



SCHOLAR COACH

Boutique Academic Coaching & Tutoring

AP PHYSICS C: MECHANICS · COURSE REVIEW AND PRACTICE

AP Physics C: Mechanics **Mastery.**

Every unit on the exam: concise concept review, worked examples, exam-style writing, practice problems with full explanations, a mixed practice test, and a reference section.

INSIDE

Words in Context · Inferences · Command of Evidence · Central Ideas and Details · Function of a Sentence · Algebra · Advanced Math · Problem-Solving and Data Analysis · Geometry and Trigonometry · Three Full-Length Practice Exams with Explanations



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Course review, practice, and a mixed test.

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Start Here: The Exam, Your Plan, and Your Tools

How the AP Physics C: Mechanics exam is built, how points are earned, and how to use this book.

F Welcome to AP Physics C: Mechanics

Exam format, unit weighting, the three practices, how free-response points are earned, two study plans, and the math habits this course demands.

Why This Matters

AP Physics C: Mechanics is a calculus-based course: you are expected to differentiate and integrate, to separate variables in a simple differential equation, and to turn a physical situation into a symbolic derivation. This front section tells you how the exam is built, where the points are, and how to use this book over four or eight weeks. Check the College Board course page for the latest details, because administration and scoring information can change.

Exam Format

The exam is on Monday, May 3, 2027 (Session 1) and is hybrid digital: multiple-choice answers are entered on a device and free-response answers are written by hand in a booklet. Both sections allow a calculator, and a reference sheet is provided; check the current sheet in the course materials because conventions and wording can be updated.

Section	Questions	Time	Share of score
I: Multiple choice	42	85 minutes	50%
II: Free response	4	95 minutes	50%

The 2027 update changed the multiple-choice count (from 40 to 42 questions) and shortened both timings; course content and skills did not change. Older books that describe three free-response questions are out of date.

Unit Weighting

Unit	Title	Share of multiple choice
1	Kinematics	10-15%
2	Force and Translational Dynamics	20-25%
3	Work, Energy, and Power	15-25%
4	Linear Momentum	10-20%
5	Torque and Rotational Dynamics	10-15%
6	Energy and Momentum of Rotating Systems	10-15%
7	Oscillations	10-15%

Practices and Free-Response Types

The course names three practices: **Creating Representations** (diagrams, graphs with scales and units, qualitative sketches), **Mathematical Routines** (derive a symbolic expression, calculate with units, compare quantities, predict how a result changes), and **Scientific Questioning and Argumentation** (design a procedure, apply a law to make a claim, justify a claim with data or principles). The four free-response questions appear in a fixed order:

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Question	Type	Points	What it rewards
1	Mathematical Routines	10	Derivations and calculus: integrals, differentials, symbolic results
2	Translation Between Representations	12	Moving between graphs, equations, diagrams and words
3	Experimental Design and Analysis	10	Procedures, linearized graphs, slope meaning, uncertainty
4	Qualitative/Quantitative Translation	8	Claims argued in words, then supported with a derivation

How points are earned

Show a **path**: start from a principle (Newton's second law, work-energy theorem, conservation of energy or momentum), then manipulate symbolically. A list of unrelated equations is not a derivation.

Define variables you introduce and keep the algebra symbolic until the end; equivalent definitions are accepted when clearly stated.

Justify by linking a physics quantity to the claim ('the net force is zero, so the acceleration is zero'). Restating a formula is not justification.

For graphs: label axes with quantities and units, use a sensible scale, draw a best-fit line, and tie the slope or intercept to the unknown. For sketches say whether each feature rises or falls.

Math Habits That Earn Credit

Write limits on every definite integral and carry initial conditions. Use units to catch mistakes: if the answer to a time question has units of meters, stop. When a force depends on velocity, write $m dv/dt = \sum F$ and separate variables. Linearize experiments: if $v^2 = 2gh$, plot v^2 versus h and the slope is $2g$. Check limiting cases: set a coefficient to zero or a variable to a large value and see whether the formula matches common sense. Use $g = 9.8 \text{ m/s}^2$ in this book; the exam uses 10 m/s^2 in its numerical problems and does not penalize 9.8 or 9.81.

Study Plans

Week	Focus	Do in this book
1	Units 1 and 2	Read the lessons, do both practice sets, redo missed items
2	Units 3 and 4	Practice sets; derive every result in Unit 3 without notes
3	Units 5 and 6	Practice sets; draw an FBD for every rotation problem
4	Unit 7, review	Practice set, mixed test, free-response clinic, glossary

Week	Focus	Do in this book
1	Unit 1	Calculus kinematics and projectile practice
2	Unit 2, first half	FBDs, Newton's laws, friction, springs
3	Unit 2, second half	Drag and circular motion, gravitation
4	Unit 3	Work, energy, power, $U(x)$ graphs
5	Unit 4	Impulse, momentum, collisions
6	Units 5 and 6	Torque, rotation, rolling, angular momentum
7	Unit 7	SHM, energy, pendulums
8	Review	Mixed test, free-response clinic, weak topics

How to Use This Book

Each unit opens with the ideas the exam tests, then gives four worked examples, a model free-response answer with a grading checklist, a playbook of habits and traps, and a practice set of ten problems (six multiple choice and four free response). Cover the solution and try each example first. For free-response practice, write your answer on paper as you would on exam day, then compare with the point-by-point answer. Keep an error log: for every miss, write the principle you should have started from.

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Practice

WARM-UP Front Practice

Show your work. A calculator is allowed.

- A particle moves along a line with velocity $v = 3t^2$ (SI units). What is its displacement between $t = 0$ and $t = 4$ s?
 (A) 12 m (B) 24 m (C) 48 m (D) 64 m
- A ball is dropped from several heights h and its speed v just before landing is measured. Which graph gives a straight line whose slope equals $2g$?
 (A) v versus h (B) v versus the square root of h
 (C) v squared versus h (D) h versus v
- A 2.0 kg block starts from rest on a frictionless surface and is pushed by a net force $F = 6.0t$ N.
 (a) Derive $v(t)$ and $x(t)$.
 (b) Calculate the work done by the force from $t = 0$ to $t = 3.0$ s using the work-energy theorem.
 (c) Verify the result by integrating $\int F dx$.
- A student has a spring, a ruler, a set of known masses and a stand. Describe an experiment to determine the spring constant, state what to plot to obtain a straight line, and explain how the constant is obtained from the graph and why the line is expected to pass through the origin (or not).

Answers and Explanations

F Welcome to AP Physics C: Mechanics

WARM-UP · FRONT PRACTICE

- D. Displacement is the integral of velocity: $\int_0^4 3t^2 dt = t^3 \Big|_0^4 = 64$ m. Choice C uses $v = 3t^2$ at $t = 4$ s (48 m/s) as though it were a displacement.
- C. From $v^2 = 2gh$, a plot of v^2 (vertical) against h (horizontal) is linear through the origin with slope $2g$. Choice A would curve; choice B has slope $\sqrt{2g}$, not $2g$.
- (a) $a = F/m = 3.0t$; $v = \int a dt = 1.5t^2$; $x = \int v dt = 0.5t^3$. (b) $v(3) = 13.5$ m/s, so $W = \Delta K = \frac{1}{2}(2.0)(13.5)^2 = 182$ J. (c) $\int F dx = \int_0^3 F v dt = \int_0^3 (6t)(1.5t^2) dt = 2.25 t^4 \Big|_0^3 = 182$ J, in agreement. Points: (a) 2, (b) 2, (c) 2.
- Procedure: hang the spring from the stand, measure the relaxed length, add one mass at a time, wait for equilibrium and record the length; compute the extension x for each mass. At equilibrium $kx = mg$. Plot weight mg (or m) versus x ; the slope of the best-fit line is k (or k/g if m is plotted). Repeat for several masses and average repeated measurements. The line should pass through the origin if the extension is measured from the relaxed length; an intercept appears if the spring has a preload or if the length rather than the extension is used. Points: procedure 2, graph 1, slope 1.

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Unit 1: Kinematics

Exam weighting: about 10-15% of the multiple-choice section. Concept review, worked examples, exam-style writing, and 10 practice items, all with full explanations.

1 Kinematics

Calculus kinematics: velocity as a derivative, displacement as an integral, graphs, relative motion, and two-dimensional projectile motion with clear sign conventions.

Why This Matters

Kinematics is 10-15% of the multiple-choice section, but it is also the first place the course asks you to use calculus as a language: velocity is a derivative, displacement is an integral, and a graph is a statement about both. Free-response questions reward a visible path from a starting relationship to a result, so a solution that jumps from $a(t)$ to a number earns less than one that writes $v = v_0 + \int a \, dt$ first. Students also lose points by applying constant-acceleration formulas when a depends on time, by reading the height of a graph when the slope or area was needed, and by using the level-ground range formula for a projectile that lands at a different height. This book uses $g = 9.8 \, \text{m/s}^2$; the exam's default is $10 \, \text{m/s}^2$ and a 9.8 or 9.81 answer is accepted. Air resistance is negligible and reference frames are inertial unless a problem says otherwise.

Core Ideas

1.1 Scalars and Vectors

A **scalar** has magnitude only (distance, speed, time, mass, energy). A **vector** has magnitude and direction (position, displacement, velocity, acceleration, force). In two dimensions write a vector with components, $\vec{A} = A_x \hat{x} + A_y \hat{y}$, where $A_x = A \cos \theta$ and $A_y = A \sin \theta$ for an angle θ measured from the $+x$ axis. The magnitude is $A = \sqrt{A_x^2 + A_y^2}$ and the direction is $\tan \theta = A_y/A_x$, with the quadrant checked by the signs. Vectors add component by component, which is why every two-dimensional problem in this unit is solved one axis at a time. In one dimension a sign carries the direction, so state your origin and positive direction before computing anything.

1.2 Displacement, Velocity, and Acceleration

Displacement $\Delta x = x_f - x_i$ depends only on the endpoints; distance adds the length of every leg. Average velocity is $\Delta x/\Delta t$. The **instantaneous** quantities are limits, which is the calculus the course builds on.

Calculus definitions of motion

$$v = \frac{dx}{dt} \text{ and } a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$v(t) = v_0 + \int_0^t a \, dt' \text{ and } x(t) = x_0 + \int_0^t v \, dt'$$

Displacement from t_1 to t_2 is $\int_{t_1}^{t_2} v \, dt$; the distance traveled is $\int_{t_1}^{t_2} |v| \, dt$.

When a is given as a function of position, use $a = v \frac{dv}{dx}$ to get $\int v \, dv = \int a \, dx$.

Integration needs initial conditions: always carry v_0 and x_0 (the constants of integration).

If a is **constant**, these integrals reduce to the familiar model $v = v_0 + at$, $\Delta x = v_0 t + \frac{1}{2}at^2$ and $v^2 = v_0^2 + 2a\Delta x$. That model is a special case, not a law. If a problem gives $x(t)$ as a polynomial, an exponential, or a sine, the acceleration is almost never constant and the formulas above must not be used. An object speeds up when v and a have the same sign and slows down when they have opposite signs; zero velocity does not imply zero acceleration (the top of a vertical toss has $v = 0$ and $a = -g$).

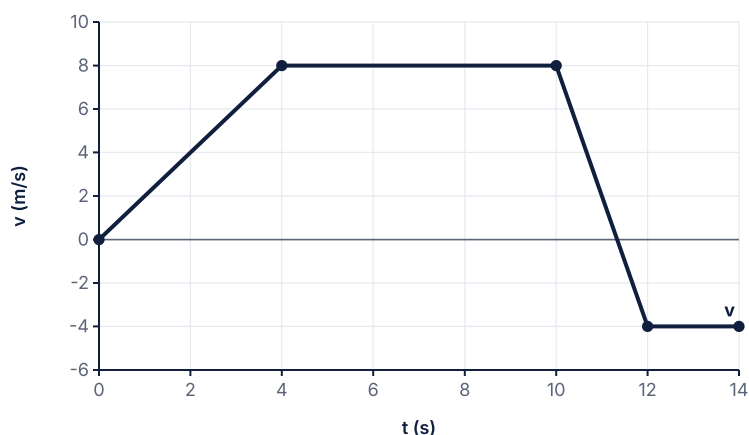
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1.3 Representing Motion

One motion can be shown with words, a motion map, equations, or graphs, and the exam moves freely among them. The slope of an x - t graph is v , the slope of a v - t graph is a , the area under a - t is Δv and the area under v - t is Δx . Area below the axis counts as negative displacement. A curved x - t graph means the velocity is changing; concave up means $a > 0$ and concave down means $a < 0$. When you sketch a derived graph, mark where it crosses zero (where the original has a maximum, minimum or inflection) and say whether each segment rises or falls.



A piecewise v - t graph: acceleration is its slope, displacement its signed area.

1.4 Reference Frames and Relative Motion

Velocity depends on the observer. If $\vec{v}_{A/B}$ is the velocity of A as measured in the frame of B , then $\vec{v}_{A/C} = \vec{v}_{A/B} + \vec{v}_{B/C}$. A boat's velocity relative to the ground equals its velocity relative to the water plus the water's velocity relative to the ground. Acceleration is the same in all frames moving at constant velocity relative to one another, which is why the laws of motion keep their form in any inertial frame. The most common error is adding speeds when the vectors are not parallel; draw the vector triangle and treat each component separately.

1.5 Motion in Two Dimensions

With position $\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y}$, velocity and acceleration are the derivatives of each component. For **projectile motion** the horizontal acceleration is zero and the vertical acceleration is $-g$, so the two axes are independent and share only the time. With launch speed v_0 at angle θ above the horizontal: $x = (v_0 \cos \theta) t$ and $y = y_0 + (v_0 \sin \theta) t - \frac{1}{2}gt^2$. At the highest point $v_y = 0$ but v_x is unchanged, so the speed there is $v_0 \cos \theta$, not zero.

Projectile checklist (no air resistance)

Horizontal: $a_x = 0$, $v_x = v_0 \cos \theta$ constant, $x = v_x t$.

Vertical: $a_y = -g$, $v_y = v_0 \sin \theta - gt$, $y = y_0 + v_0 \sin \theta t - \frac{1}{2}gt^2$.

Level ground: $R = \frac{v_0^2 \sin 2\theta}{g}$, maximum at 45° ; a launch angle θ and $90^\circ - \theta$ give the same range.

Different launch and landing heights: solve $y(t) = y_{\text{final}}$ for t first (quadratic), never use R above.

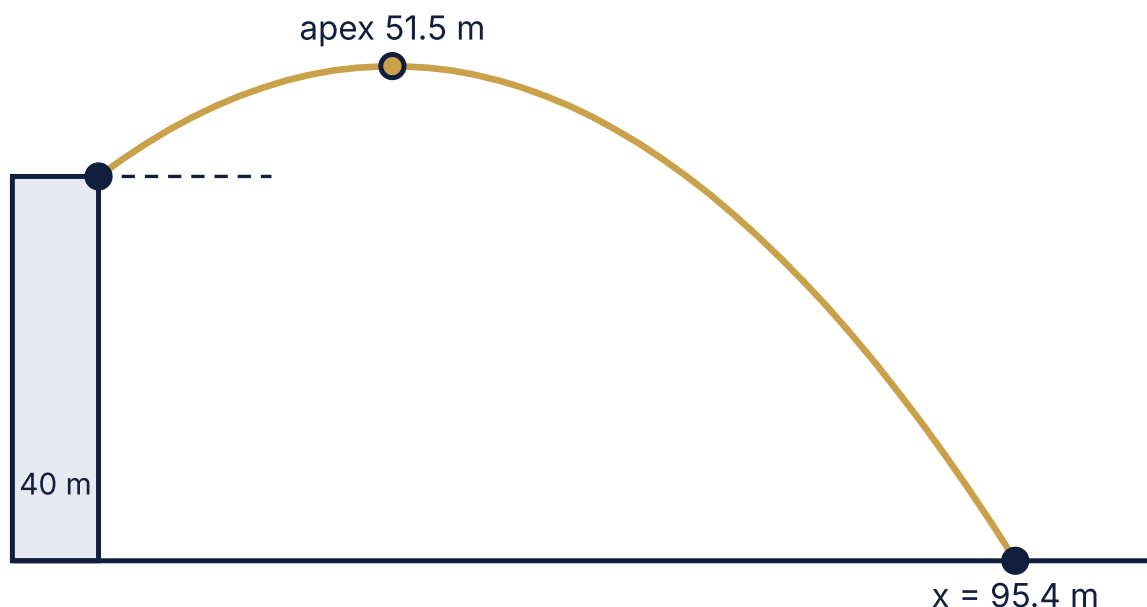
A projectile's path is a parabola because eliminating t gives y as a quadratic function of x . Notice that the velocity vector is tangent to the path while the acceleration vector is always the same downward vector $-g\hat{y}$; the component of acceleration parallel to the velocity changes the speed, and the component perpendicular to it changes the direction. The cliff launch below is worked in Example 3.

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Cliff launch: 25 m/s at 37 deg above horizontal



Path for Example 3. The trajectory is a parabola; the horizontal spacing of equal time steps is constant.

Worked Examples

EXAMPLE 1: CALCULUS KINEMATICS FROM $x(t)$

A particle moves along the x axis with $x(t) = t^3 - 9t^2 + 15t + 4$ (SI units). (a) Find $v(t)$ and $a(t)$. (b) When is the particle momentarily at rest? (c) Find its displacement and the distance it travels from $t = 0$ to $t = 6$ s.

SOLUTION

(a) Differentiate: $v = dx/dt = 3t^2 - 18t + 15$ and $a = dv/dt = 6t - 18$.

(b) Set $v = 0$: $3(t - 1)(t - 5) = 0$, so $t = 1$ s and $t = 5$ s. The velocity changes sign at each, so the particle reverses direction.

(c) Positions: $x(0) = 4$ m, $x(1) = 11$ m, $x(5) = -21$ m, $x(6) = -14$ m. Displacement $= -14 - 4 = -18$ m. Distance must add each leg because the direction reverses: $|11 - 4| + |-21 - 11| + |-14 - (-21)| = 7 + 32 + 7 = 46$ m. Distance (46 m) exceeds the magnitude of displacement (18 m) exactly because of the two turning points.

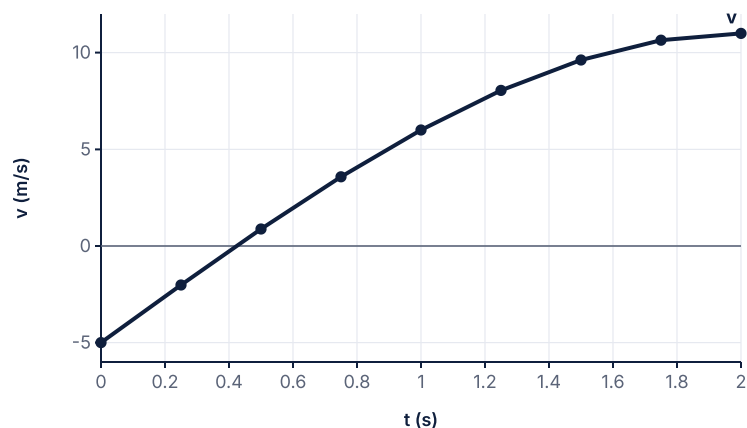
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EXAMPLE 2: INTEGRATING A(T)

A cart on a track has $a(t) = 12 - 3t^2$ (SI units), $v(0) = -5.0$ m/s and $x(0) = 2.0$ m. Find $v(t)$, $x(t)$, the velocity at $t = 2.0$ s, and the displacement during the first 2.0 s.



$v(t) = -5 + 12t - t^3$ for the first 2 s.

SOLUTION

Integrate once with the initial condition: $v(t) = -5 + \int_0^t (12 - 3t'^2) dt' = -5 + 12t - t^3$.

Integrate again: $x(t) = 2 + \int_0^t (-5 + 12t' - t'^3) dt' = 2 - 5t + 6t^2 - \frac{1}{4}t^4$.

At $t = 2.0$ s: $v = -5 + 24 - 8 = +11$ m/s. Displacement $= x(2) - x(0) = (2 - 10 + 24 - 4) - 2 = +10$ m.

Check: the cart starts moving backward, turns around when $v = 0$ (a little before 0.5 s), then moves forward, so a positive net displacement is reasonable. Constant-acceleration formulas would have been wrong here because a changes.

EXAMPLE 3: PROJECTILE FROM A CLIFF

A ball is launched from the edge of a 40 m cliff at 25 m/s, 37 degrees above the horizontal ($\sin 37^\circ = 0.60$, $\cos 37^\circ = 0.80$). Find the time of flight, the horizontal distance to the landing point, the maximum height above the ground, and the speed at impact.

SOLUTION

Components: $v_{0x} = 20$ m/s, $v_{0y} = 15$ m/s. Take up and right as positive, origin at the base of the cliff.

Landing: $0 = 40 + 15t - 4.9t^2$. By the quadratic formula $t = \frac{15 + \sqrt{15^2 + 4(4.9)(40)}}{9.8} = 4.77$ s (the negative root is rejected).

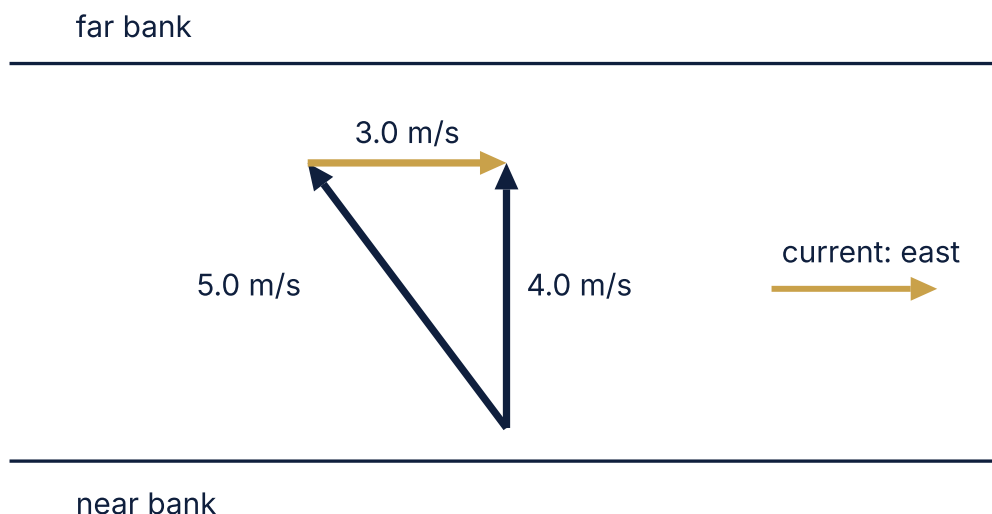
Range: $x = 20(4.77) = 95.4$ m. Maximum height: at the apex $v_y = 0$ after 1.53 s, rising $15^2 / (2 \cdot 9.8) = 11.5$ m above the launch point, so 51.5 m above the ground.

Impact: $v_y = 15 - 9.8(4.77) = -31.8$ m/s and $v_x = 20$ m/s, so speed $= \sqrt{20^2 + 31.8^2} = 37.5$ m/s at an angle of 58° below the horizontal. Energy gives the same speed: $\sqrt{25^2 + 2(9.8)(40)} = 37.5$ m/s (see Unit 3).

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Example 4(b): the vector sum of the boat's velocity relative to the water and the water's velocity relative to the ground points straight across.

EXAMPLE 4: RELATIVE MOTION

A boat can move at 5.0 m/s relative to the water. A river 80 m wide flows east at 3.0 m/s. (a) If the boat points straight north (across), how long does it take to cross and how far downstream does it land? (b) At what angle to the north should it point to land directly across, and how long does that take?

SOLUTION

(a) The crossing uses only the north component: $t = 80/5.0 = 16$ s. The current carries it east $3.0 \times 16 = 48$ m.

(b) To cancel the current, the boat's westward (upstream) component must equal 3.0 m/s: $5.0 \sin \phi = 3.0$, so $\phi = 37^\circ$ west of north. The northward component is then $5.0 \cos \phi = 4.0$ m/s and $t = 80/4.0 = 20$ s.

Pointing straight across is faster but sacrifices position; the path that lands directly opposite costs 4 s. Note $v_{\text{boat/ground}} = v_{\text{boat/water}} + v_{\text{water/ground}}$ as vectors, so the speed over the ground in (b) is 4.0 m/s, not 2.0 or 8.0 m/s.

Write It Like a 5

Model Response

Prompt: A drone moves along a straight line with velocity $v(t) = 3.0t^2 - 12t$ (SI units) and $x(0) = 5.0$ m. (a) Derive the acceleration. (b) Determine when the drone changes direction and justify that it does. (c) Calculate its displacement and the distance traveled from $t = 0$ to $t = 6.0$ s. (a) $a = dv/dt = 6.0t - 12$ m/s². (b) Setting $v = 3t(t - 4) = 0$ gives $t = 0$ and $t = 4.0$ s. For $0 < t < 4$ both factors have opposite signs so $v < 0$; for $t > 4$, $v > 0$. Since v changes sign at $t = 4.0$ s, the drone reverses direction there, and $a(4) = +12$ m/s² shows it is speeding up in the new direction. (c) Position: $x(t) = 5 + \int_0^t v dt' = 5 + t^3 - 6t^2$. Then $x(4) = -27$ m and $x(6) = 5$ m. Displacement = $x(6) - x(0) = 0$. Distance = $|x(4) - x(0)| + |x(6) - x(4)| = 32 + 32 = 64$ m. The displacement is zero because the drone returns to its start, but it traveled 64 m.

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What the grader looks for

(1) The derivative is shown or stated as $a = dv/dt$, with units. (2) The direction change is justified by a sign change of v and not by $a = 0$. (3) Position is found by integrating with the initial condition x_0 , or the integral of v is evaluated over each leg. (4) Distance is computed leg by leg with absolute values, and the answer distinguishes distance from displacement. (5) Units and signs are consistent throughout.

Coach's Playbook

- 1 Write the coordinate system first: origin, positive direction, and the sign of g .
- 2 Ask what the function is: if a varies, integrate; do not reach for $v = v_0 + at$.
- 3 Insert limits or initial conditions every time you integrate, then check the units of the result.
- 4 For a graph question decide whether the answer is a slope, an area, or a value at a point.
- 5 For projectiles separate the axes; carry only the time between them, and solve for the landing time from the actual landing height.
- 6 Check the story: sign of v against the direction of motion, speeding up or slowing down against the signs of v and a .

Common traps

(1) Applying the constant-acceleration equations when $a(t)$ or $x(t)$ is a polynomial. (2) Reporting the net displacement when the question asks for distance, or integrating v without splitting at turning points. (3) Saying the acceleration is zero at the top of a toss because the velocity is zero, or that the speed of a projectile is zero at its highest point. (4) Using $R = v_0^2 \sin 2\theta / g$ for a launch from a cliff. Also: a justification that merely restates a formula earns little; link the physics quantity to the claim.

Quick Check

Q1. What does the area under an a - t graph give? A: the change in velocity.
 Q2. A particle has $x = 4t - t^2$. Is the acceleration constant, and what is it? A: yes, $a = -2 \text{ m/s}^2$ (second derivative).
 Q3. A ball is thrown horizontally at 6 m/s from a table. What is its horizontal velocity just before landing? A: still 6 m/s, because $a_x = 0$.

Challenge Corner

A particle starts at rest at the origin with acceleration that depends on position, $a(x) = 8.0 - 2.0x$ (SI units), for $x \geq 0$. Find its maximum speed and the largest x it reaches. Use $a = v dv/dx$: $\int_0^v v dv = \int_0^x (8 - 2x') dx'$, so $\frac{1}{2}v^2 = 8x - x^2$ and $v = \sqrt{16x - 2x^2}$. Maximum speed occurs where $a = 0$, at $x = 4.0 \text{ m}$: $v_{\max} = \sqrt{64 - 32} = 5.66 \text{ m/s}$. The particle stops when $v = 0$: $16x = 2x^2$, so $x = 8.0 \text{ m}$. At that point $a = -8.0 \text{ m/s}^2$ (not zero), so it then moves back toward the origin: the motion is oscillatory, a preview of Unit 7.

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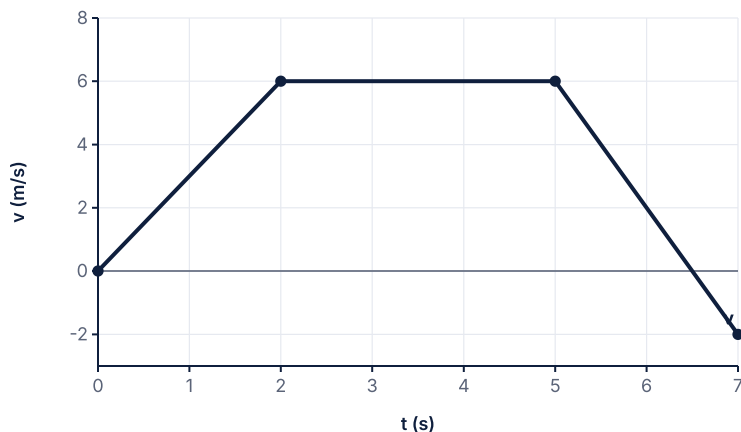
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Practice

PRACTICE Unit 1 Practice

Show your work. A calculator is allowed. Justify conclusions.

1. The velocity of a cart along a line is shown. What is the cart's displacement from $t = 0$ to $t = 7$ s?



Velocity of the cart.

- (A) 28 m (B) 29 m (C) 30 m (D) 36 m
2. A particle's position is $x = t^3 - 6t^2 + 9t$ (SI units). What is its velocity at the instant its acceleration is zero?
- (A) -3 m/s (B) 0 (C) +3 m/s (D) +9 m/s
3. A particle starts from rest with acceleration $a = 6e^{-2t}$ m/s², with t in seconds. What speed does it approach as $t \rightarrow \infty$?
- (A) 1.5 m/s (B) 3 m/s
(C) 6 m/s (D) It increases without limit
4. A ball rolls off a table 20 m above the floor with a horizontal speed of 12 m/s. How far from the table's edge, horizontally, does it land? Air resistance is negligible.
- (A) 17 m (B) 24 m (C) 49 m (D) 98 m
5. A plane flies with an airspeed of 100 m/s due north. A steady wind blows toward the east at 30 m/s. What is the plane's speed relative to the ground?
- (A) 70 m/s (B) 100 m/s (C) 104 m/s (D) 130 m/s
6. A particle's position vector is $\vec{r} = (2t^2)\hat{x} + (3t^3)\hat{y}$ (SI units). What is its acceleration at $t = 2$ s?
- (A) $8\hat{x} + 36\hat{y}$ m/s² (B) $4\hat{x} + 36\hat{y}$ m/s² (C) $4\hat{x} + 18\hat{y}$ m/s²
(D) $4\hat{x} + 54\hat{y}$ m/s²
7. A particle moves along the x axis with acceleration $a(t) = 12t - 3t^2$ (SI units) for $0 \leq t \leq 4.0$ s. At $t = 0$, $v = 2.0$ m/s and $x = 0$.
- (a) Derive expressions for $v(t)$ and $x(t)$.
(b) Determine the time at which the particle's speed is greatest, and calculate that speed.
(c) Calculate the position at $t = 4.0$ s.
(d) Sketch $a(t)$ and $v(t)$ on the same time axis, labeling the value of a at $t = 2.0$ s and explaining where $a(t)$ is a maximum.
8. The velocity of an elevator (up positive) is 0 at $t = 0$, increases linearly to +8 m/s at $t = 4$ s, stays at +8 m/s until $t = 10$ s, decreases linearly to -4 m/s at $t = 12$ s, and stays at -4 m/s until $t = 14$ s. The elevator starts at position $x = 5.0$ m.
- (a) Calculate the acceleration during each interval.
(b) Calculate the displacement from $t = 0$ to $t = 14$ s and the final position.
(c) Sketch the position-versus-time graph, labeling its shape in each interval (straight, curved up, curved down) and the time of its maximum.
(d) Is the elevator ever momentarily at rest after it starts moving? Justify and give its acceleration then.

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9. A student releases a low-friction cart from rest on a gently tilted track and records its distance x from the starting point at several times t (a motion sensor is used; assume constant acceleration). Data: $t = 0.5, 1.0, 1.5, 2.0, 2.5$ s give $x = 0.105, 0.424, 0.937, 1.635, 2.589$ m.
- Explain how to plot these data so that the graph is a straight line, and state what the slope represents.
 - Using the data, calculate the slope of the best-fit line (the line through the origin and the last point is a rough check) and the cart's acceleration.
 - The best-fit line has a small positive intercept of 0.005 m. Give a possible experimental reason and say what it does to the acceleration you found.
 - Describe one way to reduce the uncertainty in the acceleration without changing the track angle.
10. A projectile is launched from level ground with speed v_0 at angle θ above the horizontal. Air resistance is negligible.
- Derive an expression for the range R in terms of v_0 , θ and g .
 - Use calculus to show that the range is a maximum for $\theta = 45^\circ$.
 - For $v_0 = 30$ m/s, find the two launch angles that give a range of 70 m.
 - For the two angles in (c), which projectile has the greater time of flight? Justify using the vertical component of the launch velocity.

Answers Unit 1: Answers and Explanations

Read every explanation, including the ones for problems you answered correctly.

1 Kinematics

PRACTICE · UNIT 1 PRACTICE

- A. Signed areas: 6 m (0–2 s) + 18 m (2–5 s), then from 5 s the velocity falls to zero at 6.5 s (area +4.5 m) and reverses (area -0.5 m), total 28 m. Choice B (29 m) is the distance, which counts the last triangle as positive.
- A. $a = 6t - 12 = 0$ at $t = 2$ s, where $v = 3t^2 - 12t + 9 = 12 - 24 + 9 = -3$ m/s. Choice B confuses zero acceleration with zero velocity; D is the velocity at $t = 0$.
- B. $v = \int_0^\infty 6e^{-2t} dt = 3$ m/s. Choice C uses the initial acceleration as a speed; choice D assumes a positive acceleration must keep increasing the speed without bound, but the acceleration decays so the integral converges.
- B. The fall time is $\sqrt{2(20)/9.8} = 2.02$ s, so $x = 12(2.02) = 24$ m. Choice C comes from using $t = 2h/g = 4.1$ s, forgetting the square root.
- C. The velocity components are perpendicular, so the ground speed is $\sqrt{100^2 + 30^2} = 104$ m/s. Choice D adds the speeds as though the two velocities were parallel.
- B. $\vec{a} = d^2\vec{r}/dt^2 = 4\hat{x} + 18t\hat{y}$, which at $t = 2$ s is $4\hat{x} + 36\hat{y}$. Choice A is the x-component of the velocity, $4t$, not of the acceleration; choice D multiplies the y-component by an extra factor of 3.
- (a) $v = 2 + \int_0^t (12t' - 3t'^2) dt' = 2 + 6t^2 - t^3$; $x = \int_0^t v dt' = 2t + 2t^3 - t^4/4$. (b) v has a maximum where $a = 0$ and a changes from positive to negative: $12t - 3t^2 = 0$ gives $t = 4.0$ s (the end of the interval), $v = 2 + 96 - 64 = 34$ m/s. Velocity rises throughout since $a \geq 0$ on the interval. (c) $x(4) = 8 + 128 - 64 = 72$ m. (d) $a(t)$ is a downward parabola from 0 at $t = 0$, peaking at $t = 2.0$ s where $da/dt = 12 - 6t = 0$, $a = 24 - 12 = 12$ m/s², and returning to 0 at 4.0 s. $v(t)$ starts at 2 m/s, has the steepest slope at $t = 2$ s (the inflection), and increases with zero slope at $t = 4$ s. Points: part (a) 2, (b) 2, (c) 1, (d) 2.
- (a) Slopes: 0–4 s: +2 m/s²; 4–10 s: 0; 10–12 s: $-12/2 = -6$ m/s²; 12–14 s: 0. (b) Areas: $16 + 48 = 64$ m; from 10 s the velocity crosses zero at $10 + 2(8/12) = 11.33$ s: positive triangle $\frac{1}{2}(8)(1.33) = 5.33$ m, negative triangle $-\frac{1}{2}(4)(0.67) = -1.33$ m; last interval $-4 \times 2 = -8$ m. Displacement = $64 + 5.33 - 1.33 - 8 = 60$ m, so the final position is 65 m. (c) x - t : concave up (parabola) from 0 to 4 s; straight line of slope +8 m/s from 4 to 10 s; concave down from 10 to 12 s with a maximum at $t = 11.33$ s; straight line of slope -4 m/s from 12 to 14 s. (d) Yes, at $t = 10 + 2(8/12) = 11.33$ s the velocity passes through zero, where the position is maximum; the acceleration there is -6 m/s², so it is at rest only instantaneously (zero velocity does not mean zero acceleration). Points: (a) 2, (b) 3, (c) 3, (d) 2.
- (a) For constant acceleration from rest, $x = \frac{1}{2}at^2$, so plot x on the vertical axis against t^2 on the horizontal axis ($t^2 = 0.25, 1.0, 2.25, 4.0, 6.25$ s²); the slope is $a/2$. (b) A least-squares fit gives slope = 0.41 m/s² (the endpoint check $2.589/6.25 = 0.41$ agrees), so $a = 2(0.41) = 0.82$ m/s². (c) A nonzero intercept suggests the clock and position zero do not coincide with the release (or the cart had a tiny initial speed); it shifts the fit but changes the slope only slightly, so a is essentially unchanged. (d) Repeat the release several times at each distance and average, use a larger range of times, or use a longer track; a single fit through many points reduces random error. Points: (a) 2, (b) 2, (c) 1, (d) 1.
- (a) Time of flight from $y = 0$: $v_0 \sin \theta t - \frac{1}{2}gt^2 = 0$ gives $t = 2v_0 \sin \theta / g$. Then $R = v_0 \cos \theta t = 2v_0^2 \sin \theta \cos \theta / g = v_0^2 \sin 2\theta / g$. (b) $dR/d\theta = 2v_0^2 \cos 2\theta / g = 0$ gives $2\theta = 90^\circ$, so $\theta = 45^\circ$; $d^2R/d\theta^2 = -4v_0^2 \sin 2\theta / g < 0$ there, so it is a maximum, $R_{\max} = v_0^2 / g$. (c) $\sin 2\theta = Rg/v_0^2 = 70(9.8)/900 = 0.762$, so $2\theta = 49.7^\circ$ or 130.3° , giving $\theta = 24.8^\circ$ and 65.2° . (d) The 65.2° launch has the larger vertical component ($v_0 \sin \theta$), the same horizontal travel takes longer because v_x is smaller, and $t = 2v_0 \sin \theta / g$ is larger: 5.55 s versus 2.57 s. Points: (a) 3, (b) 2, (c) 2, (d) 1.

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Unit 2: Force and Translational Dynamics

Exam weighting: about 20-25% of the multiple-choice section. Concept review, worked examples, exam-style writing, and 10 practice items, all with full explanations.

2 Force and Translational Dynamics

Systems and center of mass, free-body diagrams, Newton's laws, gravitation, friction, springs, drag forces solved by separation of variables, and circular motion.

Why This Matters

Force and Translational Dynamics is the largest single unit in multiple choice (20-25%), and it feeds almost every later unit. Free-response questions here reward a clean free-body diagram, a correct component equation from Newton's second law, and increasingly a short calculus argument: the 2026 scoring guidelines for this course included a resistive-force problem solved by separation of variables. Students lose points by drawing the net force as an extra arrow, by assuming static friction equals its maximum value, by using the wrong normal force on an incline, and by writing a justification that restates an equation without linking it to the claim. Numerical work uses $g = 9.8 \text{ m/s}^2$ (the exam default is 10 m/s^2 ; 9.8 or 9.81 is accepted).

Core Ideas

2.1 Systems and Center of Mass

A **system** is the set of objects you choose to analyze; forces from objects outside the system are external, and forces between its members are internal and cancel in pairs. The **center of mass** is the mass-weighted average position. For point masses $x_{cm} = \sum m_i x_i / \sum m_i$; for a continuous object $x_{cm} = \frac{1}{M} \int x dm$, where $dm = \lambda dx$ for an object with linear mass density λ . The center of mass moves as if all the mass were there and all external forces acted there: $\vec{F}_{net,ext} = M\vec{a}_{cm}$. It need not lie inside the object. The rod below has a density that increases to the right, so its center of mass is displaced toward the denser end (Example 2).

2.2 Forces and Free-Body Diagrams

A **free-body diagram** (FBD) shows the object as a dot or box and draws one arrow for every force exerted on it by something else: weight (gravitational force, mg , toward Earth), normal force N (perpendicular to a surface), friction (parallel to a surface), tension T (along a string, away from the object), spring force, applied forces and drag. Do not draw the net force, components, or forces the object exerts on others. Label each arrow with its type and agent, and draw the lengths roughly in proportion when the object is in equilibrium or the motion is known.

Procedure for a dynamics problem

1. Choose the system and draw the FBD.
2. Choose axes (along and perpendicular to the acceleration, e.g., parallel to an incline) and resolve forces into components.
3. Write $\sum F_x = ma_x$ and $\sum F_y = ma_y$ with signs set by your axes.
4. Add constraint equations (a rope makes two accelerations equal in size) and force laws ($f_k = \mu_k N$, $F_s = -kx$).
5. Solve symbolically, then substitute; check units and limiting cases.

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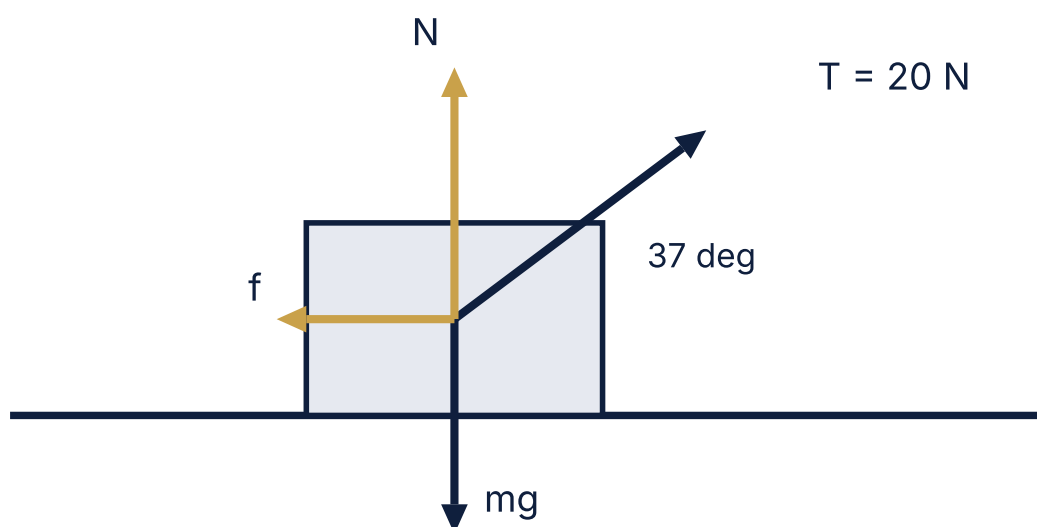
2.3 Newton's Third Law and 2.4 Newton's First Law

Forces come in **interaction pairs**: if A pushes on B with $\vec{F}$, B pushes on A with $-\vec{F}$. The two forces have equal magnitude, opposite direction, the same type, and act on *different* objects, so they never cancel in one FBD. The weight of a book and the table's normal force are *not* a third-law pair (they act on the same object and are different kinds of force); the pair of the weight is the book's gravitational pull on Earth. **Newton's first law** says that an object with zero net force keeps its velocity (which may be zero). Equilibrium means $\vec{a} = 0$, not $\vec{v} = 0$.

2.5 Newton's Second Law

In its general form $\sum \vec{F} = d\vec{p}/dt$, which reduces to $\sum \vec{F} = m\vec{a}$ when mass is constant. The net force is a vector; its components along each axis give separate equations. If the net force depends on time, $a = F(t)/m$ is integrated exactly as in Unit 1. The figure shows a 4.0 kg block pulled by a rope at 37 degrees above the horizontal on a floor; the vertical equation is $N + T \sin 37^\circ - mg = 0$, so the normal force is *less* than the weight, and the horizontal equation determines the acceleration.

Free-body diagram: block pulled by a rope



The four forces on the block: tension (at 37 degrees), normal force, weight, and friction opposing the motion to the right.

2.6 Gravitational Force

Two point masses (or spherically symmetric bodies) attract with a force of magnitude $F_G = Gm_1m_2/r^2$, where r is the distance between centers. The **gravitational field strength** at distance r from a body of mass M is $g = GM/r^2$; it is the free-fall acceleration there. At Earth's surface this gives 9.8 N/kg, and it falls as $1/r^2$ with distance from Earth's *center*. For a circular orbit the gravitational force supplies the centripetal force, so $GMm/r^2 = mv^2/r$ and $v = \sqrt{GM/r}$; the period satisfies $T^2 = 4\pi^2 r^3/(GM)$. Weightlessness in orbit means falling freely (no normal force), not an absence of gravity.

2.7 Kinetic and Static Friction

Kinetic friction opposes the sliding of surfaces and has magnitude $f_k = \mu_k N$, independent of speed and contact area. **Static** friction adjusts to whatever value prevents sliding, up to a maximum: $f_s \leq \mu_s N$. A block on a ramp that is not slipping has $f_s = mg \sin \theta$, which is generally *less* than $\mu_s N$; only at the threshold of slipping does $f_s = \mu_s N$. Friction opposes *relative sliding of the surfaces*, not necessarily the motion of the object: a box on an accelerating truck bed is pushed forward by static friction.

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2.8 Spring Forces

An ideal spring exerts a restoring force $F_s = -kx$ (Hooke's law), where x is the displacement from the relaxed length and k is the spring constant in N/m. The force always points toward equilibrium. When a spring supports a hanging mass at rest, $kx_0 = mg$ gives the equilibrium stretch. Because F depends on x , the resulting motion needs calculus (energy in Unit 3, oscillation in Unit 7).

2.9 Resistive Forces

A fluid exerts a force on a moving object that opposes the velocity and grows with speed. Two models appear in this course: **linear** drag $\vec{F}_R = -b\vec{v}$ (slow motion, viscous fluids) and **quadratic** drag $\vec{F}_R = -cv^2\hat{v}$ (faster motion through air). Because the force depends on velocity, acceleration is not constant, and Newton's second law becomes a differential equation that is solved by separating variables.

Linear drag by separation of variables

Falling, down positive: $m \frac{dv}{dt} = mg - bv$. Let $v_T = mg/b$ and $\tau = m/b$.

$$\frac{dv}{v_T - v} = \frac{dt}{\tau}, \text{ so } -\ln\left(\frac{v_T - v}{v_T}\right) = \frac{t}{\tau} \text{ and } v(t) = v_T(1 - e^{-t/\tau}).$$

Sliding horizontally with only drag: $m dv/dt = -bv$ gives $v = v_0 e^{-t/\tau}$ and total distance $x = \int_0^\infty v dt = v_0 \tau = mv_0/b$.

After one time constant τ the speed has covered 63% of the way to terminal speed (or fallen to 37% of v_0).

For quadratic drag, terminal speed occurs when $mg = cv_T^2$, so $v_T = \sqrt{mg/c}$. The differential equation $m dv/dt = mg - cv^2$ has the solution $v = v_T \tanh(gt/v_T)$, which you may verify by substitution; the key physical features are the same as for linear drag: acceleration g at the start, decreasing acceleration as speed grows, and $a \rightarrow 0$ as $v \rightarrow v_T$. Two conclusions are tested often. First, at terminal speed the net force is zero, so the velocity is constant. Second, an object thrown upward with drag takes *longer to fall back than it took to rise*, and returns slower than it was launched.

2.10 Circular Motion

For motion on a circle of radius r at speed v , the acceleration points toward the center with magnitude $a_c = v^2/r$ (the centripetal acceleration). The net force toward the center is what makes the path curve; **centripetal force** is not a new force, it is the net of the real forces (tension, gravity, normal force, friction) along the radial direction. Write the radial equation with the center as positive:

$\sum F_{\text{toward center}} = mv^2/r$. If the speed is changing there is also a tangential acceleration dv/dt . On a level curve static friction provides the radial force, so the maximum safe speed satisfies $\mu_s mg = mv^2/r$. There is no outward force on the object; the apparent outward push is inertia in a rotating frame.

Circular-motion rules

$$a_c = \frac{v^2}{r} = \omega^2 r, \text{ directed toward the center.}$$

Radial equation: $\sum F_r = m \frac{v^2}{r}$ with inward positive; do not draw mv^2/r on the FBD.

Doubling the speed at fixed radius quadruples the required net inward force; doubling the radius at fixed speed halves it.

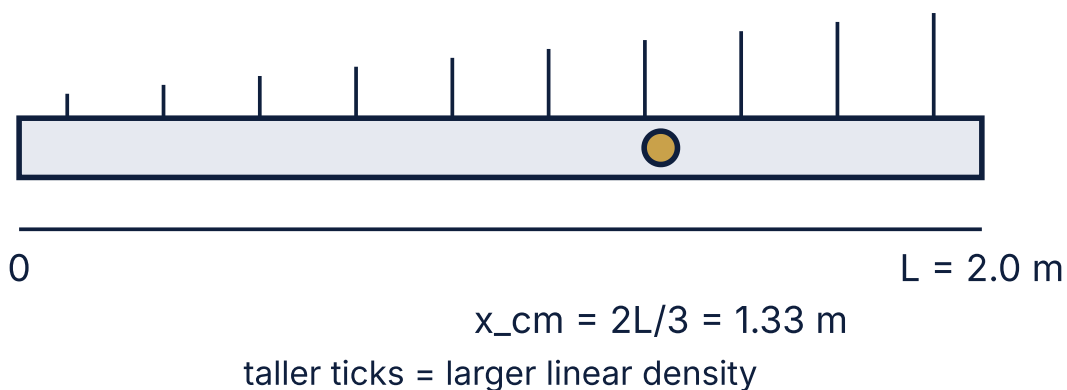
Orbits: $GMm/r^2 = mv^2/r$, giving $v = \sqrt{GM/r}$.

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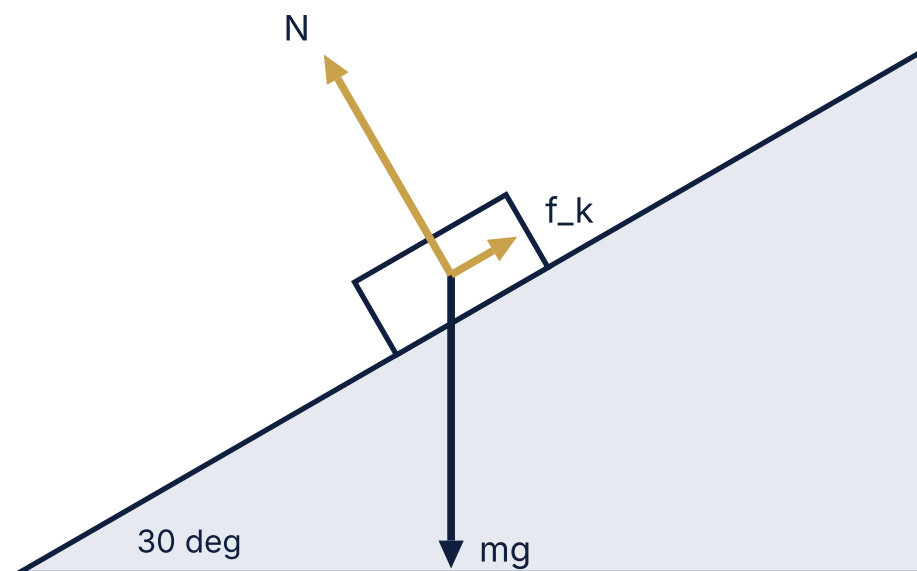
Rod with linear density $\lambda(x) = 3.0x$ (kg/m)



The center of mass (gold dot) lies in the denser half of the rod.

Worked Examples

Block sliding down a rough incline



Free-body diagram for Example 1. Kinetic friction points up the slope, opposite the sliding; the normal force is perpendicular to the surface.

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EXAMPLE 1: INCLINE WITH KINETIC FRICTION

A 5.0 kg block slides down a 30 degree incline with $\mu_k = 0.30$. Find the normal force, the friction force and the acceleration. Take down the slope as positive.

SOLUTION

FBD as drawn: weight mg straight down, normal force perpendicular to the surface, kinetic friction up the slope. Resolve along and perpendicular to the incline.

Perpendicular (no acceleration): $N - mg \cos 30^\circ = 0$, so $N = 5.0(9.8)(0.866) = 42.4 \text{ N}$.

Friction: $f_k = \mu_k N = 0.30(42.4) = 12.7 \text{ N}$ up the slope.

Along the slope: $mg \sin 30^\circ - f_k = ma$, so $a = g(\sin 30^\circ - \mu_k \cos 30^\circ) = 9.8(0.500 - 0.260) = 2.35 \text{ m/s}^2$ down the slope. Check: $24.5 - 12.7 = 11.8 \text{ N}$, and $11.8/5.0 = 2.35 \text{ m/s}^2$. The mass cancels in the symbolic expression, so a heavier block of the same material would have the same acceleration.

EXAMPLE 2: CENTER OF MASS OF A NONUNIFORM ROD

A rod of length $L = 2.0 \text{ m}$ lies along the x axis from $x = 0$ with linear mass density $\lambda(x) = 3.0x \text{ (kg/m, } x \text{ in m)}$. Find its mass and the position of its center of mass.

SOLUTION

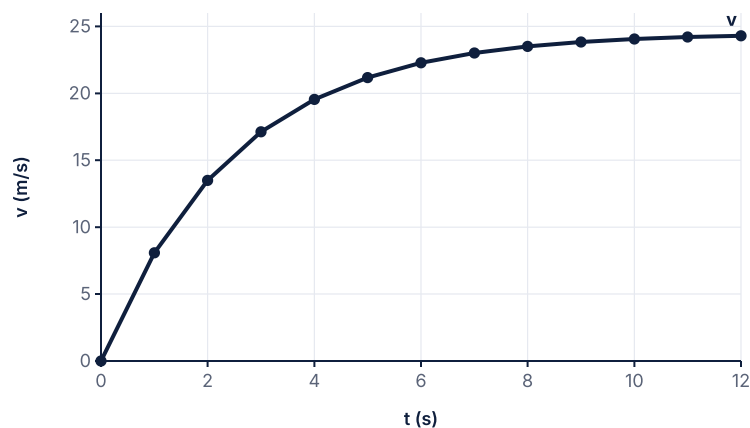
Mass: $M = \int dm = \int_0^L \lambda dx = \int_0^2 3x dx = 1.5x^2 \Big|_0^2 = 6.0 \text{ kg}$.

Center of mass: $x_{cm} = \frac{1}{M} \int_0^L x \lambda dx = \frac{1}{6.0} \int_0^2 3x^2 dx = \frac{1}{6.0} [x^3]_0^2 = \frac{8.0}{6.0} = 1.33 \text{ m}$.

In general $x_{cm} = 2L/3$ for density proportional to x . This is greater than $L/2$, as expected: more mass is concentrated at the right end.

EXAMPLE 3: LINEAR DRAG AND TERMINAL SPEED

A small ball of mass 0.050 kg falls from rest in a viscous fluid with drag $F_R = bv$, $b = 0.020 \text{ kg/s}$. (a) Find the terminal speed and the time constant. (b) Derive $v(t)$. (c) How long does it take to reach 90% of terminal speed, and how far has it fallen by then?



$v(t)$ approaches 24.5 m/s; at $t = 2.5 \text{ s}$, $v = 15.5 \text{ m/s}$ (63% of v_T).

SOLUTION

(a) At terminal speed $mg = bv_T$: $v_T = (0.050)(9.8)/0.020 = 24.5 \text{ m/s}$. The time constant is $\tau = m/b = 2.5 \text{ s}$.

(b) With down positive, $m dv/dt = mg - bv$. Separate: $\int_0^v \frac{dv'}{v_T - v'} = \int_0^t \frac{dt'}{\tau}$, so $-\ln \frac{v_T - v}{v_T} = \frac{t}{\tau}$ and $v = v_T(1 - e^{-t/\tau})$.

(c) $1 - e^{-t/\tau} = 0.90$ gives $t = \tau \ln 10 = 5.76 \text{ s}$. Distance: $y = \int_0^t v dt' = v_T [t - \tau(1 - e^{-t/\tau})] = 24.5 [5.76 - 2.5(0.90)] = 85.9 \text{ m}$.

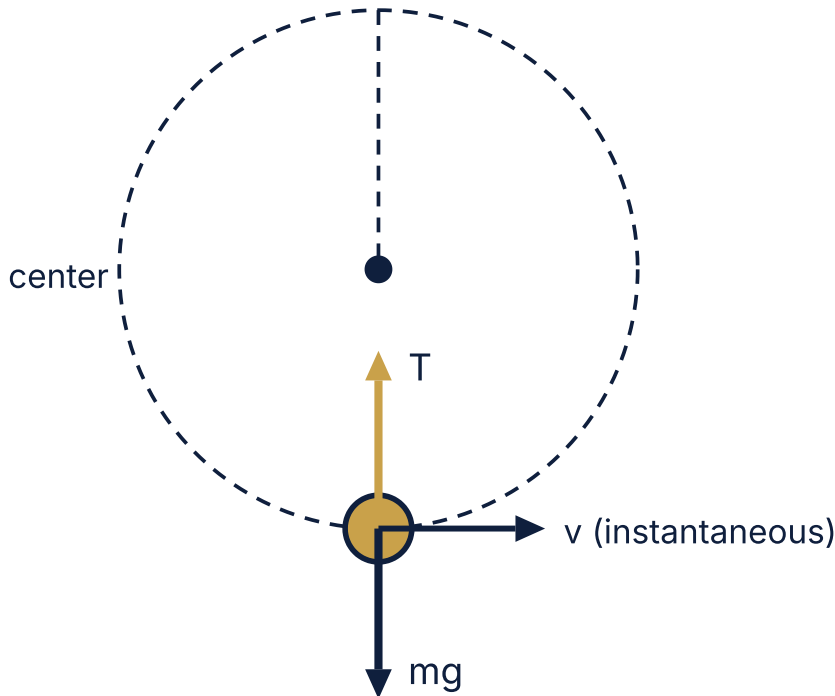
Without drag the ball would have fallen $\frac{1}{2}(9.8)(5.76)^2 = 163 \text{ m}$ in that time, so drag has substantially reduced the fall. The graph shows $v \rightarrow 24.5 \text{ m/s}$ with decreasing slope.

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Ball on a string at the bottom of a vertical circle



At the lowest point the net force points up (toward the center), so T exceeds mg .

EXAMPLE 4: CIRCULAR MOTION AT THE BOTTOM OF A LOOP

A 0.40 kg ball on a 0.80 m string moves in a vertical circle. At the lowest point its speed is 4.0 m/s. Find the tension there and the ball's acceleration.

SOLUTION

Forces at the bottom: tension T up (toward the center) and weight mg down. Take up (toward the center) as positive.

Radial equation: $T - mg = mv^2/r$, so $T = m\left(g + \frac{v^2}{r}\right) = 0.40\left(9.8 + \frac{16}{0.80}\right) = 11.9$ N.

Acceleration: $a_c = v^2/r = 20$ m/s² upward (toward the center). Because $a_c > g$, the tension is larger than the weight (3.92 N). The tension is not $mv^2/r = 8.0$ N alone; the string must also hold the ball up against gravity.

Write It Like a 5

Model Response

Prompt: A boat of mass m coasts on a lake after its engine is shut off. The water exerts a resistive force $-b\vec{v}$. The initial speed is v_0 . (a) Derive $v(t)$. (b) Derive the total distance traveled. (c) Justify that the boat never stops in finite time but travels a finite distance. (a) Newton's second law along the motion: $m dv/dt = -bv$. Separating variables, $\int_{v_0}^v dv'/v' = -(b/m) \int_0^t dt'$, so $\ln(v/v_0) = -bt/m$ and $v = v_0 e^{-bt/m}$. (b) The distance is $x = \int_0^\infty v dt = v_0 \int_0^\infty e^{-bt/m} dt = v_0 (m/b) = mv_0/b$. (c) The exponential function never reaches zero for finite t , so $v > 0$ for all t ; the boat never stops. But the area under the decaying curve converges, so the distance is the finite value mv_0/b . Equivalently, since $dv/dt = -(b/m)v$, the deceleration shrinks as the speed shrinks.

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What the grader looks for

(1) Newton's second law written with the correct sign for the resistive force. (2) Variables separated and integrated with the initial condition (definite limits or a constant evaluated from $v(0) = v_0$). (3) The final expression has the right form and units; m/b is a time. (4) Distance is found by integrating $v dt$, not by a constant-acceleration formula. (5) The justification links the mathematics to the physics (exponential never reaches zero; finite integral).

Coach's Playbook

- 1 Draw the FBD before any equation; one arrow per force on the object, none for the net force.
- 2 Choose axes along the acceleration (up the slope, toward the center of the circle).
- 3 For friction ask first: is the surface sliding (kinetic, $\mu_k N$) or not (static, solve for it)?
- 4 For velocity-dependent forces write $m dv/dt = \sum F$ with the sign of each term chosen from a picture, then separate variables.
- 5 Write the symbolic result, check units and limiting cases ($b = 0$, large t), then substitute numbers.

Common traps

(1) Treating static friction as $\mu_s N$ even when nothing is about to slip. (2) Using $N = mg$ on an incline or when a rope pulls at an angle. (3) Calling the third-law partner of the weight the normal force, or adding a 'centrifugal' arrow to the FBD. (4) In drag problems, letting the net force be mg at terminal speed, or integrating a as if it were constant. Also: forgetting that r in gravitation is measured from Earth's center, not its surface.

Quick Check

Q1. What is the net force on an object moving at constant velocity? A: zero.
 Q2. A satellite's orbital radius doubles. By what factor does its speed change? A: it is multiplied by $1/\sqrt{2}$.
 Q3. Why does a skydiver reach terminal speed? A: drag grows with speed until it equals the weight, so the net force and acceleration become zero.

Challenge Corner

A planet of mass M and radius R has surface field strength g_0 . At what height above the surface is the field strength $g_0/4$, and what is the speed of a circular orbit there in terms of g_0 and R ? Field strength is GM/r^2 and $g_0 = GM/R^2$, so $g/g_0 = (R/r)^2 = 1/4$ gives $r = 2R$, i.e., a height $h = R$ above the surface. The circular-orbit speed is $v = \sqrt{GM/r} = \sqrt{g_0 R^2/(2R)} = \sqrt{g_0 R/2}$. For Earth ($g_0 = 9.8 \text{ m/s}^2$, $R = 6.37 \times 10^6 \text{ m}$) that is 5.6 km/s. Note that the astronaut's weight has dropped to a quarter, but the apparent weightlessness arises because the spacecraft and astronaut share the same acceleration.

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Practice

PRACTICE Unit 2 Practice

Show your work. A calculator is allowed. Justify conclusions.

- A 4.0 kg block is pulled across a horizontal floor by a rope that exerts a 20 N tension at 37 degrees above the horizontal ($\sin 37 = 0.60$). What is the normal force from the floor on the block?
 (A) 27 N (B) 39 N (C) 51 N (D) 20 N
- A book rests on a table. Which force is the third-law partner of the table's normal force on the book?
 (A) The weight of the book (B) The book's gravitational pull on Earth
 (C) The book's push on the table (D) The table's weight
- Masses of 2.0 kg and 3.0 kg hang from the ends of a light string over a frictionless pulley and are released. What is the tension in the string?
 (A) 19.6 N (B) 23.5 N (C) 29.4 N (D) 49.0 N
- An object falls from rest through a fluid with drag proportional to speed, with time constant τ . What fraction of its terminal speed does it have at $t = 2\tau$?
 (A) 63% (B) 86% (C) 95% (D) 100%
- A car rounds a flat circular curve at its maximum safe speed. If the speed is doubled while the radius is unchanged, the minimum coefficient of static friction needed is multiplied by
 (A) $1/2$ (B) 2 (C) 4 (D) 8
- Planet X has twice Earth's mass and twice Earth's radius. The gravitational field strength at its surface, compared with Earth's, is
 (A) twice as large (B) the same (C) one half as large
 (D) one fourth as large
- A boat of mass 1200 kg coasts after its engine is turned off. Water exerts a drag force of magnitude bv opposing the motion, with $b = 150 \text{ kg/s}$. At $t = 0$, $v = 8.0 \text{ m/s}$.
 (a) Write the differential equation from Newton's second law and derive $v(t)$.
 (b) Calculate the time constant and the time at which the speed has fallen to 1.0 m/s.
 (c) Derive and calculate the total distance the boat travels, and the distance traveled by the time of (b).
 (d) Sketch $v(t)$ and $x(t)$ on separate axes, marking the long-time behavior of each.
- Students study quadratic air drag $F = cv^2$ on stacked paper cups released from a height of 2.5 m with a motion sensor. For stacks of mass 2, 4, 6, 8, 10 g (0.002 to 0.010 kg) the terminal speeds are 1.46, 1.88, 2.45, 2.77, 3.10 m/s.
 (a) Derive an expression for terminal speed in terms of m , g and c , and state which quantities should be plotted to obtain a straight line.
 (b) Using the data, calculate the slope of the best-fit line and use it to determine c .
 (c) Explain whether the line should pass through the origin and what a large nonzero intercept would suggest.
 (d) Identify one source of systematic error that makes the measured terminal speed smaller than the true one.
- A geostationary satellite orbits Earth once every 24 h (86 400 s) in a circular orbit. Use $G = 6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2$, $M_E = 5.97 \times 10^{24} \text{ kg}$ and $R_E = 6.37 \times 10^6 \text{ m}$.
 (a) Derive the relationship between the orbital period T and the orbital radius r for a circular orbit.
 (b) Calculate the orbital radius and the satellite's altitude above Earth's surface.
 (c) Calculate the orbital speed.
 (d) Another satellite orbits at half the radius. Without further calculation, state its speed compared with the first and justify.
- A worker drags a crate of mass 30 kg across a horizontal floor at constant speed by pulling a rope that makes an angle θ above the horizontal. The coefficient of kinetic friction is μ .
 (a) Draw a free-body diagram of the crate and derive the rope tension T as a function of θ , μ , m and g .
 (b) Use calculus to find the angle that minimizes T .
 (c) For $\mu = 0.40$, calculate that angle and the minimum tension.
 (d) Compare with pulling horizontally.

Answers Unit 2: Answers and Explanations

Read every explanation, including the ones for problems you answered correctly.

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2 Force and Translational Dynamics

PRACTICE · UNIT 2 PRACTICE

1. A. Vertical equilibrium: $N + T \sin 37^\circ = mg$, so $N = 39.2 - 12 = 27.2$ N. Choice B assumes $N = mg$, which ignores the upward pull of the rope.
2. C. The partner of a contact force exerted by the table on the book is the contact force of the book on the table, equal in magnitude and opposite in direction. Choice A is the tempting one: it equals the normal force here, but only because the book is in equilibrium, and it is not a third-law pair.
3. B. The acceleration is $a = (3 - 2)g/5 = 1.96$ m/s²; for the 2.0 kg mass, $T = m(g + a) = 2.0(11.76) = 23.5$ N. Choice A is the weight of the lighter mass, which would be the tension only if the system were in equilibrium.
4. B. $v/v_T = 1 - e^{-2} = 0.865$. Choice A is the value at $t = \tau$; choice D would require an infinite time because the approach is exponential.
5. C. Friction supplies the centripetal force: $\mu mg = mv^2/r$, so $\mu \propto v^2$ and doubling v multiplies it by 4. Choice B treats the dependence as linear.
6. C. $g = GM/R^2$ scales as $2/2^2 = 1/2$. Choice B treats the mass and radius as canceling, which would only be true if $g \propto M/R$.
7. (a) $m dv/dt = -bv$; separating, $\int_8^v dv'/v' = -(b/m) \int_0^t dt'$, so $v = v_0 e^{-bt/m} = 8.0e^{-t/8.0}$ m/s. (b) $\tau = m/b = 1200/150 = 8.0$ s. $v = 1.0$ at $e^{-t/8} = 1/8$, $t = 8 \ln 8 = 16.6$ s. (c) $x = \int_0^\infty v_0 e^{-t/\tau} dt = v_0 \tau = mv_0/b = 64$ m. By 16.6 s: $x = v_0 \tau (1 - e^{-t/\tau}) = 64(1 - 1/8) = 56$ m. (d) $v(t)$ is a decaying exponential from 8 m/s toward 0 (never touching it), concave up; $x(t)$ rises with decreasing slope toward the horizontal asymptote 64 m. Points: (a) 3, (b) 2, (c) 3, (d) 2.
8. (a) At terminal speed $mg = cv_T^2$, so $v_T^2 = (g/c)m$. Plot v_T^2 (vertical) against m (horizontal); the slope is g/c . (b) $v_T^2 = 2.13, 3.53, 6.00, 7.67, 9.61$ m²/s². Best-fit slope = 955 m²/(s² kg). Then $c = g/\text{slope} = 9.8/955 = 0.0103$ kg/m. (c) Yes, a zero-mass stack has zero terminal speed, so the line should go through the origin (the fitted intercept, 0.06 m²/s², is small). A large intercept suggests the speed was not terminal or that the sensor reading was offset. (d) Measuring before the cups reach terminal speed (the drop height too short), or a sensor that reads the average speed over a window that includes slower earlier motion. Points: (a) 2, (b) 3, (c) 2, (d) 1.
9. (a) Gravity provides the centripetal force: $GMm/r^2 = mv^2/r$ with $v = 2\pi r/T$, so $GM/r^2 = 4\pi^2 r/T^2$, giving $T^2 = 4\pi^2 r^3/(GM)$. (b) $r = [GMT^2/(4\pi^2)]^{1/3} = 4.22 \times 10^7$ m; altitude = $4.22 \times 10^7 - 6.37 \times 10^6 = 3.59 \times 10^7$ m. (c) $v = 2\pi r/T = 3.07 \times 10^3$ m/s (check: $\sqrt{GM/r} = 3.07 \times 10^3$ m/s). (d) $v = \sqrt{GM/r}$ so halving r multiplies v by $\sqrt{2}$: the inner satellite is faster, about 4.34 km/s. The satellite closer to Earth feels a larger field strength, so it needs a larger centripetal acceleration v^2/r . Points: (a) 3, (b) 2, (c) 1, (d) 2.
10. (a) FBD: weight mg down, normal force N up, friction μN opposite motion, tension at angle θ . Constant speed: $T \cos \theta = \mu N$ and $N + T \sin \theta = mg$, so $T = \mu mg/(\cos \theta + \mu \sin \theta)$. (b) T is minimum when the denominator $D = \cos \theta + \mu \sin \theta$ is maximum: $dD/d\theta = -\sin \theta + \mu \cos \theta = 0$, so $\tan \theta = \mu$; $d^2D/d\theta^2 < 0$ confirms a maximum. (c) $\theta = \arctan 0.40 = 21.8^\circ$, $D = \sqrt{1 + \mu^2} = 1.077$, $T_{\min} = 0.40(30)(9.8)/1.077 = 109$ N. (d) Horizontal pull ($\theta = 0$): $T = \mu mg = 118$ N, about 8% larger, because a slight upward angle reduces the normal force and thus friction. Points: (a) 3, (b) 2, (c) 2, (d) 1.

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Unit 3: Work, Energy, and Power

Exam weighting: about 15-25% of the multiple-choice section. Concept review, worked examples, exam-style writing, and 10 practice items, all with full explanations.

3 Work, Energy, and Power

Kinetic energy, work as an integral, potential energy from force, conservation of energy with nonconservative work, and power as the rate of energy transfer.

Why This Matters

Work, Energy, and Power carries 15-25% of multiple choice and anchors the energy-based free-response questions, including the lab-style task where a linearized graph yields a spring constant or a coefficient of friction. The calculus is the point: work is an integral of force over displacement, and a force is the negative slope of a potential-energy function. Students lose points by writing energy equations that omit a term (a spring, an initial or final state), by treating spring energy as negative, by applying conservation of mechanical energy when friction acts, and by plotting v instead of v^2 so the graph is not linear. Numerical work uses $g = 9.8 \text{ m/s}^2$.

Core Ideas

3.1 Translational Kinetic Energy

The kinetic energy of a particle of mass m and speed v is $K = \frac{1}{2}mv^2 = p^2/(2m)$. It is a scalar and never negative. Doubling the speed quadruples K ; it does not depend on direction. The kinetic energy of a system's center of mass is $\frac{1}{2}Mv_{cm}^2$, and the total kinetic energy of a system can be larger when its parts also move relative to the center of mass.

3.2 Work

Work is the transfer of energy by a force acting through a displacement. For a force $F(x)$ along a line, $W = \int_{x_1}^{x_2} F(x) dx$; in general $W = \int \vec{F} \cdot d\vec{r}$, and for a constant force $W = \vec{F} \cdot \Delta\vec{r} = F \Delta r \cos \phi$, where ϕ is the angle between force and displacement. A force perpendicular to the motion does no work (the normal force on a sliding block, the tension on a swinging pendulum bob). Work is positive when the force has a component along the displacement and negative when it opposes it. On an F - x graph the work is the signed area between the curve and the axis.

Work-Energy Theorem

$$W_{net} = \int \vec{F}_{net} \cdot d\vec{r} = \Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

Derivation: $F = m dv/dt$, so $\int F dx = \int m \frac{dv}{dt} dx = \int m v dv = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$.

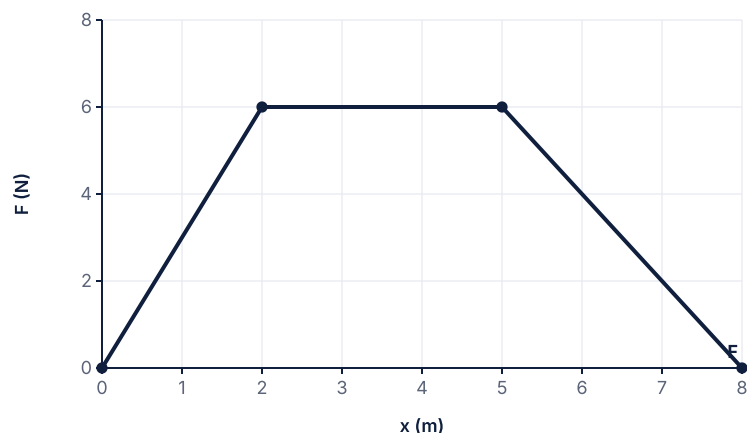
Spring (from relaxed length): $W = \int_0^x (-kx') dx' = -\frac{1}{2}kx^2$ by the spring; the agent stretching it does $+\frac{1}{2}kx^2$.

Gravity near Earth (up positive): $W_g = -mg \Delta y$.

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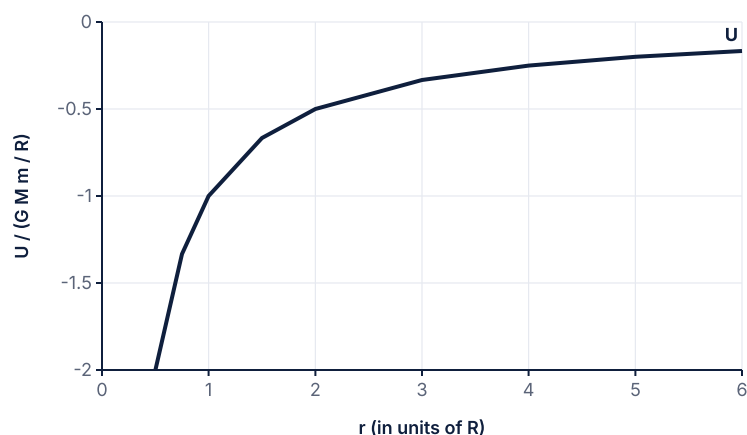


Area under F - x is the work: $6 + 18 + 9 = 33$ J for the trapezoid.

3.3 Potential Energy

When the work done by a force depends only on the endpoints, the force is **conservative** and has a potential energy: $\Delta U = -\int \vec{F} \cdot d\vec{r}$, or in one dimension $F(x) = -dU/dx$. The force points toward lower potential energy, with magnitude equal to the slope. Reference points are arbitrary; only changes in U matter. Standard results: $U_s = \frac{1}{2}kx^2$ (zero at the relaxed length, never negative, for stretch and compression alike), $U_g = mgy$ near Earth, and $U_g = -GMm/r$ for two masses with $U = 0$ at infinite separation. The last follows from $\Delta U = -\int_{\infty}^r (-GMm/r'^2) dr' = -GMm/r$. Friction and drag are **nonconservative**: the work they do depends on the path, and no potential energy can be assigned.

A graph of $U(x)$ shows the physics. At a minimum of U ($dU/dx = 0$, $d^2U/dx^2 > 0$) the equilibrium is **stable**: a small push produces a restoring force. At a maximum the equilibrium is **unstable**. With total mechanical energy $E = K + U$, the particle can only be where $E \geq U$; points where $E = U$ are **turning points** where $K = 0$. The kinetic energy at any position is the vertical gap between E and the U curve.



Gravitational potential energy $U = -GMm/r$: negative, rising toward zero as r increases.

3.4 Conservation of Energy

For a system with no external work, the total energy is constant: $K_i + U_i + W_{nc} = K_f + U_f$, where W_{nc} is the work by nonconservative forces *inside the system* (negative for friction, since it removes mechanical energy). Friction converts mechanical energy into thermal energy, $\Delta E_{th} = f_k d$ for sliding distance d . Always list the energy forms in the initial and final state and name the system; a missing spring, or an initial or final state left out, is the most common source of lost points. Static friction on a rolling object does no work.

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Energy-conservation procedure

1. Choose the system and two instants (initial and final).
2. Write K and each U (gravity, spring) at both instants, with a zero level for each U .
3. Add work by outside or nonconservative forces: $K_i + U_i + W_{ext} = K_f + U_f$ (with $W_{fric} = -f_k d$).
4. Solve; check the sign of every energy and that $K \geq 0$.

3.5 Power

Power is the rate of energy transfer, $P = dW/dt$, with average power $P_{avg} = W/\Delta t$. For a force acting on an object moving with velocity $\vec{v}$, $P = \vec{F} \cdot \vec{v} = Fv \cos \phi$. The SI unit is the watt (1 J/s). If P is constant while $F = ma$ changes, a is not constant, and one integrates $P = dK/dt$ to find $v(t)$ (Example 4). The total work from t_1 to t_2 is $\int P dt$, which is the area under a power-versus-time graph.

Worked Examples

EXAMPLE 1: WORK FROM A GRAPH OF F(X)

A 3.0 kg block on a frictionless surface starts from rest at $x = 0$ and is pushed by a force along x described by the graph above (F rises from 0 to 6 N at 2 m, stays at 6 N to 5 m, then falls to zero at 8 m). Find the work done and the block's speed at $x = 8$ m.

SOLUTION

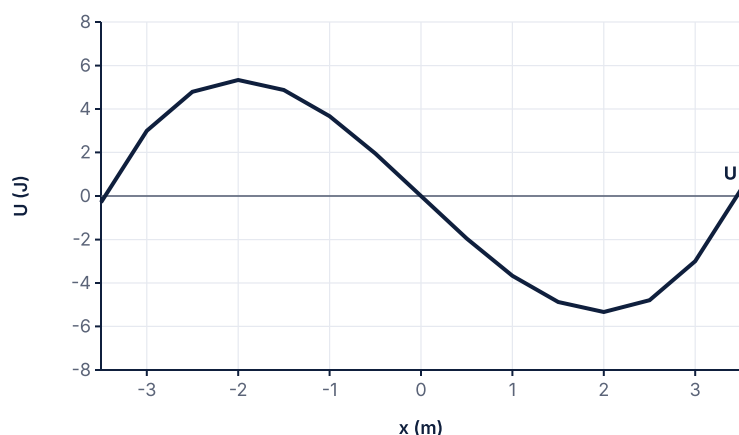
Work equals the area: $W = \frac{1}{2}(2)(6) + (3)(6) + \frac{1}{2}(3)(6) = 6 + 18 + 9 = 33$ J.

Work-energy theorem: $W = \frac{1}{2}mv^2 - 0$, so $v = \sqrt{2W/m} = \sqrt{66/3.0} = 4.7$ m/s.

The normal force and weight are perpendicular to the motion and do no work, so the push is the only force doing work.

EXAMPLE 2: FORCE FROM A POTENTIAL ENERGY FUNCTION

A 0.50 kg particle moves along x with potential energy $U(x) = \frac{1}{3}x^3 - 4x$ (joules, x in m). (a) Find $F(x)$ and the equilibrium positions, and classify them. (b) If the particle has total mechanical energy $E = 0$ and is at $x = 2.0$ m, find its speed there. Where are the turning points for $x \geq 0$?



$U(x) = x^3/3 - 4x$: minimum at $x = +2$ (stable), maximum at $x = -2$ (unstable).

SOLUTION

(a) $F = -dU/dx = -(x^2 - 4) = 4 - x^2$. Equilibrium where $F = 0$: $x = \pm 2$ m. $d^2U/dx^2 = 2x$: positive at $x = +2$ (a minimum of U , stable) and negative at $x = -2$ (a maximum, unstable).

(b) $U(2) = 8/3 - 8 = -5.33$ J, so $K = E - U = 0 - (-5.33) = 5.33$ J and $v = \sqrt{2K/m} = \sqrt{2(5.33)/0.50} = 4.6$ m/s.

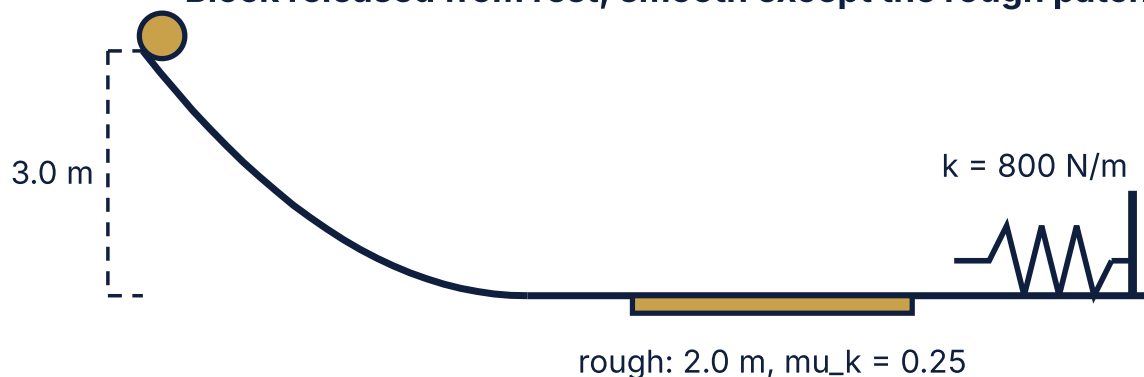
Turning points where $K = 0$: $U = E = 0$, so $x(x^2/3 - 4) = 0$, giving $x = 0$ and $x = \sqrt{12} = 3.46$ m. The particle oscillates between them with maximum speed at the minimum of U ($x = 2$ m).

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Block released from rest; smooth except the rough patch



Setup for Example 3: a 2.0 kg block drops 3.0 m, crosses a rough patch, and compresses a spring.

EXAMPLE 3: ENERGY WITH FRICTION AND A SPRING

A 2.0 kg block is released from rest 3.0 m above a horizontal surface at the top of a smooth curved track. The surface has a rough patch 2.0 m long with $\mu_k = 0.25$, followed by a spring with $k = 800 \text{ N/m}$. Find the maximum compression of the spring.

SOLUTION

System: block, track, spring, Earth. Initial: at rest at height 3.0 m. Final: at rest, spring compressed by x (take the spring's compression to be small compared to the track, so its gravitational change is negligible).

Energy: $mgh - \mu_k mgd = \frac{1}{2}kx^2$, where the second term is the thermal energy produced on the rough patch ($d = 2.0 \text{ m}$).

Numbers: $mgh = 2.0(9.8)(3.0) = 58.8 \text{ J}$ and $\mu_k mgd = 0.25(2.0)(9.8)(2.0) = 9.8 \text{ J}$. So $\frac{1}{2}kx^2 = 49.0 \text{ J}$ and $x = \sqrt{2(49.0)/800} = 0.35 \text{ m}$.

Speed on reaching the rough patch: $\sqrt{2gh} = 7.7 \text{ m/s}$; the patch removes 9.8 J, one-sixth of the energy. Mechanical energy alone is not conserved, but total energy (including thermal) is.

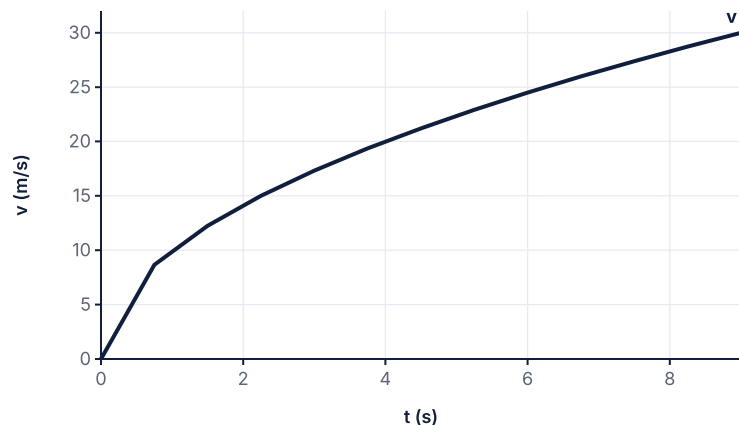
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EXAMPLE 4: POWER AND A VARIABLE FORCE

A 1200 kg vehicle starts from rest and its engine delivers a constant useful power of 60 kW to the wheels. Ignore resistive forces. Find $v(t)$, the time to reach 30 m/s, and the distance covered in that time.



$v = 10 \sqrt{t}$: steep at first, then flattening.

SOLUTION

Use $P = dK/dt$: $\frac{d}{dt}(\frac{1}{2}mv^2) = P$, so $\frac{1}{2}mv^2 = Pt$ and $v = \sqrt{2Pt/m} = \sqrt{2(60000)t/1200} = 10\sqrt{t}$ m/s.

For $v = 30$ m/s: $t = (30/10)^2 = 9.0$ s.

Distance: $x = \int_0^9 10t^{1/2} dt = 10 \cdot \frac{2}{3}t^{3/2} \Big|_0^9 = \frac{20}{3}(27) = 180$ m.

The acceleration $a = dv/dt = 5/\sqrt{t}$ is not constant (infinite at $t = 0$ in this idealization, which is why real engines are limited by traction); the constant-acceleration formulas would give the wrong distance (135 m).

Write It Like a 5**Model Response**

Prompt: A particle of mass $m = 2.0$ kg moves along the x axis under a net force $F(x) = 6x - 0.75x^2$ (N, x in m). It starts at rest at $x = 0$. (a) Calculate the work done as it moves to $x = 6.0$ m. (b) Calculate its speed there. (c) Derive a potential energy function $U(x)$ with $U(0) = 0$ and state whether the force is conservative. (a) $W = \int_0^6 (6x - 0.75x^2) dx = [3x^2 - 0.25x^3]_0^6 = 108 - 54 = 54$ J. (b) By the work-energy theorem, $W = \frac{1}{2}mv^2 - 0$, so $v = \sqrt{2(54)/2.0} = 7.3$ m/s. (c) For a conservative force $U(x) - U(0) = -\int_0^x F dx' = -3x^2 + 0.25x^3$, so $U(x) = -3x^2 + 0.25x^3$ with $U(6) = -54$ J. The force depends only on position, so the work between two points depends only on the endpoints; it is conservative. Check: $\Delta K = -\Delta U = +54$ J.

What the grader looks for

(1) A definite integral with correct limits and the antiderivative shown. (2) The theorem named or written ($W = \Delta K$) before substituting. (3) Correct speed with units. (4) For U , the minus sign in $\Delta U = -\int F dx$, the reference $U(0) = 0$, and a statement that the force is conservative because it depends only on position. (5) A consistency check: $\Delta K = -\Delta U$.

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Coach's Playbook

- 1 Define the system and two states; list every energy form in each state, including springs.
- 2 Decide the energy accounting: pure conservation, or add W_{ext} or $-f_k d$.
- 3 For a variable force, integrate; do not use $W = Fd$. For graphs, find the signed area.
- 4 For linearization, rearrange so the quantity you measure is on the axes: e.g., plot v^2 versus h , or h versus x^2 , and connect the slope to g , k or μ .
- 5 State the relation between a force and a U graph: $F = -dU/dx$, minimum is stable.
- 6 Check that K is never negative and that the signs of W , ΔK and ΔU agree.

Common traps

(1) Giving spring energy a negative sign for compression, or equating kinetic energy to the energy of one spring when two share it. (2) Using $\Delta K = mgh$ when friction or drag removes energy. (3) Plotting v instead of v^2 against height, so the graph is not a straight line. (4) Claiming work by static friction on a rolling object or by a centripetal force on circular motion (perpendicular to displacement, so zero). Also: forgetting that power is $\vec{F} \cdot \vec{v}$, not force times a distance.

Quick Check

Q1. How does kinetic energy change if the speed triples? A: it becomes 9 times larger.
 Q2. What is the sign of the work done by kinetic friction on a sliding block? A: negative.
 Q3. A $U(x)$ curve has a local minimum at x_0 . Is the equilibrium there stable or unstable? A: stable, since the force points back toward x_0 on both sides.

Challenge Corner

Use the work done by gravity to find the minimum launch speed from a planet's surface for an object to escape, in terms of G , M and R . Then evaluate for Earth ($M = 5.97 \times 10^{24}$ kg, $R = 6.37 \times 10^6$ m). To reach infinity the object needs total energy ≥ 0 (at infinity $U = 0$ and, at the minimum speed, $K = 0$). The work done against gravity from R to infinity is $\int_R^\infty (GMm/r^2) dr = GMm/R$. Setting $\frac{1}{2}mv_{esc}^2 = GMm/R$ gives $v_{esc} = \sqrt{2GM/R}$. For Earth: $\sqrt{2(6.67 \times 10^{-11})(5.97 \times 10^{24})/6.37 \times 10^6} = 1.12 \times 10^4$ m/s, about 11.2 km/s. The mass of the escaping object cancels. This is $\sqrt{2}$ times the circular-orbit speed at the surface.

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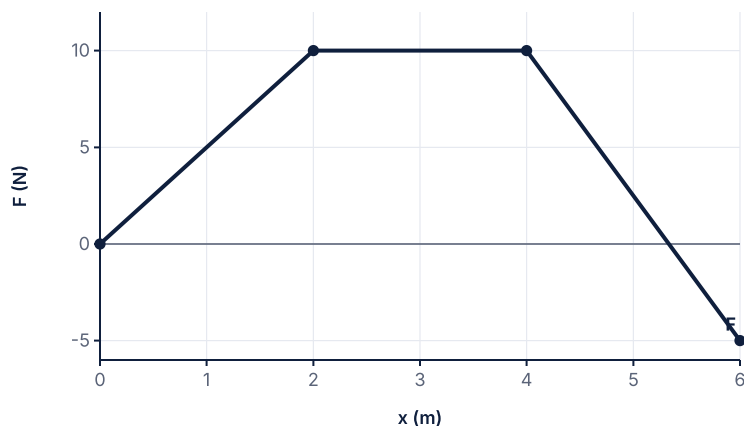
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Practice

PRACTICE Unit 3 Practice

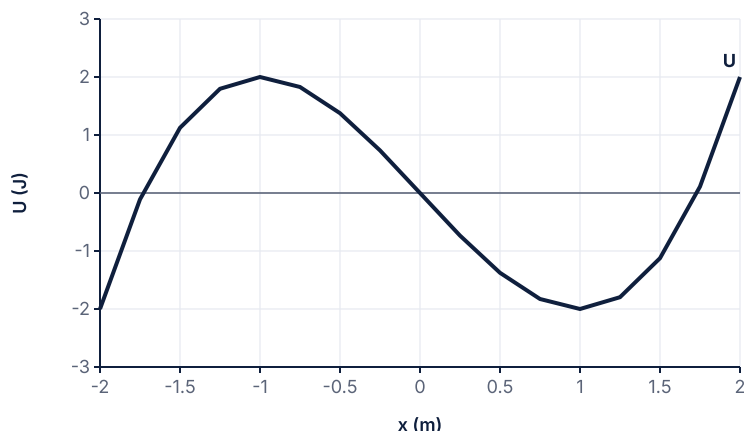
Show your work. A calculator is allowed. Justify conclusions.

1. A force along the x axis varies with position as shown. What is the net work done by the force as the object moves from $x = 0$ to $x = 6$ m?



Force versus position.

- (A) 20 J (B) 30 J (C) 35 J (D) 40 J
2. A particle moves from the origin to the point (2 m, -1 m) under a constant force $\vec{F} = (3\hat{x} + 4\hat{y})$ N. How much work does the force do?
- (A) 2 J (B) 10 J (C) 6 J (D) 14 J
3. A particle's potential energy is $U(x) = x^3 - 3x$ (SI units). Which statement about the equilibrium positions is correct?



Potential energy function.

- (A) $x = -1$ is stable and $x = +1$ is unstable (B) $x = -1$ is unstable and $x = +1$ is stable
- (C) Both are stable (D) Both are unstable
4. An elevator motor lifts a 800 kg elevator car upward at a constant speed of 2.0 m/s. Ignoring other losses, what power must the motor supply?
- (A) 1.6 kW (B) 3.2 kW (C) 7.8 kW (D) 15.7 kW
5. An ideal spring requires 2 J of work to stretch it from its relaxed length by 0.10 m. How much additional work is needed to stretch it from 0.10 m to 0.20 m?
- (A) 2 J (B) 4 J (C) 6 J (D) 8 J
6. A roller-coaster car starts from rest at a height of 20 m above the ground on a frictionless track. What is its speed when it reaches a height of 5.0 m?
- (A) 9.9 m/s (B) 14 m/s (C) 17 m/s (D) 20 m/s

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7. A particle of mass 2.0 kg moves along the x axis under a net force $F(x) = 6.0x - 0.75x^2$ (N, x in m), starting from rest at $x = 0$.
 - (a) Derive an expression for the work done on the particle from $x = 0$ to a general position x .
 - (b) Calculate the speed at $x = 6.0$ m.
 - (c) Determine the position at which the force is greatest, and its value.
 - (d) Derive a potential energy function $U(x)$ with $U(0) = 0$ and calculate $U(6.0)$. Explain how $U(6.0)$ relates to the answer in (b).
8. A student launches a small ball vertically with a spring-loaded launcher and wants to determine the spring constant. The ball has mass 0.050 kg. The spring is compressed by x (0.020 to 0.060 m) and the maximum height h of the ball above its release position is measured. Data: $x = 0.020, 0.030, 0.040, 0.050, 0.060$ m give $h = 0.080, 0.176, 0.324, 0.517, 0.760$ m. Assume all the spring energy goes into the ball.
 - (a) Derive an expression for h in terms of x , k , m and g .
 - (b) State which quantities should be plotted to obtain a straight line and what the slope represents.
 - (c) Using the data, calculate the slope of the best-fit line and the spring constant.
 - (d) Describe how air resistance and friction in the launcher would affect the slope and the value of k you determine.
9. The gravitational force on a mass m at distance r from the center of a planet of mass M is $F = -GMm/r^2$ (attractive, r outward).
 - (a) Derive the gravitational potential energy $U(r)$ with $U = 0$ at infinite separation.
 - (b) Sketch $U(r)$ and a horizontal line for total energy $E = 0$ and for $E < 0$. Label the regions the object can reach for $E < 0$.
 - (c) A planet has $M = 4.8 \times 10^{23}$ kg and $R = 3.4 \times 10^6$ m. Calculate the escape speed from its surface.
 - (d) Explain why an object launched at exactly the escape speed has zero speed at infinity.
10. A small ball of mass 0.050 kg falls through a viscous fluid with drag force $F_R = bv$, $b = 0.020$ kg/s, until it reaches terminal speed.
 - (a) Show that the power dissipated by drag is $P = bv^2$.
 - (b) Calculate the terminal speed, and show that at terminal speed the power dissipated by the drag equals the rate at which gravity does work.
 - (c) At terminal speed, state the rates of change of kinetic energy and of gravitational potential energy, and explain where the energy goes.
 - (d) Calculate the thermal power produced at terminal speed.

Answers**Unit 3: Answers and Explanations**

Read every explanation, including the ones for problems you answered correctly.

3 Work, Energy, and Power**PRACTICE · UNIT 3 PRACTICE**

1. C. Signed areas: 10 J (0–2 m) + 20 J (2–4 m), then the force falls to zero at 5.33 m (+6.67 J) and becomes negative (–1.67 J), total 35 J. Choice B (30 J) stops at $x = 4$ m; choice D counts the negative area as positive.
2. A. $W = \vec{F} \cdot \Delta \vec{r} = (3)(2) + (4)(-1) = 2$ J. Choice B adds the magnitudes of the component works (6 + 4), ignoring that the y -displacement is opposite to the y -force; choice C keeps only the x -term.
3. B. Equilibrium occurs where $dU/dx = 3x^2 - 3 = 0$, at $x = \pm 1$. $d^2U/dx^2 = 6x$ is positive at $x = +1$ (a local minimum, stable) and negative at $x = -1$ (a local maximum, unstable). Choice A reverses the maximum and minimum.
4. D. At constant speed the lifting force equals the weight, so $P = Fv = mgv = 800(9.8)(2.0) = 15.7$ kW. Choice A uses mv instead of mgv ; choice C is mg alone, forgetting to multiply by the speed.
5. C. $W = \frac{1}{2}kx^2$ scales as x^2 : stretching to 0.20 m takes 8 J in total, so the second 0.10 m needs $8 - 2 = 6$ J. Choice A assumes that equal stretches need equal work, but the force increases with x .
6. C. $\frac{1}{2}mv^2 = mg(20 - 5)$, so $v = \sqrt{2(9.8)(15)} = 17$ m/s. Choice D uses the full height of 20 m, which would give the speed at the bottom.
7. (a) $W = \int_0^x (6x' - 0.75x'^2) dx' = 3x^2 - 0.25x^3$. (b) At $x = 6$: $W = 108 - 54 = 54$ J; $v = \sqrt{2W/m} = \sqrt{54} = 7.3$ m/s. (c) $dF/dx = 6 - 1.5x = 0$ gives $x = 4.0$ m, where $F = 24 - 12 = 12$ N (second derivative $-1.5 < 0$, a maximum). (d) $U = -\int_0^x F dx' = -3x^2 + 0.25x^3$, so $U(6) = -108 + 54 = -54$ J. The decrease in potential energy equals the increase in kinetic energy, $\Delta K = 54$ J = $-\Delta U$, as required for a conservative force. Points: (a) 2, (b) 2, (c) 2, (d) 2.
8. (a) Energy conservation: $\frac{1}{2}kx^2 = mgh$, so $h = kx^2/(2mg)$. (b) Plot h (vertical) against x^2 (horizontal): a line through the origin with slope $k/(2mg)$. (c) The values of x^2 are 4.0, 9.0, 16, 25, 36×10^{-4} m². The best-fit slope is about 213 m/m² (m⁻¹), so $k = 2mg \times \text{slope} = 2(0.050)(9.8)(213) = 209$ N/m, or about 210 N/m. (d) Energy dissipated by friction or drag means the ball rises less than the spring energy allows, so the measured slope is smaller and the inferred k is lower than the true value. Points: (a) 2, (b) 2, (c) 3, (d) 1. Note: plotting h against x would give a curve, not a straight line.
9. (a) $U(r) = -\int_{\infty}^r F dr' = -\int_{\infty}^r (-GMm/r'^2) dr' = -GMm/r$. (b) U is a negative curve rising toward 0 as $r \rightarrow \infty$ (steep near small r). For $E < 0$, a horizontal line meets the curve at a turning point r_{max} ; the object is confined between the planet's surface and r_{max} (it is bound); for $E = 0$ the line never meets the curve at finite r . (c) $v_{\text{esc}} = \sqrt{2GM/R} = \sqrt{2(6.67 \times 10^{-11})(4.8 \times 10^{23})/(3.4 \times 10^6)} = 4.3 \times 10^3$ m/s. (d) At escape speed $E = K + U = 0$; at infinity $U = 0$, so $K = 0$ there: it arrives with zero speed (it barely makes it). Points: (a) 3, (b) 2, (c) 2, (d) 1.

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10. (a) The drag force on the ball is $-bv$ (upward) while the velocity is downward, so $P = \vec{F} \cdot \vec{v}$ gives $-bv^2$: the drag removes energy from the ball at the rate bv^2 . (b) At terminal speed $mg = bv_T$, so $v_T = 0.050(9.8)/0.020 = 24.5$ m/s. Gravity's power is $mgv_T = 12.0$ W, and $bv_T^2 = 0.020(24.5)^2 = 12.0$ W: equal. (c) $dK/dt = 0$ since v is constant, and $dU_g/dt = -mgv_T = -12.0$ W. The lost gravitational potential energy is transferred to thermal energy in the fluid (and ball). (d) 12.0 W. Points: (a) 2, (b) 3, (c) 2, (d) 1.

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Unit 4: Linear Momentum

Exam weighting: about 10-20% of the multiple-choice section. Concept review, worked examples, exam-style writing, and 10 practice items, all with full explanations.

4 Linear Momentum

Momentum, impulse as an integral, center of mass by integration, conservation, and collisions of every kind. Learn to set up the integral and to say why a system is isolated.

Why This Matters

Unit 4 carries roughly 10 to 15 percent of the multiple-choice section, and momentum was the Mathematical Routines free-response question on the 2025 exam. Points are lost by drawing pre-collision momentum as two arrows, missing that a changing force needs a definite integral, and assuming a post-collision direction without evidence. This book uses $g = 10 \text{ m/s}^2$, the exam's stated default; 9.8 m/s^2 is also accepted.

Core Ideas

4.1 Linear Momentum

Linear momentum is the vector $\vec{p} = m\vec{v}$, in kg m/s. A system's momentum is the vector sum of its parts' momenta, and in one dimension the sign of each term carries direction. Momentum and kinetic energy are different things: $K = p^2/2m$, so two objects with equal momentum do not share equal energy.

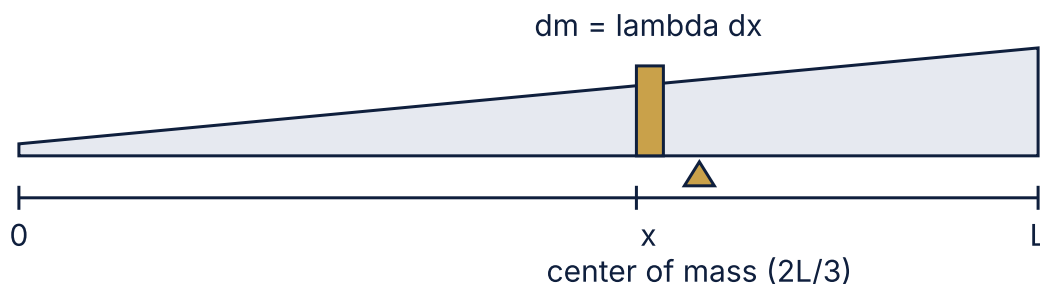
For a continuous object the sum becomes an integral. The **center of mass** is the mass-weighted average position, $x_{cm} = \frac{1}{M} \int x dm$. On a thin rod with linear mass density λ , a slice of length dx has $dm = \lambda dx$, so $M = \int \lambda dx$ and $x_{cm} = \int x \lambda dx / \int \lambda dx$. Differentiating $\vec{p}_{sys} = M\vec{v}_{cm}$ shows that the center of mass moves as if all the mass sat there and all external force acted there.

Center of mass and system momentum

$x_{cm} = \frac{\sum m_i x_i}{M}$ for point masses; $x_{cm} = \frac{1}{M} \int x dm$ for a continuous object.

Rod: $dm = \lambda dx$. If λ is larger toward one end, the center of mass sits closer to that end.

$\vec{p}_{sys} = M\vec{v}_{cm}$ and $\vec{F}_{ext,net} = M\vec{a}_{cm}$.



A rod whose linear density grows linearly from 0 at $x = 0$ to the right end (shown by the wedge). The shaded slice has $dm = \lambda dx$. For $\lambda = a x$ the center of mass is at $2L/3$, to the heavy side of the midpoint.

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4.2 Change in Momentum and Impulse

Newton's second law in its general form is $\vec{F}_{net} = d\vec{p}/dt$. Integrating over a time interval gives the **impulse-momentum theorem**. Impulse is the integral of the net force, which is the signed area under the force-time graph when the force varies. The exam expects you to set up this integral, with limits, for a force given as a function of time.

Impulse-momentum theorem

$$\vec{J} = \int_{t_1}^{t_2} \vec{F}_{net} dt = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i \text{ (final minus initial, with signs).}$$

Average force: $F_{avg} = J/\Delta t$. Same Δp over a longer time means a smaller peak and average force.

Units: $\text{N s} = \text{kg m/s}$. The area under an $F(t)$ graph is impulse, never momentum itself.

Reversing direction doubles the effect: a ball rebounding elastically at speed v receives $|J| = 2mv$.

4.3 Conservation of Linear Momentum

Choose a system so that the net external impulse on it is zero (or negligible during a brief collision). Then $\vec{p}_{sys}$ is constant, component by component: $\sum p_x$ and $\sum p_y$ are each conserved separately. Internal forces are third-law pairs, so they redistribute momentum without changing the total. In an explosion starting at rest the total stays zero, so the fragments' momenta cancel, and the lighter fragment gets the larger speed and the larger kinetic energy. Say in your answer that external impulses are negligible during the brief collision.

4.4 Elastic and Inelastic Collisions

Type	Momentum	Kinetic energy	Notes
Elastic	Conserved	Conserved	In 1-D, relative speed of approach equals relative speed of separation
Inelastic	Conserved	Decreases	Objects separate; some energy becomes thermal or deformation
Perfectly inelastic	Conserved	Maximum loss allowed	Objects share one final velocity (stick together)

In a one-dimensional elastic collision you may combine the momentum equation with $v_{1i} - v_{2i} = -(v_{1f} - v_{2f})$ to avoid solving a quadratic. When a mass m_1 moving at v sticks to a stationary m_2 , the final speed is $v' = m_1 v / (m_1 + m_2)$ and the fraction of kinetic energy lost is $m_2 / (m_1 + m_2)$. In two dimensions write one equation per axis, and use K only to classify the collision; a K that increases signals an arithmetic error.

Collision toolkit

1. Define the system and state why external impulse is negligible.
2. Pick axes and signs.
3. Write $\sum p_x$ and $\sum p_y$.
4. Classify with K_i and K_f .
5. Check signs and plausibility.

Perfectly inelastic: $K_{lost}/K_i = m_2/(m_1 + m_2)$ when m_2 starts at rest.

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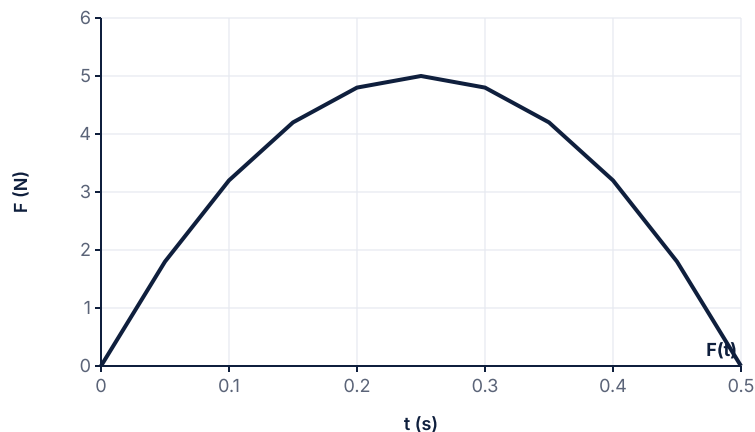
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Worked Examples

IMPULSE AS AN INTEGRAL

A 0.50 kg cart on a frictionless track starts at rest. A push gives it a net force $F(t) = 40t - 80t^2$ (SI units) for $0 \leq t \leq 0.50$ s. Find the impulse, the final speed, the average force and the peak force.



The area under this curve is the impulse, 1.67 N s.

SOLUTION

The force reaches zero at $t = 0.50$ s, so the push ends there. Impulse is the integral: $J = \int_0^{0.50} (40t - 80t^2) dt = [20t^2 - \frac{80}{3}t^3]_0^{0.50} = 5.00 - 3.33 = 1.67$ N s.

Starting from rest, $J = mv_f$, so $v_f = 1.67/0.50 = 3.3$ m/s.

$F_{avg} = J/\Delta t = 1.67/0.50 = 3.3$ N. Peak force: $dF/dt = 40 - 160t = 0$ gives $t = 0.25$ s and $F = 10 - 5 = 5.0$ N, larger than the average, as it must be for a pulse that rises and falls.

CENTER OF MASS BY INTEGRATION

A thin rod of length 1.5 m has linear density $\lambda = ax$ with $a = 4.0$ kg/m² and x measured from the light end. (a) Find its mass and center of mass. (b) A 1.5 kg ball is fixed at the heavy end. Where is the center of mass of the rod plus ball?

SOLUTION

(a) $M = \int_0^L ax dx = aL^2/2 = 4.0(2.25)/2 = 4.5$ kg. Then $x_{cm} = \frac{1}{M} \int_0^L ax^2 dx = \frac{aL^3/3}{aL^2/2} = \frac{2L}{3} = 1.0$ m from the light end, past the midpoint (0.75 m) as expected.

(b) Treat the rod as a point mass of 4.5 kg at 1.0 m: $x_{cm} = \frac{4.5(1.0) + 1.5(1.5)}{4.5 + 1.5} = \frac{6.75}{6.0} = 1.125$ m.

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A PERFECTLY INELASTIC COLLISION AND A SWING

A 0.020 kg bullet strikes a 2.0 kg block hanging from light strings and embeds in it. The block rises 0.10 m above its starting height. Find the bullet's speed and the fraction of kinetic energy lost in the collision.



Left: the bullet embeds in the block (momentum conserved). Right: block and bullet swing up a height $h = 0.10$ m (mechanical energy conserved).

SOLUTION

Two stages need two different principles. **Collision** (momentum conserved, energy not): $mv = (m + M)V$. **Swing** (energy conserved, momentum not, since the strings and gravity act): $\frac{1}{2}(m + M)V^2 = (m + M)gh$.

$V = \sqrt{2gh} = \sqrt{2(10)(0.10)} = 1.41$ m/s. Then $v = V(m + M)/m = 1.41(2.02)/0.020 = 1.4 \times 10^2$ m/s.

Fraction lost = $M/(m + M) = 2.0/2.02 \approx 0.99$, so 99 percent of the bullet's kinetic energy becomes thermal and deformation energy.

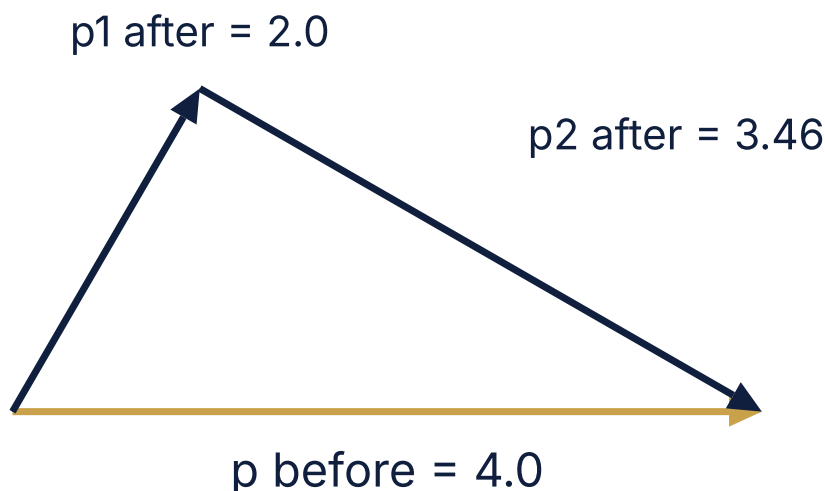
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A TWO-DIMENSIONAL COLLISION

A 1.0 kg puck moving at 4.0 m/s in the $+x$ direction strikes a 2.0 kg puck at rest on ice. Afterward the 1.0 kg puck moves at 2.0 m/s, 60° above the $+x$ axis. Find the velocity of the 2.0 kg puck and classify the collision.



Momentum vectors drawn head to tail form a closed triangle: $p_1 \text{ after} + p_2 \text{ after} = p \text{ before}$. Scale: 40 px per kg m/s.

SOLUTION

The ice makes external horizontal impulse negligible, so each component of momentum is conserved. x : $4.0 = 1.0(2.0 \cos 60^\circ) + 2.0v_x$ gives $v_x = 1.5$ m/s. y : $0 = 1.0(2.0 \sin 60^\circ) + 2.0v_y$ gives $v_y = -0.87$ m/s.

Speed = $\sqrt{1.5^2 + 0.87^2} = 1.73$ m/s at 30° below the $+x$ axis. The vector diagram closes into a triangle, which is exactly how to draw momentum before and after.

Kinetic energy: before $\frac{1}{2}(1.0)(16) = 8.0$ J; after $\frac{1}{2}(1.0)(4.0) + \frac{1}{2}(2.0)(3.0) = 2.0 + 3.0 = 5.0$ J. It decreased by 3.0 J, so the collision is inelastic.

Write It Like a 5**Model Response**

Prompt. A block of mass m rests on a frictionless surface. A horizontal force $F(t) = F_0 [1 - (t/T)^2]$ acts for $0 \leq t \leq T$.

(a) Derive the block's speed at $t = T$. (b) Derive the average force. (c) State whether the average force is greater or less than F_0 and justify.

(a) Newton's second law with $F = dp/dt$ gives $\Delta p = \int_0^T F dt$. So $mv_f - 0 =$

$F_0 \int_0^T (1 - t^2/T^2) dt = F_0 (T - T/3) = \frac{2}{3} F_0 T$, and $v_f = \frac{2F_0 T}{3m}$.

(b) $F_{\text{avg}} = J/T = \frac{2}{3} F_0$.

(c) Less than F_0 . The force starts at its maximum F_0 and falls, so it is below F_0 at every later time and so is the average.

What the grader looks for

One point for starting from $\Delta p = \int F dt$ rather than Ft . One for correct limits and antiderivative. One for the symbolic result with m in the denominator. One for impulse divided by time. One for a justification that cites the shape of the force function, not just the number $2/3$. Show the integral line.

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Coach's Playbook

- 1 Name the system and state which external impulses are negligible.
- 2 Sketch before and after with a positive direction; draw the system's total momentum as one arrow.
- 3 For a force given as a function of time, write the definite integral before plugging in numbers.
- 4 Decide what energy does by computing it, never by assuming.
- 5 Check signs, units, and that K did not increase.

Common traps

1. Drawing the pre-collision system momentum as two arrows or with the wrong length. 2. Assuming a post-collision direction without evidence. 3. Using energy conservation through a collision where objects stick. 4. Treating the area under an $F(t)$ graph as momentum instead of change in momentum, or skipping the integral for a varying force.

Quick Check

Q1. Does a longer collision time change the impulse if Δp is fixed? **A:** No; it lowers the average force. **Q2.** Which fragment of a 2-piece explosion from rest has more kinetic energy? **A:** The lighter one, since $K = p^2/2m$ and the momenta are equal in size. **Q3.** What does the area under an $F(t)$ graph give? **A:** The impulse, equal to Δp .

Challenge Corner

Sand falls vertically onto a horizontal conveyor belt at a steady 5.0 kg/s , and the belt moves at constant 2.0 m/s . Find the extra force the motor must apply and compare the motor's power with the rate at which the sand's kinetic energy grows.

Solution. Take the belt plus sand on it as the system and apply $F = dp/dt$ to the horizontal momentum. The belt's speed is constant, so $dp/dt = v dm/dt = 2.0(5.0) = 10 \text{ N}$. The motor power is $Fv = 10(2.0) = 20 \text{ W}$. The kinetic energy grows at $\frac{1}{2}v^2 dm/dt = \frac{1}{2}(4.0)(5.0) = 10 \text{ W}$. Half the motor's work is dissipated as the sand slips on the belt.

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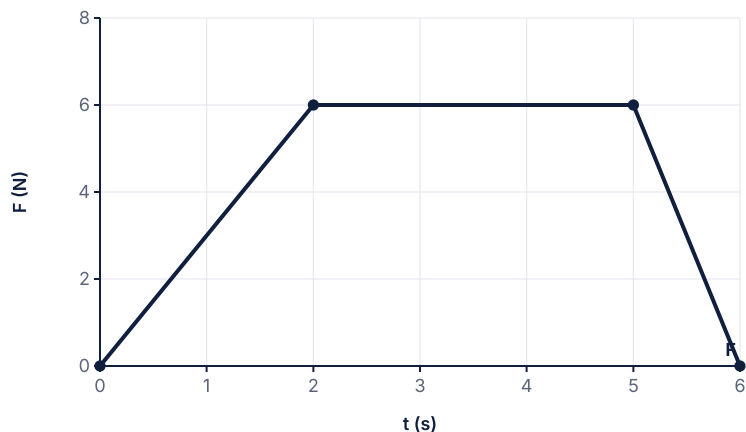
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Practice

PRACTICE Unit 4 Practice

Show your work. A calculator is allowed. Justify conclusions.

1. A 3.0 kg block at rest is pushed by a net force whose graph is shown. What is its speed at $t = 6$ s?



- (A) 4.5 m/s (B) 9.0 m/s (C) 27 m/s (D) 12 m/s
2. A thin rod of length L has linear density $\lambda = kx^2$, with x measured from one end. How far from that end is the center of mass?
 (A) $L/2$ (B) $2L/3$ (C) $3L/4$ (D) $4L/5$
3. A mass m moving at speed v makes a head-on elastic collision with a stationary mass $3m$. What is the velocity of m afterward?
 (A) $v/2$, reversed in direction (B) $v/2$, same direction
 (C) $v/4$, reversed in direction (D) Zero
4. A 1.0 kg cart moving at speed v collides with a stationary 9.0 kg cart, and they stick together. What fraction of the initial kinetic energy is lost?
 (A) 10% (B) 45% (C) 50% (D) 90%
5. The momentum of a particle is $p(t) = 4t^3 - 6t$ (SI units). What is the net force at $t = 1.0$ s?
 (A) 6 N (B) -2 N (C) 12 N (D) 10 N
6. A 3.0 kg object at rest breaks apart. A 1.0 kg piece moves off at 6.0 m/s. How much kinetic energy was released?
 (A) 9 J (B) 18 J (C) 27 J (D) 54 J
7. A block of mass $m = 2.0$ kg starts at rest on a frictionless surface. A force $F(t) = F_0 e^{-t/\tau}$ acts in the $+x$ direction starting at $t = 0$, with $F_0 = 8.0$ N and $\tau = 0.50$ s.
 (a) Derive an expression for the velocity $v(t)$ in terms of m , F_0 , τ and t .
 (b) Calculate the velocity at $t = \tau$ and the limiting velocity as $t \rightarrow \infty$.
 (c) Sketch v versus t , labeling the asymptote, and explain why the slope decreases.
 (d) Derive an expression for the position $x(t)$.
8. A thin rod of length L and total mass M has linear density $\lambda(x) = \lambda_0(1 + x/L)$, where x is measured from the left end and λ_0 is a constant.
 (a) Derive the total mass M in terms of λ_0 and L .
 (b) Derive the position of the center of mass in terms of L .
 (c) A small clay lump of mass $M/3$ is stuck to the right end. Calculate the new center-of-mass position.
 (d) The rod and lump are free in space and a brief external impulse is applied at the center of mass. Explain, without calculation, why the center of mass keeps moving in a straight line even if the object also rotates.

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9. Two carts ride on a low-friction track. Cart 1 ($m_1 = 0.40$ kg) is launched toward stationary cart 2 ($m_2 = 0.60$ kg) with spring bumpers. Photogates record the following speeds (positive to the right).
- Trial: v_{1i} , v_{1f} , v_{2f} in m/s
- 1: 0.50, -0.10, 0.40
- 2: 0.80, -0.17, 0.63
- 3: 1.10, -0.21, 0.87
- (a) For each trial calculate the total momentum before and after. Does the data support conservation of momentum? Justify.
- (b) For each trial calculate the ratio K_f/K_i and decide whether the collision was elastic, inelastic or perfectly inelastic.
- (c) Describe how to use a graph of v_{2f} versus v_{1i} to test the prediction for an elastic collision, giving the predicted slope.
- (d) Identify one source of systematic error that explains any small momentum loss.
10. A 0.50 kg puck moving at 6.0 m/s in the $+x$ direction on frictionless ice hits a stationary 1.5 kg puck. After the collision the 0.50 kg puck moves at 2.0 m/s in the $+y$ direction.
- (a) Draw a vector diagram of the momenta before and after, showing the total initial momentum as a single arrow.
- (b) Calculate the velocity (magnitude and direction) of the 1.5 kg puck afterward.
- (c) Calculate the kinetic energy change of the system and classify the collision.
- (d) A student claims the 1.5 kg puck must move in the $+x$ direction because it was hit from the left. Refute with a principle.

Answers Unit 4: Answers and Explanations

Read every explanation, including the ones for problems you answered correctly.

4 Linear Momentum

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1. B. Impulse is the area: $\frac{1}{2}(2)(6) + 3(6) + \frac{1}{2}(1)(6) = 27$ N s, and $v = J/m = 9.0$ m/s. Choice C forgets to divide by the mass, and D multiplies the peak force by the time and divides by the mass.
2. C. $x_{cm} = \int kx^3 dx / \int kx^2 dx = (L^4/4)/(L^3/3) = 3L/4$. Choice B is the answer for $\lambda \propto x$, and A is what a uniform rod would give.
3. A. Elastic: $v_{1f} = \frac{m-3m}{m+3m}v = -v/2$. The minus sign means the light object rebounds. Choice D is correct only for equal masses, and B ignores the sign.
4. D. $V = v/10$, so $K_f = \frac{1}{2}(10)(v/10)^2 = 0.05v^2$ versus $K_i = 0.5v^2$. That is 10 percent retained and 90 percent lost. Choice A is the fraction retained, and C applies to equal masses.
5. A. $F = dp/dt = 12t^2 - 6$, which is 6 N at $t = 1$. Choice B evaluates p itself instead of differentiating, and C drops the -6 .
6. C. Momentum conservation gives the 2.0 kg piece 3.0 m/s the other way. $K = \frac{1}{2}(1)(36) + \frac{1}{2}(2)(9) = 18 + 9 = 27$ J. Choice B counts only the lighter piece.
7. (a) 2 points: $mv = \int_0^t F_0 e^{-t'/\tau} dt' = F_0 \tau (1 - e^{-t/\tau})$, so $v(t) = \frac{F_0 \tau}{m} (1 - e^{-t/\tau})$.
- (b) 2 points: $F_0 \tau / m = 8.0(0.50)/2.0 = 2.0$ m/s. At $t = \tau$: $2.0(1 - 0.368) = 1.26$ m/s. Limit: 2.0 m/s (total impulse $F_0 \tau = 4.0$ N s).
- (c) 2 points: curve starts at the origin, rises, and flattens toward a horizontal asymptote at 2.0 m/s. The slope is the acceleration F/m , and the force decays exponentially, so the curve rises ever more slowly but never exceeds the asymptote.
- (d) 2 points: $x = \int_0^t v dt' = \frac{F_0 \tau}{m} [t - \tau (1 - e^{-t/\tau})]$.
8. (a) 2 points: $M = \int_0^L \lambda_0 (1 + x/L) dx = \lambda_0 (L + L/2) = \frac{3}{2} \lambda_0 L$.
- (b) 3 points: $x_{cm} = \frac{1}{M} \int_0^L \lambda_0 (x + x^2/L) dx = \frac{\lambda_0 (L^2/2 + L^2/3)}{\frac{3}{2} \lambda_0 L} = \frac{5L/6}{3/2} = \frac{5}{9} L$.
- (c) 2 points: $x'_{cm} = \frac{M(5L/9) + (M/3)L}{4M/3} = \frac{8/9}{4/3} L = \frac{2}{3} L$.
- (d) 1 point: Internal forces cancel in pairs, so the system's momentum changes only through external force; with no net external force after the impulse, $\vec{p}_{sys} = M\vec{v}_{cm}$ is constant, so $\vec{v}_{cm}$ is constant, regardless of rotation about the center of mass.
9. (a) 3 points: Before: 0.200, 0.320, 0.440 kg m/s. After: $0.40(-0.10) + 0.60(0.40) = 0.200$; 0.310; 0.438 kg m/s. Differences are 0 to 3 percent, within photogate and rounding uncertainty, so momentum is conserved to experimental accuracy.
- (b) 3 points: $K_i = 0.050, 0.128, 0.242$ J; $K_f = 0.050, 0.125, 0.236$ J. Ratios 1.00, 0.975, 0.975. Nearly elastic (inelastic by a small amount in trials 2 and 3); not perfectly inelastic because the carts separate.
- (c) 2 points: Elastic prediction: $v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} = 0.80 v_{1i}$. Plot v_{2f} (vertical) against v_{1i} with labeled axes and units; a straight line through the origin of slope 0.80 supports an elastic collision. The best fit of the data gives 0.78, about 2 percent below prediction.
- (d) 1 point: Residual track friction or air drag removes a little momentum from the carts, or the gate flags give slightly inaccurate speeds. Either lowers the measured outgoing momentum and kinetic energy.

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10. (a) 2 points: One arrow of length 3.0 kg m/s along $+x$ for the initial total; head-to-tail arrows 1.0 kg m/s along $+y$ (puck 1) and the closing vector (puck 2) with components (3.0, -1.0) that sum to the initial arrow.
- (b) 3 points: x : $3.0 = 1.5v_x$ so $v_x = 2.0$ m/s. y : $0 = 0.50(2.0) + 1.5v_y$ so $v_y = -0.67$ m/s. Speed 2.1 m/s at 18° below the $+x$ axis.
- (c) 3 points: $K_i = 9.0$ J. $K_f = \frac{1}{2}(0.5)(4.0) + \frac{1}{2}(1.5)(4.44) = 1.0 + 3.33 = 4.33$ J. Change = -4.7 J. Inelastic, because momentum is conserved but kinetic energy fell.
- (d) 2 points: Total p_y was zero before, so it must stay zero. Puck 1 gained upward momentum, so puck 2 must gain equal downward momentum; its direction follows from conservation of each momentum component, not from where it was struck.

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Unit 5: Torque and Rotational Dynamics

Exam weighting: about 10-15% of the multiple-choice section. Concept review, worked examples, exam-style writing, and 10 practice items, all with full explanations.

5 Torque and Rotational Dynamics

Rotational kinematics, torque, rotational inertia by integration, the parallel-axis theorem, equilibrium, and Newton's second law for rotation, with calculus throughout.

Why This Matters

Unit 5 carries about 10 to 15 percent of the multiple-choice section. Rotation mirrors translation, and the calculus is the new part: setting up $\int r^2 dm$ for a rod or disk, and integrating $\alpha(t)$ to get $\omega(t)$. Students lose points by using the wrong rotation axis for I , mixing linear and angular quantities, and writing torque and force equations with inconsistent signs. This book uses $g = 10 \text{ m/s}^2$, the exam default.

Core Ideas

5.1 Rotational Kinematics

Angular position θ is in radians, with counterclockwise positive unless you state otherwise. **Angular velocity** is $\omega = d\theta/dt$ and **angular acceleration** is $\alpha = d\omega/dt$. For a given $\alpha(t)$ you integrate: $\omega(t) = \omega_0 + \int_0^t \alpha dt'$ and $\theta(t) = \theta_0 + \int_0^t \omega dt'$. For constant α the familiar set holds: $\omega = \omega_0 + \alpha t$, $\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$, and $\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$. On an ω - t graph, the slope is α and the area is $\Delta\theta$. Speeding up means ω and α share a sign.

5.2 Connecting Linear and Rotational Motion

A point at distance r from the axis of a rigid body has $s = r\theta$, $v = \omega r$, tangential acceleration $a_t = \alpha r$ and centripetal acceleration $a_c = \omega^2 r$ directed toward the axis. The two are perpendicular. A rope that does not slip on a pulley moves with the rim: speed ωR , acceleration αR . Every point shares the same ω and α .

5.3 Torque

Torque measures a force's ability to change rotation about a chosen axis: $\vec{\tau} = \vec{r} \times \vec{F}$, with magnitude $\tau = rF \sin \phi$, where ϕ is the angle between $\vec{r}$ (from axis to the point of application) and $\vec{F}$. That is F times the lever arm $r \sin \phi$, or r times the perpendicular component of F . A force through the axis gives zero torque. Direction follows the right-hand rule; in a plane, call counterclockwise positive. Torque depends on the axis, so name it.

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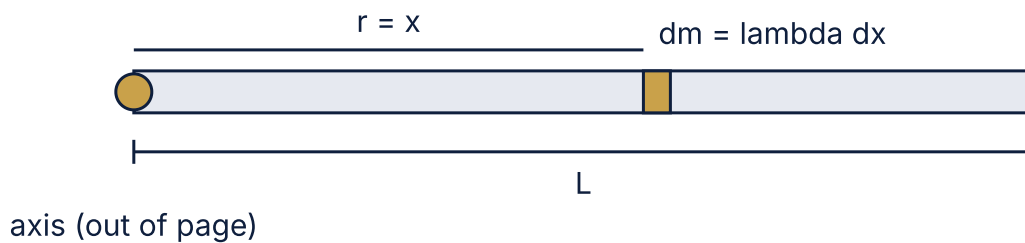
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A force F applied at the end of a wrench of length r , at angle ϕ to the wrench. Only the component $F \sin(\phi)$, perpendicular to r , produces torque: $\tau = r F \sin(\phi)$.

5.4 Rotational Inertia

Rotational inertia is the rotational analogue of mass: $I = \sum m_i r_i^2$ for point masses and $I = \int r^2 dm$ for a continuous body, where r is the distance to the axis. To integrate, choose a slice and write dm in the coordinate: $dm = \lambda dx$ for a rod, $dm = \sigma 2\pi r dr$ for a ring of a disk. For a uniform rod about its end, $I = \int_0^L x^2 (M/L) dx = ML^2/3$. For a uniform disk, summing rings gives $I = \int_0^R r^2 \frac{M}{\pi R^2} 2\pi r dr = \frac{1}{2} MR^2$.



Setup for the rod: the slice dm sits a distance x from the axis, so it contributes $dI = x^2 dm = x^2 \lambda dx$. Integrate from 0 to L .

Body (axis)	I
Uniform rod, through center, perpendicular	$\frac{1}{12} ML^2$
Uniform rod, through one end	$\frac{1}{3} ML^2$
Solid disk or cylinder, through center	$\frac{1}{2} MR^2$
Thin hoop, through center	MR^2
Solid sphere, through center	$\frac{2}{5} MR^2$
Thin spherical shell, through center	$\frac{2}{3} MR^2$

Rotational inertia rules

$I = \int r^2 dm$ about the chosen axis. Point masses add: $I_{tot} = \sum I_i$. Use $dm = \lambda dx$ (rod) or σdA (plate).

Parallel-axis theorem: $I = I_{cm} + Md^2$, where d is the distance between the parallel axes and I_{cm} is about the center of mass. The minimum I for a given direction is at the center of mass.

Do not count on a table of shapes being supplied: be ready to integrate for rods, disks and nonuniform densities.

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5.5 Rotational Equilibrium and Newton's First Law in Rotational Form

A rigid body in equilibrium has $\sum \vec{F} = 0$ and $\sum \vec{\tau} = 0$ about any axis. The torque condition holds for every axis, so choose the axis where the most unknown forces act (a hinge); they then drop out. A uniform beam's weight acts at its center. Use force balance for the hinge force.

5.6 Newton's Second Law in Rotational Form

For a rigid body about a fixed axis, $\sum \tau = I\alpha$, with torques and α signed consistently. In connected problems write three things: $\sum F = ma$ for each translating object, $\sum \tau = I\alpha$ for each rotating object, and the no-slip constraint $a = \alpha R$. A massive pulley has unequal tensions, since $(T_1 - T_2)R$ is what accelerates it.

Rotational dynamics checklist

$\sum \tau = I\alpha$ (about a fixed axis or the center of mass), $\sum F = ma$, and the constraint $a = \alpha R$.

Pick the positive sense of rotation to match the positive linear motion of the connected mass.

Worked Examples

ROTATIONAL KINEMATICS WITH A CHANGING ACCELERATION

A wheel has $\omega_0 = 2.0$ rad/s at $t = 0$ and angular acceleration $\alpha = 3.0t$ (SI units, $\theta_0 = 0$). Find ω and θ at $t = 4.0$ s, and the speed and acceleration components of a point 0.30 m from the axis.

SOLUTION

$\omega(t) = 2.0 + \int_0^t 3.0t' dt' = 2.0 + 1.5t^2$, so $\omega(4.0) = 2.0 + 24 = 26$ rad/s.

$\theta(t) = \int_0^t (2.0 + 1.5t'^2) dt' = 2.0t + 0.50t^3$, so $\theta(4.0) = 8.0 + 32 = 40$ rad.

For $r = 0.30$ m: $v = \omega r = 7.8$ m/s; $a_t = \alpha r = 12(0.30) = 3.6$ m/s²; $a_c = \omega^2 r = 676(0.30) = 2.0 \times 10^2$ m/s². The centripetal part dominates.

ROTATIONAL INERTIA BY INTEGRATION

A rod of mass $M = 3.0$ kg and length $L = 0.90$ m has linear density $\lambda = ax$, with x measured from the left end. Find I about the left end, about the center of mass, and about the right end.

SOLUTION

Mass: $M = \int_0^L ax dx = aL^2/2$, so $a = 2M/L^2$. Left end: $I = \int_0^L x^2(ax) dx = aL^4/4 = \frac{1}{2}ML^2 = \frac{1}{2}(3.0)(0.81) = 1.2$ kg m².

The center of mass is at $2L/3$ (from $\int x dm/M$). Parallel-axis theorem: $I_{cm} = I_{end} - M(2L/3)^2 = (\frac{1}{2} - \frac{4}{9})ML^2 = \frac{1}{18}ML^2 = 0.14$ kg m².

Right end is $L/3$ from the center of mass: $I = I_{cm} + M(L/3)^2 = (\frac{1}{18} + \frac{1}{9})ML^2 = \frac{1}{6}ML^2 = 0.41$ kg m².

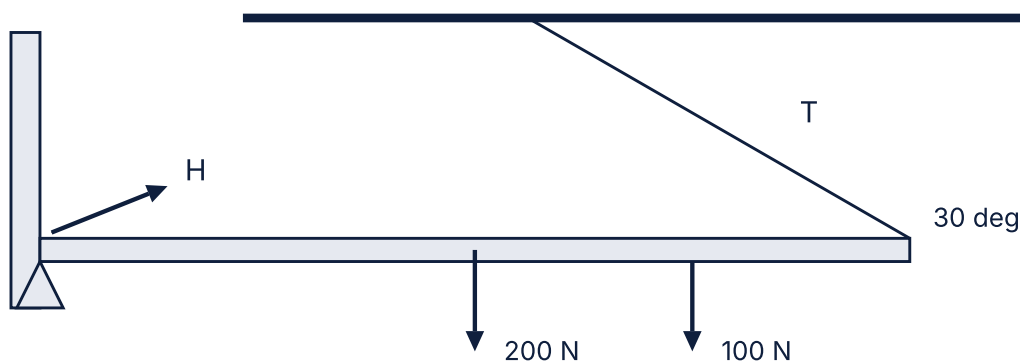
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ROTATIONAL EQUILIBRIUM

A uniform beam of mass 20 kg and length 4.0 m is hinged to a wall and held horizontal by a cable attached to its far end, making 30° with the beam. A 10 kg load hangs 3.0 m from the wall. Find the cable tension and the hinge force.



The beam is 4.0 m long (300 px). The weights act at 2.0 m (beam) and 3.0 m (load). The cable is at 30 degrees above the beam; H is the hinge force, found in the solution.

SOLUTION

Take torques about the hinge so the unknown hinge force drops out. Cable torque (counterclockwise): $T \sin 30^\circ (4.0)$. Weights (clockwise): $(20)(10)(2.0) + (10)(10)(3.0) = 700 \text{ N}\cdot\text{m}$. So $T(0.50)(4.0) = 700$ and $T = 350 \text{ N}$.

Forces: horizontal, the cable pulls toward the wall with $T \cos 30^\circ = 303 \text{ N}$, so the hinge pushes away from the wall with $H_x = 303 \text{ N}$.

Vertical: $H_y + T \sin 30^\circ = 300$, so $H_y = 300 - 175 = 125 \text{ N}$ upward.

Hinge force magnitude $\sqrt{303^2 + 125^2} = 328 \text{ N}$, directed 22° above the horizontal, away from the wall.

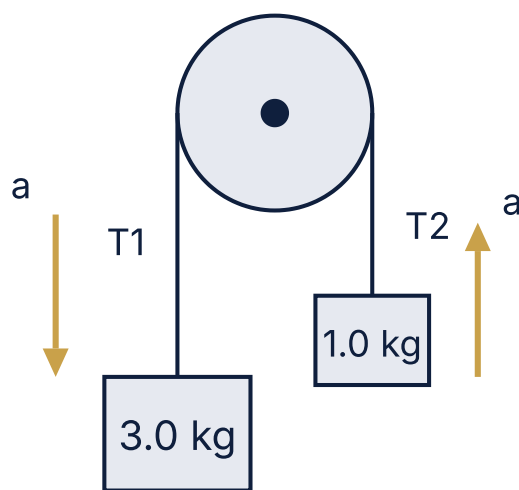
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NEWTON'S SECOND LAW FOR ROTATION: A MASSIVE PULLEY

A 3.0 kg block and a 1.0 kg block hang from a rope over a pulley that is a solid disk of mass 2.0 kg and radius 0.20 m. The rope does not slip. Find the acceleration, both tensions, and the pulley's angular acceleration.



pulley: $M = 2.0 \text{ kg}$, $R = 0.20 \text{ m}$

Atwood machine with a massive pulley. The tensions on the two sides differ; the net torque $(T_1 - T_2)R$ accelerates the pulley.

SOLUTION

Take the heavier block moving down as positive. Block 1: $m_1g - T_1 = m_1a$. Block 2: $T_2 - m_2g = m_2a$. Pulley: $(T_1 - T_2)R = I\alpha = \frac{1}{2}MR^2(a/R)$, so $T_1 - T_2 = \frac{1}{2}Ma$.

Add the three: $(m_1 - m_2)g = (m_1 + m_2 + \frac{1}{2}M)a$, so $a = \frac{20}{3.0 + 1.0 + 1.0} = 4.0 \text{ m/s}^2$.

$T_1 = 3.0(10 - 4.0) = 18 \text{ N}$ and $T_2 = 1.0(10 + 4.0) = 14 \text{ N}$. $\alpha = a/R = 20 \text{ rad/s}^2$. Check: $(18 - 14)(0.20) = 0.80 \text{ N m}$ and $I\alpha = (0.040)(20) = 0.80 \text{ N m}$. A massless pulley would wrongly give 5.0 m/s^2 .

Write It Like a 5**Model Response**

Prompt. A uniform rod of mass M and length L hangs from a frictionless pivot at one end. It is released from rest in a horizontal position. (a) Derive I about the pivot. (b) Derive the angular acceleration just after release. (c) Determine the vertical force the pivot exerts just after release.

(a) With $dm = (M/L) dx$: $I = \int_0^L x^2 \frac{M}{L} dx = \frac{1}{3}ML^2$.

(b) The weight acts at the center, a distance $L/2$ from the pivot and perpendicular to the rod, so $\tau = Mg(L/2)$. Then $\alpha = \tau/I = \frac{MgL/2}{ML^2/3} = \frac{3g}{2L}$.

(c) The center of mass accelerates downward with $a_{cm} = \alpha L/2 = 3g/4$. Newton's second law in the vertical direction: $Mg - F = M(3g/4)$, so $F = Mg/4$ upward.

What the grader looks for

A correct dm with limits. The weight's torque at $L/2$. $\alpha = \tau/I$ with the right I . The link $a_{cm} = \alpha L/2$. A force equation with consistent signs, not $F = Mg$ by habit.

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Coach's Playbook

- 1 Name the axis before computing any torque or I .
- 2 For I of a nonuniform object, write dm first, then $r^2 dm$, then limits.
- 3 Draw a free-body diagram with each force at its point of application.
- 4 In equilibrium, choose the axis at the unknown force you do not need.
- 5 Write translational and rotational equations, add the constraint $a = \alpha R$, and check signs.

Common traps

1. Using I_{cm} when the axis is elsewhere. 2. Using the full force instead of its perpendicular component. 3. Equal tensions on a massive pulley. 4. Sign inconsistency between torque and force equations, or putting g in as the acceleration of a hanging mass.

Quick Check

Q1. Which has larger I about its center, a hoop or a disk of equal mass and radius? **A:** The hoop, MR^2 versus $\frac{1}{2}MR^2$.
Q2. Torque from a force pointing at the axis? **A:** Zero. **Q3.** What does the area under an ω - t graph give? **A:** $\Delta\theta$.

Challenge Corner

A uniform rod of mass 0.80 kg and length 1.2 m, pivoted at one end, is released from horizontal rest. Find α and the pivot force just after release.
Solution. $I = \frac{1}{3}(0.80)(1.2)^2 = 0.384 \text{ kg m}^2$ and $\tau = (0.80)(10)(0.60) = 4.8 \text{ N m}$, so $\alpha = 12.5 \text{ rad/s}^2$. Then $a_{cm} = \alpha(0.60) = 7.5 \text{ m/s}^2$ downward, and $F = m(g - a_{cm}) = 0.80(2.5) = 2.0 \text{ N}$ upward, which is $Mg/4$. The rod is at rest, so there is no centripetal acceleration and no horizontal pivot force.

Practice

PRACTICE Unit 5 Practice

Show your work. A calculator is allowed. Justify conclusions.

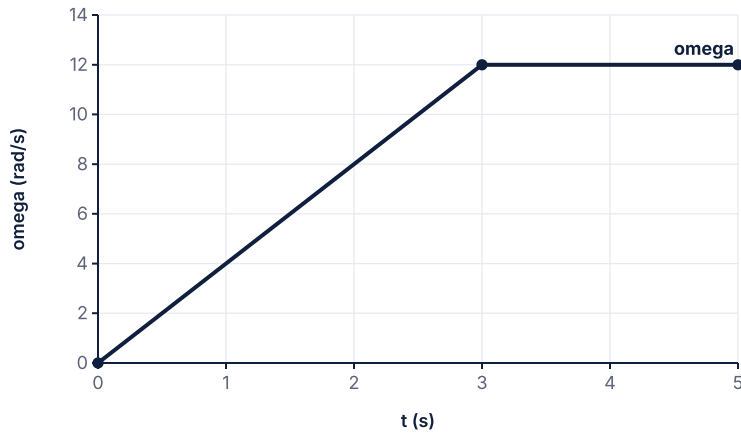
1. A wheel's angular position is $\theta(t) = 5 + 4t - t^3$ (SI units). What is its angular velocity at $t = 2$ s?
 (A) -12 rad/s (B) -8 rad/s (C) 4 rad/s (D) -4 rad/s
2. A uniform disk of mass M and radius R rotates about a perpendicular axis through a point on its rim. What is I ?
 (A) $\frac{1}{2}MR^2$ (B) MR^2 (C) $\frac{3}{2}MR^2$ (D) $2MR^2$
3. A 20 N force acts at the end of a 0.50 m wrench, at 30° to the wrench. What is the torque about the pivot end?
 (A) 5.0 N m (B) 10 N m (C) 8.7 N m (D) 2.5 N m
4. A massless plank pivots at its center. A 40 kg child sits 2.0 m from the pivot. Where must a 50 kg child sit on the other side to balance it?
 (A) 1.25 m (B) 2.0 m (C) 2.5 m (D) 1.6 m
5. The same tangential force F is applied at the rim to a solid disk and to a hoop of equal mass and radius, both starting from rest. What is $\alpha_{\text{disk}}/\alpha_{\text{hoop}}$?
 (A) 2 (B) 1 (C) $1/2$ (D) 4

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6. The angular velocity of a wheel is shown. Through how many radians does it turn from $t = 0$ to $t = 5$ s?



- (A) 30 rad (B) 42 rad (C) 60 rad (D) 12 rad

7. A uniform solid disk of mass $M = 2.0$ kg and radius $R = 0.25$ m rotates freely about a fixed horizontal axle through its center. A light string wrapped around the rim supports a $m = 1.0$ kg block hanging below.
- Derive $I = \frac{1}{2}MR^2$ for the disk by integrating over rings of radius r .
 - Derive an expression for the block's acceleration in terms of m , M and g .
 - Calculate the acceleration, the string tension, and the disk's angular acceleration.
 - Explain why the tension is less than mg .
8. Students measure the rotational inertia of a turntable. A string wrapped around a pulley on the turntable's axle (radius $r = 0.050$ m) is pulled by a hanging mass that is small enough that the hanging mass's acceleration is negligible in computing tension. Hanging masses of increasing size give the torque τ and angular acceleration α below.
- τ (N m): 0.050, 0.100, 0.150, 0.200, 0.250
- α (rad/s²): 0.8, 1.8, 2.7, 3.8, 4.8
- Describe how the torque is determined from measured quantities, and name the instrument for each.
 - Which quantity should be plotted on each axis so that a linear fit gives the rotational inertia as a slope? Write the equation that justifies it, including bearing friction.
 - Use the data to determine I and the friction torque.
 - Explain why the best-fit line does not pass through the origin.
9. A uniform beam of mass $M = 6.0$ kg and length $L = 2.0$ m is hinged to a wall and held horizontal by a cable attached to its outer end at 30° above the beam. A $m = 4.0$ kg sign hangs $d = 1.5$ m from the wall.
- Draw a free-body diagram of the beam.
 - Derive a symbolic expression for the cable tension T in terms of M , m , L , d , g and the cable angle θ .
 - Calculate T and the two components of the hinge force.
 - If the cable were attached at the same point at a steeper angle, would the tension increase or decrease? Justify without calculation.
10. A thin rod of mass $M = 2.0$ kg and length $L = 1.0$ m has linear density $\lambda = \lambda_0(x/L)^2$ with x measured from the pivot end.
- Derive λ_0 in terms of M and L .
 - Derive the rotational inertia about the end at $x = 0$ in terms of M and L .
 - Derive the position of the center of mass, and use the parallel-axis theorem to find I_{cm} .
 - A force perpendicular to the rod acts 0.80 m from the pivot. What force gives the rod an angular acceleration of 2.0 rad/s² about the pivot?

Answers Unit 5: Answers and Explanations

Read every explanation, including the ones for problems you answered correctly.

5 Torque and Rotational Dynamics

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PRACTICE · UNIT 5 PRACTICE

1. B. $\omega = d\theta/dt = 4 - 3t^2 = -8 \text{ rad/s}$ at $t = 2$. Choice A is the angular acceleration, $\alpha = -6t$, and C ignores the t^3 term.
2. C. Parallel-axis theorem: $I = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$. Choice B treats the disk as a hoop, and D is the hoop's value about its rim.
3. A. $\tau = rF \sin 30^\circ = 0.50(20)(0.50) = 5.0 \text{ N m}$. Choice B uses $\sin \phi = 1$, and C uses cosine, which is the parallel component.
4. D. $40(2.0) = 50x$ gives $x = 1.6 \text{ m}$; the heavier child sits closer. Choice C inverts the ratio, and B ignores the unequal masses.
5. A. $\alpha = FR/I$. The disk has half the rotational inertia of the hoop, so its α is double. Choice B wrongly assumes equal torque means equal α .
6. B. The angle is the area: $\frac{1}{2}(3)(12) + 2(12) = 18 + 24 = 42 \text{ rad}$. Choice C multiplies the final ω by 5 s, ignoring the speed-up phase.
7. (a) 2 points: $dm = \sigma 2\pi r dr$ with $\sigma = M/\pi R^2$. $I = \int_0^R r^2 \frac{M}{\pi R^2} 2\pi r dr = \frac{2M}{R^2} \frac{r^4}{4} = \frac{1}{2}MR^2$.
 (b) 3 points: Block: $mg - T = ma$. Disk: $TR = I\alpha = \frac{1}{2}MR^2(a/R)$, so $T = \frac{1}{2}Ma$. Adding: $mg = (m + M/2)a$, so $a = \frac{g}{1 + M/2m}$.
 (c) 3 points: $a = 10/(1 + 1.0) = 5.0 \text{ m/s}^2$. $T = \frac{1}{2}(2.0)(5.0) = 5.0 \text{ N}$. $\alpha = a/R = 20 \text{ rad/s}^2$.
 (d) 2 points: The block accelerates downward, so the net force $mg - T$ on it is downward and nonzero; therefore $T < mg$. Here $T = 5.0 \text{ N}$ versus $mg = 10 \text{ N}$.
8. (a) 2 points: $\tau = rT \approx rmg$ for small hanging masses: mass by a balance, r by calipers (or a ruler) and g known. The tension is approximately mg because the mass accelerates slowly; a more careful analysis uses $T = m(g - a)$.
 (b) 2 points: $\tau - \tau_f = I\alpha$, so $\tau = I\alpha + \tau_f$. Plot τ (vertical) against α (horizontal), with labeled axes and units, so that the slope is I and the vertical intercept is τ_f .
 (c) 3 points: A best-fit line gives slope $\approx 0.050 \text{ kg m}^2$ (for example, using the end points, $(0.250 - 0.050)/(4.8 - 0.8) = 0.050$) and intercept $\approx 0.011 \text{ N m}$. Therefore $I \approx 0.050 \text{ kg m}^2$ and $\tau_f \approx 0.011 \text{ N m}$.
 (d) 1 point: Friction in the bearing opposes rotation, so a nonzero torque τ_f is needed before the turntable begins to accelerate; the intercept on the torque axis equals that torque.
9. (a) 2 points: weight Mg down at the center, mg down at d , T at the outer end at 30° above horizontal toward the wall, hinge force with horizontal and vertical components at the wall end.
 (b) 3 points: Torques about the hinge: $TL \sin \theta = Mg \frac{L}{2} + mgd$, so $T = \frac{g(ML/2 + md)}{L \sin \theta}$.
 (c) 3 points: $T = \frac{10(6.0 + 6.0)}{2.0(0.50)} = 120 \text{ N}$. $H_x = T \cos 30^\circ = 104 \text{ N}$ away from the wall. $H_y = (60 + 40) - 120(0.50) = 40 \text{ N}$ upward.
 (d) 2 points: The weights' torque about the hinge is unchanged and the lever arm of the cable's perpendicular component is $L \sin \theta$. A steeper angle makes $\sin \theta$ larger, so less tension supplies the same torque; T decreases (for example, 69 N at 60°).
10. (a) 1 point: $M = \int_0^L \lambda_0 x^2/L^2 dx = \lambda_0 L/3$, so $\lambda_0 = 3M/L$.
 (b) 3 points: $I = \int_0^L x^2 \lambda_0 \frac{x^2}{L^2} dx = \frac{\lambda_0 L^5}{5} = \frac{3}{5}ML^2$, which is 1.2 kg m^2 .
 (c) 3 points: $x_{cm} = \frac{1}{M} \int x \lambda dx = \frac{\lambda_0 L^3/4}{\lambda_0 L/3} = \frac{3}{4}L$. Then $I_{cm} = \frac{3}{5}ML^2 - M(\frac{3}{4}L)^2 = \frac{3}{80}ML^2 = 0.075 \text{ kg m}^2$.
 (d) 3 points: $\tau = I\alpha = 1.2(2.0) = 2.4 \text{ N m}$. With $\sin \phi = 1$, $F = \tau/r = 2.4/0.80 = 3.0 \text{ N}$.

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Unit 6: Energy and Momentum of Rotating Systems

Exam weighting: about 10-15% of the multiple-choice section. Concept review, worked examples, exam-style writing, and 10 practice items, all with full explanations.

6 Energy and Momentum of Rotating Systems

Rotational kinetic energy, torque and work, angular momentum and its conservation, rolling without slipping, and orbits. Learn where friction belongs in rolling problems.

Why This Matters

Unit 6 carries about 10 to 15 percent of the multiple-choice section, and its ideas anchor the hardest free-response questions. A rolling disk and ring was the lowest-scoring question on the 2025 exam, and a torque-and-work crank problem followed in 2026. Students lose points by treating static friction as $\mu_s N$, by looking for friction in an energy equation, and by omitting the translational or rotational term in a rolling-energy statement. This book uses $g = 10 \text{ m/s}^2$, the exam default.

Core Ideas

6.1 Rotational Kinetic Energy

A rigid body rotating about a fixed axis has $K_{rot} = \frac{1}{2}I\omega^2$, found by adding $\frac{1}{2}m_i v_i^2$ over all particles with $v_i = \omega r_i$. A body whose center of mass also moves has both terms: $K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$.

6.2 Torque and Work

A torque does work as it turns the body: $W = \int \tau d\theta$, which is $\tau\Delta\theta$ for a constant torque, and the power is $P = \tau\omega$. The work-energy theorem becomes $W_{net} = \Delta K_{rot}$. On a τ - θ graph the area is the work. A force that acts at a point that is momentarily at rest does no work, which is why static friction does none on a body that rolls without slipping.

Rotational work and energy

$$K_{rot} = \frac{1}{2}I\omega^2, \quad W = \int_{\theta_1}^{\theta_2} \tau d\theta, \quad P = \tau\omega, \quad W_{net} = \Delta K_{rot}.$$

$$\text{Total energy of rolling: } K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2.$$

6.3 Angular Momentum and Angular Impulse

For a rigid body about a fixed axis, $L = I\omega$. For a single particle, $\vec{L} = \vec{r} \times \vec{p}$ with magnitude $L = mvr \sin \phi = mvd$, where d is the perpendicular distance from the reference point to the line of motion. A particle moving in a straight line has constant L about any fixed point. Newton's second law for rotation in its general form is $\sum \vec{\tau} = d\vec{L}/dt$. Integrating gives the **angular impulse**, $\int \tau dt = \Delta L$, the area under a τ - t graph.

6.4 Conservation of Angular Momentum

If the net external torque about an axis is zero, L about that axis is constant. A collision between a particle and a pivoted body usually conserves L about the pivot but not linear momentum, since the pivot force is external yet produces no torque about the pivot. When I changes, ω changes inversely, and kinetic energy changes too: $K = L^2/2I$. A skater who pulls in her arms speeds up and gains kinetic energy because her muscles do work; parts that stick together lose kinetic energy.

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Conservation of angular momentum

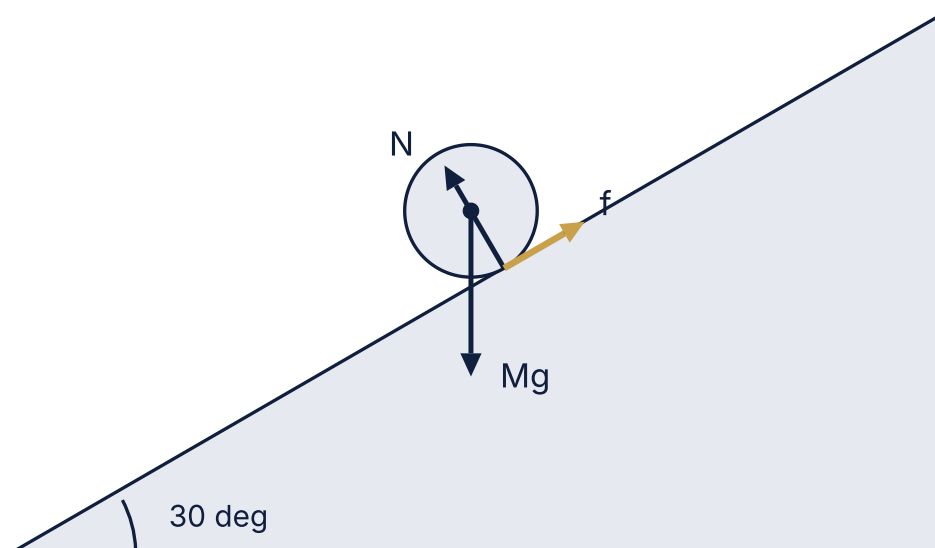
$\sum \tau_{ext} = 0 \Rightarrow L_i = L_f$, so $I_i \omega_i = I_f \omega_f$ for rigid bodies, and mvd summed over particles about the chosen axis.

State the axis, and state why the external torque about it is zero. Kinetic energy is not automatically conserved: $K = L^2/2I$.

6.5 Rolling

Rolling without slipping means the contact point is instantaneously at rest, so $v_{cm} = \omega R$ and $a_{cm} = \alpha R$. Static friction acts, but it does no work. If $I_{cm} = \beta MR^2$ ($\beta = 1$ for a hoop, $\frac{1}{2}$ for a disk, $\frac{2}{5}$ for a solid sphere), then $K = \frac{1}{2}(1 + \beta)Mv_{cm}^2$, and the fraction of kinetic energy that is translational is $1/(1 + \beta)$. Down an incline of angle θ , translation gives $Mg \sin \theta - f = Ma$ and rotation gives $fR = I\alpha = \beta MR^2(a/R)$.

Together, $a = \frac{g \sin \theta}{1 + \beta}$ and $f = \frac{\beta}{1 + \beta} Mg \sin \theta$, directed **up the slope**. Rolling is possible only if $f \leq \mu_s N$, and friction is generally less than $\mu_s N$: use the equations, not the maximum.



A cylinder rolling down a 30 degree incline without slipping. Static friction f points up the slope, N is perpendicular to the surface, and Mg is down. Friction supplies the torque about the center and does no work.

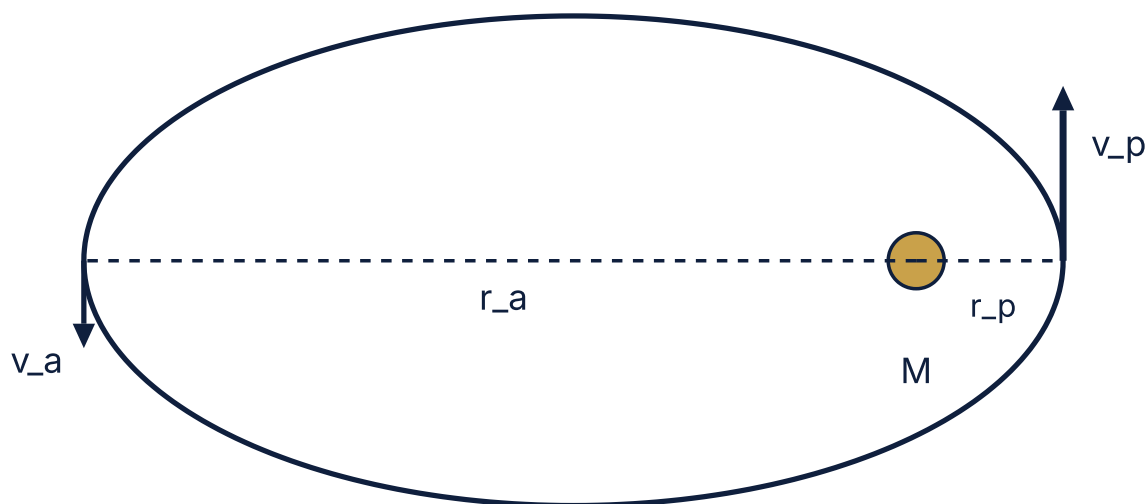
6.6 Motion of Orbiting Satellites

For a satellite of mass m in a circular orbit of radius r around mass M , gravity supplies the centripetal force: $GMm/r^2 = mv^2/r$, so $v = \sqrt{GM/r}$ and $T^2 = 4\pi^2 r^3/GM$. With $U = -GMm/r$ (zero at infinite separation), a circular orbit has $E = -GMm/2r$, so $K = -E$ and $U = 2E$. A bound orbit has $E < 0$. The escape speed from radius r is $\sqrt{2GM/r}$. Gravity is a central force, so $L = mrv \sin \phi$ about the central body is constant, and in an elliptical orbit the satellite is fastest at its closest point: $r_p v_p = r_a v_a$ at perigee and apogee, where velocity is perpendicular to $\vec{r}$. Combine this with energy conservation to find unknowns.

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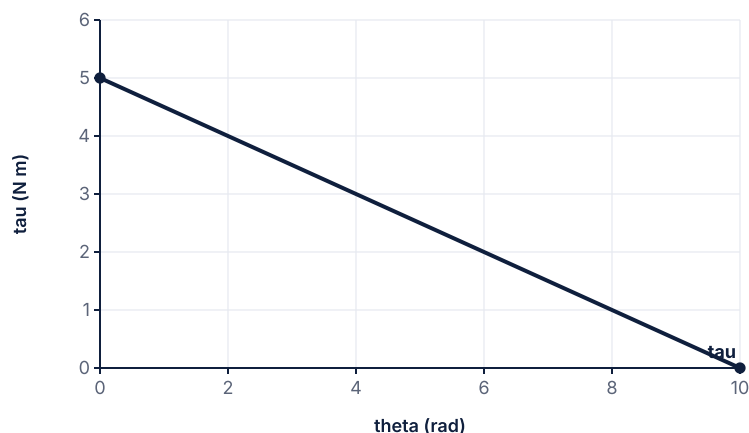


An elliptical orbit about mass M (gold) at one focus. At perigee the satellite is closest and fastest; at apogee it is farthest and slowest. Angular momentum $m r v$ is the same at both points.

Worked Examples

TORQUE AND WORK

A flywheel of rotational inertia 0.50 kg m^2 starts at rest. A motor applies torque $\tau = 5.0 - 0.50\theta$ (SI units) from $\theta = 0$ to $\theta = 10 \text{ rad}$. Find the work done and the final angular speed.



The area under the torque-angle graph is the work: 25 J.

SOLUTION

$W = \int_0^{10} (5.0 - 0.50\theta) d\theta = 5.0(10) - 0.25(100) = 25 \text{ J}$. This is the area of the triangle on the graph, $\frac{1}{2}(10)(5.0)$.

$W_{\text{net}} = \Delta K$ gives $25 = \frac{1}{2}(0.50)\omega^2$, so $\omega = \sqrt{100} = 10 \text{ rad/s}$.

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CONSERVATION OF ANGULAR MOMENTUM

A frictionless turntable has $I = 4.0 \text{ kg m}^2$ and spins at 3.0 rad/s . A 2.0 kg lump of clay is dropped onto it and sticks 0.50 m from the axis. Find the new angular speed and the kinetic energy lost.

SOLUTION

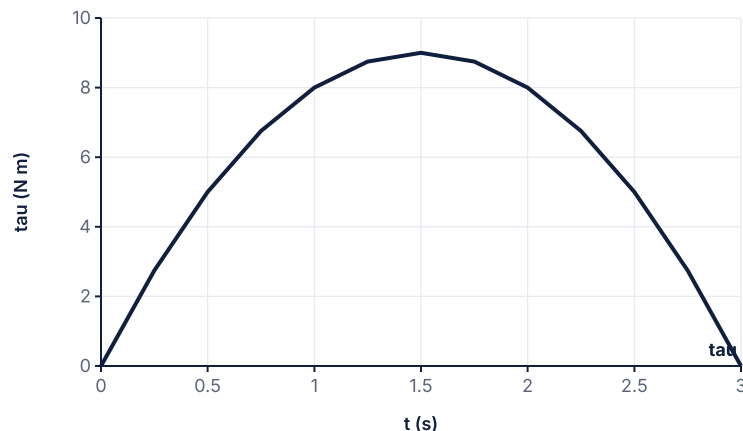
The clay's weight and the bearing's force give no torque about the axis, so L is conserved even though the collision is inelastic.

$$I_f = 4.0 + 2.0(0.50)^2 = 4.5 \text{ kg m}^2.$$

$4.0(3.0) = 4.5\omega_f$ gives $\omega_f = 2.7 \text{ rad/s}$. Energy: $K_i = \frac{1}{2}(4.0)(9.0) = 18 \text{ J}$ and $K_f = \frac{1}{2}(4.5)(2.67)^2 = 16 \text{ J}$. The 2.0 J loss (11 percent) is thermal.

ANGULAR IMPULSE

A torque $\tau(t) = 12t - 4t^2$ (SI units) acts on a flywheel with $I = 0.60 \text{ kg m}^2$ at rest, from $t = 0$ to $t = 3.0 \text{ s}$. Find the final angular velocity and the work done.



The area under the torque-time graph is the angular impulse, 18 N m s .

SOLUTION

Angular impulse equals ΔL : $\int_0^3 (12t - 4t^2) dt = [6t^2 - \frac{4}{3}t^3]_0^3 = 54 - 36 = 18 \text{ N m s}$.

$\omega = \Delta L/I = 18/0.60 = 30 \text{ rad/s}$. The work done equals the kinetic energy gained: $W = \frac{1}{2}(0.60)(30)^2 = 270 \text{ J}$.

ROLLING DOWN AN INCLINE

A uniform solid cylinder of mass M rolls without slipping from rest down a 30° incline. Find its acceleration, the friction force, the minimum μ_s , and its speed after descending 1.0 m vertically. Compare with a hoop.

Body	beta	a (m/s ²)	v after 1.0 m drop (m/s)	min μ_s
Solid cylinder	0.50	3.3	3.7	0.19
Hoop	1.0	2.5	3.2	0.29

SOLUTION

For a cylinder $\beta = \frac{1}{2}$, so $a = \frac{g \sin 30^\circ}{1.5} = \frac{5.0}{1.5} = 3.3 \text{ m/s}^2$. Friction: $f = \beta Ma = \frac{1}{2}M(3.3) = 1.7M$ in SI units, up the slope.

Rolling needs $\mu_s N \geq f$ with $N = Mg \cos 30^\circ = 8.66M$; $\mu_s \geq 1.67/8.66 = 0.19$. This is a minimum, not the value of friction.

Energy (static friction does no work): $Mgh = \frac{1}{2}(1 + \beta)Mv^2$, so $v = \sqrt{\frac{2gh}{1.5}} = \sqrt{13.3} = 3.7 \text{ m/s}$. For the hoop, $\beta = 1$: $a = 2.5 \text{ m/s}^2$, $v = \sqrt{gh} = 3.2 \text{ m/s}$, and $\mu_s \geq 0.29$.

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Write It Like a 5

Model Response

Prompt. A uniform rod of mass M and length L lies on a frictionless horizontal table and is pivoted about a vertical axle at one end. A clay ball of mass m moving horizontally at speed v , perpendicular to the rod, strikes the free end and sticks. (a) Derive the angular speed just after. (b) Explain why linear momentum is not conserved but angular momentum is. (c) Determine the fraction of kinetic energy retained.

(a) Take the pivot as the axis. The ball's angular momentum is mvL since its velocity is perpendicular to the lever arm L . After the collision, $I = \frac{1}{3}ML^2 + mL^2$. Conservation gives $mvL = (\frac{1}{3}ML^2 + mL^2)\omega$, so $\omega = \frac{3mv}{L(M + 3m)}$.

(b) The axle exerts an external horizontal force during the impact, so linear momentum is not conserved. That force acts at the axis, so its torque about the axis is zero, and the net external torque is zero: L is conserved.

(c) $K_f/K_i = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}mv^2} = \frac{3m}{M + 3m}$, which is less than 1 because the ball sticks.

What the grader looks for

The axis named. $L_{ball} = mvL$ with the perpendicular lever arm. The combined I and a clean result. The axle force linked to zero torque. A kinetic-energy ratio with a reason for the loss.

Coach's Playbook

- 1 Choose the principle: collisions with pivots use L ; speeds after a drop use energy.
- 2 For rolling, write energy and the translation and rotation equations separately. Get friction from the force equations, never from energy.
- 3 Always state the axis and list the torques about it.
- 4 For orbits, write $r_p v_p = r_a v_a$ and the energy equation, then solve the pair.

Common traps

1. Using $f = \mu_s N$ for a body that rolls without slipping, or placing friction down the slope. 2. Looking for friction in an energy equation, or giving static friction work. 3. Leaving out the translational or the rotational term in K for rolling. 4. Putting g in as the acceleration, or writing translational and rotational equations with inconsistent signs.

Quick Check

Q1. What fraction of a rolling hoop's kinetic energy is rotational? **A:** One half. **Q2.** Is the work done by static friction on a rolling wheel positive? **A:** No, it is zero. **Q3.** A satellite moves from perigee to apogee; does L change? **A:** No, gravity gives no torque about the central body.

Challenge Corner

A satellite passes perigee at $r_p = 7.0 \times 10^6$ m with speed 9.0×10^3 m/s around a planet of mass 6.0×10^{24} kg ($G = 6.67 \times 10^{-11}$ in SI). Find its apogee distance and speed.

Solution. $GM = 4.00 \times 10^{14}$. Angular momentum per mass: $r_p v_p = 6.3 \times 10^{10}$ m²/s, so $v_a = 6.3 \times 10^{10}/r_a$. Energy per mass: $\frac{1}{2}(9.0 \times 10^3)^2 - GM/r_p = -1.667 \times 10^7$ J/kg. Set $\frac{1}{2}(6.3 \times 10^{10}/r)^2 - GM/r = -1.667 \times 10^7$. The roots are $r = 7.0 \times 10^6$ m (the perigee, as a check) and $r_a = 1.7 \times 10^7$ m. Then $v_a = 3.7 \times 10^3$ m/s.

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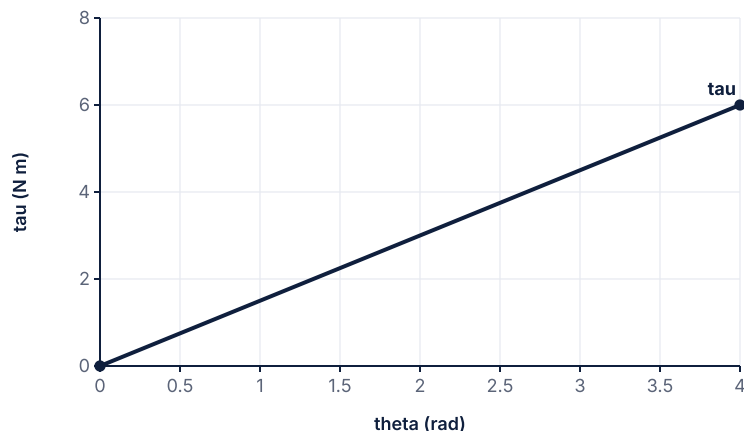
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Practice

PRACTICE Unit 6 Practice

Show your work. A calculator is allowed. Justify conclusions.

1. A uniform solid disk of mass 4.0 kg and radius 0.50 m rotates at 6.0 rad/s about its central axis. What is its rotational kinetic energy?
 (A) 4.5 J (B) 18 J (C) 9.0 J (D) 36 J
2. A wheel with $I = 0.25 \text{ kg m}^2$ starts at rest. The torque graph is shown. What is its angular speed after turning 4.0 rad?



- (A) 9.8 rad/s (B) 6.9 rad/s (C) 48 rad/s (D) 96 rad/s
3. A skater spins with arms extended, then pulls them in so that her rotational inertia drops to half. Compared with her initial kinetic energy, her final kinetic energy is
 (A) half (B) the same (C) four times (D) twice
4. A solid sphere ($I = \frac{2}{5}MR^2$) rolls without slipping on a level floor. What fraction of its kinetic energy is translational?
 (A) 2/5 (B) 5/7 (C) 7/10 (D) 1/2
5. A satellite in an elliptical orbit has apogee distance three times its perigee distance. What is v_p/v_a ?
 (A) 1/3 (B) 9 (C) 3 (D) $\sqrt{3}$
6. A ball rolls without slipping down an incline. Which statement about the friction on the ball is correct?
 (A) Friction does negative work, so mechanical energy decreases.
 (B) Friction points down the slope because it opposes the motion.
 (C) Friction has magnitude $\mu_s N$.
 (D) Friction points up the slope and does no work.
7. A uniform disk A (mass 2.0 kg, radius 0.30 m) spins freely about its central vertical axle at 40 rad/s. A second uniform disk B (mass 1.0 kg, radius 0.30 m), initially at rest, is dropped on top of it so they share the axle, and they spin together.
 (a) Derive an expression for the final angular speed in terms of the rotational inertias.
 (b) Calculate the final angular speed.
 (c) Calculate the fraction of the initial kinetic energy lost, and explain where it goes.
 (d) Sketch angular speed of disk A versus time through the event, and justify the shape.
8. A uniform hoop of mass M and radius R is released from rest on a rough incline at 37° (take $\sin 37^\circ = 0.60$, $\cos 37^\circ = 0.80$) and rolls without slipping.
 (a) Draw a free-body diagram of the hoop.
 (b) Derive the acceleration of its center and the friction force in terms of M , g and θ .
 (c) Calculate the minimum coefficient of static friction needed.
 (d) A student proposes using energy conservation to find the friction force. Explain why this fails.
 (e) A solid disk of the same mass and radius is released on the same incline. Which reaches the bottom first? Justify.

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9. Students release a solid cylinder from rest at several heights h (measured vertically) on an incline and measure its speed v at the bottom with a photogate. They want to determine the shape factor β in $I = \beta MR^2$.
- h (m): 0.050, 0.100, 0.150, 0.200, 0.250
- v^2 (m^2/s^2): 0.66, 1.33, 2.01, 2.65, 3.35
- (a) Derive an expression for v^2 in terms of h , g and β for rolling without slipping.
- (b) Which quantities should be plotted to obtain a straight line, and what does the slope represent?
- (c) Use the data to determine β . Does it match a solid cylinder?
- (d) Identify one reason the students' measured speeds might be systematically low.
10. A satellite orbits a planet of mass $M = 6.0 \times 10^{24}$ kg ($G = 6.67 \times 10^{-11}$ in SI). At perigee its distance is $r_p = 7.0 \times 10^6$ m and its speed is $v_p = 9.0 \times 10^3$ m/s, directed perpendicular to the radius.
- (a) Justify that the satellite's angular momentum about the planet's center is constant.
- (b) Derive, but do not solve, an equation that determines the apogee distance r_a .
- (c) Calculate r_a and the speed at apogee.
- (d) Show that the orbit is bound but not circular.

Answers**Unit 6: Answers and Explanations**

Read every explanation, including the ones for problems you answered correctly.

6 Energy and Momentum of Rotating Systems**PRACTICE · UNIT 6 PRACTICE**

1. C. $I = \frac{1}{2}(4.0)(0.25) = 0.50$ kg m^2 , so $K = \frac{1}{2}(0.50)(36) = 9.0$ J. Choice B uses $I = MR^2$, the hoop value.
2. A. Work = area = $\frac{1}{2}(4)(6) = 12$ J, and $12 = \frac{1}{2}(0.25)\omega^2$ gives $\omega^2 = 96$, so $\omega = 9.8$ rad/s. Choice B omits the factor of one half, and D stops at ω^2 .
3. D. L is conserved, so ω doubles and $K = L^2/2I$ doubles. Choice B confuses conservation of L with conservation of K ; the skater does positive work pulling her arms inward.
4. B. $K_{\text{trans}} = \frac{1}{2}Mv^2$ and $K_{\text{rot}} = \frac{1}{2}(\frac{2}{5}MR^2)(v/R)^2 = \frac{1}{5}Mv^2$, so the translational fraction is $0.5/0.7 = 5/7$. Choice A is just the shape factor, not an energy fraction.
5. C. $r_p v_p = r_a v_a$ gives $v_p/v_a = r_a/r_p = 3$. Choice B squares the ratio and D takes its square root; neither follows from $L = mrv$.
6. D. Friction must supply the torque that spins the ball up, so it points up the slope; the contact point is at rest, so it does no work and mechanical energy is conserved. Choice B is the sliding-friction intuition, and C holds only at the limit of slipping.
7. (a) 2 points: No external torque about the axle, so $I_A \omega_0 = (I_A + I_B) \omega_f$ and $\omega_f = \frac{I_A}{I_A + I_B} \omega_0$.
- (b) 2 points: $I_A = \frac{1}{2}(2.0)(0.090) = 0.090$ kg m^2 and $I_B = 0.045$ kg m^2 . $\omega_f = (0.090/0.135)(40) = 27$ rad/s.
- (c) 3 points: $K_i = \frac{1}{2}(0.090)(1600) = 72$ J and $K_f = \frac{1}{2}(0.135)(26.7)^2 = 48$ J. The loss is 24 J, or $1/3$ of K_i . It becomes thermal energy as kinetic friction between the disks acts until they match speeds.
- (d) 3 points: A horizontal line at 40 rad/s, then a smooth, rapid decrease while the disks slip, then a horizontal line at 27 rad/s. The drop is smooth rather than a gap because friction exerts a torque for a finite time on A, reducing its ω while increasing B's.
8. (a) 2 points: Mg down from the center, normal force perpendicular to the incline, static friction up the slope at the contact point.
- (b) 3 points: Translation: $Mg \sin \theta - f = Ma$. Rotation: $fR = I\alpha = MR^2(a/R)$, so $f = Ma$. Then $a = \frac{1}{2}g \sin \theta$ and $f = \frac{1}{2}Mg \sin \theta$ (up the slope).
- (c) 2 points: $a = 3.0$ m/s², $f = 3.0M$, $N = Mg \cos \theta = 8.0M$, so $\mu_s \geq f/N = 0.375$ (equal to $\frac{1}{2} \tan \theta$).
- (d) 1 point: Static friction acts at a point at rest and does no work, so it does not appear in the energy equation; energy yields the speed, not the force.
- (e) 2 points: The disk has smaller β , so $a = g \sin \theta / (1 + \frac{1}{2}) = 4.0$ m/s² $>$ 3.0 m/s², and it reaches the bottom first.
9. (a) 2 points: $mgh = \frac{1}{2}mv^2 + \frac{1}{2}\beta mR^2(v/R)^2$, so $v^2 = \frac{2g}{1+\beta}h$.
- (b) 2 points: Plot v^2 (vertical, m^2/s^2) versus h (horizontal, m). The slope equals $2g/(1+\beta)$. A plot of v versus h would curve.
- (c) 3 points: The best fit has slope about 13.4 m/s² (for example, using the endpoints, $(3.35 - 0.66)/0.200 = 13.45$). Then $1 + \beta = 20/13.4 = 1.49$ and $\beta \approx 0.49$, which agrees with the solid cylinder's 0.50 within the data's precision.
- (d) 1 point: Rolling resistance or air drag removes energy so that v is less than ideal; or the photogate measures an average speed across the flag that is slightly below the speed at the bottom of the slope.
10. (a) 2 points: Gravity is a central force directed along $\vec{r}$ from the planet's center, so its torque $\vec{r} \times \vec{F}$ about that point is zero; therefore $L = mrv \sin \phi$ is constant.
- (b) 3 points: $r_p v_p = r_a v_a$ gives $v_a = r_p v_p / r_a$, and energy conservation: $\frac{1}{2}v_p^2 - GM/r_p = \frac{1}{2}(r_p v_p / r_a)^2 - GM/r_a$.
- (c) 3 points: $GM = 4.00 \times 10^{14}$. The right side of the equation equals the energy per mass -1.67×10^7 J/kg. Solving the resulting quadratic in $1/r_a$ (one root is r_p) gives $r_a = 1.7 \times 10^7$ m and $v_a = 3.7 \times 10^3$ m/s.
- (d) 2 points: $E/m = -1.67 \times 10^7 < 0$, so the orbit is bound. The circular speed at r_p is $\sqrt{GM/r_p} = 7.6 \times 10^3$ m/s, and $v_p = 9.0 \times 10^3$ is greater, so the satellite moves outward and the orbit is not circular (also, v_p is below the escape speed 1.07×10^4 m/s).

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Unit 7: Oscillations

Exam weighting: about 10-15% of the multiple-choice section. Concept review, worked examples, exam-style writing, and 10 practice items, all with full explanations.

7 Oscillations

Simple harmonic motion with calculus, the period of springs and pendulums including physical pendulums, graphs of x , v and a , and how energy moves between kinetic and potential forms.

Why This Matters

Oscillations make up about 10 to 15 percent of the multiple-choice section, and the unit also feeds free-response questions: a spring and block whose energy you must graph, a pendulum timed in a lab, or a derivation that starts from a differential equation. Students lose points by using the spring formula for a pendulum, by claiming the period depends on amplitude, by drawing negative kinetic energy, and by treating every pendulum as a point mass. This book uses $g = 10 \text{ m/s}^2$; 9.8 is also accepted.

Core Ideas

7.1 Defining Simple Harmonic Motion (SHM)

A **restoring force** acts to return an object to its **equilibrium position**, the place where the net force along the motion is zero. Motion is **simple harmonic** when the net force is proportional to the displacement from equilibrium and opposite to it. An ideal spring is the model case. Newton's second law with $F_x = -kx$ gives a differential equation, and its solutions are sinusoids.

The SHM differential equation

Condition: $F_x = -kx$ for displacement x from equilibrium, so $m \frac{d^2x}{dt^2} = -kx$.

Standard form: $\frac{d^2x}{dt^2} = -\omega^2 x$, with $\omega^2 = k/m$ for a spring-mass system.

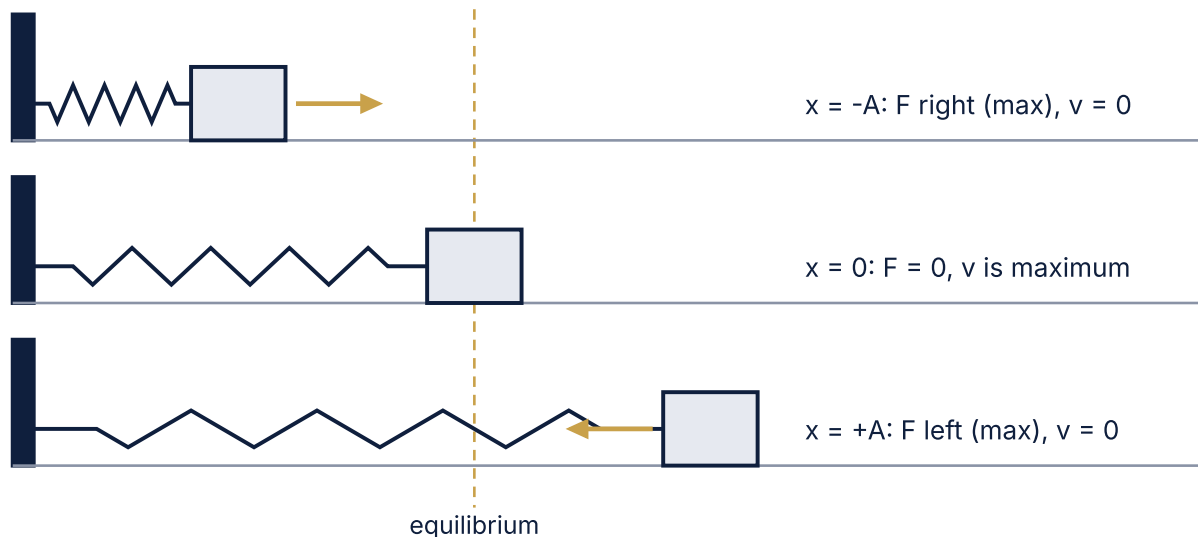
General solution: $x(t) = A \cos(\omega t + \phi)$. The amplitude A and phase constant ϕ come from the starting conditions, not from the system.

You are expected to **verify** a proposed solution. Differentiate twice: $v = -A\omega \sin(\omega t + \phi)$ and $a = -A\omega^2 \cos(\omega t + \phi) = -\omega^2 x$. The last equality is the SHM equation, which proves the sinusoid works. The same derivatives give $v_{\max} = A\omega$ and $a_{\max} = A\omega^2$.

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One spring-mass system at three positions. Gold arrows show the spring force on the block; it always points toward equilibrium.

At the turning points the block is momentarily at rest and force and acceleration are largest; at equilibrium the net force is zero but the speed is greatest. A block on a vertical spring is still SHM: gravity only shifts equilibrium to where $kx_0 = mg$.

7.2 Frequency and Period of SHM

Period, frequency, angular frequency

$T = \frac{1}{f}$ and $\omega = 2\pi f = \frac{2\pi}{T}$, with T in seconds, f in hertz and ω in rad/s.

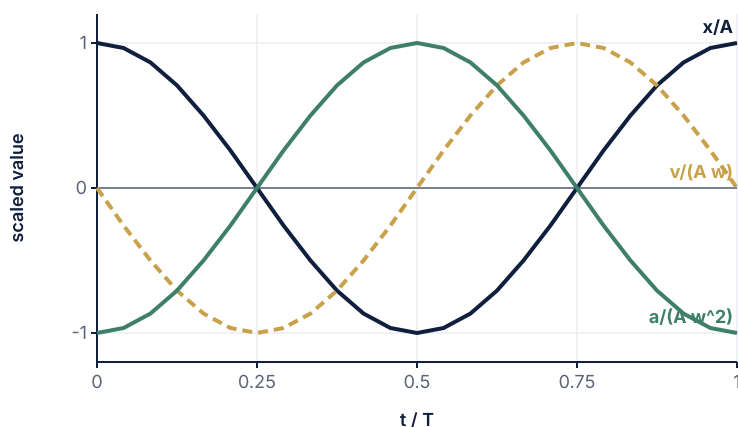
Spring and block: $\omega = \sqrt{k/m}$, so $T = 2\pi\sqrt{m/k}$. Neither depends on amplitude or on g .

Doubling the mass multiplies T by $\sqrt{2}$; quadrupling k halves T .

Springs combine by how they share the displacement. Two springs attached on opposite sides of a block (each stretched as the other compresses) or side by side act in **parallel**: $k_{\text{eff}} = k_1 + k_2$, a stiffer system with a shorter period. Springs joined end to end act in **series**: $\frac{1}{k_{\text{eff}}} = \frac{1}{k_1} + \frac{1}{k_2}$, a softer system with a longer period. Confusing the two is a documented exam error.

7.3 Representing and Analyzing SHM

The graphs of x , v and a are sinusoids shifted by quarter cycles: the slope of $x(t)$ is v and the slope of $v(t)$ is a . The plot is scaled so all three share an axis. Note that a mirrors x , the graphical form of $a = -\omega^2 x$.



Release from rest at $+A$: $x/A = \cos$, $v/(A\omega) = -\sin$, $a/(A\omega^2) = -\cos$ over one period.

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Reading and building SHM descriptions

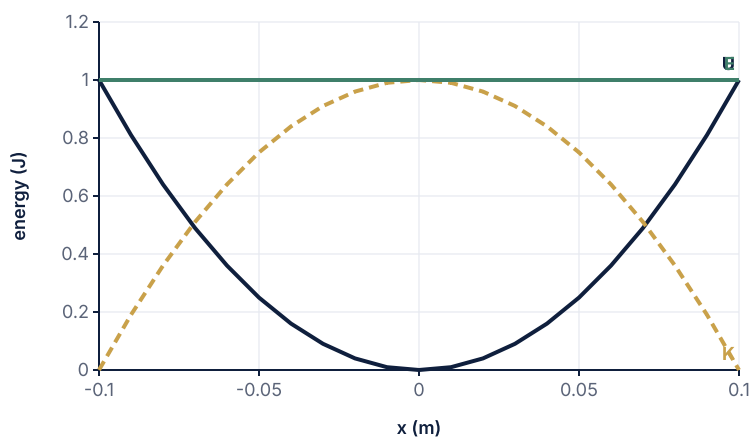
From $x(t) = A \cos(\omega t + \phi)$: read A from the peak, T from the repeat time, and ϕ from the value at $t = 0$.

Speed at any position: $v = \pm \omega \sqrt{A^2 - x^2}$. Starting at $+A$ from rest gives $\phi = 0$; starting at equilibrium moving toward $+x$ gives $x = A \sin \omega t$.

A sketch must show the correct period, amplitude, and a stated rise or fall at each feature.

7.4 Energy of Simple Harmonic Oscillators

For an ideal spring on a frictionless surface the mechanical energy is conserved, $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$. At a turning point all of it is spring energy, so $E = \frac{1}{2}kA^2$. Equating this to the kinetic energy at equilibrium gives $v_{\max} = A\sqrt{k/m} = A\omega$, matching the calculus result. Setting $dE/dt = 0$ recovers $a = -(k/m)x$. With $x = A \cos \omega t$, $U = \frac{1}{2}kA^2 \cos^2 \omega t$ and $K = \frac{1}{2}kA^2 \sin^2 \omega t$.



$k = 200 \text{ N/m}$, $A = 0.10 \text{ m}$. Spring energy U (parabola), kinetic energy K and total E against position.

Energy facts for an ideal oscillator

$E = \frac{1}{2}kA^2$ is constant; U and K are never negative and trade off twice per period.

$K = \frac{1}{2}k(A^2 - x^2)$. $K = U$ at $x = \pm A/\sqrt{2}$, not at $x = \pm A/2$.

Versus time, K and U each oscillate at *twice* the oscillator's frequency. Real damping (friction or drag) removes energy, so A shrinks and T is slightly altered; sketch a decaying envelope and say so.

The same idea extends to any stable equilibrium. Near a minimum of a potential energy curve $U(x)$ at x_0 , a Taylor expansion gives $U \approx U(x_0) + \frac{1}{2}U''(x_0)(x - x_0)^2$, so small oscillations are simple harmonic with effective constant $k_{\text{eff}} = U''(x_0)$. The Challenge Corner uses this.

7.5 Simple and Physical Pendulums

A **simple pendulum** is a point mass on a light string of length L . Gravity's component along the arc, $-mg \sin \theta$, is the restoring force. The

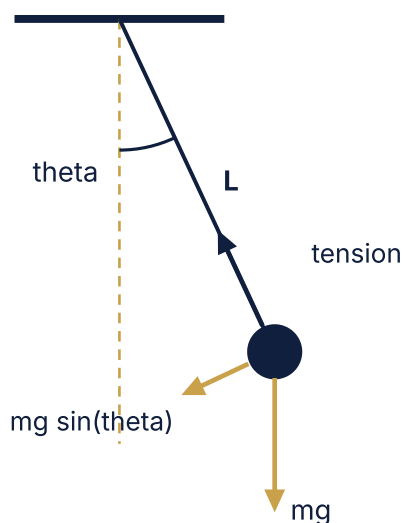
torque form is $-mgL \sin \theta = I\alpha = mL^2 \frac{d^2\theta}{dt^2}$. For small angles (about 15 degrees or less), $\sin \theta \approx \theta$ in radians, and the equation becomes

$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$, which is SHM with $\omega = \sqrt{g/L}$.

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Restoring force along the arc:
 $-mg \sin(\theta)$, toward $\theta = 0$

Small angle: $\sin(\theta) = \theta$,
 so $T = 2\pi \sqrt{L/g}$

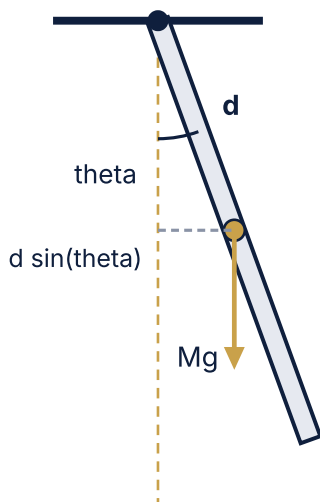
Simple pendulum displaced by angle θ . The tension has no component along the arc, so only $mg \sin(\theta)$ restores the bob.

Pendulum periods (small angles)

Simple pendulum: $T = 2\pi \sqrt{L/g}$. Independent of mass and, for small angles, of amplitude; longer or on a weaker g means a longer period.

Physical pendulum (any rigid body swinging about a pivot): $T = 2\pi \sqrt{\frac{I}{Mgd}}$, where I is the rotational inertia about the pivot, M the total mass and d the pivot-to-center-of-mass distance.

Use the parallel-axis theorem $I = I_{\text{cm}} + Md^2$ to find I about the pivot. The simple pendulum is the special case $I = ML^2$, $d = L$.



Pivot at the top; gold dot is the center of mass.

Torque about pivot:
 $\tau = -Mg d \sin(\theta)$

Small angle: $I \alpha = -Mgd \theta$
 so $T = 2\pi \sqrt{I / (Mgd)}$

Physical pendulum: a rod swinging about a pivot. Weight acts at the center of mass; its lever arm is $d \sin(\theta)$.

Here $\tau = -Mgd \sin \theta \approx -Mgd \theta = I d^2 \theta / dt^2$, so $\omega^2 = Mg d / I$. Use radians for $\sin \theta \approx \theta$; for large amplitudes the period grows.

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Worked Examples

EXAMPLE 1: VERIFY AND USE A POSITION FUNCTION

A block oscillates with $x(t) = (0.080 \text{ m}) \cos(5.0t + \pi/3)$, t in seconds. (a) Show that this satisfies the SHM equation and identify ω , T and f . (b) Find $v_{\max}$ and $a_{\max}$. (c) Find x , v and a at $t = 0$ and $t = 0.20 \text{ s}$.

$t \text{ (s)}$	$x \text{ (m)}$	$v \text{ (m/s)}$	$a \text{ (m/s}^2\text{)}$
0	0.0400	-0.346	-1.000
0.20	-0.0367	-0.355	0.917

SOLUTION

(a) $v = dx/dt = -(0.080)(5.0) \sin(5.0t + \pi/3)$ and $a = dv/dt = -(0.080)(5.0)^2 \cos(5.0t + \pi/3) = -(25)x$. Since $d^2x/dt^2 = -\omega^2x$ with $\omega^2 = 25 \text{ s}^{-2}$, it is SHM with $\omega = 5.0 \text{ rad/s}$, $T = 2\pi/\omega = 1.26 \text{ s}$, $f = 0.80 \text{ Hz}$.

(b) $v_{\max} = A\omega = 0.080(5.0) = 0.40 \text{ m/s}$ and $a_{\max} = A\omega^2 = 0.080(25) = 2.0 \text{ m/s}^2$.

(c) At $t = 0$: $x = 0.080 \cos(\pi/3) = 0.040 \text{ m}$, $v = -0.40 \sin(\pi/3) = -0.346 \text{ m/s}$ (moving toward equilibrium), $a = -25(0.040) = -1.00 \text{ m/s}^2$. At $t = 0.20 \text{ s}$ the phase is $1.0 + 1.047 = 2.047 \text{ rad}$, so the table values follow. Check the signs: a always has the opposite sign of x .

EXAMPLE 2: SPRINGS IN PARALLEL AND IN SERIES

A 0.40 kg block rests on a frictionless table. Spring 1 ($k_1 = 60 \text{ N/m}$) and spring 2 ($k_2 = 140 \text{ N/m}$) are used two ways: (i) one on each side of the block, both attached to walls; (ii) joined end to end with the free end attached to a wall. Find the period for each arrangement.

Arrangement	$k_{\text{eff}} \text{ (N/m)}$	$T \text{ (s)}$
Parallel (both sides)	200	0.281
Series (end to end)	42	0.613

SOLUTION

(i) Displacing the block stretches one spring and compresses the other, and both push the same way, so they are in parallel: $k_{\text{eff}} = 60 + 140 = 200 \text{ N/m}$. $T = 2\pi\sqrt{0.40/200} = 0.281 \text{ s}$.

(ii) End to end, the same force acts on each spring and the extensions add, so they are in series: $1/k_{\text{eff}} = 1/60 + 1/140$, giving $k_{\text{eff}} = 42 \text{ N/m}$. $T = 2\pi\sqrt{0.40/42} = 0.613 \text{ s}$.

EXAMPLE 3: A VERTICAL SPRING-MASS SYSTEM

A 0.80 kg mass hangs at rest from a vertical spring with $k = 160 \text{ N/m}$. It is pulled 0.040 m below equilibrium and released from rest. (a) How far did the spring stretch to reach equilibrium? (b) Find the period and the maximum speed. (c) Show that the total energy of the oscillation is 0.128 J.

Position	Speed	Energy form (about equilibrium)
$y = +0.040 \text{ m}$ (bottom)	0	0.128 J in $\frac{1}{2}ky^2$
$y = 0$	0.566 m/s	0.128 J of kinetic energy

SOLUTION

(a) At equilibrium $kx_0 = mg$, so $x_0 = 0.80(10)/160 = 0.050 \text{ m}$.

(b) Gravity only moves the equilibrium point; the period is $T = 2\pi\sqrt{m/k} = 2\pi\sqrt{0.80/160} = 0.444 \text{ s}$, the same as on a horizontal surface. With $A = 0.040 \text{ m}$ and $\omega = \sqrt{160/0.80} = 14.1 \text{ rad/s}$, $v_{\max} = A\omega = 0.566 \text{ m/s}$ at the equilibrium position.

(c) About the new equilibrium, gravity and spring energy together act like $\frac{1}{2}ky^2$, so $E = \frac{1}{2}kA^2 = \frac{1}{2}(160)(0.040)^2 = 0.128 \text{ J}$, which equals $\frac{1}{2}mv_{\max}^2$.

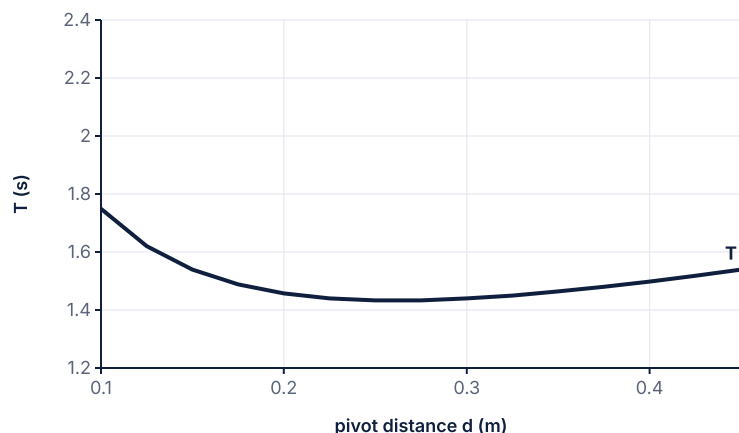
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EXAMPLE 4: PHYSICAL PENDULUM AND THE BEST PIVOT

A uniform rod of mass M and length $L = 0.90$ m is hung from a pivot a distance d from its center. (a) For a pivot at one end, find T and compare with a simple pendulum of length 0.90 m. (b) What pivot distance d gives the shortest period, and what is that period?



Period of a 0.90 m uniform rod against pivot distance from its center; the minimum is near $d = 0.26$ m.

SOLUTION

(a) At the end, $d = L/2 = 0.45$ m and $I = \frac{1}{3}ML^2$. $T = 2\pi\sqrt{\frac{ML^2/3}{Mg(L/2)}} = 2\pi\sqrt{\frac{2L}{3g}} = 2\pi\sqrt{0.060} = 1.54$ s. The simple pendulum gives $2\pi\sqrt{0.90/10} = 1.88$ s. The rod behaves like a simple pendulum of length $2L/3$.

(b) For a pivot at distance d , $I = \frac{1}{12}ML^2 + Md^2$, so $T^2 = \frac{4\pi^2}{g} \left(\frac{L^2}{12d} + d \right)$. Set $\frac{d}{dd} \left(\frac{L^2}{12d} + d \right) = -\frac{L^2}{12d^2} + 1 = 0$, giving $d = \frac{L}{\sqrt{12}} = 0.260$ m. The second derivative is positive, so this is a minimum.

At that d , $\frac{L^2}{12d} = d$, so $T_{\min} = 2\pi\sqrt{2d/g} = 2\pi\sqrt{0.0520} = 1.43$ s. As $d \rightarrow 0$ the period grows without bound.

Write It Like a 5**Model Response**

Prompt. A block of mass m is attached to an ideal horizontal spring of constant k and slides on a frictionless surface.

(a) Write the differential equation for the block's position x measured from equilibrium. (b) Show that $x = A \cos(\omega t)$ is a solution and give ω . (c) Justify whether the period changes if the amplitude is doubled. **Response.** (a) The only horizontal force is the spring force $-kx$, so Newton's second law gives $m d^2x/dt^2 = -kx$, or $d^2x/dt^2 = -(k/m)x$. (b) Differentiating, $dx/dt = -A\omega \sin \omega t$ and $d^2x/dt^2 = -A\omega^2 \cos \omega t = -\omega^2 x$. This matches the equation in (a) only if $\omega^2 = k/m$, so $x = A \cos \omega t$ is a solution with $\omega = \sqrt{k/m}$. (c) The period is $T = 2\pi/\omega = 2\pi\sqrt{m/k}$. It contains only m and k , and A does not appear, so doubling the amplitude does not change the period. A larger amplitude brings a larger restoring force that offsets the longer path.

What the grader looks for

(1) Force with correct sign in second-derivative form. (2) Two derivatives, then *compared* with part (a). (3) ω in terms of k and m . (4) A justification that names the absent variable A .

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Coach's Playbook

- 1 Write the net force along the motion about equilibrium first; if it is not $-kx$ (or $-$ constant times θ for small angles), it is not SHM.
- 2 Pick the right period formula by system: spring-mass, simple pendulum, or physical pendulum. Say which in one phrase.
- 3 For energy questions, find E at a turning point ($\frac{1}{2}kA^2$) and subtract U to get K anywhere else.
- 4 For graph sketches, mark the starting value, the period, and the amplitude, and state whether each feature rises or falls.
- 5 Check limits: v cannot exceed $A\omega$ and $|a|$ cannot exceed $A\omega^2$.

Common traps

1) Using $T = 2\pi\sqrt{m/k}$ for a pendulum (or $\sqrt{L/g}$ for a spring). 2) Saying the period depends on amplitude or, for a pendulum, on the bob's mass. 3) Drawing kinetic energy as negative or as a curve that goes below zero, or treating spring energy as negative for compression. 4) Adding spring constants for springs in series, or taking the reciprocal sum for springs in parallel.

Quick Check

Q1. What is the net force and acceleration at equilibrium? **A:** Both are zero, yet the speed is at its maximum. **Q2.** A spring-mass system is moved to the Moon. Does the period change? **A:** No; $T = 2\pi\sqrt{m/k}$ has no g . **Q3.** At what position does $K = U$? **A:** $x = \pm A/\sqrt{2}$, about 0.71 A .

Challenge Corner

A 0.50 kg particle moves along $x > 0$ with potential energy $U(x) = \frac{a}{x} + bx$, where $a = 0.50$ J m and $b = 2.0$ J/m. Find the equilibrium position, show it is stable, and find the period of small oscillations. **Solution.** $dU/dx = -a/x^2 + b = 0$ gives $x_0 = \sqrt{a/b} = 0.50$ m. The second derivative $U'' = 2a/x^3 = 2(0.50)/(0.50)^3 = 8.0$ N/m is positive, so the equilibrium is stable and $k_{\text{eff}} = 8.0$ N/m. Then $\omega = \sqrt{k_{\text{eff}}/m} = \sqrt{8.0/0.50} = 4.0$ rad/s and $T = 2\pi/\omega = 1.57$ s.

Practice

PRACTICE Unit 7 Practice

Show your work. A calculator is allowed. Justify conclusions.

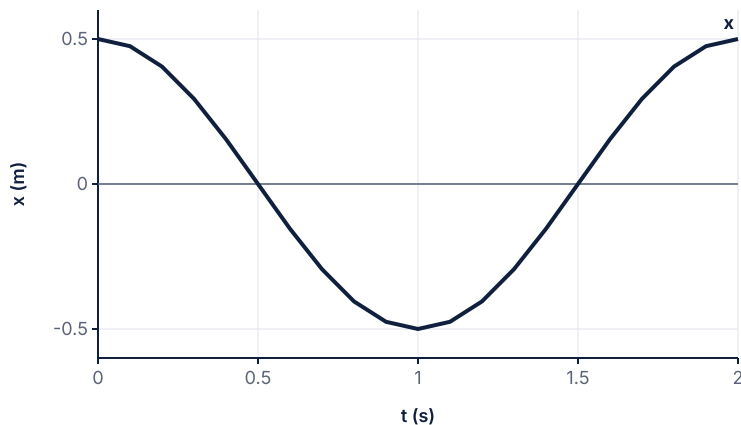
1. A particle's position is $x(t) = 0.12 \cos(8t)$ in meters, with t in seconds. What is the largest magnitude of its acceleration?
 (A) 0.96 m/s² (B) 7.7 m/s² (C) 15 m/s²
 (D) 0.015 m/s²
2. A block of mass m on a frictionless table is attached to walls on opposite sides by two identical springs, each of constant k . With one spring alone the period would be T_0 . What is the period with both springs attached?
 (A) $\sqrt{2} T_0$ (B) $2T_0$ (C) $T_0 / 2$ (D) $T_0 / \sqrt{2}$
3. An ideal spring-mass oscillator has amplitude A . The amplitude is doubled by starting it from rest at twice the distance. Which description is correct?
 (A) The period is unchanged, the maximum speed doubles, and the total energy becomes four times larger.
 (B) The period doubles, the maximum speed doubles, and the total energy doubles.
 (C) The period is unchanged, the maximum speed is unchanged, and the total energy doubles.
 (D) The period doubles, the maximum speed quadruples, and the total energy quadruples.
4. A simple pendulum 1.0 m long hangs from the ceiling of an elevator that is accelerating upward at 2.5 m/s². For small swings, what is its period? (Use $g = 10$ m/s².)
 (A) 1.99 s (B) 2.29 s (C) 1.78 s (D) 1.50 s

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5. A uniform rod of length L swings about a pivot at one end. A simple pendulum has a string of the same length L . For small angles, the ratio $T_{\text{rod}}/T_{\text{simple}}$ is
 (A) 1.22 (B) 0.82 (C) 0.67 (D) 1.00
6. The graph shows the position of an oscillator for one period. Which statement is true at $t = 0.5$ s?



Position of an oscillator for one period.

- (A) The velocity is zero and the acceleration is at its maximum.
 (B) The velocity is positive with maximum speed, and the acceleration is zero.
 (C) The velocity and acceleration are both zero.
 (D) The velocity is negative with maximum speed, and the acceleration is zero.
7. A student wants to find the spring constant of a spring by timing oscillations. Available: the spring, a support stand, a set of known masses, a meter stick and a stopwatch. (a) Describe an experimental procedure, including how to reduce timing error. (b) The data below were taken by hanging each mass and timing 10 full periods. Use them to complete a table of T^2 against m , and state what quantities to graph and why the graph is linear. (c) The best-fit line through the data has slope $0.983 \text{ s}^2/\text{kg}$ and vertical intercept 0.0106 s^2 . Determine the spring constant. (d) Explain physically why the intercept is not zero.
- $m = 0.10 \text{ kg}$: 10 periods took 3.30 s
 $m = 0.20 \text{ kg}$: 10 periods took 4.54 s
 $m = 0.30 \text{ kg}$: 10 periods took 5.54 s
 $m = 0.40 \text{ kg}$: 10 periods took 6.35 s
 $m = 0.50 \text{ kg}$: 10 periods took 7.10 s
 $m = 0.60 \text{ kg}$: 10 periods took 7.74 s
8. A 0.50 kg block on a frictionless table is attached to a spring of constant 50 N/m and released from rest at $x = 0.20 \text{ m}$ from equilibrium. (a) Using $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ and the fact that E is constant, derive the equation for d^2x/dt^2 . (b) Find the angular frequency, the period and the maximum speed. (c) Find the speed when $x = 0.10 \text{ m}$. (d) Describe, with values, a sketch of the kinetic energy against time for one period.
9. A thin uniform rod of mass M and length $L = 1.2 \text{ m}$ is pivoted on a horizontal axis a distance $L/4$ from one end and set swinging through small angles. (a) Use the parallel-axis theorem to find the rotational inertia about the pivot in terms of M and L . (b) Write the torque equation and derive the period. (c) Calculate the period. (d) What length of simple pendulum would have the same period?
10. A vertical spring-mass system (mass 0.40 kg , constant 40 N/m) and a simple pendulum of length 0.40 m are both taken from Earth ($g = 10 \text{ m/s}^2$) to the Moon ($g = 1.6 \text{ m/s}^2$). (a) Predict and justify, using formulas, whether each period increases, decreases or stays the same. (b) Calculate the period of each on Earth and on the Moon. (c) A student claims the spring-mass system oscillates about the same point on both worlds. Evaluate the claim.

Answers Unit 7: Answers and Explanations

Read every explanation, including the ones for problems you answered correctly.

7 Oscillations

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PRACTICE · UNIT 7 PRACTICE

1. B. $a = d^2x/dt^2 = -A\omega^2 \cos \omega t$, so $a_{\max} = 0.12(8)^2 = 7.68 \text{ m/s}^2$. The 0.96 m/s^2 choice is $A\omega$, the maximum speed, which forgets the second derivative.
2. D. Moving the block stretches one spring and compresses the other, so they act in parallel: $k_{\text{eff}} = 2k$, and $T \propto 1/\sqrt{k}$ gives $T_0/\sqrt{2}$. The choice $\sqrt{2}T_0$ would be right only if the springs made the system softer, as series springs do.
3. A. $T = 2\pi\sqrt{m/k}$ has no A ; $v_{\max} = A\omega$ doubles; $E = \frac{1}{2}kA^2$ quadruples. The tempting choice that doubles the energy forgets the square in $\frac{1}{2}kA^2$.
4. C. In the elevator the effective gravity is $g + a = 12.5 \text{ m/s}^2$, so $T = 2\pi\sqrt{1.0/12.5} = 1.78 \text{ s}$. The 1.99 s choice ignores the acceleration, and the 2.29 s choice subtracts it (as if accelerating downward).
5. B. For the rod $I = \frac{1}{3}ML^2$ and $d = L/2$, so $T = 2\pi\sqrt{2L/(3g)}$, and the ratio with $2\pi\sqrt{L/g}$ is $\sqrt{2/3} = 0.82$. The ratio 1.22 is the inverse, and 1.00 treats the rod as a point mass at its end.
6. D. At $t = 0.5 \text{ s}$ the graph passes through $x = 0$ with a steep negative slope, so v is at its negative extreme while $a = -\omega^2 x = 0$. The choice with zero velocity describes a turning point such as $t = 0$ or 1 s .
7. (a) Hang a mass, pull it a small distance down and release it so it oscillates vertically; time 10 full periods with the stopwatch starting and stopping as the mass passes equilibrium, divide by 10 to get T , and repeat for each of several masses. Timing many periods reduces the effect of reaction time. Keep the amplitude small and the same each time. (b) T values: $0.330, 0.454, 0.554, 0.635, 0.710, 0.774 \text{ s}$; T^2 : $0.109, 0.206, 0.307, 0.403, 0.504, 0.599 \text{ s}^2$. Graph T^2 (vertical) against m (horizontal), because $T^2 = \frac{4\pi^2}{k}m$ is a straight line through the origin if the spring is ideal. (c) Slope $= 4\pi^2/k$, so $k = 4\pi^2/0.983 = 40.2 \text{ N/m}$. (d) The intercept shows that the spring itself has mass and part of it moves with the load: $T^2 = \frac{4\pi^2}{k}(m + m_s)$ with $m_s \approx 0.0106/0.983 = 0.011 \text{ kg}$, about one third of the spring's mass. (Point values: (a) 3, (b) 3, (c) 2, (d) 2.)
8. (a) Set $dE/dt = 0$: $mv \frac{dv}{dt} + kx \frac{dx}{dt} = 0$. With $v = dx/dt$ and $a = dv/dt$, this is $mva + kxv = 0$, so for $v \neq 0$, $a = \frac{d^2x}{dt^2} = -\frac{k}{m}x$. (b) $\omega = \sqrt{50/0.50} = 10 \text{ rad/s}$, $T = 2\pi/10 = 0.628 \text{ s}$, $v_{\max} = A\omega = 0.20(10) = 2.0 \text{ m/s}$. (c) $v = \omega\sqrt{A^2 - x^2} = 10\sqrt{0.04 - 0.01} = 1.73 \text{ m/s}$; check by energy: $E = \frac{1}{2}(50)(0.04) = 1.0 \text{ J}$, $U = \frac{1}{2}(50)(0.01) = 0.25 \text{ J}$, $K = 0.75 \text{ J}$, $v = \sqrt{2(0.75)/0.50} = 1.73 \text{ m/s}$. (d) $K = E \sin^2 \omega t = (1.0 \text{ J}) \sin^2(10t)$. It starts at 0, rises to 1.0 J at $T/4 = 0.157 \text{ s}$, returns to 0 at $T/2 = 0.314 \text{ s}$, and repeats; it never goes below zero and has period $T/2$. (Points: (a) 3, (b) 3, (c) 2, (d) 2.)
9. (a) The pivot is $d = L/2 - L/4 = L/4$ from the center. $I = \frac{1}{12}ML^2 + M(L/4)^2 = (\frac{1}{12} + \frac{1}{16})ML^2 = \frac{7}{48}ML^2$. (b) For angle θ from vertical, $\tau = -Mgd \sin \theta \approx -Mg(L/4)\theta = I \frac{d^2\theta}{dt^2}$, so $\frac{d^2\theta}{dt^2} = -\frac{Mg(L/4)}{(7/48)ML^2}\theta = -\frac{12g}{7L}\theta$. This is SHM with $\omega^2 = 12g/(7L)$ and $T = 2\pi\sqrt{\frac{7L}{12g}}$. (c) $T = 2\pi\sqrt{\frac{7(1.2)}{12(10)}} = 2\pi\sqrt{0.070} = 1.66 \text{ s}$. (d) $T = 2\pi\sqrt{L_s/g}$ gives $L_s = 7L/12 = 0.70 \text{ m}$. (Points: (a) 2, (b) 3, (c) 2, (d) 1.)
10. (a) The spring-mass period $T = 2\pi\sqrt{m/k}$ contains no g , so it stays the same. The pendulum period $T = 2\pi\sqrt{L/g}$ has g in the denominator, so a smaller g makes it increase. (b) Spring: $T = 2\pi\sqrt{0.40/40} = 0.628 \text{ s}$ on both. Pendulum: Earth $2\pi\sqrt{0.40/10} = 1.26 \text{ s}$; Moon $2\pi\sqrt{0.40/1.6} = 3.14 \text{ s}$. (c) The claim is false. The equilibrium stretch is $x_0 = mg/k$: 0.10 m on Earth and 0.016 m on the Moon, so the center of oscillation shifts upward, even though the period does not change. (Points: (a) 3, (b) 3, (c) 2.)

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Workshop: Experimental Design, Graphs, and Statistics

Skills that earn points on every free-response section.

W Workshop: Calculus, Graphs, and Lab Analysis

The calculus, linearization and derivation habits behind the Mathematical Routines and Lab free-response questions: integrate rates, separate variables, read slopes and intercepts, weigh uncertainty, and write steps a grader can follow.

Why This Matters

Every free-response question on this exam asks you to do mathematics with physics attached. Two of the four questions are built around it: the Mathematical Routines question rewards a derivation that uses a derivative, an integral, or a separable differential equation, and the Experimental Design and Analysis question rewards a linearized graph whose slope and intercept are tied to a physical constant. Past scoring reports repeat the same losses: students did not notice that calculus was required, plotted v instead of v^2 , left out a limit of integration, or listed equations without a path from one to the next. This workshop collects those habits in one place so that every later unit can lean on them. Use $g = 10 \text{ m/s}^2$ for numerical work; 9.8 m/s^2 is also accepted and is noted where it matters.

Core Ideas

The calculus toolkit physics actually uses

Almost every calculus step in this course is one of five moves. **Differentiate** a position or momentum function: $v = dx/dt$, $a = dv/dt$, $F = dp/dt$. **Integrate** a rate to get an accumulated quantity: $\Delta v = \int a \, dt$, $J = \int F \, dt$, $W = \int F \, dx$. **Chain rule** to change the variable of differentiation: $a = \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}$. **Separate variables** when a force depends on speed. **Verify** a proposed solution by substituting it back into the differential equation.

Power-function rules and what they mean

$$\frac{d}{dt} t^n = n t^{n-1} \text{ and } \int t^n \, dt = \frac{t^{n+1}}{n+1} \text{ for } n \neq -1; \int \frac{dt}{t} = \ln t.$$

$$\text{Exponential and sine: } \frac{d}{dt} e^{-bt} = -b e^{-bt}, \frac{d}{dt} \sin \omega t = \omega \cos \omega t, \frac{d}{dt} \cos \omega t = -\omega \sin \omega t.$$

A definite integral is an area with sign: below the axis counts negative. $\int_{t_1}^{t_2} a \, dt$ is the change in velocity, not the velocity; add the starting value $v(t_1)$ to get $v(t_2)$.

Check units at every line. The integrand times its differential must have the units of the answer: $(\text{N})(\text{m}) = \text{J}$, $(\text{m/s}^2)(\text{s}) = \text{m/s}$.

Slope and area carry units. On a graph of y against x , the slope has units of y/x and the area has units of $y \cdot x$. Slope of $x(t)$ is velocity, area under $a(t)$ is velocity change, slope of $U(x)$ is minus the force, area under $F(x)$ is work. When a graph question says "what does the area represent," multiply the axis units and name the quantity that has those units.

Chain rule in dynamics. When force depends on position, a is naturally a function of x , so write $a = v \, dv/dx$ and separate: $v \, dv = a(x) \, dx$. Integrating from the start gives $\frac{1}{2} v^2 - \frac{1}{2} v_0^2 = \int a \, dx$, which is the work-energy theorem per unit mass. When force depends on speed, use $a = dv/dt$ and separate v from t instead.

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Separable equations for drag and for SHM

Linear drag, $m \frac{dv}{dt} = -bv$: move all v to one side, $\frac{dv}{v} = -\frac{b}{m} dt$, integrate from v_0 at $t = 0$: $\ln \frac{v}{v_0} = -\frac{b}{m} t$, so $v = v_0 e^{-t/\tau}$ with $\tau = m/b$.

Quadratic drag, $m \frac{dv}{dt} = -cv^2$: $\frac{dv}{v^2} = -\frac{c}{m} dt$ gives $\frac{1}{v} = \frac{1}{v_0} + \frac{c}{m} t$. The speed falls as a power law, not an exponential.

SHM: $m \frac{d^2x}{dt^2} = -kx$ is not separable in the same way. You are expected to substitute $x = A \cos(\omega t + \phi)$, differentiate twice, and show the equation holds when $\omega^2 = k/m$.

Always state the initial condition you used to fix the constant of integration.

A definite integral with matching limits (for example v from v_0 to v on the left, t from 0 to t on the right) is the cleanest way to handle the constant, because it cannot be forgotten. Writing $\int v dx$ with v still a function of t is an error; every variable under an integral sign must be a function of the variable of integration.

Linearizing data and reading constants

A straight line is the only graph shape the eye and the grader can judge reliably, so lab analysis rewrites the model as $y = mx + b$ with y and x chosen from the measured quantities. The slope and intercept are then expressions made of the unknown constants.

Model	Plot y vs x	Slope	Intercept
$v^2 = v_0^2 + 2gh$	v^2 vs h	$2g$	v_0^2
$T = 2\pi\sqrt{L/g}$	T^2 vs L	$4\pi^2/g$	0
$T = 2\pi\sqrt{m/k}$	T^2 vs m	$4\pi^2/k$	0
$d = \frac{kx^2}{2\mu mg}$	d vs x^2	$k/(2\mu mg)$	0
$x = x_0 + \frac{1}{2}at^2$	x vs t^2	$a/2$	x_0
$y = Cx^n$	$\ln y$ vs $\ln x$	n	$\ln C$

The recipe is the same every time. (1) Write the theoretical relationship. (2) Solve it so one measured quantity sits alone on the left and the other appears to the first power on the right. (3) Name the plotted variables, with units, and say what the slope and intercept equal. (4) After plotting, choose two points on the best-fit line, not two data points, far apart, and compute the slope with units. (5) Solve for the constant. Graders accept any equivalent linearization, including reciprocals, if the variables are clearly defined and the slope is tied to the constant correctly.

Uncertainty and best-fit-line reasoning

A measurement is a value with a spread. Repeated trials reveal the spread; the instrument resolution sets a floor (about half the smallest division for a ruler). Draw error bars when the uncertainty matters, and judge a best-fit line by whether it passes within the error bars of most points with points scattered above and below it along its whole length, not by whether it touches the dots.

- ◆ **Best-fit line:** passes near the center of the data and balances scatter on both sides. A line forced through the origin is justified only if the model says the intercept is zero; a clear nonzero intercept is information, not a nuisance.
- ◆ **Uncertainty in slope:** draw the steepest and shallowest lines that still pass through the error bars; half their difference estimates the slope uncertainty. Propagate it to the constant, since a constant like $g = 4\pi^2/\text{slope}$ has the same fractional uncertainty as the slope.
- ◆ **Random versus systematic:** random error scatters points about the line and is reduced by repeated trials. Systematic error (a clock that starts late, a ruler that begins at 1 cm) shifts the whole data set and shows up as a surprising intercept.
- ◆ **Compare, do not just subtract:** a result agrees with an accepted value when the value lies within the uncertainty range. Report a percent difference only together with the uncertainty.

Translating between representations

The exam asks you to move among an equation, a graph, a table, a diagram, and a sentence, usually with a derivation attached. Treat each move as a small calculation. From $x(t)$ to a $v(t)$ graph: take the slope at several times and plot it. From $a(t)$ to $v(t)$: accumulate area, starting from the known initial value. From $F(x)$ to a speed: integrate for work, then use $W = \Delta K$. From $U(x)$ to force: $F = -dU/dx$, so the force points toward lower U . From a sketch to a sentence: state the feature and its direction, for example "the speed rises, then levels off at a constant terminal value," never merely "it changes." A kinetic energy graph never goes below zero, and a spring energy graph is never negative for stretch or compression.

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Writing derivations that earn credit

Derivation habits graders reward

Start from a named principle or an equation you can justify ($\sum F = ma$, $F = dp/dt$, $W = \int F dx$), not from the answer.

Define every symbol that is not given, and say which are constants and which depend on the variable of integration.

Show each algebra or calculus step in order, one idea per line, with limits written on definite integrals.

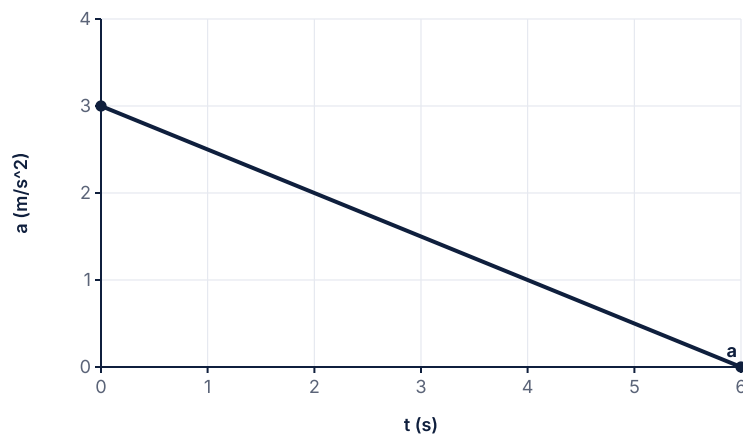
End with the requested expression in the requested variables, then check limits or units: does it reduce correctly when a parameter goes to zero or infinity?

Do not substitute numbers until the symbolic result is finished unless the prompt asks for a number only; a final symbolic expression lets the grader see which step went wrong. When a part says "justify" or "explain," a numerical value alone earns nothing; name the physical principle and link the quantities in a full sentence, such as "static friction does no work on a body that rolls without slipping, so energy conservation cannot supply it."

Worked Examples

EXAMPLE 1: VELOCITY AND POSITION FROM A CHANGING ACCELERATION

A cart starts at $x = 0$ with $v = 2.0$ m/s. For 6.0 s its acceleration is $a(t) = 3.0 - 0.50t$ in m/s^2 with t in seconds (graph shown). Find (a) the change in velocity, (b) the velocity at 6.0 s, (c) the displacement, and (d) the time of maximum speed.



Area under $a(t)$ equals the change in velocity.

SOLUTION

(a) The area under $a(t)$ is a triangle: $\frac{1}{2}(6.0)(3.0) = 9.0$ m/s. By integral, $\int_0^6 (3.0 - 0.50t) dt = 3.0(6) - 0.25(36) = 9.0$ m/s. Units: $(\text{m/s}^2)(\text{s}) = \text{m/s}$.

(b) $v(6) = v_0 + \Delta v = 2.0 + 9.0 = 11$ m/s. Generally $v(t) = 2.0 + 3.0t - 0.25t^2$.

(c) $\Delta x = \int_0^6 (2.0 + 3.0t - 0.25t^2) dt = 2.0(6) + 1.5(36) - \frac{0.25}{3}(216) = 12 + 54 - 18 = 48$ m.

(d) Speed is greatest where $a = 0$ and changes sign: $3.0 - 0.50t = 0$ gives $t = 6.0$ s, the end of the interval. Because $a > 0$ throughout, the cart never slows down.

EXAMPLE 2: LINEAR DRAG BY SEPARATION OF VARIABLES

A 0.50 kg block slides on a rough, oily surface that exerts $F = -bv$ with $b = 0.20$ kg/s and no other horizontal force. It starts at $v_0 = 8.0$ m/s. (a) Derive $v(t)$. (b) Find the time constant and the time for the speed to reach 2.0 m/s. (c) Find the total distance traveled.

SOLUTION

(a) $m dv/dt = -bv$, so $\frac{dv}{v} = -\frac{b}{m} dt$. Integrate with matching limits: $\int_{v_0}^v \frac{dv'}{v'} = -\frac{b}{m} \int_0^t dt'$, giving $\ln(v/v_0) = -bt/m$ and $v = v_0 e^{-bt/m}$.

(b) $\tau = m/b = 0.50/0.20 = 2.5$ s. Set $2.0 = 8.0 e^{-t/2.5}$: $t = 2.5 \ln 4 = 3.47$ s.

(c) Use $v = dx/dt$: $x = \int_0^\infty v_0 e^{-t/\tau} dt = v_0 \tau = 8.0(2.5) = 20$ m. Check by the chain rule: $mv dv/dx = -bv$ gives $dv = -(b/m)dx$, so $\Delta x = mv_0/b = 20$ m, agreeing and showing the block stops only after an infinite time but a finite distance.

Units: b/m has units $(\text{kg/s})/\text{kg} = 1/\text{s}$, so the exponent is dimensionless.

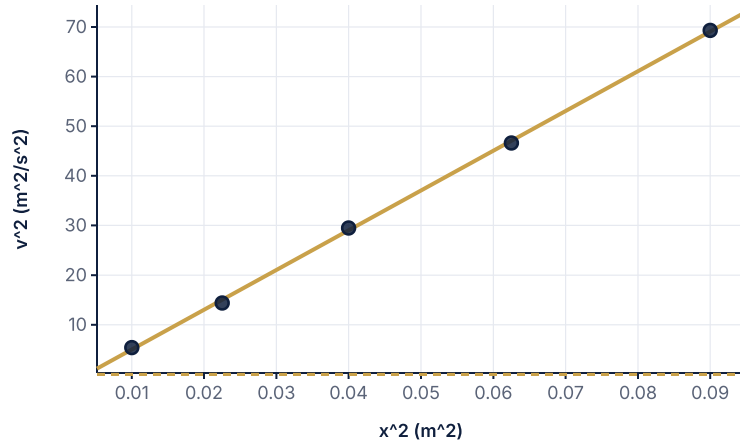
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EXAMPLE 3: LINEARIZING A SPRING-LAUNCH EXPERIMENT

A 0.250 kg block is launched by a spring compressed a distance x on a frictionless track, then crosses a rough strip of length $L = 0.60$ m and passes a speed sensor. The data are in the table. (a) Show that $v^2 = \frac{k}{m}x^2 - 2\mu gL$. (b) Plot v^2 against x^2 and find k and μ from the line.



Slope gives k/m ; the negative intercept gives the friction loss.

x (m)	x^2 (m ²)	v^2 (m ² /s ²)
0.100	0.0100	5.4
0.150	0.0225	14.4
0.200	0.0400	29.5
0.250	0.0625	46.6
0.300	0.0900	69.3

SOLUTION

(a) Energy: $\frac{1}{2}kx^2 - \mu mgL = \frac{1}{2}mv^2$. Multiply by $2/m$: $v^2 = \frac{k}{m}x^2 - 2\mu gL$, which has the form $y = (\text{slope})x^2 + (\text{intercept})$.

(b) The best-fit line gives slope $= 8.0 \times 10^2 \text{ s}^{-2}$ and intercept $= -3.0 \text{ m}^2/\text{s}^2$.

$k = m(\text{slope}) = 0.250(801) = 2.0 \times 10^2 \text{ N/m}$.

$\mu = -\text{intercept}/(2gL) = 3.0/(2 \cdot 10 \cdot 0.60) = 0.25$. With $g = 9.8$ the same data give $\mu = 0.26$.

A plot of v against x would be a curve and give no direct constants; v^2 against x^2 makes both constants readable.

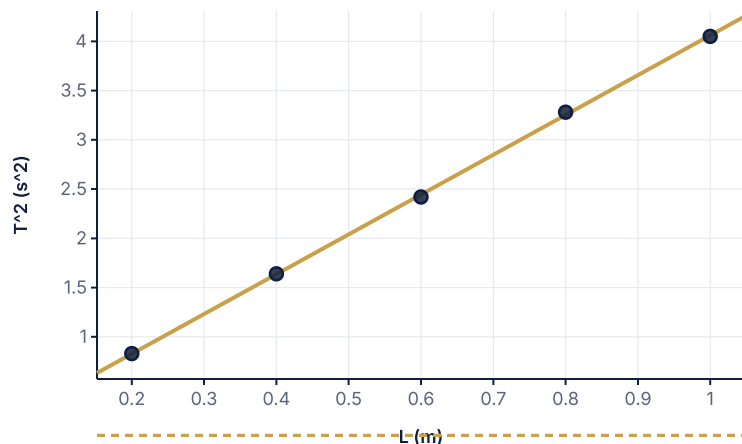
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EXAMPLE 4: PENDULUM DATA, BEST FIT, AND UNCERTAINTY

Students time a simple pendulum for several lengths and record T^2 . Each period has an uncertainty of ± 0.02 s, so $\delta(T^2) = 2T \delta T$. Use the table to find g with an uncertainty, and say whether the result agrees with 9.8 m/s^2 and with 10 m/s^2 .



Best-fit line for the pendulum data.

L (m)	T^2 (s^2)	uncertainty in T^2 (s^2)
0.20	0.83	0.04
0.40	1.64	0.05
0.60	2.42	0.06
0.80	3.28	0.07
1.00	4.05	0.08

SOLUTION

Model: $T^2 = \frac{4\pi^2}{g}L$, so the slope is $4\pi^2/g$ and the intercept should be 0.

Best-fit line: slope = $4.04 \text{ s}^2/\text{m}$, intercept = 0.02 s^2 (zero within error). So $g = 4\pi^2/4.04 = 9.77 \text{ m/s}^2$.

Steepest line through the lower-left error bar of the first point and the upper-right bar of the last: slope $4.17 \text{ s}^2/\text{m}$, giving $g = 9.46$.

Shallowest: slope $3.88 \text{ s}^2/\text{m}$, giving $g = 10.18$.

Half the range is $(10.18 - 9.46)/2 \approx 0.36$, so $g = 9.8 \pm 0.4 \text{ m/s}^2$.

Both 9.8 and 10 lie inside 9.4 to 10.2, so the data agree with either value; they cannot distinguish them. A near-zero intercept also suggests no large systematic timing error.

Write It Like a 5**Model Response**

Prompt. A block of mass m starts from rest at $x = 0$ on a frictionless surface. A horizontal force $F(x) = F_0 e^{-x/\ell}$ acts on it, where F_0 and ℓ are constants. (a) Derive the block's speed as a function of x . (b) Determine the largest speed the block can ever reach. (c) Justify the sign of the work done. **Response.** (a) The force is the only horizontal force, so the work-energy theorem applies: $W = \Delta K$. The work is $W = \int_0^x F_0 e^{-x'/\ell} dx' = F_0 \ell (1 - e^{-x/\ell})$. Starting from rest, $\frac{1}{2}mv^2 = F_0 \ell (1 - e^{-x/\ell})$, so $v = \sqrt{\frac{2F_0 \ell}{m} (1 - e^{-x/\ell})}$. (b) The force is always positive, so the block speeds up for all x and the speed increases toward a limit. As $x \rightarrow \infty$, $e^{-x/\ell} \rightarrow 0$ and $v \rightarrow \sqrt{2F_0 \ell / m}$; the speed approaches this value but never exceeds it. Check: $F_0 \ell$ has units $\text{N m} = \text{J}$, and $2J/\text{kg}$ gives m^2/s^2 . (c) The force points along the displacement, so F and dx have the same sign at every step, each piece of the integral is positive, and the total work is positive, consistent with the kinetic energy increasing.

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What the grader looks for

(1) A named principle ($W = \Delta K$ with $W = \int F dx$) before any calculation. (2) An integral with correct limits and variable. (3) The result in terms of the given symbols only. (4) The limit $x \rightarrow \infty$ evaluated, with a statement that the speed rises toward but does not pass the limit. (5) A justification that links sign of work, direction of force, and change in kinetic energy.

Coach's Playbook

- 1 Read the verb. "Derive" means symbols and a visible path; "calculate" means a number with units; "justify" means a physical reason in a sentence.
- 2 Ask what is varying: time, position, or speed. That decides whether you integrate dt , dx , or separate variables.
- 3 Write limits on every definite integral and define the dummy variable inside the integral.
- 4 Before computing, check dimensions of the setup; before submitting, check the answer's units and a limiting case.
- 5 For a lab, write the model, rearrange to $y = mx + b$, say what the slope and intercept equal, and only then describe measurements.
- 6 Use only the equipment listed; an invented instrument loses the procedure point.

Common traps

1) Plotting v against h (or T against L) and calling the curve a straight line; plot v^2 and T^2 instead. 2) Treating $\int a dt$ as the velocity itself and forgetting the initial velocity. 3) Integrating $v dt$ with v still a function of position, or leaving out limits so the constant of integration vanishes. 4) Describing a sketch or graph feature as "changes" without saying whether it rises or falls, or sketching an energy that goes negative.

Quick Check

Q1. What are the units of the area under an F versus t graph? **A:** N s, which is impulse and equals Δp . **Q2.** A line of T^2 against m for a spring has slope $6.0 \text{ s}^2/\text{kg}$. What is k ? **A:** $k = 4\pi^2/6.0 = 6.6 \text{ N/m}$. **Q3.** What does a nonzero intercept on a v^2 versus h graph tell you? **A:** The squared speed at $h = 0$, so an initial speed (or a start/zero-point offset).

Practice**PRACTICE Workshop Practice**

Show your work. A calculator is allowed. Use $g = 10 \text{ m/s}^2$ (9.8 is also accepted).

1. A cart starts from rest and moves with constant acceleration. A student plots position x against t^2 and finds a straight line through the origin with slope 1.8 m/s^2 . What is the cart's acceleration?
(A) 0.90 m/s^2 **(B)** 1.8 m/s^2 **(C)** 3.6 m/s^2
(D) 7.2 m/s^2
2. A sliding puck experiences a drag acceleration $a = -kv$ with $k = 0.40 \text{ s}^{-1}$ and starts at $v_0 = 10 \text{ m/s}$. How far does it travel before stopping?
(A) 4.0 m **(B)** 12.5 m **(C)** 25 m **(D)** 125 m
3. A mass on a spring is timed for several masses. A plot of T^2 against m is a straight line through the origin with slope $8.0 \text{ s}^2/\text{kg}$. What is the spring constant?
(A) 0.20 N/m **(B)** 4.9 N/m **(C)** 8.0 N/m **(D)** 39 N/m

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4. A ball is thrown downward and its speed v is measured after it falls a distance h below the release point. The graph of v^2 against h is a straight line with slope 20 m/s^2 and vertical intercept $4.0 \text{ m}^2/\text{s}^2$. Which statement is correct (take $g = 10 \text{ m/s}^2$)?
- (A) The release speed was 2.0 m/s and the slope equals $2g$
 (B) The release speed was 4.0 m/s and the slope equals g
 (C) The release speed was zero and the intercept is a timing error
 (D) The release speed was 2.0 m/s and the slope equals $g/2$
5. A 2.0 kg block starts at rest at $x = 0$ on a frictionless surface and is pushed by a force $F(x) = 12x - 3x^2$ (SI units) along the x axis.
- (a) Calculate the work done on the block from $x = 0$ to $x = 3.0 \text{ m}$ and the block's speed there.
 (b) Find the position at which the speed is largest and calculate that speed.
 (c) Explain, using the sign of F , what happens to the speed beyond that position.
6. A 2.0 kg block moves on a surface that exerts a force $F = -cv^2$ with $c = 0.50 \text{ kg/m}$. Its speed at $t = 0$ is 6.0 m/s .
- (a) Starting from Newton's second law, derive an expression for $1/v$ as a function of time.
 (b) Calculate the speed at $t = 4.0 \text{ s}$.
 (c) Show that $x(t) = (m/c) \ln \left(1 + \frac{cv_0 t}{m} \right)$ and decide whether the block stops at a finite distance. Justify.
7. A student drops a steel ball through a photogate placed at several heights h below the release point and records the speed v at the gate. Only a meterstick, the photogate timer, and the ball are provided.
- (a) Which two quantities should be plotted to give a straight line, and what does the slope equal?
 (b) Using the data $h = 0.10, 0.20, 0.30, 0.40, 0.50 \text{ m}$ and $v = 1.38, 1.97, 2.40, 2.77, 3.13 \text{ m/s}$, calculate the slope and the value of g it implies.
 (c) Describe how the intercept would reveal a systematic error, and one change that reduces random error.
8. A particle of mass 0.50 kg starts at rest. A force $F(t) = F_0 \sin(\pi t/T)$ acts along a straight line for $0 \leq t \leq T$, with $F_0 = 30 \text{ N}$ and $T = 0.20 \text{ s}$.
- (a) Derive an expression for the impulse in terms of F_0 and T , and then calculate its value.
 (b) Calculate the final speed and the average force over the interval.
 (c) Explain how the area under the F - t graph connects to the change in momentum.

Answers**Workshop: Answers and Explanations****W Workshop: Calculus, Graphs, and Lab Analysis****PRACTICE · WORKSHOP PRACTICE**

1. C. For $x = \frac{1}{2}at^2$ the slope of x against t^2 is $a/2$, so $a = 2(1.8) = 3.6 \text{ m/s}^2$. Choice B reads the slope as the acceleration, forgetting the factor of one half.
2. C. Chain rule: $v dv/dx = -kv$ gives $dv = -k dx$, so $x = v_0/k = 10/0.40 = 25 \text{ m}$. Choice B (12.5 m) applies constant-acceleration kinematics with the initial deceleration kv_0 , wrongly treating the drag as constant.
3. B. $T^2 = (4\pi^2/k)m$, so slope $= 4\pi^2/k$ and $k = 4\pi^2/8.0 = 4.9 \text{ N/m}$. Choice C treats the slope as k itself; choice A inverts the relation without the $4\pi^2$.
4. A. $v^2 = v_0^2 + 2gh$ gives intercept $v_0^2 = 4.0$, so $v_0 = 2.0 \text{ m/s}$, and slope $2g = 20 \text{ m/s}^2$. Choice B reads the intercept as the speed itself instead of its square.
5. (a) $W = \int_0^3 (12x - 3x^2) dx = [6x^2 - x^3]_0^3 = 54 - 27 = 27 \text{ J}$ (1 pt for setup with limits, 1 pt for value). $\frac{1}{2}(2.0)v^2 = 27$, so $v = \sqrt{27} = 5.2 \text{ m/s}$ (1 pt).
 (b) Speed is largest where $F = 0$ after being positive: $12x - 3x^2 = 0$ gives $x = 4.0 \text{ m}$ (1 pt). $W = [6x^2 - x^3]_0^4 = 96 - 64 = 32 \text{ J}$, $v = \sqrt{32} = 5.7 \text{ m/s}$ (1 pt).
 (c) For $x > 4.0 \text{ m}$, $F < 0$ opposes the motion, so the work is negative and the kinetic energy and speed decrease (1 pt, must say the speed falls).
6. (a) $m dv/dt = -cv^2$ (1 pt). Separate: $dv/v^2 = -(c/m) dt$ (1 pt). Integrate with limits: $[-1/v]_{v_0}^v = -(c/m)t$, so $1/v = 1/v_0 + (c/m)t$ (1 pt).
 (b) $c/m = 0.25 \text{ m}^{-1}$; $1/v = 1/6.0 + 0.25(4.0) = 1.167 \text{ s/m}$, so $v = 0.86 \text{ m/s}$ (1 pt).
 (c) $v = \frac{v_0}{1 + cv_0 t/m}$, and $x = \int_0^t v dt' = \frac{m}{c} \ln \left(1 + \frac{cv_0 t}{m} \right)$ (1 pt). At 4.0 s this is $4.0 \ln 7 = 7.8 \text{ m}$. As $t \rightarrow \infty$ the logarithm grows without bound, so the block never stops at a finite distance although its speed tends to zero (1 pt).
7. (a) From $v^2 = 2gh$, plot v^2 (vertical) against h (horizontal); the slope equals $2g$ (1 pt for variables, 1 pt for slope). Only the meterstick, photogate, and ball are used.
 (b) $v^2 = 1.90, 3.88, 5.76, 7.67, 9.80 \text{ m}^2/\text{s}^2$. Slope from the endpoints of the best-fit line $= (9.80 - 1.90)/(0.50 - 0.10) \approx 19.6 \text{ m/s}^2$ (1 pt); $g = \text{slope}/2 = 9.8 \text{ m/s}^2$ (1 pt).
 (c) A best-fit intercept near $-0.07 \text{ m}^2/\text{s}^2$ is essentially zero, so there is no large offset; a significant nonzero intercept would mean the height or release point was measured from the wrong zero, or the gate was not triggered at the stated height (1 pt). Repeat each drop several times and average, or use larger heights, to reduce random error (1 pt).

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8. (a) $J = \int_0^T F_0 \sin(\pi t/T) dt = F_0 \frac{T}{\pi} [-\cos(\pi t/T)]_0^T = \frac{2F_0 T}{\pi}$ (1 pt for setup and limits, 1 pt for result). Numerically $2(30)(0.20)/\pi = 3.8 \text{ N s}$ (1 pt).
- (b) $J = \Delta p = mv$, so $v = 3.82/0.50 = 7.6 \text{ m/s}$ (1 pt). Average force $= J/T = 2F_0/\pi = 19 \text{ N}$, below the 30 N peak (1 pt).
- (c) The area under $F(t)$ is the integral $\int F dt$, which equals the impulse and, by $F = dp/dt$, the change in momentum; this holds for any shape of the force, with area below the axis counting as negative (1 pt).

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Mixed Practice Test

A mixed set of multiple-choice and free-response questions across all units, with answers, explanations, and scoring guidance.

T Mixed Practice Test

Twelve multiple-choice questions and three free-response questions covering all seven units, weighted like the exam. Use $g = 10 \text{ m/s}^2$.

Time: about 55 minutes. Part A: 12 multiple choice (Units 1 to 7: 1, 3, 2, 2, 1, 1, 2 items). Part B: three free-response questions worth 10, 10 and 8 points. A calculator is allowed. Score with the guide that follows each answer.

PRACTICE TEST Mixed Practice Test

Part A: choose the best answer for items 1 to 12. Part B: items 13 to 15 are free response; show all work and justify claims.

- A particle moves along the x -axis with $x(t) = t^3 - 6t^2 + 9t$ in meters, t in seconds. What total distance does it travel from $t = 0$ to $t = 4 \text{ s}$?
(A) 4 m (B) 8 m (C) 12 m (D) 16 m
- A 4.0 kg block slides down a rough incline at 37 degrees to the horizontal ($\sin 37 = 0.6$, $\cos 37 = 0.8$) with kinetic friction coefficient 0.25. What is its acceleration? ($g = 10 \text{ m/s}^2$)
(A) 6.0 m/s^2 (B) 4.0 m/s^2 (C) 6.5 m/s^2 (D) 2.0 m/s^2
- A 0.50 kg object falls from rest through air that exerts a drag force of magnitude bv with $b = 2.5 \text{ kg/s}$. What is its speed after 0.20 s? ($g = 10 \text{ m/s}^2$)
(A) 2.0 m/s (B) 0.74 m/s (C) 1.0 m/s (D) 1.26 m/s
- A 0.50 kg bob on a 0.80 m string swings through the lowest point of its arc at 3.0 m/s. What is the tension in the string there? ($g = 10 \text{ m/s}^2$)
(A) 10.6 N (B) 5.0 N (C) 5.6 N (D) 0.63 N
- A 2.0 kg block, initially at rest on a frictionless horizontal surface, is pulled along $+x$ by a force $F = 4x + 3x^2$ (newtons, x in meters). What is its speed at $x = 2.0 \text{ m}$?
(A) 2.8 m/s (B) 4.0 m/s (C) 6.3 m/s (D) 8.0 m/s
- A spring ($k = 800 \text{ N/m}$) is compressed 0.10 m and launches a 0.20 kg block up a frictionless ramp. To what maximum height above the launch point does the block rise? ($g = 10 \text{ m/s}^2$)
(A) 0.5 m (B) 1.0 m (C) 2.0 m (D) 4.0 m
- A 4.0 kg cart moving at 1.0 m/s along a horizontal track receives a force $F = 3t^2$ (newtons, t in seconds) in the direction of motion for the interval $0 \leq t \leq 2.0 \text{ s}$. What is its speed afterward?
(A) 3.0 m/s (B) 2.0 m/s (C) 4.0 m/s (D) 7.0 m/s
- A 3.0 kg cart moving at 4.0 m/s collides with a 1.0 kg cart at rest. They stick together. How much kinetic energy is converted to other forms?
(A) 18 J (B) 24 J (C) 6.0 J (D) 0 J
- A uniform disk of mass 2.0 kg and radius 0.30 m rotates freely about its central axis. A constant tangential force of 3.0 N is applied at the rim, starting from rest. Through what angle does it turn in 2.0 s?
(A) 10 rad (B) 20 rad (C) 40 rad (D) 67 rad
- A solid sphere rolls without slipping from rest down an incline, dropping a vertical height of 0.70 m. What is its speed at the bottom? ($g = 10 \text{ m/s}^2$; $I = \frac{2}{5}mR^2$)
(A) 3.7 m/s (B) 3.1 m/s (C) 3.2 m/s (D) 2.6 m/s

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11. A 0.50 kg block oscillates on a spring of constant 50 N/m with amplitude 0.20 m. What is its speed when it is 0.10 m from equilibrium?
 (A) 1.7 m/s (B) 1.0 m/s (C) 2.0 m/s (D) 0.87 m/s
12. A thin hoop of mass M and radius $R = 0.50$ m hangs from a nail through its rim (a pivot at the rim) and swings through small angles in its own plane. What is its period? ($g = 10$ m/s²; $I_{\text{cm}} = MR^2$)
 (A) 1.40 s (B) 1.99 s (C) 1.72 s (D) 0.99 s
13. [10 points] A 5.0 kg sled slides along level, frictionless ice with initial speed $v_0 = 6.0$ m/s. Air resistance exerts a force of magnitude cv^2 opposite the motion, with $c = 0.40$ kg/m. (a) [1 point] Write Newton's second law for the sled's velocity. (b) [4 points] Using separation of variables, show that $v(t) = \frac{v_0}{1 + cv_0 t/m}$. (c) [2 points] At what time is the speed 3.0 m/s? (d) [3 points] Integrate $v(t)$ to find the distance traveled in terms of t , and evaluate the distance at the time found in (c).
14. [10 points] A student launches a block along a level board with a spring launcher. A photogate at the start of the board gives the block's speed v_0 , and the block slides a distance d to rest. Available: the block, the board, the launcher, a photogate, a meter stick. (a) [3 points] Describe a procedure to collect data for several values of d , and explain how to control variables. (b) [2 points] Use the work-energy theorem to derive an expression for v_0^2 in terms of d , the kinetic friction coefficient μ_k and g . (c) [3 points] Data are given below. Explain what to plot, and use the best-fit slope of 6.25 m/s² to determine μ_k . (d) [2 points] Explain why plotting v_0 against d would not produce a straight line, and what a large nonzero intercept would suggest.
- $d = 0.20$ m: $v_0 = 1.13$ m/s
 $d = 0.40$ m: $v_0 = 1.60$ m/s
 $d = 0.60$ m: $v_0 = 1.95$ m/s
 $d = 0.80$ m: $v_0 = 2.24$ m/s
 $d = 1.00$ m: $v_0 = 2.52$ m/s
 $d = 1.20$ m: $v_0 = 2.74$ m/s
15. [8 points] A satellite of mass m orbits a planet of mass M in an elliptical orbit. At its closest approach (periapsis) the distance from the planet's center is r_p and the speed is v_p ; at its farthest point (apoapsis) the distance is $r_a > r_p$. (a) [2 points] Without equations, justify whether the speed at apoapsis is greater than, less than or equal to v_p . (b) [2 points] Derive v_a in terms of v_p , r_p and r_a . (c) [3 points] Use energy conservation to derive v_p in terms of G , M , r_p and r_a . (d) [1 point] Show that your result for (c) agrees with the circular-orbit speed when $r_a = r_p$.

Key**Mixed Practice Test: Answers, Explanations, and Scoring****T****Mixed Practice Test****PRACTICE TEST · MIXED PRACTICE TEST**

1. C. Unit 1. $v = 3t^2 - 12t + 9 = 3(t-1)(t-3)$ is zero at $t = 1$ and 3 s, so the particle reverses. x is 0, 4, 0, 4 m at $t = 0, 1, 3, 4$ s, giving $4 + 4 + 4 = 12$ m. The tempting 4 m is the net displacement.
2. B. Unit 2. Along the slope: $a = g \sin \theta - \mu g \cos \theta = 10(0.6) - 0.25(10)(0.8) = 4.0$ m/s². The 6.0 m/s² choice omits friction, and 6.5 m/s² swaps sine and cosine.
3. D. Unit 2. Terminal speed is $mg/b = 2.0$ m/s and the time constant is $m/b = 0.20$ s, so $v = 2.0(1 - e^{-1}) = 1.26$ m/s. The 2.0 m/s choice is the terminal speed itself, which is only approached asymptotically.
4. A. Unit 2. At the bottom the net upward force provides the centripetal force: $T - mg = mv^2/L$, so $T = 0.50(10 + 9.0/0.80) = 10.6$ N. The 5.0 N choice is only the weight and 5.6 N is only mv^2/L .
5. B. Unit 3. $W = \int_0^2 (4x + 3x^2) dx = 2x^2 + x^3 \Big|_0^2 = 16$ J, so $v = \sqrt{2W/m} = 4.0$ m/s. The 6.3 m/s choice multiplies the final force (20 N) by the distance, treating the force as constant.
6. C. Unit 3. $\frac{1}{2}kx^2 = \frac{1}{2}(800)(0.10)^2 = 4.0$ J and this becomes $mgh = 0.20(10)h$, so $h = 2.0$ m. The 4.0 m choice omits the factor of one half in the spring energy.
7. A. Unit 4. Impulse $J = \int_0^2 3t^2 dt = 8.0$ N s, so $\Delta v = J/m = 2.0$ m/s and $v = 1.0 + 2.0 = 3.0$ m/s. The 2.0 m/s choice forgets the initial speed, and 7.0 m/s uses the final force $F(2) = 12$ N as if it acted over the whole interval.
8. D. Unit 4. Momentum conservation gives $v = 12/4.0 = 3.0$ m/s. Kinetic energy goes from 24 J to 18 J, so 6.0 J is lost. The 18 J choice is the final energy and 24 J the initial; 0 J wrongly assumes kinetic energy is conserved because momentum is.
9. B. Unit 5. $I = \frac{1}{2}MR^2 = 0.090$ kg m² and $\tau = FR = 0.90$ N m, so $\alpha = 10$ rad/s² and $\theta = \frac{1}{2}\alpha t^2 = 20$ rad. The 67 rad choice divides force (not torque) by I , and 10 rad uses $I = MR^2$.
10. C. Unit 6. $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2(1 + \frac{2}{5})$ gives $v = \sqrt{10gh/7} = 3.16$ m/s. The 3.7 m/s choice ignores rotational energy, and 2.6 m/s is the result for a hoop.
11. A. Unit 7. $\omega = \sqrt{k/m} = 10$ rad/s and $v = \omega\sqrt{A^2 - x^2} = 10\sqrt{0.04 - 0.01} = 1.73$ m/s. The 2.0 m/s choice is the maximum speed, and 1.0 m/s is ωx .
12. B. Unit 7. $I = MR^2 + MR^2 = 2MR^2$ about the rim and $d = R$, so $T = 2\pi\sqrt{I/(Mgd)} = 2\pi\sqrt{2R/g} = 1.99$ s. The 1.40 s choice uses $I = MR^2$ (no parallel-axis term), which treats the hoop as if it were pivoted at its center of mass.

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13. (a) 1 point: $m \frac{dv}{dt} = -cv^2$. (b) 4 points: separate, $\frac{dv}{v^2} = -\frac{c}{m} dt$ (1); integrate both sides from v_0 to v and 0 to t : $-\frac{1}{v} + \frac{1}{v_0} = -\frac{c}{m} t$ (2); solve $\frac{1}{v} = \frac{1}{v_0} + \frac{c}{m} t$ so $v = \frac{v_0}{1 + cv_0 t/m}$ (1). (c) 2 points: $cv_0/m = 0.40(6.0)/5.0 = 0.48 \text{ s}^{-1}$; $3.0 = 6.0/(1 + 0.48t)$ gives $1 + 0.48t = 2$ (1) and $t = 2.08 \text{ s}$ (1). (d) 3 points: $x = \int_0^t v dt = \frac{m}{c} \ln\left(1 + \frac{cv_0 t}{m}\right)$ (2), with $m/c = 12.5 \text{ m}$; at $t = 2.08 \text{ s}$, $x = 12.5 \ln 2 = 8.66 \text{ m}$ (1). Model response: write the law with the minus sign, separate variables before integrating, keep limits, and verify that $v(0) = v_0$ and that v decreases without ever reaching zero.
14. (a) 3 points: vary the launcher setting to get different v_0 (1); read v_0 from the photogate and d from the meter stick for each launch (1); same block and board surface for every trial, repeat each setting and average (1). (b) 2 points: friction does work $-\mu_k mg d$ (1) and equals the change in kinetic energy $0 - \frac{1}{2}mv_0^2$, so $v_0^2 = 2\mu_k g d$ (1). (c) 3 points: plot v_0^2 (vertical) against d (horizontal) (1); the slope is $2\mu_k g$ (1); $\mu_k = 6.25/(2 \times 10) = 0.31$ (1). (Values of v_0^2 : 1.28, 2.56, 3.80, 5.02, 6.35, 7.51 m^2/s^2 .) (d) 2 points: since $v_0 = \sqrt{2\mu_k g d}$, a graph of v_0 against d is a square-root curve (1). A large nonzero intercept would indicate that the block lost or gained energy before the gate, or other forces such as air drag or a tilted board are acting (1).
15. (a) 2 points: less than v_p (1). Gravity is a central force, so it exerts no torque about the planet and the angular momentum is constant; the satellite's distance is larger at apoapsis, so its speed must be smaller (1). (Equivalent energy argument: potential energy is higher at apoapsis, so kinetic energy is lower.) (b) 2 points: at both points the velocity is perpendicular to the radius, so $mv_p r_p = mv_a r_a$ (1) and $v_a = v_p r_p/r_a$ (1). (c) 3 points: $\frac{1}{2}mv_p^2 - \frac{GMm}{r_p} = \frac{1}{2}mv_a^2 - \frac{GMm}{r_a}$ (1); substitute v_a : $\frac{1}{2}v_p^2 \left(1 - \frac{r_p^2}{r_a^2}\right) = GM \left(\frac{1}{r_p} - \frac{1}{r_a}\right)$ (1); simplify: $v_p = \sqrt{\frac{2GM r_a}{r_p(r_a + r_p)}}$ (1). (d) 1 point: with $r_a = r_p$ the result is $\sqrt{2GM r_p/(2r_p^2)} = \sqrt{GM/r_p}$, the circular-orbit speed.

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Free-Response Clinic

Exam-style free-response questions with point-by-point scoring guides and full-credit model responses.

X Free-Response Clinic

Five free-response questions in the exam's four styles, plus one extra. Practice point values are Scholar Coach estimates; attempt each in 20 to 25 minutes.

The exam's free-response section has four questions in a fixed order: Mathematical Routines (10 points), Translation Between Representations (12), Experimental Design and Analysis (10) and Qualitative/Quantitative Translation (8). Questions 1 to 4 below follow that order, and Question 5 is extra practice with calculus on continuous mass distributions. Point values and scoring guides are practice rubrics written for this book. Show every integral setup, label graph axes with units, and justify each claim with a named law.

FREE-RESPONSE CLINIC Free-Response Clinic

Show all work. Calculator allowed. Use $g = 10 \text{ m/s}^2$. Scoring guides follow each question in the answer section.

- Question 1 (Mathematical Routines, practice value 10 points). A 2.0 kg cart sits at rest on a frictionless track. From $t = 0$ to $t = \tau = 0.20 \text{ s}$ a motor pushes it with a force along the track $F(t) = F_0 \sin(\pi t / \tau)$, with $F_0 = 30 \text{ N}$, and the force is zero afterward.
 - [3 points] Write and evaluate the definite integral for the impulse delivered to the cart.
 - [1 point] Find the cart's speed after the push.
 - [2 points] The cart then collides with a stationary 3.0 kg cart and the two stick together. Find their common speed.
 - [2 points] Find the kinetic energy lost in the collision.
 - [2 points] Find the average force during the push and explain why it is smaller than F_0 .
- Question 2 (Translation Between Representations, practice value 12 points). A 0.30 kg particle moves along the x -axis with potential energy $U(x) = 24x^4 - 12x^2$ in joules, x in meters. Selected values: $U(0) = 0$, $U(0.25) = -0.66$, $U(0.50) = -1.50$, $U(0.75) = 0.84 \text{ J}$.
 - [2 points] Find the force $F_x(x)$ on the particle.
 - [3 points] Find all equilibrium positions and classify each as stable or unstable, with justification.
 - [3 points] The particle has total energy $E = -0.50 \text{ J}$. Find the turning points of its motion and describe a sketch of its kinetic energy against x . Explain why it cannot reach $x < 0$.
 - [2 points] Find the particle's maximum speed.
 - [2 points] If E is only slightly above -1.50 J , find the period of the small oscillations.
- Question 3 (Experimental Design and Analysis, practice value 10 points). A uniform meter stick ($L = 1.00 \text{ m}$) has small holes so it can be hung from a horizontal pin at a distance d from its center and swung through small angles in a plane. Available: the meter stick, a support with a pin, a stopwatch, and a protractor. The data below are the measured periods.

$d=0.10 \text{ m}$, $T=1.939 \text{ s}$; $d=0.15 \text{ m}$, $T=1.695 \text{ s}$; $d=0.20 \text{ m}$, $T=1.584 \text{ s}$; $d=0.30 \text{ m}$, $T=1.523 \text{ s}$; $d=0.40 \text{ m}$, $T=1.564 \text{ s}$; $d=0.45 \text{ m}$, $T=1.596 \text{ s}$

 - [3 points] Describe a procedure to obtain the data, including how to improve the timing precision and keep the angle small.
 - [3 points] Starting from $T = 2\pi\sqrt{I/(Mgd)}$ with $I = \frac{1}{12}ML^2 + Md^2$, show how to linearize the relationship, name the quantities to plot on each axis, and state what the slope represents.
 - [2 points] The best-fit line through the plotted data has slope $3.99 \text{ s}^2 \text{ m}^{-1}$ (the plotted y is T^2d in $\text{s}^2 \text{ m}$ and the horizontal quantity d^2 is in m^2). Determine g .
 - [2 points] Predict the vertical intercept of the line from the theory and say what it is expected to be for this stick.

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4. Question 4 (Qualitative/Quantitative Translation, practice value 8 points). A 0.20 kg puck moves in a circle of radius $r_0 = 0.80$ m at speed $v_0 = 1.5$ m/s on a frictionless horizontal table. It is attached to a string that passes through a hole at the center of the table. A person slowly pulls the string down through the hole until the radius is $r_0/2$.
- (a) [2 points] In words only, justify whether the puck's speed increases, decreases or stays the same.
- (b) [3 points] Derive an expression for the string tension as a function of the radius r during the pull, in terms of m , r and the angular momentum L , and evaluate the tension at r_0 and at $r_0/2$.
- (c) [3 points] Calculate the work done by the person by integrating the tension, and show your result agrees with the change in the puck's kinetic energy.
5. Question 5 (extra practice, Mathematical Routines, 10 points). A thin rod of length L lies along the x -axis from $x = 0$ to $x = L$ with linear mass density $\lambda = ax$, where a is a constant. (a) [2 points] Find the total mass M in terms of a and L . (b) [2 points] Find the center of mass position in terms of L . (c) [3 points] Find the rotational inertia about the axis through $x = 0$, perpendicular to the rod, in terms of M and L . (d) [3 points] The rod ($L = 0.60$ m) is pivoted at $x = 0$, released from rest horizontal, and swings down. Find its angular speed and the speed of its far end when it is vertical. ($g = 10$ m/s²)

Key Free-Response Clinic: Scoring Guides and Model Responses

Free-Response Clinic

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1. (a) 3 points: $J = \int_0^\tau F dt$ (1); $= F_0 \int_0^{0.20} \sin(\pi t/0.20) dt = F_0 \frac{\tau}{\pi} [-\cos u]_0^\pi$ (1); $= \frac{2F_0\tau}{\pi} = 3.82$ N s (1). (b) 1 point: $v = J/m = 1.91$ m/s. (c) 2 points: momentum conserved, $3.82 = (2.0 + 3.0)v_f$ (1), $v_f = 0.764$ m/s (1). (d) 2 points: $K_i = \frac{1}{2}(2.0)(1.91)^2 = 3.65$ J (1); $K_f = \frac{1}{2}(5.0)(0.764)^2 = 1.46$ J, so 2.19 J is lost (1). (e) 2 points: $F_{\text{avg}} = J/\tau = 19.1$ N (1); the force is zero at both ends and peaks at 30 N at the midpoint, so its mean is $2/\pi \approx 0.64$ of the peak (1). Model response: state the integral before evaluating, keep units on the impulse, and treat the collision as a separate system with momentum conserved.
2. (a) 2 points: $F_x = -dU/dx$ (1) $= -(96x^3 - 24x) = 24x - 96x^3$ N (1). (b) 3 points: set $F = 0$: $24x(1 - 4x^2) = 0$, so $x = 0, \pm 0.50$ m (1). $U'' = 288x^2 - 24$: at $x = \pm 0.50$, $U'' = 48$ N/m > 0 , a minimum, stable (1); at $x = 0$, $U'' = -24 < 0$, a maximum, unstable (1). (c) 3 points: $K = 0$ where $24x^4 - 12x^2 = -0.50$; with $u = x^2$, $24u^2 - 12u + 0.5 = 0$ gives $u = 0.0459$ or 0.454 , so $x = \pm 0.214$ m and ± 0.674 m (1). Sketch: on $0.214 < x < 0.674$, $K = E - U$ is zero at both ends, peaks at 1.0 J at $x = 0.50$ m, never negative (1). The barrier at $x = 0$ has $U = 0 > E$, so the particle would need negative kinetic energy there; it stays in the right-hand well (1). (d) 2 points: maximum $K = E - U_{\text{min}} = -0.50 + 1.50 = 1.0$ J (1); $v = \sqrt{2K/m} = 2.58$ m/s (1). (e) 2 points: $k_{\text{eff}} = U''(0.50) = 48$ N/m (1); $T = 2\pi\sqrt{m/k_{\text{eff}}} = 2\pi\sqrt{0.30/48} = 0.497$ s (1). Model response: tie each result back to the shape of U : minima for stable points, the energy line E for turning points.
3. (a) 3 points: hang the stick from the pin at a hole at distance d (1); displace by a small angle (about 10 degrees, measured with the protractor) and time 10 full oscillations, then divide by 10 (1); repeat for several pin positions d , keeping the angle and plane of swing the same (1). (b) 3 points: $T^2 = \frac{4\pi^2}{g} \left(\frac{L^2}{12d} + d \right)$ (1), so $T^2d = \frac{4\pi^2}{g}d^2 + \frac{\pi^2 L^2}{3g}$ (1). Plot T^2d (vertical) against d^2 (horizontal); the slope is $4\pi^2/g$ (1). (c) 2 points: $g = 4\pi^2/\text{slope}$ (1) $= 39.48/3.99 = 9.9$ m/s² (1). (d) 2 points: the intercept is $\pi^2 L^2/(3g)$ (1) $= 9.87/(3 \times 9.8) = 0.34$ s² m using $L = 1.00$ m and the accepted $g = 9.8$ m/s²; the line fit gives 0.339 (1). Model response: define the plotted variables clearly, since any equivalent linearization with variables defined earns credit; include units on the slope.
4. (a) 2 points: the speed increases (1). The string force points toward the hole, so it exerts no torque about the hole and the puck's angular momentum is constant; with a smaller radius the puck must move faster to keep mvr the same (1). (b) 3 points: $T = mv^2/r$ (1) with $v = L/(mr)$ so $T = \frac{L^2}{mr^3}$ (1). Here $L = mv_0r_0 = 0.24$ kg m²/s: at r_0 , $T = 0.56$ N; at $r_0/2 = 0.40$ m, $v = 3.0$ m/s and $T = 4.5$ N, eight times larger (1). (c) 3 points: $W = \int_{r_0/2}^{r_0} (-T) dr = \int_{r_0/2}^{r_0} \frac{L^2}{mr^3} dr$ (1) $= \frac{L^2}{2m} \left(\frac{1}{(0.40)^2} - \frac{1}{(0.80)^2} \right) = 0.675$ J (1). The kinetic energy changes from $\frac{1}{2}(0.20)(1.5)^2 = 0.225$ J to $\frac{1}{2}(0.20)(3.0)^2 = 0.90$ J, a gain of 0.675 J, matching (1).
5. (a) 2 points: $M = \int_0^L ax dx$ (1) $= aL^2/2$ (1). (b) 2 points: $x_{\text{cm}} = \frac{1}{M} \int_0^L x(ax) dx$ (1) $= \frac{aL^3/3}{aL^2/2} = \frac{2L}{3}$ (1). (c) 3 points: $I = \int x^2 \lambda dx$ (1) $= a \int_0^L x^3 dx = aL^4/4$ (1) $= \frac{1}{2}ML^2$ using $a = 2M/L^2$ (1). (d) 3 points: energy conservation $Mgx_{\text{cm}} = \frac{1}{2}I\omega^2$ (1); $\omega^2 = \frac{2Mg(2L/3)}{ML^2/2} = \frac{8g}{3L} = 44.4$ s⁻², so $\omega = 6.67$ rad/s (1); far-end speed $v = \omega L = 4.0$ m/s (1). Model response: show the integrals with limits, express the answer without a , and note the center of mass is farther out than for a uniform rod because the mass is concentrated at the far end.

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Reference: Formulas, Skills Checklist, and Glossary

Quick-reference pages to review the night before the exam.

R1 Equations, Constants and Reference

A unit-by-unit summary of the relationships, constants and calculus facts used in Physics C: Mechanics, in our own wording.

How to Use This Reference

These pages restate the relationships you need, grouped by unit, in our own wording. On the exam a reference sheet is provided, and its layout and contents can change between years, so confirm what it lists in your course materials and do not rely on memorizing this page alone. Several results below (for example the rotational inertias of common shapes) may not appear on the sheet; learn how to derive them. Throughout, $g = 10 \text{ m/s}^2$ unless a problem says otherwise (9.8 is also accepted).

Equations by Unit

Unit 1: Kinematics

Calculus definitions: $v = \frac{dx}{dt}$, $a = \frac{dv}{dt}$; going backward, $v = v_0 + \int a \, dt$ and $x = x_0 + \int v \, dt$.

Constant acceleration only: $v = v_0 + at$, $x = x_0 + v_0t + \frac{1}{2}at^2$, $v^2 = v_0^2 + 2a\Delta x$.

Projectiles: treat x and y separately; $a_x = 0$ and $a_y = -g$. Relative motion adds velocities between frames.

Unit 2: Force and Translational Dynamics

Newton's second law: $\sum \vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$. Weight = mg . Center of mass: $x_{\text{cm}} = \frac{\sum m_i x_i}{M} = \frac{1}{M} \int x \, dm$.

Friction: kinetic $f_k = \mu_k N$; static $f_s \leq \mu_s N$. Spring: $F_s = -kx$. Resistive forces: $-bv$ (slow) or $-cv^2$ (fast); solve with separation of variables.

Circular motion: $a_c = \frac{v^2}{r}$ toward the center. Gravity between masses: $F = \frac{Gm_1m_2}{r^2}$.

Unit 3: Work, Energy, and Power

$K = \frac{1}{2}mv^2$; $W = \int \vec{F} \cdot d\vec{x}$; work-energy theorem $W_{\text{net}} = \Delta K$. Power $P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}$.

Potential energy: $U_s = \frac{1}{2}kx^2$, $U_g = mgh$ near the surface, and in general $F_x = -\frac{dU}{dx}$. Without nonconservative work, $K + U$ is constant.

Unit 4: Linear Momentum

$\vec{p} = m\vec{v}$; impulse $\vec{J} = \int \vec{F} \, dt = \Delta\vec{p}$. Momentum of an isolated system is conserved in every collision.

Elastic: kinetic energy also conserved. Perfectly inelastic: objects stick and kinetic energy loss is maximum.

Unit 5: Torque and Rotational Dynamics

$\omega = \frac{d\theta}{dt}$, $\alpha = \frac{d\omega}{dt}$; with constant α the kinematic forms mirror Unit 1. Links: $v = \omega r$, $a_t = \alpha r$, $a_c = \omega^2 r$.

Torque $\tau = rF \sin \phi$; rotational inertia $I = \sum mr^2 = \int r^2 \, dm$; $\sum \tau = I\alpha$. Parallel-axis: $I = I_{\text{cm}} + Md^2$.

Common I_{cm} : rod $\frac{1}{12}ML^2$; disk $\frac{1}{2}MR^2$; ring MR^2 ; solid sphere $\frac{2}{5}MR^2$.

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Unit 6: Energy and Momentum of Rotating Systems

$K_{\text{rot}} = \frac{1}{2}I\omega^2$; $W = \int \tau d\theta$; $P = \tau\omega$. Rolling without slipping: $v_{\text{cm}} = \omega R$ and $K = \frac{1}{2}Mv_{\text{cm}}^2 + \frac{1}{2}I\omega^2$.

Angular momentum $L = I\omega$ (or $\vec{r} \times \vec{p}$); $\sum \tau = \frac{dL}{dt}$, so L is constant when the net torque is zero.

Orbits: $U_g = -\frac{GMm}{r}$; circular orbit speed $v = \sqrt{GM/r}$; energy and angular momentum are conserved.

Unit 7: Oscillations

SHM: $\frac{d^2x}{dt^2} = -\omega^2x$, $x = A \cos(\omega t + \phi)$, $T = \frac{1}{f} = \frac{2\pi}{\omega}$.

Spring-mass: $\omega = \sqrt{k/m}$; energy $E = \frac{1}{2}kA^2$. Simple pendulum: $T = 2\pi\sqrt{L/g}$. Physical pendulum: $T = 2\pi\sqrt{I/(Mgd)}$ (small angles).

Constants, Prefixes and Trig Values

Quantity	Value or note
Gravitational field near Earth	$g = 10 \text{ m/s}^2$ (9.8 accepted); also 10 N/kg
Universal gravitation constant	$G = 6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2$
Prefixes	tera 10^{12} , giga 10^9 , mega 10^6 , kilo 10^3 , centi 10^{-2} , milli 10^{-3} , micro 10^{-6} , nano 10^{-9} , pico 10^{-12}
Units	hertz (1/s), joule (N m), watt (J/s), newton (kg m/s ²)
sin and cos at 30 degrees	0.50 and 0.87
sin and cos at 37 degrees	0.60 and 0.80
sin and cos at 45 degrees	0.71 and 0.71
sin and cos at 53 degrees	0.80 and 0.60
sin and cos at 60 degrees	0.87 and 0.50

Calculus toolkit

Power rule $\frac{d}{dx}x^n = nx^{n-1}$ and $\int x^n dx = \frac{x^{n+1}}{n+1}$; $\frac{d}{dt}\sin \omega t = \omega \cos \omega t$ and $\frac{d}{dt}\cos \omega t = -\omega \sin \omega t$; $\frac{d}{dt}e^{-bt} = -be^{-bt}$ and $\int \frac{dv}{v} = \ln v$. Check units after every integral: impulse in N s, work in J.

R2 Skills Checklist and Glossary

A checklist of the skills the exam rewards and a glossary of the key terms for Physics C: Mechanics.

Skills Checklist

Tick each skill only after you have done it on a problem you had not seen. The exam rewards three practices: creating representations, mathematical routines and scientific argumentation.

- ◆ Representations: draw a free-body diagram with labeled forces and a chosen positive direction (practice 1.A).
- ◆ Graphs: label axes with units and scales, plot data, and sketch qualitative graphs (x , v , a , K , U) stating where each rises or falls (1.B, 1.C).
- ◆ Derivations: reach a symbolic result by a logical path, defining every symbol and showing each integral or derivative setup (2.A).
- ◆ Calculation: compute unknowns with units and a sensible number of digits (2.B).
- ◆ Comparison: say how a quantity changes between scenarios, including factors of change such as a square root (2.C, 2.D).
- ◆ Calculus: integrate $F(t)$ for impulse, $F(x)$ for work, $a(t)$ for velocity; separate variables for $dv/dt = -cv^2$.
- ◆ Experiments: list only the equipment given, plan repeated trials, linearize (for example v^2 against height), and tie the slope to the unknown (3.A).
- ◆ Justification: name the law or definition, link it to the physical quantities, and finish with a clear claim (3.B, 3.C).

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- ◆ Rotation: write translational and rotational second-law equations with consistent signs; treat static friction as unknown, not $\mu_s N$.
- ◆ Energy: list every state and every energy term, including each spring, before writing the conservation equation.

Last-week habits

Do one timed free-response question daily, rewrite one weak derivation from memory, and review the traps in each unit. Use $g = 10 \text{ m/s}^2$ consistently.

Glossary

Term	Meaning
Acceleration	Rate of change of velocity, dv/dt .
Amplitude	Maximum displacement from equilibrium in an oscillation.
Angular acceleration	Rate of change of angular velocity, $d\omega/dt$.
Angular frequency	Rate of phase change in rad/s; $\omega = 2\pi f$.
Angular impulse	Integral of net torque over time; equals change in angular momentum.
Angular momentum	Rotational analog of momentum, $L = I\omega$ or $\mathbf{r} \times \mathbf{p}$.
Apoapsis	Point of an orbit farthest from the central body.
Center of mass	Point where a system's mass acts as if concentrated.
Centripetal force	Net inward force producing circular motion.
Conservative force	Force whose work depends only on endpoints; has a potential energy.
Damping	Loss of oscillation energy to friction or drag, shrinking amplitude.
Displacement	Change in position, a vector.
Elastic collision	Collision conserving kinetic energy as well as momentum.
Equilibrium	State of zero net force (and torque).
Escape speed	Minimum speed to leave a planet's gravity without further propulsion.
Frequency	Cycles per second, in hertz.
Friction, kinetic	Force opposing sliding, $f = \mu_k N$.
Friction, static	Force preventing sliding, at most $\mu_s N$.
Hooke's law	Spring force proportional to stretch, $F = -kx$.
Impulse	Integral of force over time; equals change in momentum.
Inertial frame	Reference frame in which Newton's first law holds.
Inelastic collision	Collision in which kinetic energy is not conserved.
Kinetic energy	Energy of motion, $\frac{1}{2}mv^2$.
Lever arm	Perpendicular distance from the axis to the line of force.
Linear mass density	Mass per unit length, $\lambda = dm/dx$.
Mechanical energy	Sum of kinetic and potential energy.
Moment of inertia	Another name for rotational inertia.
Momentum	Mass times velocity.
Net force	Vector sum of all forces on an object.
Normal force	Contact force perpendicular to a surface.
Parallel-axis theorem	$I = I_{\text{cm}} + Md^2$ about a parallel axis.
Period	Time for one full cycle.

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Term	Meaning
Periapsis	Point of an orbit closest to the central body.
Phase constant	Angle that sets the starting point of a sinusoid.
Physical pendulum	Rigid body swinging about a pivot.
Potential energy	Energy stored by position or configuration.
Power	Rate of doing work, dW/dt .
Resistive force	Force opposing motion through a fluid, depending on speed.
Restoring force	Force directed back toward equilibrium.
Rolling without slipping	Rolling with $v_{cm} = \omega R$ at the contact point.
Rotational inertia	Resistance to angular acceleration, sum of $m r^2$.
Rotational kinetic energy	Energy of rotation, $\frac{1}{2} I \omega^2$.
Scalar	Quantity with magnitude only.
Separation of variables	Method placing each variable on its own side before integrating.
Simple harmonic motion	Motion with acceleration proportional and opposite to displacement.
Simple pendulum	Point mass on a light string.
Small-angle approximation	$\sin(\theta) = \theta$ for angles in radians when small.
Spring constant	Stiffness k in N/m.
System	Chosen set of objects for analysis.
Terminal speed	Speed at which drag balances the driving force.
Torque	Rotational effect of a force, $r F \sin(\phi)$.
Turning point	Position where an oscillator momentarily stops.
Vector	Quantity with magnitude and direction.
Velocity	Rate of change of position, dx/dt .
Work	Integral of force along displacement.
Work-energy theorem	Net work equals change in kinetic energy.

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