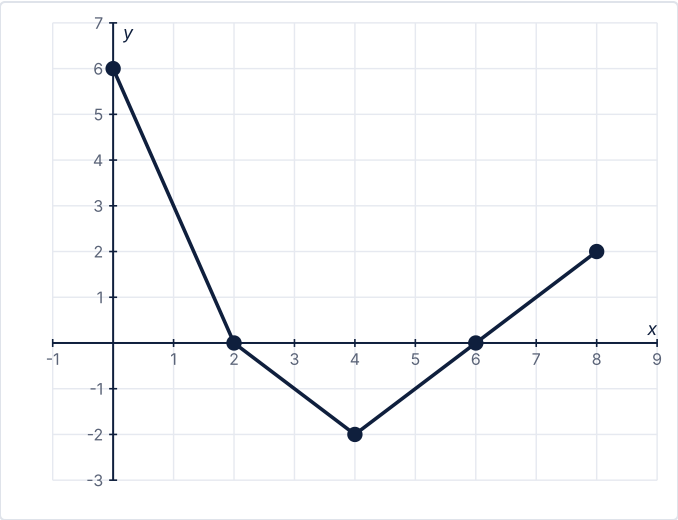
34. The function f is differentiable on $[0, 8]$ with $f(0) = 1$. The graph of f', the derivative of f, consists of four line segments and is shown in the figure.

(a) Find $f(4)$ and $f(6)$.

(b) For $0 < x < 8$, find all values of x at which f has a relative maximum or a relative minimum. Justify your answer.

(c) Find the x-coordinate of each point of inflection of the graph of f on $0 < x < 8$. Justify your answer.

(d) Find the absolute minimum value of f on $[0, 8]$. Justify your answer.



Graph of f'.

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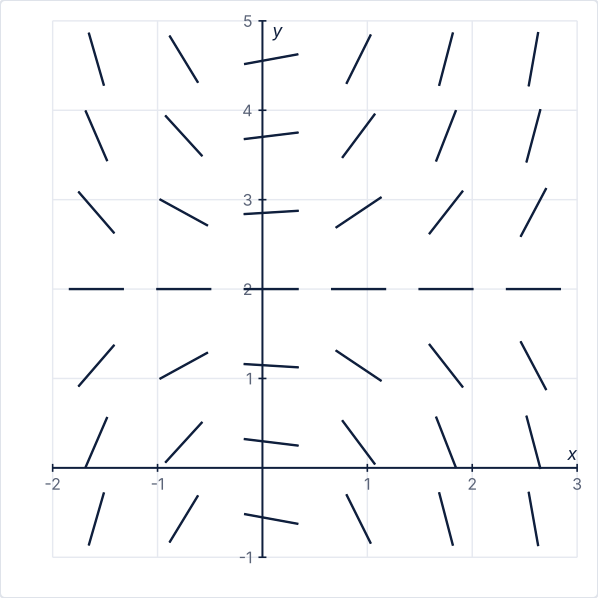
35. Consider the differential equation $\frac{dy}{dx} = x(y-2)$. A slope field for the equation is shown in the figure. Let $y = f(x)$ be the particular solution with the initial condition $f(1) = 4$.

(a) At which points of the xy-plane do the slope segments appear horizontal? What is the sign of $\frac{dy}{dx}$ at the point $(-1, 4)$? Give reasons.

(b) Write an equation for the line tangent to the graph of f at $x = 1$, and use it to approximate $f(1.2)$.

(c) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Is your approximation in part (b) an underestimate or an overestimate of $f(1.2)$? Give a reason.

(d) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 4$.



Slope field for dy/dx = x(y - 2).

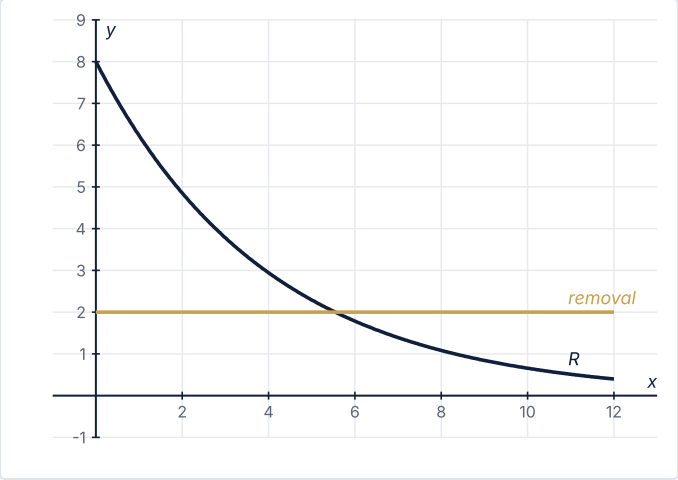
36. A medication enters a patient's bloodstream at the rate $R(t) = 8e^{-t/4}$ milligrams per hour for $0 \le t \le 12$, where t is in hours. The medication is removed from the bloodstream at a constant rate of 2 milligrams per hour. Let $Q(t)$ be the amount of medication, in milligrams, in the bloodstream at time t. There are 5 milligrams in the bloodstream at $t = 0$. The graph of R and the removal rate are shown.

(a) Find $R'(4)$. Using correct units, explain the meaning of $R'(4)$ in the context of the problem.

(b) Find the total amount of medication that enters the bloodstream during the first 8 hours. Give an exact answer.

(c) At what time t in $[0, 12]$ is $Q(t)$ greatest? Justify your answer.

(d) Find the exact value of the greatest amount of medication in the bloodstream.



Entry rate R(t) and removal rate.

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Key Practice Exam 1 (Calculus AB Style): Answers and Explanations

Complete solutions. Free-response keys show the scoring breakdown.

Exam 1 Practice Exam 1 (Calculus AB Style)

SECTION I, PART A · MULTIPLE CHOICE [NO CALCULATOR]

1. B. Factor: $\frac{(x-3)(x+3)}{(x-3)(x+1)} = \frac{x+3}{x+1}$, which tends to $\frac{6}{4} = \frac{3}{2}$ as $x \to 3$.
2. A. The denominator has the higher degree (3 versus 2), so the quotient tends to 0.
3. E. $f'(x) = 3\sec^2(3x)$, so $f'(\pi/12) = 3\sec^2(\pi/4) = 3 \cdot 2 = 6$.
4. D. Differentiate: $2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$. At $(1,2)$: $4 + 5\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{4}{5}$.
5. B. Chain rule: $h'(2) = f'(g(2))\,g'(2) = f'(3) \cdot g'(2) = 5 \cdot (-2) = -10$.
6. C. Inflection points occur where f' changes from increasing to decreasing or the reverse. The graph of f' has one local maximum and one local minimum, so f'' changes sign twice.
7. D. The average rate is $\frac{27-1}{2} = 13$. Solve $3c^2 = 13$: $c = \sqrt{13/3} \approx 2.08$, which lies in $(1,3)$.
8. A. With $x^2 + y^2 = 169$: $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$. At $x = 5$, $y = 12$: $5(2) + 12\frac{dy}{dt} = 0$, so $\frac{dy}{dt} = -\frac{5}{6}$ (sliding down at $\frac{5}{6}$ ft/s).
9. D. $f'(x) = 3(x-1)(x-3)$, so check $x = 0, 1, 3, 5$: $f = 1, 5, 1, 21$. The largest is 21 at the endpoint $x = 5$.
10. A. Let $u = -x^2$, so $x\,dx = -\frac{1}{2}du$: $-\frac{1}{2}\left[e^{-x^2}\right]_0^1 = \frac{1}{2}\left(1 - e^{-1}\right)$.
11. D. By the Fundamental Theorem with the chain rule, $F'(x) = \sqrt{1+x^4} \cdot 2x$, so $F'(1) = 2\sqrt{2}$.
12. C. $g(5)$ is the signed area from 0 to 5: 4 (rectangle on $[0,2]$) $+0$ (triangle above cancels triangle below on $[2,4]$) -2 (rectangle on $[4,5]$) $= 2$.
13. A. The curves meet at $x = \pm 2$. Area $= \int_{-2}^{2}(8 - 2x^2)\,dx = \left[8x - \frac{2}{3}x^3\right]_{-2}^{2} = \frac{64}{3}$.
14. E. Washers: $V = \pi\int_0^1 \left(x^2 - x^4\right)dx = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$.
15. E. The segments are horizontal along the line $y = x$, positive below that line (for example at $(2,0)$) and negative above it (for example at $(0,2)$). Only $\frac{dy}{dx} = x - y$ fits.
16. B. Separate: $y\,dy = x\,dx$, so $\frac{y^2}{2} = \frac{x^2}{2} + C$. Using $y(1) = 2$: $y^2 = x^2 + 3$. Then $y(3) = \sqrt{12} = 2\sqrt{3}$.
17. E. Equal derivatives at 1: $2a = 4$, so $a = 2$. Continuity: $a + 1 = 4 + b$, so $b = -1$. Then $a + b = 1$.
18. C. $\frac{3+5}{2}(2) + \frac{5+4}{2}(3) + \frac{4+8}{2}(4) = 8 + 13.5 + 24 = 45.5$.
19. C. $v = (t-1)(t-3)$ changes sign at 1 and 3. Each of $[0,1]$, $[1,3]$, $[3,4]$ contributes $\frac{4}{3}$ of distance, so the total is 4 (the displacement is only $\frac{4}{3}$).
20. B. $y' = \frac{1}{3}x^{-2/3}$, so the slope at 27 is $\frac{1}{27}$. Then $L(28) = 3 + \frac{1}{27}(1) = \frac{82}{27}$.

SECTION I, PART B · MULTIPLE CHOICE [CALCULATOR]

21. A. $\frac{1}{2-0}\int_0^2 \sqrt{1+x^3}\,dx = \frac{3.241}{2} \approx 1.621$ (the value 3.241 is the integral alone).
22. B. The velocity changes sign at $t = \sqrt{\pi}$, so distance $= \int_0^2 |\sin(t^2)|\,dt \approx 0.985$. The value 0.805 is the displacement $\int_0^2 \sin(t^2)\,dt$.
23. E. The curves intersect at $x \approx -1.677$ and $x \approx 0.834$. Area $= \int_{-1.677}^{0.834}\left(3 - x^2 - e^x\right)dx \approx 3.652$.
24. C. Disks: $V = \pi\int_1^4 (\ln x)^2\,dx \approx \pi(2.597) \approx 8.158$.
25. D. Amount $= 40 + \int_0^8 \left(9 + 4\sin\frac{t}{2} - 10\right)dt = 40 + 5.229 \approx 45.229$ liters.
26. D. With $h = \frac{500}{\pi r^2}$, $S = 2\pi r^2 + \frac{1000}{r}$. Setting $S' = 0$ gives $r = \sqrt[3]{250/\pi} \approx 4.301$, and $S \approx 348.734$. (The value 4.301 is the optimal radius, not the area.)
27. C. The value 8 lies strictly between $f(0) = 3$ and $f(2) = 9$, between $f(2) = 9$ and $f(5) = 6$, and between $f(5) = 6$ and $f(9) = 14$. Those three intervals do not overlap, so there are at least 3 solutions.
28. E. Step 1: slope $= 1 + 2 = 3$, so $y_1 = 2 + 0.25(3) = 2.75$ at $x = 1.25$. Step 2: slope $= 1.5625 + 2.75 = 4.3125$, so $y_2 = 2.75 + 0.25(4.3125) = 3.828$.
29. A. The average rate is $e^{-3} \approx 0.0498$. Solve $f'(c) = e^{-c}(1-c) = 0.0498$ on $(0,3)$ to get $c \approx 0.880$ (it is the only solution, because f' decreases from 1 to 0 on $(0,1)$ and is negative for $x > 1$).
30. B. $f'(x) = \frac{1}{2}e^{x/2} - 2 = 0$ gives $e^{x/2} = 4$, so $x = 2\ln 4 \approx 2.773$. Compare values: $f(0) = 1$, $f(2.773) \approx -1.545$, $f(5) \approx 2.182$.

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SECTION II, PART A · FREE RESPONSE [CALCULATOR]

31. (a) Use the closest data on either side of $t = 16$: $R'(16) \approx \frac{R(20) - R(12)}{20 - 12} = \frac{8 - 13}{8} = -0.625$ gallons per minute per minute.

(b) $\int_0^{30} R(t)\,dt \approx \frac{10+14}{2}(5) + \frac{14+13}{2}(7) + \frac{13+8}{2}(8) + \frac{8+4}{2}(10) = 60 + 94.5 + 84 + 60 = 298.5$. The integral is the total number of gallons that flow into the tank during the first 30 minutes (about 298.5 gallons).

(c) Gallons drained: $\int_0^{30} D(t)\,dt = 120 + \frac{360}{\pi} \approx 234.592$. Amount at $t = 30$: $50 + 298.5 - 234.592 \approx 113.908$ gallons.

(d) The rate of change of the amount is $R(20) - D(20) = 8 - \left(4 + 6\sin\frac{\pi}{3}\right) = 8 - 9.196 = -1.196 < 0$. Since this rate is negative, the amount of water is decreasing at $t = 20$ (at about 1.196 gallons per minute).

Scoring: 9 points: (a) 2 (1 for the difference quotient with correct values, 1 for the answer with units), (b) 2 (1 for the trapezoidal sum, 1 for the meaning with units), (c) 3 (1 for the amount drained, 1 for the setup $50 + \int R - \int D$, 1 for the answer), (d) 2 (1 for the net rate, 1 for the conclusion with reason).

32. (a) $a(t) = v'(t) = 3.6\cos(1.2t)$, so $a(2) = 3.6\cos(2.4) \approx -2.655$ meters per second squared. Also $v(2) = 3\sin(2.4) \approx 2.026 > 0$. Velocity is positive and acceleration is negative (opposite signs), so the speed is decreasing at $t = 2$.

(b) $x(4) = 2 + \int_0^4 3\sin(1.2t)\,dt = 2 + 2.5(1 - \cos 4.8) \approx 4.281$ meters.

(c) Total distance $= \int_0^6 |3\sin(1.2t)|\,dt \approx 10.979$ meters.

(d) $v(t) = 0$ at $t = \frac{\pi}{1.2} \approx 2.618$ and $t = \frac{2\pi}{1.2} \approx 5.236$. Velocity changes from positive to negative at $t = 2.618$, so that is the only candidate for a maximum position in the interior. Position values: $x(0) = 2$, $x(2.618) = 2 + \int_0^{2.618} v\,dt = 7$, $x(6) \approx 2.979$. The greatest value is 7, so the particle is farthest to the right at $t \approx 2.618$ seconds.

Scoring: 9 points: (a) 2 (1 for the acceleration, 1 for the conclusion with reason), (b) 2 (1 for the integral setup with initial position, 1 for the answer), (c) 2 (1 for the integral of $|v|$, 1 for the answer), (d) 3 (1 for the critical times, 1 for comparing candidate positions, 1 for the answer with justification).

SECTION II, PART B · FREE RESPONSE [NO CALCULATOR]

33. (a) Setting $4x - x^2 = x$ gives $x = 0$ and $x = 3$, and $f \geq g$ on $[0, 3]$. Area $= \int_0^3 \left(3x - x^2\right)dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3 = \frac{27}{2} - 9 = \frac{9}{2}$.

(b) Washers: $V = \pi\int_0^3 \left[(4x - x^2)^2 - x^2\right]dx = \pi\int_0^3 \left(15x^2 - 8x^3 + x^4\right)dx = \pi\left[5x^3 - 2x^4 + \frac{x^5}{5}\right]_0^3 = \pi\left(135 - 162 + \frac{243}{5}\right) = \frac{108\pi}{5}$.

(c) The side of each square is $3x - x^2$, so $V = \int_0^3 \left(3x - x^2\right)^2 dx = \int_0^3 \left(9x^2 - 6x^3 + x^4\right)dx = \left[3x^3 - \frac{3}{2}x^4 + \frac{x^5}{5}\right]_0^3 = 81 - \frac{243}{2} + \frac{243}{5} = \frac{81}{10}$.

(d) The outer radius is $4x - x^2 + 1$ and the inner radius is $x + 1$, so $V = \pi\int_0^3 \left[\left(4x - x^2 + 1\right)^2 - (x + 1)^2\right]dx$.

Scoring: 9 points: (a) 2 (1 for the integral with limits and integrand, 1 for the answer), (b) 3 (1 for the washer integrand, 1 for the antiderivative, 1 for the answer), (c) 2 (1 for the integral, 1 for the answer), (d) 2 (1 for the correct radii, 1 for the integral with limits and π).

34. (a) $f(4) = f(0) + \int_0^4 f'(x)\,dx = 1 + (6 - 2) = 5$, since the triangle on $[0, 2]$ has area 6 and the triangle on $[2, 4]$ lies below the axis with area 2. Then $f(6) = f(4) + \int_4^6 f'(x)\,dx = 5 - 2 = 3$.

(b) $f'(x) = 0$ at $x = 2$ and $x = 6$. At $x = 2$, f' changes from positive to negative, so f has a relative maximum at $x = 2$. At $x = 6$, f' changes from negative to positive, so f has a relative minimum at $x = 6$.

(c) f' is decreasing on $(0, 4)$ and increasing on $(4, 8)$, so the graph of f is concave down on $(0, 4)$ and concave up on $(4, 8)$. The concavity changes at $x = 4$, so the point of inflection is at $x = 4$.

(d) The only critical points are $x = 2$ and $x = 6$. Candidates: $f(0) = 1$, $f(2) = 7$, $f(6) = 3$, $f(8) = 3 + 2 = 5$. The absolute minimum value is 1, at $x = 0$.

Scoring: 9 points: (a) 2 (1 for each value), (b) 2 (1 for each extremum, with sign-change reasoning), (c) 2 (1 for $x = 4$, 1 for the concavity justification), (d) 3 (1 for identifying candidates, 1 for evaluating them, 1 for the conclusion).

35. (a) Segments are horizontal where $x(y - 2) = 0$: along the line $y = 2$ and along the line $x = 0$. At $(-1, 4)$, $\frac{dy}{dx} = (-1)(4 - 2) = -2 < 0$, so the slope is negative.

(b) At $(1, 4)$ the slope is $1 \cdot (4 - 2) = 2$. Tangent line: $y = 4 + 2(x - 1)$. So $f(1.2) \approx 4 + 2(0.2) = 4.4$.

(c) $\frac{d^2y}{dx^2} = (y - 2) + x\frac{dy}{dx} = (y - 2) + x^2(y - 2) = (y - 2)\left(1 + x^2\right)$. At $(1, 4)$ this equals $4 > 0$, so the graph of f is concave up near $x = 1$ and lies above its tangent line. The approximation 4.4 is an underestimate.

(d) Separate: $\frac{dy}{y - 2} = x\,dx$, so $\ln|y - 2| = \frac{x^2}{2} + C$. Using $f(1) = 4$: $\ln 2 = \frac{1}{2} + C$, so $C = \ln 2 - \frac{1}{2}$. Then $|y - 2| = 2e^{(x^2 - 1)/2}$, and since $y - 2 > 0$ at $(1, 4)$, $y = 2 + 2e^{(x^2 - 1)/2}$.

Scoring: 9 points: (a) 2 (1 for the zero-slope lines, 1 for the sign at $(-1, 4)$), (b) 2 (1 for the tangent line, 1 for the approximation), (c) 2 (1 for $\frac{d^2y}{dx^2}$, 1 for the conclusion with reason), (d) 3 (1 for separating variables, 1 for antiderivatives, 1 for the constant and the final solution).

36. (a) $R'(t) = -2e^{-t/4}$, so $R'(4) = -2e^{-1} = -\frac{2}{e}$ milligrams per hour per hour. At $t = 4$ hours, the rate at which medication enters the bloodstream is decreasing by about 0.736 milligrams per hour each hour.

(b) $\int_0^8 8e^{-t/4}\,dt = \left[-32e^{-t/4}\right]_0^8 = 32 - 32e^{-2} = 32\left(1 - e^{-2}\right)$ milligrams.

(c) $Q'(t) = 8e^{-t/4} - 2 = 0$ gives $e^{-t/4} = \frac{1}{4}$, so $t = 4\ln 4 \approx 5.545$, which is in $[0, 12]$. For $t < 4\ln 4$, $Q'(t) > 0$, and for $t > 4\ln 4$, $Q'(t) < 0$. Since Q' changes from positive to negative at its only critical point, Q is greatest at $t = 4\ln 4$.

(d) $Q(4\ln 4) = 5 + \int_0^{4\ln 4}\left(8e^{-t/4} - 2\right)dt = 5 + \left[32\left(1 - \frac{1}{4}\right)\right] - 2(4\ln 4) = 5 + 24 - 8\ln 4 = 29 - 16\ln 2$ milligrams (about 17.910).

Scoring: 9 points: (a) 2 (1 for $R'(4)$, 1 for the interpretation with units), (b) 2 (1 for the integral, 1 for the exact value), (c) 3 (1 for setting $Q' = 0$, 1 for the solution, 1 for the sign-change justification), (d) 2 (1 for the setup with $Q(0) = 5$, 1 for the value).

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Exam 2

Practice Exam 2 (Calculus BC Style)

A full-length style practice test for Calculus BC with 30 multiple-choice and 6 free-response questions. Time yourself and review every answer afterward.

Exam Instructions

This practice exam follows the Calculus BC format and includes series, parametric, vector and polar equations, Euler's method, logistic models, improper integrals, integration by parts and Taylor error bounds. Section I has 30 multiple-choice questions: Part A (20 questions, no calculator) and Part B (10 questions, graphing calculator allowed). The real exam has 45 multiple-choice questions in 105 minutes; this shortened version suggests about 70 minutes for Section I. Section II has 6 free-response questions worth 9 points each: Part A (2 questions, graphing calculator allowed) and Part B (4 questions, no calculator). Allow about 90 minutes for Section II. Use a calculator only in Section I Part B and Section II Part A. Do not use a calculator in Section I Part A or in Section II Part B. For free-response questions, show your work: answers without supporting work or justification may earn little credit. Unless a problem says otherwise, give decimal answers rounded to three places. Exact answers are expected when no calculator is allowed.

SECTION I, PART A

Multiple Choice [No calculator]

Choose the best answer for each question. Do not use a calculator. [No calculator]

1. What is the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n\,4^n}$?

(A) $(-3, 5)$ (B) $(-3, 5]$ (C) $[-3, 5)$ (D) $[-3, 5]$ (E) $(-4, 4)$

2. $\int_0^1 xe^{2x}\,dx =$

(A) $\frac{e^2+1}{4}$ (B) $\frac{e^2-1}{4}$ (C) $\frac{e^2}{2}$ (D) $\frac{e^2+1}{2}$ (E) $\frac{e^2}{4}$

3. $\int_e^{\infty} \frac{dx}{x(\ln x)^2}$

(A) $\frac{1}{2}$ (B) $\frac{1}{e}$ (C) e
(D) The integral diverges. (E) 1

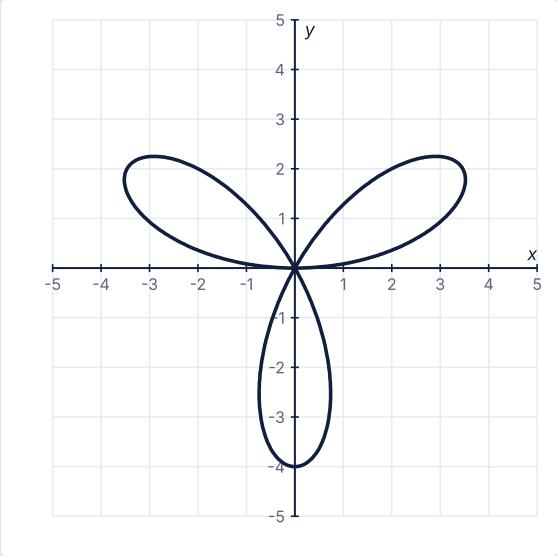
4. A curve is given parametrically by $x = t^2 + 1$ and $y = t^3 - 3t$. What is $\frac{dy}{dx}$ at $t = 2$?

(A) $\frac{4}{9}$ (B) $\frac{9}{4}$ (C) $\frac{9}{2}$ (D) 9 (E) 4

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5. What is the area of one petal of the polar curve $r = 4\sin(3\theta)$?



The curve r = 4 sin(3 theta).

(A) $\frac{2\pi}{3}$ **(B)** $\frac{8\pi}{3}$ **(C)** 2π **(D)** $\frac{4\pi}{3}$ **(E)** 4π

6. If $f(x) = x\,e^{3x}$, then $f^{(4)}(0) =$

(A) $\frac{9}{2}$ **(B)** 27 **(C)** 108 **(D)** 81 **(E)** 324

7. A particle moves in the plane with position $\langle 2t^2,\ t^3\rangle$ for $t \geq 0$. Its speed at $t = 1$ is

(A) 7 **(B)** 5 **(C)** $\sqrt{13}$ **(D)** $\sqrt{52}$ **(E)** 25

8. Let $\frac{dy}{dx} = 2x - y$ with $y(1) = 3$. Euler's method with two steps of size $h = 0.5$ gives an approximation of $y(2)$ equal to

(A) $\frac{5}{2}$ **(B)** 3 **(C)** $\frac{13}{4}$ **(D)** $\frac{11}{4}$ **(E)** $\frac{9}{4}$

9. A population satisfies $\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{800}\right)$ with $0 < P(0) < 400$. For what value of P is the population growing most rapidly?

(A) 400 **(B)** 200 **(C)** 240 **(D)** 600 **(E)** 800

10. Which of the following series converges?

(A) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ **(B)** $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ **(C)** $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

(D) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$ **(E)** $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

11. Let $P_3(x)$ be the third-degree Taylor polynomial for f about $x = 1$. If $|f^{(4)}(x)| \leq 24$ for all x in $[1, 2]$, then the Lagrange error bound for $|f(1.5) - P_3(1.5)|$ is

(A) $\frac{3}{2}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{1}{16}$ **(D)** $\frac{1}{8}$ **(E)** $\frac{1}{32}$

12. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ is approximated by the sum S_3 of its first three terms. By the alternating series error bound, $|S - S_3|$ is at most

(A) $\frac{1}{27}$ **(B)** $\frac{1}{64}$ **(C)** $\frac{1}{125}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{8}$

13. What is the length of the curve $x = 3t$, $y = 2t^{3/2}$ for $0 \leq t \leq 3$?

(A) $\frac{14}{3}$ **(B)** 7 **(C)** 21 **(D)** 14 **(E)** 28

14. $\int_2^3 \frac{4}{x^2 - 1}\,dx =$

(A) $2\ln\frac{3}{2}$ **(B)** $\ln\frac{3}{2}$ **(C)** $4\ln\frac{3}{2}$ **(D)** $2\ln 2$ **(E)** $\ln 3$

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15. $\lim_{x \to 0} \dfrac{e^{3x} - 1 - 3x}{x^2} =$

(A) 0 **(B)** $\dfrac{3}{2}$ **(C)** 3 **(D)** 9 **(E)** $\dfrac{9}{2}$

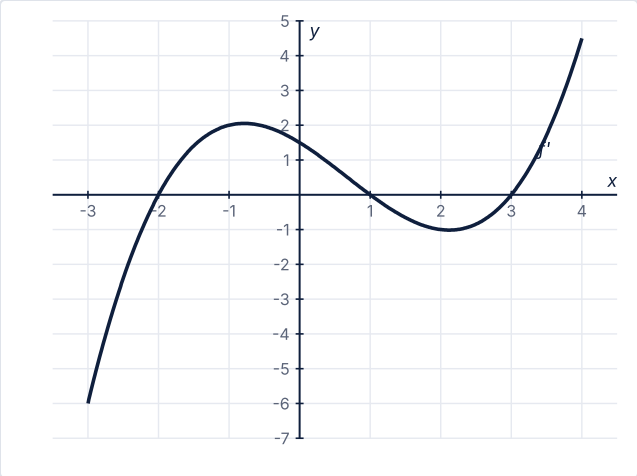
16. $\lim_{n \to \infty} \sum_{k=1}^{n} \dfrac{2}{n} \left(1 + \dfrac{2k}{n}\right)^3$ is equal to

(A) $\dfrac{15}{4}$ **(B)** $\dfrac{81}{4}$ **(C)** 16 **(D)** 20 **(E)** 40

17. If $\dfrac{dy}{dx} = y \cos x$ and $y(0) = 1$, then $y\left(\dfrac{\pi}{2}\right) =$

(A) 1 **(B)** $e^{\pi/2}$ **(C)** e **(D)** e^{2} **(E)** 2

18. The graph of f', the derivative of f, is shown for $-3 \le x \le 4$. At which value of x does f have a local maximum?



Graph of f'.

(A) $x = -2$ **(B)** $x = 1$ **(C)** $x = -1$ **(D)** $x = 0$ **(E)** $x = 3$

19. Oil flows into a barrel at a rate $r(t)$, in liters per minute, that is a decreasing function of t. Selected values of r are in the table. A trapezoidal sum with the three subintervals shown approximates the amount of oil, in liters, that flows in during $0 \le t \le 9$. The approximation is

t (min)	0	2	5	9
$r(t)$ (L/min)	6	4	3	1

(A) 28.5 **(B)** 21 **(C)** 36 **(D)** 57 **(E)** 14.25

20. If $f(x) = \ln\left(x^2 + 1\right)$, then $f''(2) =$

(A) $\dfrac{6}{25}$ **(B)** $\dfrac{4}{5}$ **(C)** $-\dfrac{4}{25}$ **(D)** $\dfrac{2}{5}$ **(E)** $-\dfrac{6}{25}$

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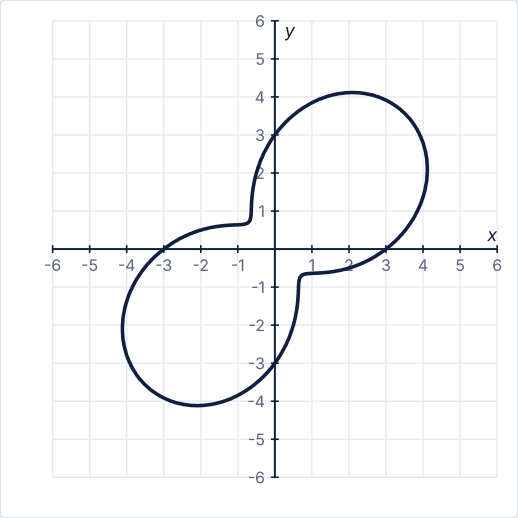
SECTION I, PART B Multiple Choice [Calculator]

Choose the best answer for each question. A graphing calculator may be used. [Calculator]

21. [Calculator] What is the length of the curve $x = t^2$, $y = t^3$ for $0 \le t \le 2$?

(A) 9.073 **(B)** 8.944 **(C)** 12.000 **(D)** 68.267 **(E)** 17.944

22. [Calculator] What is the area of the region enclosed by the polar curve $r = 3 + 2\sin(2\theta)$ for $0 \le \theta \le 2\pi$?



The curve r = 3 + 2 sin(2 theta).

(A) 17.279 **(B)** 28.274 **(C)** 34.558 **(D)** 31.416 **(E)** 69.115

23. [Calculator] The Maclaurin series for e^{-x^2} is used to approximate $\int_0^{0.5} e^{-x^2}\,dx$ with the first three nonzero terms. The approximation is

(A) 0.391 **(B)** 0.458 **(C)** 0.500 **(D)** 0.545 **(E)** 0.461

24. [Calculator] The length of the polar curve $r = \theta$ for $0 \le \theta \le 2$ is

(A) 2.000 **(B)** 2.958 **(C)** 2.236 **(D)** 3.333 **(E)** 4.472

25. [Calculator] $\int_0^{\infty} e^{-x^2}\,dx$ is approximately

(A) 0.500 **(B)** 0.785 **(C)** 1.000 **(D)** 0.886 **(E)** 1.772

26. [Calculator] A particle moves in the plane with $x(t) = t^2$ and $y(t) = \sin(2t)$. Its speed at $t = 1.5$ is

(A) 3.000 **(B)** 3.594 **(C)** 4.980 **(D)** 12.920 **(E)** 1.980

27. [Calculator] A population satisfies $\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{1200}\right)$ with $P(0) = 200$, where t is in years. At what time t does $P(t) = 600$?

(A) 1.609 **(B)** 6.438 **(C)** 2.197 **(D)** 1.386 **(E)** 3.219

28. [Calculator] The region bounded by $y = e^{-x}$, the x-axis, $x = 0$ and $x = 2$ is revolved about the x-axis. The volume of the solid is

(A) 1.542 **(B)** 0.491 **(C)** 0.865 **(D)** 2.716 **(E)** 3.084

29. [Calculator] A bacterial culture contains 400 cells at time $t = 0$. For $t \ge 0$ (hours) the culture grows at the rate $120e^{-0.1t}$ cells per hour. How many cells are in the culture at $t = 5$?

(A) 472 **(B)** 600 **(C)** 872 **(D)** 1000 **(E)** 1592

30. [Calculator] A liquid cools so that its temperature in degrees Celsius is $T(t) = 20 + 60e^{-0.1t}$, where t is in minutes. At what time t is the temperature decreasing at the rate of 2 degrees Celsius per minute?

(A) 1.099 **(B)** 5.493 **(C)** 9.163 **(D)** 10.986 **(E)** 17.918

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SECTION II, PART A **Free Response [Calculator]**

Each question is worth 9 points. A graphing calculator may be used; show the setup of any integral or equation you evaluate. [Calculator]

31. A river's flow rate is measured at a gauge. For $0 \le t \le 14$, where t is in minutes, the flow rate $R(t)$, in cubic meters per minute, is a differentiable function of t. Selected values of $R(t)$ are given in the table. At a second gauge, the flow rate is modeled by $M(t) = 12 + 9\sin\left(\frac{\pi t}{10}\right)$ cubic meters per minute for $0 \le t \le 14$.

(a) Use a trapezoidal sum with the four subintervals indicated by the table to approximate $\int_0^{14} R(t)\,dt$. Using correct units, explain the meaning of $\int_0^{14} R(t)\,dt$ in the context of the problem.

(b) Is there a time t with $7 < t < 14$ at which $R'(t) = -1$? Justify your answer.

(c) Find the average value of $M(t)$ on the interval $0 \le t \le 14$. Indicate units of measure.

(d) Find the absolute maximum value of $M(t)$ on the interval $0 \le t \le 14$. Justify your answer.

t (min)	0	3	7	10	14
$R(t)$ (m³/min)	12	19	21	15	14

32. A particle moves in the xy-plane so that its position at time t, for $0 \le t \le 4$, is $(x(t), y(t))$, with $x(0) = 1$ and $y(0) = 2$. The velocity vector is $\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$, where $\frac{dx}{dt} = 4\cos\left(\frac{t}{2}\right)$ and $\frac{dy}{dt} = 2te^{-0.4t}$. Values of the components, rounded to three decimal places, are given in the table.

(a) Use a left Riemann sum with the four subintervals of equal length indicated by the table to approximate $y(4)$.

(b) Find the speed of the particle at time $t = 3$. Find the acceleration vector at time $t = 3$. Is the speed of the particle increasing or decreasing at $t = 3$? Give a reason for your answer.

(c) Find $x(4)$.

(d) Find the total distance traveled by the particle over the time interval $0 \le t \le 4$.

t	0	1	2	3	4
$\frac{dx}{dt}$	4.000	3.510	2.161	0.283	-1.665
$\frac{dy}{dt}$	0.000	1.341	1.797	1.807	1.615

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SECTION II, PART B Free Response [No calculator]

Each question is worth 9 points. Do not use a calculator. Show all work and justify your answers. [No calculator]

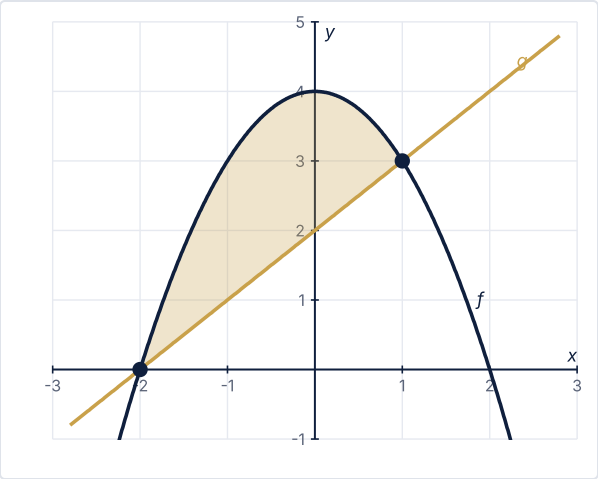
33. Let R be the region in the xy-plane bounded by the graphs of $f(x) = 4 - x^2$ and $g(x) = x + 2$, as shown in the figure.

(a) Find the area of R.

(b) Region R is the base of a solid. For each x, the cross section of the solid perpendicular to the x-axis is a square. Find the volume of the solid.

(c) Write, but do not evaluate, an integral expression for the volume of the solid generated when R is revolved about the horizontal line $y = -1$.

(d) Find the average value of $f(x) - g(x)$ on the interval $-2 \le x \le 1$.



Region R between f and g.

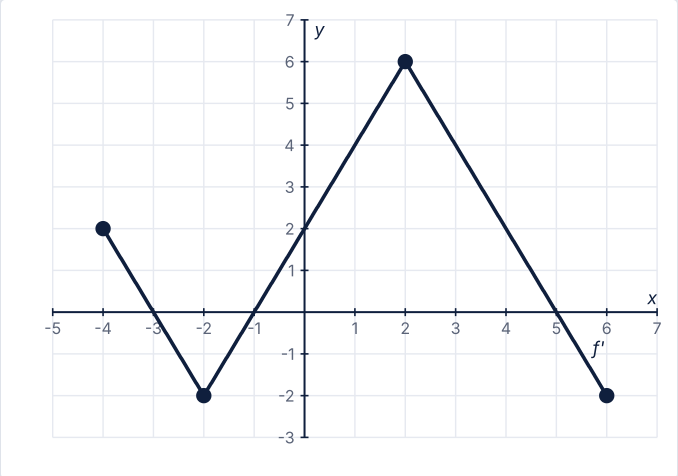
34. The function f is continuous on the closed interval $[-4, 6]$ and differentiable on the open interval $(-4, 6)$, with $f(-1) = 3$. The graph of f', the derivative of f, consists of three line segments and is shown in the figure.

(a) Find the x-coordinate of each local maximum and each local minimum of f on $(-4, 6)$. Justify your answers.

(b) On what intervals is the graph of f concave up? Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answers.

(c) Find the absolute minimum value of f on $[-4, 6]$. Justify your answer.

(d) Let $p(x) = f(x) - 2x$. Find the x-coordinate of each critical point of p in $(-4, 6)$, and classify each as the location of a local maximum, a local minimum, or neither.



Graph of f'.

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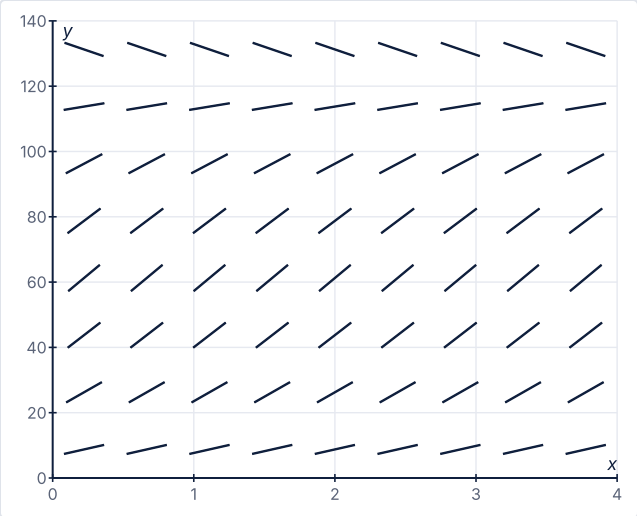
35. The number of fish $P(t)$ in a pond, where t is measured in years, satisfies the logistic differential equation $\frac{dP}{dt} = 0.01P(120 - P)$ with $P(0) = 20$. A slope field for the differential equation is shown in the figure.

(a) Find $\lim_{t\to\infty} P(t)$. Justify your answer using the differential equation or the slope field.

(b) Find $\frac{d^2P}{dt^2}$ in terms of P. For what value of P is the number of fish increasing most rapidly? Justify your answer.

(c) Use Euler's method, starting at $t = 0$ with two steps of equal size $h = 1$, to approximate $P(2)$.

(d) Find the particular solution $P = P(t)$ to the differential equation with $P(0) = 20$.



Slope field for dP/dt = 0.01 P (120 - P). Horizontal axis: t. Vertical axis: P.

36. Let $f(x) = \int_0^x e^{-t^2}\,dt$. The Maclaurin series for e^u is $1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots + \frac{u^n}{n!} + \cdots$.

(a) Write the first four nonzero terms and the general term of the Maclaurin series for $f(x)$.

(b) Use the ratio test to find the radius of convergence of the Maclaurin series for $f(x)$.

(c) The first three nonzero terms of the series are used to approximate $f\left(\frac{1}{2}\right)$. Find the approximation, and show that its error is less than $\frac{1}{1000}$.

(d) Use the series to find $\lim_{x\to 0} \dfrac{f(x) - x}{x^3}$.

Key Practice Exam 2 (Calculus BC Style): Answers and Explanations

Complete solutions. Free-response keys show the scoring breakdown.

Exam 2 Practice Exam 2 (Calculus BC Style)

SECTION I, PART A · MULTIPLE CHOICE [NO CALCULATOR]

1. C. The ratio test gives $|x-1|/4 < 1$, so $-3 < x < 5$. At $x = 5$ the series is $\sum \frac{1}{n}$ (diverges); at $x = -3$ it is $\sum \frac{(-1)^n}{n}$ (converges). Interval: $[-3, 5)$.

2. A. Integration by parts with $u = x$, $dv = e^{2x}dx$: $\left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right]_0^1 = \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4}$.

3. E. With $u = \ln x$, $\int_e^b \frac{dx}{x(\ln x)^2} = 1 - \frac{1}{\ln b}$, which tends to 1 as $b \to \infty$. The integral converges to 1.

4. B. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t}$. At $t = 2$ this is $\frac{9}{4}$.

5. D. One petal is traced for $0 \le \theta \le \frac{\pi}{3}$. Area $= \frac{1}{2}\int_0^{\pi/3} 16\sin^2(3\theta)\,d\theta = 8 \cdot \frac{\pi}{6} = \frac{4\pi}{3}$. (The value 4π is the area of all three petals.)

6. C. Since $e^{3x} = \sum \frac{(3x)^n}{n!}$, the x^4 term of xe^{3x} is $x \cdot \frac{27x^3}{6} = \frac{9}{2}x^4$. Then $f^{(4)}(0) = 4! \cdot \frac{9}{2} = 108$.

7. B. $\mathbf{v}(t) = \langle 4t,\, 3t^2\rangle$, so $\mathbf{v}(1) = \langle 4, 3\rangle$ and the speed is $\sqrt{16+9} = 5$.

8. D. At $(1, 3)$ the slope is -1, so $y(1.5) \approx 3 - 0.5 = 2.5$. At $(1.5, 2.5)$ the slope is 0.5, so $y(2) \approx 2.5 + 0.25 = 2.75 = \frac{11}{4}$.

9. A. For a logistic model with carrying capacity $K = 800$, $\frac{dP}{dt} = 0.3P - \frac{0.3}{800}P^2$ is a downward parabola in P with its maximum at $P = \frac{K}{2} = 400$.

10. E. Since $\frac{n}{n^3+1} < \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges, the comparison test shows convergence. The first is a p-series with $p = \frac{1}{2}$, the second and fourth have terms that do not tend to 0, and the third diverges by the integral test.

11. C. $|R_3(1.5)| \le \frac{24}{4!}|1.5 - 1|^4 = 1 \cdot \frac{1}{16} = \frac{1}{16}$.

12. B. The series alternates, its terms decrease in magnitude, and they tend to 0. The error is no larger than the first omitted term, $\frac{1}{4^3} = \frac{1}{64}$.

13. D. $\frac{dx}{dt} = 3$, $\frac{dy}{dt} = 3t^{1/2}$, so speed $= \sqrt{9+9t} = 3\sqrt{1+t}$. Length $= \int_0^3 3(1+t)^{1/2}dt = 2(1+t)^{3/2}\Big|_0^3 = 2(8-1) = 14$.

14. A. $\frac{4}{x^2-1} = \frac{2}{x-1} - \frac{2}{x+1}$. So the integral is $2\ln\frac{x-1}{x+1}\Big|_2^3 = 2\left(\ln\frac{1}{2} - \ln\frac{1}{3}\right) = 2\ln\frac{3}{2}$.

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15. E. The limit has the form $\frac{0}{0}$. Applying L'Hopital's Rule twice gives $\lim \frac{3e^{3x}-3}{2x} = \lim \frac{9e^{3x}}{2} = \frac{9}{2}$.

16. D. This is a right Riemann sum with $\Delta x = \frac{2}{n}$ and $x_k = 1 + \frac{2k}{n}$ on $[1, 3]$ for $f(x) = x^3$. The limit is $\int_1^3 x^3 dx = \frac{81-1}{4} = 20$.

17. C. Separate: $\int \frac{dy}{y} = \int \cos x \, dx$, so $\ln y = \sin x + C$. With $y(0) = 1$, $C = 0$, so $y = e^{\sin x}$ and $y(\frac{\pi}{2}) = e$.

18. B. The zeros of f' are $x = -2$, 1, 3. Only at $x = 1$ does f' change from positive to negative, so f changes from increasing to decreasing there.

19. A. Trapezoids: $\frac{6+4}{2}(2) + \frac{4+3}{2}(3) + \frac{3+1}{2}(4) = 10 + 10.5 + 8 = 28.5$.

20. E. $f'(x) = \frac{2x}{x^2+1}$, so $f''(x) = \frac{2(x^2+1)-2x \cdot 2x}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}$. At $x = 2$: $\frac{-6}{25}$.

SECTION I, PART B · MULTIPLE CHOICE [CALCULATOR]

21. A. Length $= \int_0^2 \sqrt{(2t)^2 + (3t^2)^2} \, dt \approx 9.073$. (The value 8.944 is only the straight-line distance between the endpoints.)

22. C. Area $= \frac{1}{2} \int_0^{2\pi} (3 + 2\sin 2\theta)^2 d\theta = \frac{1}{2}(18\pi + 4\pi) = 11\pi \approx 34.558$.

23. E. $e^{-x^2} \approx 1 - x^2 + \frac{x^4}{2}$. Integrating from 0 to 0.5: $0.5 - \frac{0.125}{3} + \frac{0.03125}{10} \approx 0.4615 \approx 0.461$.

24. B. Length $= \int_0^2 \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta = \int_0^2 \sqrt{\theta^2 + 1} \, d\theta \approx 2.958$.

25. D. The integral converges. Evaluating $\int_0^b e^{-x^2} dx$ for large b (for example $b = 6$) gives 0.886.

26. B. $x'(1.5) = 3$ and $y'(1.5) = 2\cos 3 \approx -1.980$. Speed $= \sqrt{9 + 3.920} = \sqrt{12.920} \approx 3.594$.

27. E. The solution is $P = \frac{1200}{1+5e^{-0.5t}}$ (since $\frac{1200-200}{200} = 5$). Setting $P = 600$ gives $5e^{-0.5t} = 1$, so $t = 2\ln 5 \approx 3.219$.

28. A. Disks: $V = \pi \int_0^2 e^{-2x} dx = \frac{\pi}{2}(1 - e^{-4}) \approx 1.542$.

29. C. Cells $= 400 + \int_0^5 120e^{-0.1t} dt = 400 + 1200\left(1 - e^{-0.5}\right) \approx 872.163$, about 872 cells.

30. D. $T'(t) = -6e^{-0.1t} = -2$ gives $e^{-0.1t} = \frac{1}{3}$, so $t = 10\ln 3 \approx 10.986$ minutes.

SECTION II, PART A · FREE RESPONSE [CALCULATOR]

31. (a) $\int_0^{14} R(t) \, dt \approx \frac{12+19}{2}(3) + \frac{19+21}{2}(4) + \frac{21+15}{2}(3) + \frac{15+14}{2}(4) = 46.5 + 80 + 54 + 58 = 238.5$. The integral is the total volume of water that passes the first gauge during the 14 minutes, about 238.5 cubic meters.

(b) Yes. R is differentiable, so it is continuous on $[7, 14]$ and differentiable on $(7, 14)$. The average rate of change is $\frac{R(14)-R(7)}{14-7} = \frac{14-21}{7} = -1$. By the Mean Value Theorem there is a t in $(7, 14)$ with $R'(t) = -1$.

(c) Average value $= \frac{1}{14} \int_0^{14} M(t) \, dt = \frac{205.501}{14} \approx 14.679$ cubic meters per minute.

(d) $M'(t) = \frac{9\pi}{10} \cos\left(\frac{\pi t}{10}\right) = 0$ gives $t = 5$ on $[0, 14]$. Candidates: $M(0) = 12$, $M(5) = 21$, $M(14) \approx 3.440$. The absolute maximum value is 21 (at $t = 5$), because it is the largest of the values at the critical point and the endpoints.

Scoring: 9 points: (a) 2 (1 for the trapezoidal sum, 1 for the meaning with units), (b) 2 (1 for the average rate -1 with hypotheses, 1 for the conclusion citing the theorem), (c) 2 (1 for the integral expression divided by 14, 1 for the answer with units), (d) 3 (1 for $M'(t) = 0$ giving $t = 5$, 1 for comparing candidates, 1 for the answer with justification).

32. (a) $y(4) \approx y(0) + 1 \cdot (y'(0) + y'(1) + y'(2) + y'(3)) = 2 + (0 + 1.341 + 1.797 + 1.807) = 6.945$.

(b) Speed $= \sqrt{(0.283)^2 + (1.807)^2} \approx 1.829$. Acceleration $= \langle -2\sin(1.5), \; 2e^{-1.2}(1 - 1.2) \rangle \approx \langle -1.995, \; -0.120 \rangle$. Since $\mathbf{v}(3) \cdot \mathbf{a}(3) \approx (0.283)(-1.995) + (1.807)(-0.120) \approx -0.782 < 0$, the speed is decreasing at $t = 3$.

(c) $x(4) = 1 + \int_0^4 4\cos\left(\frac{t}{2}\right) dt = 1 + 8\sin 2 \approx 8.274$.

(d) Distance $= \int_0^4 \sqrt{\left(4\cos\frac{t}{2}\right)^2 + (2te^{-0.4t})^2} \, dt \approx 11.464$.

Scoring: 9 points: (a) 2 (1 for the sum of the correct four values, 1 for the answer), (b) 3 (1 for the speed, 1 for the acceleration vector, 1 for the conclusion with the reason $\mathbf{v} \cdot \mathbf{a} < 0$), (c) 2 (1 for the integral, 1 for the answer), (d) 2 (1 for the integral of the speed, 1 for the answer).

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SECTION II, PART B · FREE RESPONSE [NO CALCULATOR]

33. (a) Setting $4 - x^2 = x + 2$ gives $x^2 + x - 2 = 0$, so $x = -2$ or $x = 1$. On $[-2, 1]$, $f \geq g$. Area $= \int_{-2}^{1} (4 - x^2 - x - 2)\, dx = \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1} = \frac{7}{6} - \left(-\frac{10}{3}\right) = \frac{9}{2}$.

(b) The side of each square is $f(x) - g(x) = 2 - x - x^2$. Volume $= \int_{-2}^{1} (2 - x - x^2)^2\, dx = \frac{81}{10}$.

(c) Outer radius $f(x) + 1 = 5 - x^2$, inner radius $g(x) + 1 = x + 3$. Volume $= \pi \int_{-2}^{1} \left[(5 - x^2)^2 - (x + 3)^2\right] dx$.

(d) Average value $= \frac{1}{1-(-2)} \int_{-2}^{1} (f(x) - g(x))\, dx = \frac{1}{3} \cdot \frac{9}{2} = \frac{3}{2}$.

Scoring: 9 points: (a) 3 (1 for the limits found from the intersection, 1 for the integrand, 1 for the answer), (b) 2 (1 for squaring the difference in the integrand, 1 for the answer), (c) 2 (1 for the correct radii, 1 for the limits and constant π), (d) 2 (1 for the average value formula, 1 for the answer).

34. (a) The zeros of f' are $x = -3$, $x = -1$, $x = 5$. f' changes from positive to negative at $x = -3$ and at $x = 5$, so f has local maxima at $x = -3$ and $x = 5$. f' changes from negative to positive at $x = -1$, so f has a local minimum at $x = -1$.

(b) f is concave up where f' is increasing: on $(-2, 2)$. f' changes from decreasing to increasing at $x = -2$ and from increasing to decreasing at $x = 2$, so the graph of f has points of inflection at $x = -2$ and $x = 2$.

(c) The only critical point where a minimum can occur in the interior is $x = -1$. $f(-1) = 3$. Since $\int_{-4}^{-1} f'(x)\, dx = 1 - 2 = -1$ (a triangle of area 1 above the axis and a triangle of area 2 below it), $f(-4) = f(-1) - (-1) = 4$. Since $\int_{-1}^{6} f'(x)\, dx = 9 + 9 - 1 = 17$, $f(6) = 3 + 17 = 20$. The absolute minimum value is $f(-1) = 3$.

(d) $p'(x) = f'(x) - 2 = 0$ when $f'(x) = 2$: on $(-2, 2)$, $2x + 2 = 2$ gives $x = 0$; on $(2, 6)$, $10 - 2x = 2$ gives $x = 4$. (On $(-4, -2)$ the equation $-2x - 6 = 2$ gives $x = -4$, not in the open interval.) $p' = 2x$ changes from negative to positive at $x = 0$, so p has a local minimum at $x = 0$. $p' = 8 - 2x$ changes from positive to negative at $x = 4$, so p has a local maximum at $x = 4$.

Scoring: 9 points: (a) 2 (1 for the maxima with reason, 1 for the minimum with reason), (b) 2 (1 for the interval with reason, 1 for the inflection points), (c) 3 (1 for the candidates, 1 for the endpoint values using integrals of f', 1 for the answer), (d) 2 (1 for the critical points, 1 for the classification with sign change).

35. (a) Since $P(0) = 20$ is between 0 and 120, $\frac{dP}{dt} = 0.01P(120 - P) > 0$, so P is increasing. The constant function $P = 120$ is a solution, and solution curves cannot cross it, so $P < 120$ for all t. An increasing function that is bounded above has a limit, and the limit must be an equilibrium. Hence $\lim_{t \to \infty} P(t) = 120$.

(b) $\frac{dP}{dt} = 1.2P - 0.01P^2$, so $\frac{d^2P}{dt^2} = (1.2 - 0.02P)\frac{dP}{dt} = 0.01(120 - 2P) \cdot \frac{dP}{dt}$. For $0 < P < 120$, $\frac{dP}{dt} > 0$, so $\frac{d^2P}{dt^2} = 0$ only when $P = 60$. It is positive for $P < 60$ and negative for $60 < P < 120$, so $\frac{dP}{dt}$ is greatest at $P = 60$.

(c) At $(0, 20)$: slope $= 0.01(20)(100) = 20$, so $P(1) \approx 20 + 20 = 40$. At $(1, 40)$: slope $= 0.01(40)(80) = 32$, so $P(2) \approx 40 + 32 = 72$.

(d) $\frac{dP}{P(120-P)} = 0.01\, dt$. Since $\frac{1}{P(120-P)} = \frac{1}{120}\left(\frac{1}{P} + \frac{1}{120-P}\right)$, integrating gives $\frac{1}{120} \ln\left|\frac{P}{120-P}\right| = 0.01t + C$, so $\ln \frac{P}{120-P} = 1.2t + C_1$ for $0 < P < 120$. At $t = 0$: $\ln \frac{20}{100} = C_1$, so $\frac{P}{120-P} = \frac{1}{5}e^{1.2t}$. Solving for P: $P(t) = \frac{120}{1+5e^{-1.2t}}$.

Scoring: 9 points: (a) 2 (1 for the limit, 1 for the justification), (b) 2 (1 for the second derivative, 1 for $P = 60$ with justification), (c) 2 (1 for each Euler step, or 1 for the method and 1 for the answer), (d) 3 (1 for separating variables and partial fractions, 1 for antiderivatives with the constant, 1 for the solution using the initial condition).

36. (a) Replacing u by $-t^2$: $e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots + \frac{(-1)^n t^{2n}}{n!} + \cdots$. Integrating from 0 to x: $f(x) = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots + \frac{(-1)^n x^{2n+1}}{n!\,(2n+1)} + \cdots$.

(b) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{2n+3}}{(n+1)!(2n+3)} \cdot \frac{n!(2n+1)}{|x|^{2n+1}} = x^2 \cdot \frac{2n+1}{(n+1)(2n+3)} \to 0$ as $n \to \infty$ for every x. The limit is less than 1 for all x, so the radius of convergence is ∞.

(c) $f\left(\frac{1}{2}\right) \approx \frac{1}{2} - \frac{1}{3 \cdot 8} + \frac{1}{10 \cdot 32} = \frac{1}{2} - \frac{1}{24} + \frac{1}{320} \approx 0.4615$. At $x = \frac{1}{2}$ the series alternates, the magnitudes of the terms decrease to 0, so the error is less than the first omitted term: $\frac{1}{42 \cdot 128} = \frac{1}{5376} < \frac{1}{1000}$.

(d) $\frac{f(x) - x}{x^3} = -\frac{1}{3} + \frac{x^2}{10} - \frac{x^4}{42} + \cdots$, which tends to $-\frac{1}{3}$ as $x \to 0$.

Scoring: 9 points: (a) 3 (1 for the series of e^{-t^2}, 1 for the first four terms of f, 1 for the general term), (b) 2 (1 for the ratio, 1 for the conclusion), (c) 2 (1 for the approximation, 1 for the error bound with the alternating series reasoning), (d) 2 (1 for dividing the series, 1 for the limit).

Exam 3 Practice Exam 3 (Calculus AB Style)

A full-length practice exam in the Calculus AB format: 30 multiple-choice questions and 6 free-response questions. Take it under timed conditions, then check your work against the full solutions.

Exam Instructions

This practice exam follows the Calculus AB format. Section I has 30 multiple-choice questions: Part A (20 questions, no calculator) and Part B (10 questions, graphing calculator allowed). The real exam has 45 multiple-choice questions in 105 minutes; this shortened version suggests about 70 minutes for Section I. Section II has 6 free-response questions worth 9 points each: Part A (2 questions, graphing calculator allowed) and Part B (4 questions, no calculator). Allow about 90 minutes for Section II. Use a calculator only in Section I Part B and Section II Part A. Do not use a calculator in Section I Part A or in Section II Part B. For free-response questions, show your work: answers without supporting work or justification may earn little credit. Unless a problem says otherwise, give decimal answers rounded to three places. Exact answers are expected when no calculator is allowed.

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SECTION I, PART A Multiple Choice [No calculator]

Choose the best answer for each question. Do not use a calculator. [No calculator]

1. $\lim_{x \to 0} \frac{\sin 5x}{\tan 2x}$ is

(A) 0 (B) $\frac{2}{5}$ (C) $\frac{5}{2}$ (D) 5 (E) 10

2. If $f(x) = x^2 e^{3x}$, then $f'(1) =$

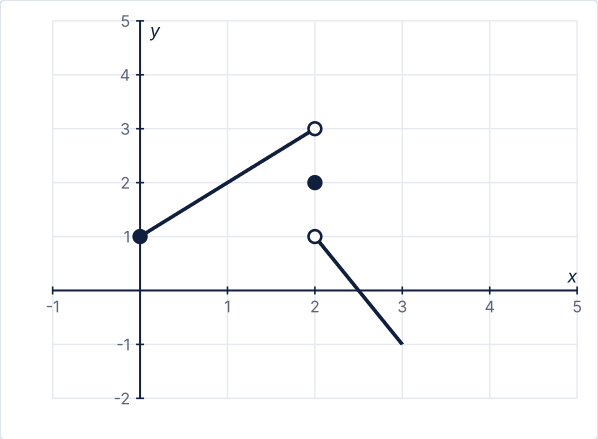
(A) $3e^3$ (B) $5e^3$ (C) $6e^3$ (D) $8e^3$ (E) $9e^3$

3. The table gives values of the differentiable functions f and g and their derivatives at $x = 2$. If $h(x) = \frac{f(x)}{g(x)}$, then $h'(2) =$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	6	4	3	-2

(A) -2 (B) 0 (C) $\frac{4}{3}$ (D) 2 (E) $\frac{8}{3}$

4. The graph of f is shown. What is the value of $\lim_{x \to 2^-} f(x) + \lim_{x \to 2^+} f(x) + f(2)$?



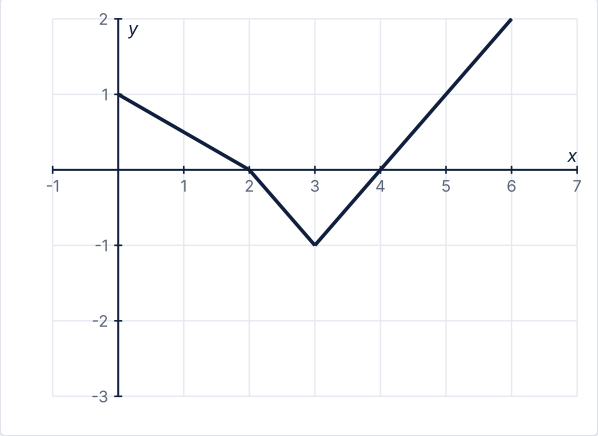
Graph of f.

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

5. If $y^3 + xy = 10$, then at the point $(1, 2)$ the value of $\frac{dy}{dx}$ is

(A) $-\frac{13}{2}$ (B) -2 (C) $-\frac{1}{6}$ (D) $-\frac{2}{13}$ (E) $\frac{2}{13}$

6. The graph of f', the derivative of the continuous function f, is shown for $0 < x < 6$. At what value of x does f have a local maximum?



Graph of f', the derivative of f, on 0 < x < 6.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

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7. Water is poured into a container shaped like an inverted right circular cone whose radius is always half its depth. Water enters at a constant rate of 8π cubic centimeters per minute. How fast, in centimeters per minute, is the depth of the water increasing at the instant the depth is 4 centimeters?

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) 4

8. What is the absolute maximum value of $f(x) = x^2e^{-x}$ on the closed interval $[0, 5]$?

(A) $\frac{4}{e^2}$ (B) $\frac{2}{e}$ (C) $\frac{4}{e}$ (D) $\frac{8}{e}$ (E) 4

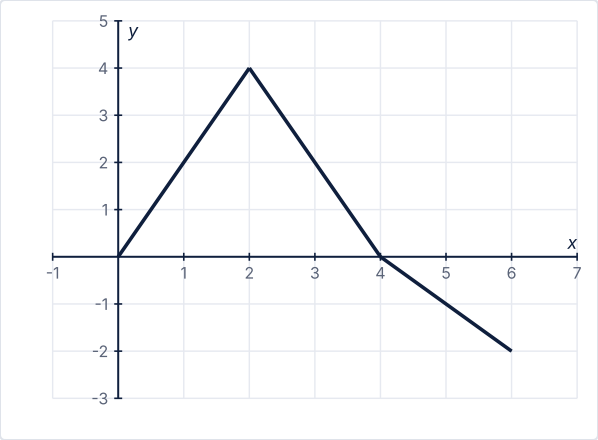
9. $\int_1^e \frac{(\ln x)^2}{x}\,dx =$

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{e}{3}$ (D) 1 (E) e

10. Let $G(x) = \int_x^{x^3} \sqrt{t^2+8}\,dt$. Then $G'(1) =$

(A) 0 (B) 3 (C) 6 (D) 9 (E) 12

11. The graph of the continuous function f is shown on $[0, 6]$ and consists of three line segments. What is the average value of f on $[0, 6]$?



Graph of f, which consists of three line segments.

(A) $\frac{2}{3}$ (B) 1 (C) $\frac{4}{3}$ (D) $\frac{3}{2}$ (E) 6

12. What is the area of the region in the first quadrant bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{2}$?

(A) $\frac{4}{3}$ (B) 2 (C) $\frac{8}{3}$ (D) 4 (E) $\frac{16}{3}$

13. The region in the first quadrant bounded by $y = x^2$, the line $y = 4$, and the y-axis is revolved about the line $y = 4$. What is the volume of the solid generated?

(A) $\frac{16\pi}{3}$ (B) 8π (C) $\frac{32\pi}{3}$ (D) 12π (E) $\frac{256\pi}{15}$

14. If $\frac{dy}{dx} = y^2\cos x$ and $y = 1$ when $x = 0$, then $y\left(\frac{\pi}{6}\right) =$

(A) $\frac{2}{3}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) 4

15. A function $y(t)$ satisfies the differential equation $\frac{dy}{dt} = \frac{y-50}{5}$ with $y(0) = 70$. What is $y(5)$?

(A) $20e$ (B) $50 + 20e$ (C) $50e$ (D) $20 + 50e$ (E) $70e$

16. The function f is twice differentiable with $f''(x) = x(x-2)^2(x+1)$. At which values of x does the graph of f have a point of inflection?

(A) $x = -1$ only (B) $x = 0$ only (C) $x = -1$ and $x = 0$ only
(D) $x = 0$ and $x = 2$ only (E) $x = -1$, $x = 0$, and $x = 2$

17. A particle moves along the x-axis with position $x(t) = t^3 - 6t^2 + 9t$ for $t \geq 0$. At which time is the speed of the particle increasing?

(A) $t = \frac{1}{4}$ (B) $t = \frac{1}{2}$ (C) $t = \frac{3}{2}$ (D) $t = \frac{5}{2}$ (E) $t = \frac{11}{4}$

18. The function f is differentiable on the closed interval $[1, 5]$ with $f(1) = 3$ and $f(5) = 11$. Which of the following must be true?

(A) f is increasing on $[1, 5]$
(B) $f'(x) = 2$ for all x in $[1, 5]$
(C) $f(c) = 2$ for some c in $(1, 5)$
(D) $f'(c) = 0$ for some c in $(1, 5)$
(E) $f'(c) = 2$ for some c in $(1, 5)$

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19. The function f is twice differentiable with $f(3) = 5$, $f'(3) = 2$, and $f''(x) > 0$ for all x. The tangent line to the graph of f at $x = 3$ is used to approximate $f(3.5)$. The approximation is

(A) 6, an underestimate **(B)** 6, an overestimate **(C)** 5.5, an underestimate
(D) 7, an underestimate **(E)** 7, an overestimate

20. $\displaystyle\int_0^4 |x^2 - 4x + 3|\,dx =$

(A) $\frac{4}{3}$ **(B)** 2 **(C)** $\frac{8}{3}$ **(D)** 4 **(E)** $\frac{16}{3}$

SECTION I, PART B Multiple Choice [Calculator]

Choose the best answer for each question. A graphing calculator may be used. Give decimal values to three places. [Calculator]

21. $\displaystyle\int_0^2 \frac{x}{1 + x^3}\,dx =$

(A) 0.286 **(B)** 0.500 **(C)** 0.667 **(D)** 0.724 **(E)** 1.333

22. The rate $r(t)$, in vehicles per minute, at which vehicles pass a checkpoint is a continuous function of time t, and values are given in the table. A right Riemann sum with the four subintervals shown is used to approximate the number of vehicles that pass the checkpoint from $t = 0$ to $t = 15$. The approximation is

t (minutes)	0	4	7	11	15
$r(t)$ (vehicles per minute)	12	15	11	8	10

(A) 150 **(B)** 165 **(C)** 167 **(D)** 169 **(E)** 210

23. A particle moves along the x-axis with position $x(t) = t\cos t$ for $t \geq 0$. What is the speed of the particle at $t = 2$?

(A) -2.235 **(B)** -0.416 **(C)** 0.416 **(D)** 1.403 **(E)** 2.235

24. What is the area of the region bounded by the graphs of $y = \cos x$ and $y = x^2$?

(A) 0.547 **(B)** 0.824 **(C)** 1.095 **(D)** 1.648 **(E)** 2.190

25. The region bounded by the graph of $y = \ln(x + 1)$, the x-axis, and the line $x = 3$ is revolved about the x-axis. What is the volume of the solid generated?

(A) 8.158 **(B)** 10.388 **(C)** 16.317 **(D)** 25.629 **(E)** 32.634

26. What is the average value of $f(x) = \ln(x^2 + 1)$ on the interval $[0, 3]$?

(A) 0.379 **(B)** 1.135 **(C)** 1.151 **(D)** 2.303 **(E)** 3.406

27. A particle moves along a line with velocity $v(t) = 3\sin t - t$ for $0 \leq t \leq 4$. What is the total distance traveled by the particle?

(A) -3.040 **(B)** 2.279 **(C)** 5.394 **(D)** 7.748 **(E)** 10.788

28. What is the absolute minimum value of $f(x) = x^2 + 5\cos x$ on the interval $[0, 3]$?

(A) 1.884 **(B)** 2.125 **(C)** 4.050 **(D)** 5.000 **(E)** 9.000

29. The function g is differentiable for all real numbers, and values of g are given in the table. What is the least number of values of x in the interval $0 < x < 8$ for which the Mean Value Theorem or Rolle's Theorem guarantees $g'(x) = 0$?

x	0	2	4	6	8
$g(x)$	2	5	5	1	1

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

30. A cup of liquid at $90°$C is placed in a room at $20°$C. Its temperature T satisfies $\frac{dT}{dt} = -k(T - 20)$, where t is in minutes and k is a positive constant. If $T = 60$ when $t = 5$, what is T when $t = 10$?

(A) 22.857 **(B)** 30.000 **(C)** 35.000 **(D)** 40.000 **(E)** 42.857

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SECTION II, PART A **Free Response [Calculator]**

Show your work. A graphing calculator may be used. Unless told otherwise, give decimal answers to three places. [Calculator]

31. Water flows into a storage tank at a rate $R(t)$, in gallons per hour, where t is measured in hours. R is a differentiable function, and selected values of R are given in the table. Water drains from the tank at the rate $D(t) = 2 + 3\sin\left(\frac{t}{4}\right)$ gallons per hour. The tank contains 80 gallons of water at time $t = 0$.

(a) Use a trapezoidal sum with the four subintervals indicated by the table to approximate $\int_0^{12} R(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} R(t)\,dt$ in the context of the problem.

(b) Use your answer from part (a) to estimate the number of gallons of water in the tank at time $t = 12$.

(c) Must there be at least two values of t in the interval $0 < t < 12$ for which $R(t) = 10$? Justify your answer.

(d) Let $A(t)$ be the number of gallons of water in the tank at time t. Find $A'(5)$. Is the amount of water in the tank increasing or decreasing at time $t = 5$? Give a reason for your answer.

t (hours)	0	2	5	9	12
$R(t)$ (gallons per hour)	8	11	14	9	6

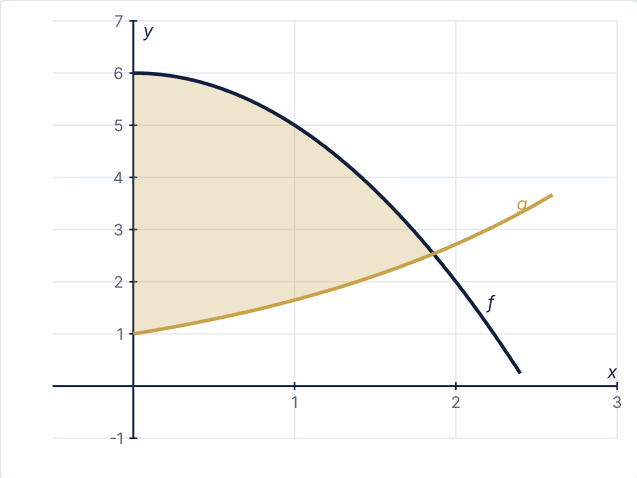
32. Let R be the region in the first quadrant bounded by the y-axis and the graphs of $f(x) = 6 - x^2$ and $g(x) = e^{x/2}$, as shown in the figure.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) Region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of the solid.

(d) Write, but do not evaluate, an expression involving one or more integrals for the volume of the solid generated when R is revolved about the horizontal line $y = -1$.



The region R, bounded by the y-axis and the graphs of f and g.

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SECTION II, PART B **Free Response [No calculator]**

Show your work and justify your answers. Do not use a calculator. [No calculator]

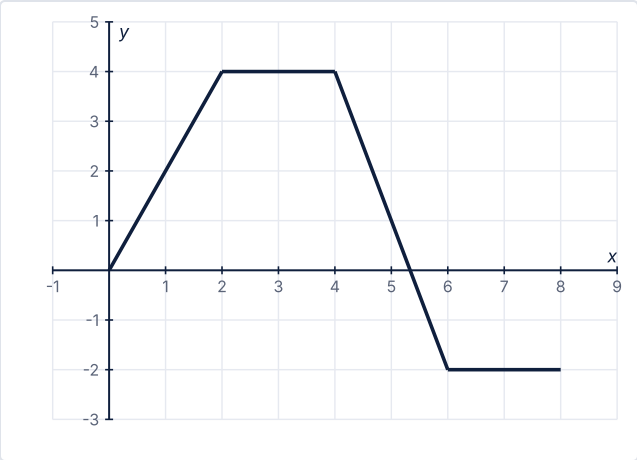
33. A particle moves along the x-axis. Its velocity $v(t)$, in meters per second, is continuous and its graph for $0 \le t \le 8$ consists of line segments, as shown in the figure. The particle is at position $x = 1$ at time $t = 0$.

(a) Find the acceleration of the particle at time $t = 5$. Is the speed of the particle increasing or decreasing at $t = 5$? Give a reason for your answer.

(b) Find the position of the particle at time $t = 6$.

(c) Find the total distance traveled by the particle from $t = 0$ to $t = 8$.

(d) At what time t, for $0 \le t \le 8$, is the particle farthest to the right? Justify your answer.



Graph of v, the velocity of the particle, for 0 <= t <= 8.

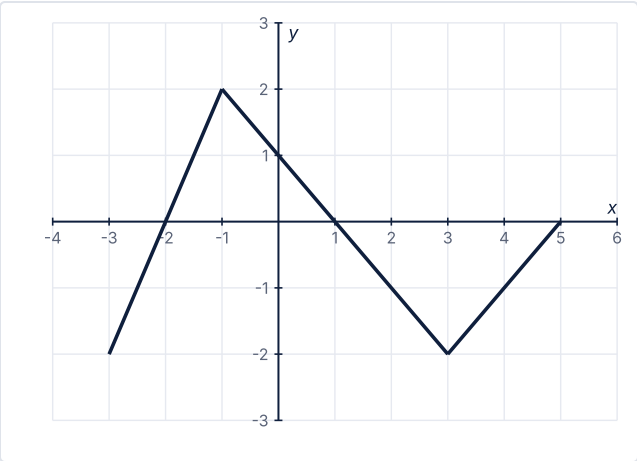
34. The function f is continuous on the closed interval $[-3, 5]$ and differentiable on the open interval $(-3, 5)$, with $f(1) = 3$. The graph of f', the derivative of f, consists of three line segments, as shown in the figure.

(a) Find the x-coordinate of each relative extremum of f on the interval $(-3, 5)$. For each, state whether it is a relative minimum or a relative maximum, and justify your answer.

(b) On what open intervals is the graph of f concave down? Find the x-coordinate of each point of inflection of the graph of f, and give a reason for your answer.

(c) Find $f(-2)$ and $f(5)$.

(d) Find the absolute minimum value and the absolute maximum value of f on $[-3, 5]$. Justify your answer.



Graph of f', consisting of three line segments.

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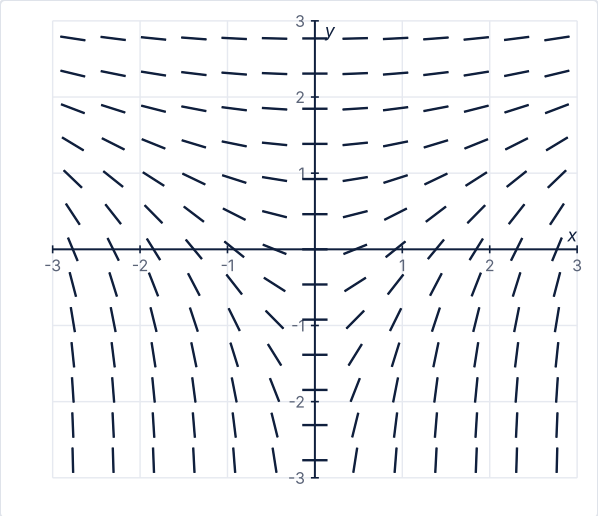
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35. Consider the differential equation $\frac{dy}{dx} = xe^{-y}$. A slope field for the equation is shown in the figure. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(2) = 0$.

(a) Write an equation for the line tangent to the graph of f at $x = 2$. Use this line to approximate $f(2.1)$.

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Is your approximation in part (a) less than or greater than $f(2.1)$? Give a reason for your answer.

(c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(2) = 0$.



Slope field for dy/dx = x e^(-y).

36. Consider the curve given by the equation $y^2 - 2xy + 2x^2 = 5$.

(a) Show that $\frac{dy}{dx} = \frac{y-2x}{y-x}$.

(b) Find an equation of the line tangent to the curve at the point $(1, 3)$.

(c) Find the x-coordinates of all points on the curve at which the tangent line is horizontal.

(d) A point moves along the curve so that $\frac{dx}{dt} = 4$ at the instant the point is at $(1, 3)$. Find $\frac{dy}{dt}$ at that instant.

Key Practice Exam 3 (Calculus AB Style): Answers and Explanations

Complete solutions. Free-response keys show the scoring breakdown.

Exam 3 Practice Exam 3 (Calculus AB Style)

SECTION I, PART A · MULTIPLE CHOICE [NO CALCULATOR]

1. C. Write $\frac{\sin 5x}{\tan 2x} = \frac{\sin 5x}{5x} \cdot \frac{2x}{\tan 2x} \cdot \frac{5x}{2x} \to 1 \cdot 1 \cdot \frac{5}{2}$.

2. B. Product rule: $f'(x) = 2xe^{3x} + 3x^2e^{3x}$, so $f'(1) = 2e^3 + 3e^3 = 5e^3$.

3. E. Quotient rule: $h'(2) = \frac{f'(2)g(2) - f(2)g'(2)}{[g(2)]^2} = \frac{4 \cdot 3 - 6 \cdot (-2)}{9} = \frac{24}{9} = \frac{8}{3}$.

4. E. From the graph, the left limit is 3, the right limit is 1, and $f(2) = 2$. The sum is $3 + 1 + 2 = 6$.

5. D. Differentiate: $3y^2y' + y + xy' = 0$, so $y' = -\frac{y}{3y^2+x} = -\frac{2}{12+1} = -\frac{2}{13}$.

6. B. f' changes from positive to negative at $x = 2$, so f has a local maximum there. At $x = 4$, f' changes from negative to positive (a local minimum), and f' does not change sign at the other values.

7. D. With $r = \frac{h}{2}$, $V = \frac{1}{3}\pi r^2 h = \frac{\pi}{12}h^3$. Then $\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$, so $8\pi = \frac{\pi}{4}(16)\frac{dh}{dt}$ and $\frac{dh}{dt} = 2$.

8. A. $f'(x) = (2x - x^2)e^{-x} = 0$ at $x = 0, 2$. Compare $f(0) = 0$, $f(2) = 4e^{-2} \approx 0.541$, $f(5) = 25e^{-5} \approx 0.168$. The maximum is $\frac{4}{e^2}$.

9. A. Let $u = \ln x$, $du = \frac{dx}{x}$. The limits become 0 and 1, so the integral is $\int_0^1 u^2\,du = \frac{1}{3}$.

10. C. $G'(x) = 3x^2\sqrt{x^6+8} - \sqrt{x^2+8}$, so $G'(1) = 3(3) - 3 = 6$.

11. B. $\int_0^6 f\,dx = \frac{1}{2}(4)(4) - \frac{1}{2}(2)(2) = 8 - 2 = 6$ (triangles above and below the axis). The average value is $\frac{6}{6-0} = 1$.

12. A. The graphs meet where $\sqrt{x} = \frac{x}{2}$, at $x = 0$ and $x = 4$. Area $= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)dx = \frac{16}{3} - 4 = \frac{4}{3}$.

13. E. Each cross-section is a disk of radius $4 - x^2$ for $0 \le x \le 2$: $V = \pi\int_0^2 (4 - x^2)^2 dx = \pi\left(32 - \frac{64}{3} + \frac{32}{5}\right) = \frac{256\pi}{15}$.

14. D. Separate: $\frac{dy}{y^2} = \cos x\,dx$, so $-\frac{1}{y} = \sin x + C$. With $y(0) = 1$, $C = -1$, so $y = \frac{1}{1-\sin x}$ and $y\left(\frac{\pi}{6}\right) = \frac{1}{1/2} = 2$.

15. B. The quantity $y - 50$ satisfies $\frac{d}{dt}(y - 50) = \frac{1}{5}(y - 50)$, so $y - 50 = 20e^{t/5}$ (since $y(0) - 50 = 20$). Then $y(5) = 50 + 20e$.

16. C. f'' changes sign at $x = -1$ (positive to negative) and at $x = 0$ (negative to positive). At $x = 2$ the factor $(x - 2)^2$ does not change sign, so f'' stays positive and there is no inflection.

17. C. $v(t) = 3(t - 1)(t - 3)$ and $a(t) = 6t - 12$. Speed increases when v and a have the same sign. At $t = \frac{3}{2}$, $v < 0$ and $a < 0$. At the other times shown, v and a have opposite signs.

18. E. f is continuous on $[1, 5]$ and differentiable on $(1, 5)$, so by the Mean Value Theorem $f'(c) = \frac{11-3}{5-1} = 2$ for some c in $(1, 5)$. The other statements need not hold.

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19. A. The tangent line gives $5 + 2(0.5) = 6$. Since $f'' > 0$, the graph is concave up and lies above its tangent lines, so 6 is an underestimate.

20. D. $x^2 - 4x + 3 = (x-1)(x-3)$ is negative on $(1, 3)$. With $F(x) = \frac{x^3}{3} - 2x^2 + 3x$: $F(1) - F(0) = \frac{4}{3}$, $-(F(3) - F(1)) = \frac{4}{3}$, $F(4) - F(3) = \frac{4}{3}$, so the total is $\frac{4}{3} \cdot 3 = 4$ (each of the three pieces has area $\frac{4}{3}$).

SECTION I, PART B · MULTIPLE CHOICE [CALCULATOR]

21. D. A graphing calculator's numerical integration gives 0.724.

22. B. Right sum $= 4(15) + 3(11) + 4(8) + 4(10) = 60 + 33 + 32 + 40 = 165$. (The left sum is 169 and the trapezoidal sum is 167.)

23. E. $v(t) = \cos t - t \sin t$, so $v(2) \approx -0.416 - 1.819 = -2.235$. Speed is $|v(2)| = 2.235$ (the negative value is the velocity).

24. C. The graphs intersect at $x = \pm 0.824$. Area $= \int_{-0.824}^{0.824} (\cos x - x^2)\, dx = 1.095$.

25. A. By the disk method, $V = \pi \int_0^3 [\ln(x+1)]^2 dx \approx \pi(2.597) = 8.158$. (Without squaring the radius you would get 7.995.)

26. B. Average value $= \frac{1}{3-0} \int_0^3 \ln(x^2+1)\, dx = \frac{3.406}{3} = 1.135$.

27. D. $v(t) = 0$ at $t = 2.279$ (and at $t = 0$), and $v < 0$ afterward. Distance $= \int_0^{2.279} v\, dt - \int_{2.279}^4 v\, dt = 2.355 + 5.394 = 7.748$. (The value -3.040 is the displacement.)

28. A. $f'(x) = 2x - 5\sin x = 0$ at $x = 0$ and $x = 2.125$. Compare $f(0) = 5$, $f(2.125) = 1.884$, $f(3) = 4.050$. The minimum is 1.884.

29. C. Since $g(2) = g(4)$, Rolle's Theorem gives $g'(c) = 0$ for some c in $(2, 4)$. Since $g(6) = g(8)$, it gives another in $(6, 8)$. These intervals do not overlap, so at least 2 values are guaranteed, and the table guarantees no more.

30. E. $T - 20 = 70e^{-kt}$. From $40 = 70e^{-5k}$, $e^{-5k} = \frac{4}{7}$. Then $T(10) = 20 + 70\left(\frac{4}{7}\right)^2 = 20 + 22.857 = 42.857$.

SECTION II, PART A · FREE RESPONSE [CALCULATOR]

31. (a) $\int_0^{12} R(t)\, dt \approx \frac{2(8+11)}{2} + \frac{3(11+14)}{2} + \frac{4(14+9)}{2} + \frac{3(9+6)}{2} = 19 + 37.5 + 46 + 22.5 = 125$. The integral is the total number of gallons of water that flow into the tank from $t = 0$ to $t = 12$ hours; the approximation is 125 gallons.

(b) $\int_0^{12} D(t)\, dt = 47.880$ gallons drain out. Estimated amount at $t = 12$: $80 + 125 - 47.880 = 157.120$ gallons.

(c) Yes. R is differentiable, so it is continuous on $[0, 12]$. Since $R(0) = 8 < 10 < 11 = R(2)$, the Intermediate Value Theorem gives a value t in $(0, 2)$ with $R(t) = 10$. Since $R(9) = 9 < 10 < 14 = R(5)$, the Intermediate Value Theorem gives another value t in $(5, 9)$ with $R(t) = 10$. The two intervals do not overlap, so these values are different.

(d) $A'(t) = R(t) - D(t)$, so $A'(5) = 14 - D(5) = 14 - 4.847 = 9.153$ gallons per hour. Because $A'(5) > 0$, the amount of water in the tank is increasing at $t = 5$.

Scoring: 9 points: (a) 2 (trapezoidal sum 1, meaning with units 1), (b) 2 (integral of D 1, estimate 1), (c) 2 (continuity and IVT on two disjoint intervals), (d) 3 (A'(t)=R-D 1, value 1, conclusion with reason 1).

32. The graphs of f and g meet at $x = k = 1.861$, where $6 - x^2 = e^{x/2}$ (found with a calculator), and $f(x) \geq g(x)$ on $[0, k]$.

(a) Area $= \int_0^k \left(6 - x^2 - e^{x/2}\right) dx = 5.946$.

(b) Washers with outer radius $f(x)$ and inner radius $g(x)$: $V = \pi \int_0^k \left[(6 - x^2)^2 - e^x\right] dx = 126.447$.

(c) The side of each square is $f(x) - g(x)$: $V = \int_0^k \left(6 - x^2 - e^{x/2}\right)^2 dx = 23.019$.

(d) The outer radius is $f(x) + 1$ and the inner radius is $g(x) + 1$: $V = \pi \int_0^k \left[(7 - x^2)^2 - \left(e^{x/2} + 1\right)^2\right] dx$.

Scoring: 9 points: (a) 3 (value of k 1, integrand 1, answer 1), (b) 2 (setup 1, answer 1), (c) 2 (setup 1, answer 1), (d) 2 (radii 1, integral with limits 1).

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SECTION II, PART B · FREE RESPONSE [NO CALCULATOR]

33. (a) On $4 < t < 6$ the graph is a line through $(4, 4)$ and $(6, -2)$, so $a(5) = v'(5) = \frac{-2-4}{6-4} = -3$ meters per second squared. Also $v(5) = 4 - 3 = 1 > 0$. Since $v(5) > 0$ and $a(5) < 0$ have opposite signs, the speed is decreasing at $t = 5$.

(b) $\int_0^6 v\,dt = \frac{1}{2}(2)(4) + (2)(4) + \frac{1}{2}\left(\frac{4}{3}\right)(4) - \frac{1}{2}\left(\frac{2}{3}\right)(2) = 4 + 8 + \frac{8}{3} - \frac{2}{3} = 14$, since $v = 0$ at $t = \frac{16}{3}$. Then $x(6) = 1 + 14 = 15$.

(c) Distance $= \int_0^8 |v|\,dt = 4 + 8 + \frac{8}{3} + \frac{2}{3} + (2)(2) = \frac{58}{3}$ meters. (The last term is the rectangle on $6 \le t \le 8$.)

(d) The velocity changes from positive to negative only at $t = \frac{16}{3}$. Candidates: $x(0) = 1$, $x\left(\frac{16}{3}\right) = 1 + 4 + 8 + \frac{8}{3} = \frac{47}{3}$, and $x(8) = 15 - 4 = 11$. The largest is $\frac{47}{3}$, so the particle is farthest to the right at $t = \frac{16}{3}$.

Scoring: 9 points: (a) 2 (acceleration 1, conclusion with reason 1), (b) 2 (integral setup 1, position 1), (c) 2 (use of |v| 1, value 1), (d) 3 (identifies t=16/3 with sign change 1, compares candidates 1, conclusion 1).

34. (a) $f'(x) = 0$ at $x = -2$ and $x = 1$ (the graph crosses the axis there). At $x = -2$, f' changes from negative to positive, so f has a relative minimum at $x = -2$. At $x = 1$, f' changes from positive to negative, so f has a relative maximum at $x = 1$.

(b) f' is decreasing on $(-1, 3)$ (the middle segment has slope -1), so $f'' < 0$ and the graph of f is concave down on $(-1, 3)$. The graph of f' changes from increasing to decreasing at $x = -1$ and from decreasing to increasing at $x = 3$, so f'' changes sign at both, and the points of inflection are at $x = -1$ and $x = 3$.

(c) $f(-2) = f(1) - \int_{-2}^{1} f'(x)\,dx = 3 - \left[\frac{1}{2}(1)(2) + \frac{1}{2}(2+0)(2)\right] = 3 - 3 = 0$. $f(5) = f(1) + \int_1^5 f'(x)\,dx = 3 + \left[-\frac{1}{2}(2)(2) - \frac{1}{2}(2)(2)\right] = 3 - 4 = -1$.

(d) Candidates: endpoints and critical points. $f(-3) = f(-2) - \int_{-3}^{-2} f'\,dx = 0 - (-1) = 1$, $f(-2) = 0$, $f(1) = 3$, $f(5) = -1$. Because f is continuous on $[-3, 5]$, the Extreme Value Theorem applies: the absolute minimum value is -1 (at $x = 5$) and the absolute maximum value is 3 (at $x = 1$).

Scoring: 9 points: (a) 2 (locations 1, classification with sign-change reasoning 1), (b) 2 (intervals 1, inflection points with reason 1), (c) 2 (one for each value), (d) 3 (finds f(-3) 1, compares candidates 1, correct min and max with justification 1).

35. (a) At $(2, 0)$, $\frac{dy}{dx} = 2e^0 = 2$. Tangent line: $y = 2(x - 2)$. Then $f(2.1) \approx 2(0.1) = 0.2$.

(b) $\frac{d^2y}{dx^2} = e^{-y} - xe^{-y}\frac{dy}{dx} = e^{-y} - x^2e^{-2y} = e^{-y}\left(1 - x^2e^{-y}\right)$. At $(2, 0)$, $\frac{d^2y}{dx^2} = 1(1 - 4) = -3 < 0$, so the graph of f is concave down near $x = 2$ and lies below its tangent line. The approximation 0.2 is greater than $f(2.1)$.

(c) Separate variables: $e^y\,dy = x\,dx$. Integrate: $e^y = \frac{x^2}{2} + C$. Use $f(2) = 0$: $1 = 2 + C$, so $C = -1$. Then $e^y = \frac{x^2}{2} - 1$ and $y = \ln\left(\frac{x^2}{2} - 1\right)$, for $x > \sqrt{2}$.

Scoring: 9 points: (a) 2 (tangent line 1, approximation 1), (b) 2 (second derivative 1, conclusion with reason 1), (c) 5 (separation 1, antiderivatives 2, constant 1, solution for y 1).

36. (a) Differentiate implicitly: $2y\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} + 4x = 0$. Then $(2y - 2x)\frac{dy}{dx} = 2y - 4x$, so $\frac{dy}{dx} = \frac{y-2x}{y-x}$.

(b) At $(1, 3)$: $\frac{dy}{dx} = \frac{3-2}{3-1} = \frac{1}{2}$. Tangent line: $y - 3 = \frac{1}{2}(x - 1)$.

(c) The tangent is horizontal when $\frac{dy}{dx} = 0$, so $y = 2x$ (with $y \ne x$). Substitute into the curve: $4x^2 - 4x^2 + 2x^2 = 5$, so $x^2 = \frac{5}{2}$ and $x = \pm\frac{\sqrt{10}}{2}$. At these points $y - x = x \ne 0$, so the slope is defined and equals 0.

(d) $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{1}{2} \cdot 4 = 2$.

Scoring: 9 points: (a) 3 (differentiates each term 2, solves 1), (b) 2 (slope 1, line 1), (c) 2 (sets numerator to 0 1, solves 1), (d) 2 (chain rule 1, value 1).

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