



Radicals and Rational Expressions

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

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Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

15.1 Simplifying Radicals and Radical Operations

Learn to simplify square roots, combine and multiply radicals, and remove radicals from denominators. You will write exact answers in simplest form and check them with decimals.

Why This Matters

Square roots show up whenever you find a distance, the side of a square from its area, or the diagonal of a screen. A calculator will give you a long decimal like 8.485281..., but an exact answer such as $6\sqrt{2}$ is shorter, more precise, and much easier to use in the next step of a problem. Teachers ask for "simplest radical form" because it lets everyone write the same answer for the same number.

Students usually get stuck in three places: not knowing which perfect square to pull out, trying to add radicals the way they add whole numbers (for example, thinking $\sqrt{2} + \sqrt{3} = \sqrt{5}$), and forgetting that a radical is not allowed to stay in a denominator. In this section you will fix all three. These skills return in the quadratic formula, the distance formula, trigonometry, and physics formulas, so learning them well now pays off for years.

Core Ideas

A **square root** of a number n is a number that multiplies by itself to give n . The symbol $\sqrt{n}$ means the nonnegative one. So $\sqrt{49} = 7$. A number like 49 is a **perfect square** because its root is a whole number. Most numbers are not perfect squares, and their roots are never-ending decimals, so we keep them as radicals. Here are the perfect squares worth memorizing:

n	1	2	3	4	5	6	7	8	9	10	11	12
n^2	1	4	9	16	25	36	49	64	81	100	121	144

Product and Quotient Rules

For $a \geq 0$ and $b \geq 0$:

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \quad \text{and} \quad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad (b > 0).$$

These rules work in both directions. Use them left to right to pull out perfect squares, and right to left to combine radicals into one.

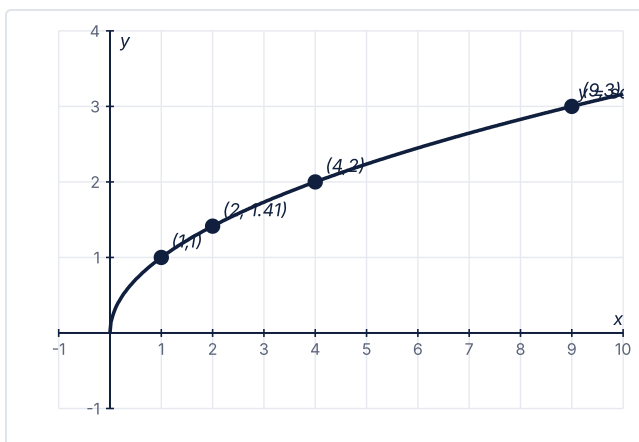
To **simplify** a radical, find the largest perfect square that divides the number under the root, split it off with the product rule, and take its root. For $\sqrt{72}$, the largest perfect square factor is 36, so $\sqrt{72} = \sqrt{36} \cdot \sqrt{2} = 6\sqrt{2}$. A radical is in **simplest form** when the number under the root has no perfect square factor other than 1, there is no fraction under the root, and no radical sits in a denominator.

For variables, every pair of identical factors comes out as one. Since $x^4 = (x^2)^2$, we have $\sqrt{x^4} = x^2$ (assume variables are nonnegative in this section). An odd power leaves one factor behind: $\sqrt{x^5} = \sqrt{x^4 \cdot x} = x^2\sqrt{x}$.

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The graph of $y = \sqrt{x}$. Perfect squares land on whole-number heights. The root of 2 is about 1.41, a point between them.

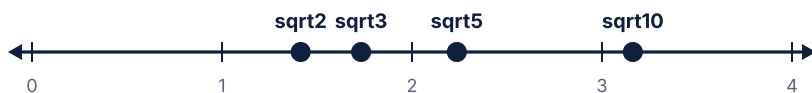
Adding and Subtracting Radicals

Radicals with the **same radicand** (the number under the root) are **like radicals** and combine like terms:

$$a\sqrt{n} + b\sqrt{n} = (a + b)\sqrt{n}.$$

Unlike radicals such as $\sqrt{2}$ and $\sqrt{3}$ cannot be combined. Always simplify each radical first, because radicals that look different may turn out to be like radicals.

Multiplying radicals uses the product rule on the outside numbers and the inside numbers separately: $(a\sqrt{m})(b\sqrt{n}) = ab\sqrt{mn}$. When you multiply two sums, use the distributive property (FOIL) just as you would with $(x + 2)(x + 3)$.



Square roots are real numbers with a place on the number line. Note that $\sqrt{2} + \sqrt{3}$ is about 3.15, not $\sqrt{5}$ (about 2.24).

Rationalizing the Denominator

To remove a radical from a denominator, multiply the top and bottom by the same thing.

One term: multiply by the radical itself: $\frac{a}{\sqrt{n}} = \frac{a\sqrt{n}}{n}$.

Two terms: multiply by the **conjugate**, which changes the sign between the terms: $(p + \sqrt{q})(p - \sqrt{q}) = p^2 - q$. The middle terms cancel and the radical disappears.

Worked Examples

EXAMPLE 1

Simplify $\sqrt{72}$ and $\sqrt{150}$.

SOLUTION

Find the largest perfect square factor. For 72 it is 36, and for 150 it is 25.

$$\sqrt{72} = \sqrt{36} \cdot \sqrt{2} = 6\sqrt{2}.$$

$$\sqrt{150} = \sqrt{25} \cdot \sqrt{6} = 5\sqrt{6}.$$

Check: $(6\sqrt{2})^2 = 36 \cdot 2 = 72$. Correct.

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EXAMPLE 2

Simplify $\sqrt{45x^5}$, where $x \geq 0$.

SOLUTION

Split into perfect squares: $45x^5 = 9 \cdot 5 \cdot x^4 \cdot x$.

$$\sqrt{45x^5} = \sqrt{9} \cdot \sqrt{x^4} \cdot \sqrt{5x} = 3x^2\sqrt{5x}.$$

The leftover $5x$ has no square factor, so we are done.

EXAMPLE 3

Simplify $3\sqrt{12} + 5\sqrt{27} - \sqrt{75}$.

SOLUTION

Simplify each radical first: $\sqrt{12} = 2\sqrt{3}$, $\sqrt{27} = 3\sqrt{3}$, $\sqrt{75} = 5\sqrt{3}$.

Substitute: $3(2\sqrt{3}) + 5(3\sqrt{3}) - 5\sqrt{3} = 6\sqrt{3} + 15\sqrt{3} - 5\sqrt{3}$.

Combine like radicals: $(6 + 15 - 5)\sqrt{3} = 16\sqrt{3}$.

EXAMPLE 4

Multiply and simplify $(2 + \sqrt{5})(3 - \sqrt{5})$.

SOLUTION

Use FOIL: $2 \cdot 3 = 6$, $2 \cdot (-\sqrt{5}) = -2\sqrt{5}$, $\sqrt{5} \cdot 3 = 3\sqrt{5}$, $\sqrt{5} \cdot (-\sqrt{5}) = -5$.

Add: $6 - 5 + (-2\sqrt{5} + 3\sqrt{5}) = 1 + \sqrt{5}$.

EXAMPLE 5

Rationalize the denominator: $\frac{6}{\sqrt{8}}$.

SOLUTION

Simplify first: $\sqrt{8} = 2\sqrt{2}$, so the fraction is $\frac{6}{2\sqrt{2}} = \frac{3}{\sqrt{2}}$.

Multiply top and bottom by $\sqrt{2}$: $\frac{3\sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{3\sqrt{2}}{2}$.

Check numerically: $6/2.828 = 2.121$ and $3(1.414)/2 = 2.121$.

EXAMPLE 6

Rationalize the denominator: $\frac{4}{3 - \sqrt{5}}$.

SOLUTION

The conjugate of $3 - \sqrt{5}$ is $3 + \sqrt{5}$. Multiply top and bottom by it.

Denominator: $(3 - \sqrt{5})(3 + \sqrt{5}) = 9 - 5 = 4$.

Numerator: $4(3 + \sqrt{5})$.

So the result is $\frac{4(3 + \sqrt{5})}{4} = 3 + \sqrt{5}$.

Coach's Playbook

- Factor out the biggest square.** List perfect squares (4, 9, 16, 25, 36, 49) and test them against the number. If you pick a smaller one, such as 4 for 72, just repeat until nothing is left to pull out.
- Simplify before you combine.** Before adding or subtracting radicals, simplify each one. $\sqrt{12}$ and $\sqrt{27}$ look unrelated but are $2\sqrt{3}$ and $3\sqrt{3}$.
- Rationalize last.** Simplify the radicals in the fraction first, reduce, and only then multiply by the radical or conjugate. The numbers stay small.
- Check with decimals.** Type both the original and your answer into a calculator (or estimate by hand). They must match to several digits.

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- ◆ Prime factor if stuck: $180 = 2^2 \cdot 3^2 \cdot 5$, so $\sqrt{180} = 2 \cdot 3 \cdot \sqrt{5} = 6\sqrt{5}$. Each pair comes out as one factor.
- ◆ Think of "like radicals" as like terms: $4\sqrt{3}$ and $9\sqrt{3}$ are like $4x$ and $9x$.
- ◆ The conjugate trick is the same as difference of squares: $(a - b)(a + b) = a^2 - b^2$.
- ◆ Remember that $(\sqrt{n})^2 = n$. This one fact finishes many problems.

Common traps

1. Writing $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$. It is false: $\sqrt{9+16} = 5$, not $3+4$. 2. Adding unlike radicals, such as $2\sqrt{3} + 4\sqrt{5} = 6\sqrt{8}$. 3. Pulling out a number but forgetting to take its root, for instance $\sqrt{72} = 36\sqrt{2}$. 4. Multiplying only the denominator when rationalizing, which changes the value of the fraction.

Mistake Clinic

Spot the error

A student writes: $\sqrt{50} + \sqrt{18} = \sqrt{68}$. **Where it goes wrong:** Radicals can only be added when they have the same radicand, and adding the numbers under the roots is not allowed. The decimals show the problem: $7.07 + 4.24 = 11.31$, but $\sqrt{68} \approx 8.25$. **Correct solution:** Simplify first. $\sqrt{50} = 5\sqrt{2}$ and $\sqrt{18} = 3\sqrt{2}$, so the sum is $8\sqrt{2} \approx 11.31$.

Spot the error

A student writes: $\frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}} \cdot \sqrt{3} = \frac{5}{3}$. **Where it goes wrong:** Only the top was multiplied by $\sqrt{3}$ in the middle step, then the radical was dropped. Multiplying the numerator alone changes the value. You must multiply top and bottom, since $\frac{\sqrt{3}}{\sqrt{3}} = 1$. **Correct solution:** $\frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$.

Spot the error

A student writes: $(1 + \sqrt{6})^2 = 1 + 6 = 7$. **Where it goes wrong:** Squaring a sum is not the same as squaring each term. The middle term $2 \cdot 1 \cdot \sqrt{6}$ was skipped. **Correct solution:** $(1 + \sqrt{6})^2 = 1 + 2\sqrt{6} + 6 = 7 + 2\sqrt{6}$. Check: $(3.449)^2 \approx 11.90$ and $7 + 4.899 = 11.90$.

Real-World Connection

A square patio has an area of 200 square feet. Its side is $\sqrt{200} = 10\sqrt{2}$ feet, about 14.14 feet. The exact form helps: the diagonal of any square is the side times $\sqrt{2}$, so this patio's diagonal is exactly $10\sqrt{2} \cdot \sqrt{2} = 20$ feet.

Challenge Corner

Write $\frac{\sqrt{7} + \sqrt{3}}{\sqrt{7} - \sqrt{3}}$ in simplest form with a rational denominator. **Solution.** Multiply top and bottom by the conjugate $\sqrt{7} + \sqrt{3}$. The denominator becomes $7 - 3 = 4$. The numerator is $(\sqrt{7} + \sqrt{3})^2 = 7 + 2\sqrt{21} + 3 = 10 + 2\sqrt{21}$. So the result is $\frac{10 + 2\sqrt{21}}{4} = \frac{5 + \sqrt{21}}{2}$. Check: the original is about $4.378/0.914 = 4.79$, and $(5 + 4.583)/2 = 4.79$.

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Practice

PRACTICE Simplifying Radicals and Radical Operations

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is $\sqrt{98}$ in simplest radical form?
 (A) $49\sqrt{2}$ (B) $7\sqrt{2}$ (C) $14\sqrt{7}$ (D) $2\sqrt{49}$
- What is $\sqrt{20} \cdot \sqrt{5}$?
 (A) 5 (B) 10 (C) 25 (D) 100
- Simplify $2\sqrt{3} + 5\sqrt{3}$.
 (A) $7\sqrt{3}$ (B) $7\sqrt{6}$ (C) $10\sqrt{3}$ (D) $10\sqrt{6}$
- What is $\frac{\sqrt{27}}{\sqrt{3}}$?
 (A) 9 (B) $\sqrt{24}$ (C) 24 (D) 3
- Rationalize the denominator: $\frac{10}{\sqrt{5}}$.
 (A) 2 (B) $10\sqrt{5}$ (C) $2\sqrt{5}$ (D) $5\sqrt{2}$
- What is $(\sqrt{3} + 2)(\sqrt{3} - 2)$?
 (A) -1 (B) 1 (C) 7 (D) $7 - 4\sqrt{3}$
- Simplify $\sqrt{75} - \sqrt{12}$.
 (A) $\sqrt{63}$ (B) $3\sqrt{3}$ (C) 3 (D) $7\sqrt{3}$
- Simplify $\sqrt{48a^7}$, where $a \geq 0$.
 (A) $16a^3\sqrt{3a}$ (B) $4a^2\sqrt{3a}$ (C) $4a^3\sqrt{3a}$ (D) $4a^3\sqrt{3}$
- Rationalize the denominator: $\frac{2}{\sqrt{6}-2}$.
 (A) $\sqrt{6} - 2$ (B) $2\sqrt{6} + 4$ (C) $\frac{\sqrt{6} + 2}{2}$ (D) $\sqrt{6} + 2$
- Simplify $\sqrt{180}$.
- Multiply and simplify $(3\sqrt{2})(4\sqrt{6})$.
- Expand and simplify $(\sqrt{5} + \sqrt{2})^2$.
- Rationalize the denominator and simplify: $\frac{12}{\sqrt{7} - \sqrt{3}}$.
- A square garden has area 200 square meters. Find its perimeter in simplest radical form.
- Write $\frac{1}{\sqrt{5} - \sqrt{3}} + \frac{1}{\sqrt{5} + \sqrt{3}}$ as a single simplified number.

15.2 Simplifying Rational Expressions

Factor the top and bottom of a fraction, cancel common factors, and state which values of the variable are not allowed. You will also see why a cancelled factor leaves a hole in the graph.

Why This Matters

A rational expression is a fraction whose top and bottom are polynomials, such as $\frac{x+3}{x-2}$. Rational expressions describe average cost (total cost divided by items), speed (distance divided by time), and the gain in many electronics formulas. To work with them you need one skill above all: **factoring**. Simplifying a rational expression is exactly like reducing $\frac{12}{18}$ to $\frac{2}{3}$, except that the common factors are hiding inside polynomials.

Students most often get stuck when they cancel terms instead of factors (crossing out the x in $\frac{x+3}{x}$), when they forget which values of x make the denominator zero, and when they miss that $x - 5$ and $5 - x$ are opposites. This section gives you a reliable routine that avoids all three: factor, cancel, and state the restrictions.

Core Ideas

A fraction is undefined when its denominator is zero, because you cannot divide by zero. So every rational expression comes with **restrictions**: values of the variable that are not allowed. To find them, set the original denominator equal to zero and solve. For $\frac{x+5}{x^2-3x-10}$ the denominator factors as $(x-5)(x+2)$, so the restrictions are $x \neq 5$ and $x \neq -2$.

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Cancelling Common Factors

If A , B , and C are polynomials and $B \neq 0$, $C \neq 0$, then

$$\frac{AC}{BC} = \frac{A}{B}.$$

You may cancel only **factors** (things multiplied), never **terms** (things added). In $\frac{x(x+3)}{x(x-2)}$ the x cancels. In $\frac{x+3}{x-2}$ nothing cancels.

Here is the routine for simplifying. First, **factor** the numerator and the denominator completely. Look for a greatest common factor first, then for a difference of squares, a trinomial that factors, or grouping. Second, **cancel** every factor that appears in both. Third, **state the restrictions** that came from the original denominator, even for factors that cancelled.

Factoring Patterns to Know

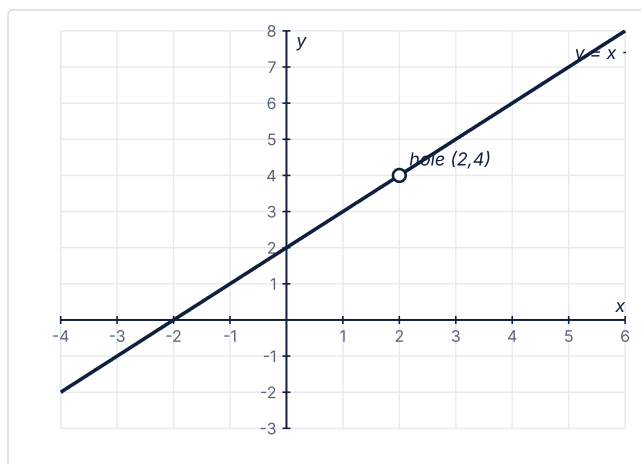
Difference of squares: $a^2 - b^2 = (a - b)(a + b)$.

Trinomial: $x^2 + bx + c = (x + m)(x + n)$ where $mn = c$ and $m + n = b$.

Opposites: $a - b = -(b - a)$, so $\frac{a - b}{b - a} = -1$ whenever $a \neq b$.

Original expression	Factored form	Simplified	Restrictions
$\frac{x^2 - 4}{x - 2}$	$\frac{(x - 2)(x + 2)}{x - 2}$	$x + 2$	$x \neq 2$
$\frac{x^2 + 3x}{x^2 - 9}$	$\frac{x(x + 3)}{(x - 3)(x + 3)}$	$\frac{x}{x - 3}$	$x \neq 3, -3$
$\frac{5 - x}{x - 5}$	$\frac{-(x - 5)}{x - 5}$	-1	$x \neq 5$

Why keep the restriction after cancelling? Look at $f(x) = \frac{x^2 - 4}{x - 2}$. It simplifies to $x + 2$, so its graph is the line $y = x + 2$. But $f(2)$ is still undefined, because the original denominator is zero there. The graph therefore has a **hole** at $x = 2$, at the height the line would have, which is $y = 4$.



The graph of $(x^2 - 4)/(x - 2)$ is the line $y = x + 2$ with a hole at $(2, 4)$.

Equivalent Only With Restrictions

The simplified expression equals the original one **for every allowed value of x** . Write the restrictions next to your answer, for example $\frac{x + 3}{x - 2}$, $x \neq 2, 3$, so that the two expressions are truly equivalent.

Worked Examples

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EXAMPLE 1

Simplify $\frac{6x^2y}{9xy^3}$ and state the restrictions.

SOLUTION

Divide the numbers by their greatest common factor, 3: $\frac{2x^2y}{3xy^3}$.

Cancel powers of each variable: $x^2/x = x$ and $y/y^3 = 1/y^2$.

The result is $\frac{2x}{3y^2}$, with $x \neq 0$ and $y \neq 0$.

EXAMPLE 2

Find the values of x for which $\frac{x+5}{x^2-3x-10}$ is undefined.

SOLUTION

Factor the denominator: $x^2 - 3x - 10 = (x - 5)(x + 2)$.

Set each factor equal to zero: $x = 5$ or $x = -2$.

The expression is undefined at $x = 5$ and $x = -2$. Nothing cancels, since $x + 5$ is not a factor of the denominator.

EXAMPLE 3

Simplify $\frac{x^2-9}{x^2-5x+6}$ and state the restrictions.

SOLUTION

Factor the top as a difference of squares: $(x - 3)(x + 3)$.

Factor the bottom: $(x - 2)(x - 3)$.

Cancel $x - 3$: the result is $\frac{x+3}{x-2}$.

Restrictions come from the original bottom: $x \neq 2$ and $x \neq 3$.

EXAMPLE 4

Simplify $\frac{2x^2+7x+3}{2x^2-x-1}$.

SOLUTION

Factor the top: $2x^2 + 7x + 3 = (2x + 1)(x + 3)$.

Factor the bottom: $2x^2 - x - 1 = (2x + 1)(x - 1)$.

Cancel $2x + 1$: the result is $\frac{x+3}{x-1}$, with $x \neq -\frac{1}{2}$ and $x \neq 1$.

Check with $x = 2$: original $\frac{8+14+3}{8-2-1} = \frac{25}{5} = 5$, and simplified $\frac{5}{1} = 5$.

EXAMPLE 5

Simplify $\frac{x^2-4x}{16-x^2}$.

SOLUTION

Factor the top: $x(x - 4)$. Factor the bottom: $16 - x^2 = (4 - x)(4 + x)$.

Rewrite $4 - x = -(x - 4)$. The bottom is $-(x - 4)(x + 4)$.

Cancel $x - 4$: $\frac{x}{-(x+4)} = -\frac{x}{x+4}$, with $x \neq 4, -4$.

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EXAMPLE 6

Simplify $\frac{x^3 + 2x^2 - 9x - 18}{x^2 - x - 6}$.

SOLUTION

Factor the top by grouping: $x^2(x + 2) - 9(x + 2) = (x + 2)(x^2 - 9) = (x + 2)(x - 3)(x + 3)$.

Factor the bottom: $(x - 3)(x + 2)$.

Cancel $(x + 2)(x - 3)$: the result is $x + 3$, with $x \neq 3$ and $x \neq -2$.

Check with $x = 1$: top = $1 + 2 - 9 - 18 = -24$, bottom = -6 , quotient 4, and $1 + 3 = 4$.

Coach's Playbook

- 1 Factor completely.** GCF first, then special patterns, then trinomials, then grouping. Do both top and bottom before you cancel anything.
 - 2 Write the restrictions early.** Set the original denominator factors to zero right away, while the denominator is still visible.
 - 3 Cancel factors only.** Circle matching factors. If something is added or subtracted in the way, it is a term and cannot be cancelled.
 - 4 Test with a number.** Pick an allowed value such as $x = 2$ and compute the original and your answer. They must match.
- ◆ Keep a negative sign in mind: if a factor looks "backwards", like $5 - x$, rewrite it as $-(x - 5)$.
 - ◆ A fraction whose top and bottom are equal factors reduces to 1. A fraction whose top and bottom are opposites reduces to -1 .
 - ◆ If the numerator has four terms, think grouping. If it has two squares, think difference of squares.
 - ◆ A restriction never goes away. Cancelling does not make a forbidden value allowed.

Common traps

1. Cancelling terms, such as turning $\frac{x+3}{x+5}$ into $\frac{3}{5}$. 2. Forgetting a restriction because the factor cancelled. 3. Missing opposites, so $\frac{x-4}{4-x}$ is left alone instead of becoming -1 . 4. Taking restrictions from the simplified answer instead of the original denominator.

Mistake Clinic**Spot the error**

A student simplifies $\frac{x^2 + 6}{x^2 + 2}$ and writes $\frac{6}{2} = 3$ by cancelling x^2 . **Where it goes wrong:** The x^2 is added to 6 and to 2, so it is a term, not a factor. Test $x = 1$: the original is $\frac{7}{3}$, not 3. **Correct solution:** Neither the top nor the bottom factors further, so $\frac{x^2 + 6}{x^2 + 2}$ is already in simplest form.

Spot the error

A student simplifies $\frac{x^2 - 25}{x^2 - 3x - 10}$ and gets $\frac{x + 5}{x + 2}$, then says the only restriction is $x \neq -2$. **Where it goes wrong:** The simplification is right, but the restriction is taken from the simplified answer. The original bottom is $(x - 5)(x + 2)$, so $x = 5$ is also forbidden. **Correct solution:** $\frac{(x - 5)(x + 5)}{(x - 5)(x + 2)} = \frac{x + 5}{x + 2}$ with $x \neq 5$ and $x \neq -2$.

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Spot the error

A student simplifies $\frac{3-x}{x-3}$ and says the answer is 1 because the top and bottom "match". **Where it goes wrong:** The top is the opposite of the bottom. Test $x = 5$: $\frac{3-5}{5-3} = \frac{-2}{2} = -1$. **Correct solution:** $3-x = -(x-3)$, so $\frac{-(x-3)}{x-3} = -1$ for $x \neq 3$.

Real-World Connection

A school print shop charges 120 dollars to set up plus 2 dollars per poster. The average cost per poster is $\frac{2n+120}{n}$ dollars, with $n > 0$. Factoring the top gives $\frac{2(n+60)}{n}$. For $n = 60$ posters the average is $\frac{2(120)}{60} = 4$ dollars. The restriction $n \neq 0$ matches real life: you cannot average the cost of zero posters.

Challenge Corner

Simplify $\frac{x^3-27}{x^2-9}$ and find its value at $x = 0$. **Solution.** Use the difference of cubes $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ to get $x^3 - 27 = (x-3)(x^2 + 3x + 9)$. The bottom is $(x-3)(x+3)$. Cancel $x-3$: the result is $\frac{x^2 + 3x + 9}{x+3}$, with $x \neq 3, -3$. At $x = 0$: $\frac{9}{3} = 3$. The original gives $\frac{-27}{-9} = 3$ as well.

Practice**PRACTICE Simplifying Rational Expressions**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- For which value of x is $\frac{5}{x-7}$ undefined?
 (A) -7 (B) 5 (C) 7 (D) 0
- Simplify $\frac{12x^3}{18x^2}$.
 (A) $\frac{2x^2}{3}$ (B) $\frac{2x^3}{3}$ (C) $\frac{2x}{3}$ (D) $\frac{2}{3}$
- Simplify $\frac{x+4}{x^2-16}$.
 (A) $\frac{1}{x+4}$ (B) $\frac{1}{x-4}$ (C) $x-4$ (D) $-\frac{1}{4}$
- What values of x are not allowed in $\frac{x-3}{(x-3)(x+1)}$?
 (A) $x \neq -1$ only (B) $x \neq 3$ only (C) $x \neq 3$ and $x \neq 1$
 (D) $x \neq 3$ and $x \neq -1$
- Simplify $\frac{x^2-25}{x^2+10x+25}$.
 (A) $\frac{x-5}{x+5}$ (B) $\frac{x+5}{x-5}$ (C) $\frac{x-5}{5}$ (D) -1
- Simplify $\frac{x^2+5x+6}{x^2+3x+2}$.
 (A) 3 (B) $\frac{x+3}{x+2}$ (C) $\frac{x+3}{x+1}$ (D) $\frac{x+6}{x+2}$
- What does $\frac{6-x}{x-6}$ simplify to, for $x \neq 6$?
 (A) 1 (B) -1 (C) 0 (D) 6
- Simplify $\frac{3x^2-12}{x^2+x-6}$.
 (A) $\frac{x+2}{x+3}$ (B) $\frac{3(x-2)}{x+3}$ (C) $\frac{3x+2}{x+3}$ (D) $\frac{3(x+2)}{x+3}$

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9. The graph of $f(x) = \frac{x^2 - 9}{x - 3}$ is a line with a hole. Where is the hole?
 (A) (3, 0) (B) (-3, 0) (C) (3, 6) (D) (0, 3)
10. Simplify $\frac{15a^2b}{25ab^3}$.
11. Simplify $\frac{x^2 - 7x + 12}{x^2 - 16}$ and list all restrictions.
12. Simplify $\frac{2x^2 - 5x - 3}{2x^2 + x}$ and list all restrictions.
13. Simplify $\frac{x^3 - 5x^2 - 4x + 20}{x^2 - 4}$.
14. A rectangle has area $x^2 + 6x + 8$ and width $x + 2$. Write an expression for its length.
15. The expression $\frac{2x^2 - 8}{x^2 - x - 2}$ is simplified. Find all values of x where the original is undefined, and the value of the simplified expression at $x = 3$.

15.3 Multiplying, Dividing, Adding, and Subtracting Rational Expressions

Use factoring and common denominators to multiply, divide, add, and subtract rational expressions. You will also clear complex fractions in one step and check results with a test number.

Why This Matters

Once you can simplify a rational expression, the next step is to do arithmetic with them. Rational expressions follow exactly the same rules as ordinary fractions: multiply tops and bottoms, flip and multiply to divide, and find a common denominator to add or subtract. The only new ingredient is that the numerators and denominators are polynomials, so factoring makes everything shorter.

This matters whenever two rates combine: two people working together, a boat going upstream and downstream, resistors in parallel, or two pipes filling a tank. Students typically get stuck by adding denominators (writing $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$), by forgetting to distribute a minus sign over a whole numerator when subtracting, and by flipping the wrong fraction when dividing. Fix those three habits and this topic becomes routine.

Core Ideas

Multiplying and Dividing

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD} \quad \text{and} \quad \frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C} \quad (C \neq 0).$$

Factor everything first, cancel common factors across the top and the bottom, then multiply what is left. For division, flip only the second fraction.

When you multiply, there is no need for a common denominator. Instead, factor each numerator and denominator, and cancel any top factor against any bottom factor, even when they come from different fractions. This is the same idea as simplifying $\frac{4}{9} \cdot \frac{3}{8}$ by cancelling the 4 with the 8 and the 3 with the 9 before multiplying.

Adding and Subtracting

Fractions can be added or subtracted only when they have the same denominator:

$$\frac{A}{D} \pm \frac{B}{D} = \frac{A \pm B}{D}.$$

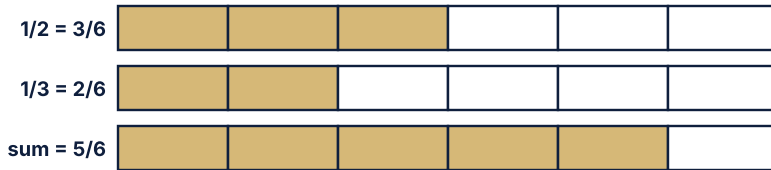
The **least common denominator (LCD)** contains every different factor of the denominators, each raised to its highest power. Rewrite each fraction with the LCD by multiplying the top and bottom by the missing factors.

To see why a common denominator is needed, think about pieces of the same size. A half and a third cannot be added directly, because the pieces differ in size. Cut both into sixths: $\frac{1}{2} = \frac{3}{6}$ and $\frac{1}{3} = \frac{2}{6}$, so together they make $\frac{5}{6}$. Rational expressions work the same way, with the LCD playing the role of the six.

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Same-size pieces make addition possible. The LCD is the common piece size.

Denominators	Factored	LCD
x and $x + 2$	$x, (x + 2)$	$x(x + 2)$
$x^2 - 9$ and $x + 3$	$(x - 3)(x + 3), (x + 3)$	$(x - 3)(x + 3)$
$x^2 + 5x + 6$ and $x^2 + 3x + 2$	$(x + 2)(x + 3), (x + 1)(x + 2)$	$(x + 1)(x + 2)(x + 3)$

A **complex fraction** has fractions in its numerator, its denominator, or both. There are two reliable methods. Method 1: simplify the top into one fraction, simplify the bottom into one fraction, then divide. Method 2: multiply the top and bottom of the whole thing by the LCD of all the small fractions, which clears them in one step.

Clearing a Complex Fraction

Multiply the numerator and the denominator by the LCD of every small fraction inside. Because you multiply by $\frac{\text{LCD}}{\text{LCD}} = 1$, the value does not change, and the small fractions disappear. Always carry the restrictions from the original denominators, and from the original bottom fraction too.

Worked Examples

EXAMPLE 1

Multiply: $\frac{x^2 - 4}{x + 3} \cdot \frac{x^2 + 3x}{x + 2}$.

SOLUTION

Factor: $\frac{(x - 2)(x + 2)}{x + 3} \cdot \frac{x(x + 3)}{x + 2}$.

Cancel $x + 2$ and $x + 3$, leaving $(x - 2) \cdot x$.

The product is $x(x - 2)$, with $x \neq -3, -2$.

EXAMPLE 2

Divide: $\frac{x^2 - 9}{2x} \div \frac{x + 3}{4x^2}$.

SOLUTION

Flip the second fraction and multiply: $\frac{(x - 3)(x + 3)}{2x} \cdot \frac{4x^2}{x + 3}$.

Cancel $x + 3$, and reduce $\frac{4x^2}{2x} = 2x$.

The quotient is $2x(x - 3)$, with $x \neq 0, -3$.

EXAMPLE 3

Add: $\frac{3}{x} + \frac{5}{x + 2}$.

SOLUTION

The LCD is $x(x + 2)$.

Rewrite: $\frac{3(x + 2)}{x(x + 2)} + \frac{5x}{x(x + 2)}$.

Add the numerators: $3x + 6 + 5x = 8x + 6$.

The sum is $\frac{8x + 6}{x(x + 2)}$. Check at $x = 1$: $3 + \frac{5}{3} = \frac{14}{3}$ and $\frac{14}{3}$. It matches.

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EXAMPLE 4

Subtract: $\frac{x}{x-3} - \frac{18}{x^2-9}$.

SOLUTION

Factor the second denominator: $(x-3)(x+3)$. This is the LCD.

Rewrite the first fraction: $\frac{x(x+3)}{(x-3)(x+3)}$.

Subtract: $\frac{x^2+3x-18}{(x-3)(x+3)}$. Factor the top: $(x+6)(x-3)$.

Cancel $x-3$: the result is $\frac{x+6}{x+3}$, with $x \neq 3, -3$.

EXAMPLE 5

Simplify the complex fraction $\frac{\frac{1}{x} + \frac{1}{3}}{\frac{1}{x} - \frac{1}{3}}$.

SOLUTION

The small denominators are x and 3 , so the LCD is $3x$.

Multiply top and bottom by $3x$: the top becomes $3+x$ and the bottom becomes $3-x$.

The result is $\frac{3+x}{3-x}$, with $x \neq 0$ and $x \neq 3$.

EXAMPLE 6

Add and simplify: $\frac{1}{x^2+5x+6} + \frac{1}{x^2+3x+2}$.

SOLUTION

Factor: $(x+2)(x+3)$ and $(x+1)(x+2)$. The LCD is $(x+1)(x+2)(x+3)$.

Rewrite: $\frac{x+1}{(x+1)(x+2)(x+3)} + \frac{x+3}{(x+1)(x+2)(x+3)}$.

Add the tops: $2x+4 = 2(x+2)$. Cancel $x+2$.

The sum is $\frac{2}{(x+1)(x+3)}$, with $x \neq -1, -2, -3$.

Coach's Playbook

- 1 Identify the operation.** Multiply and divide need factoring and cancelling. Add and subtract need an LCD.
 - 2 Factor every denominator.** Do this before looking for the LCD so no shared factor is missed.
 - 3 Build the LCD.** Write each different factor once, with the highest power it has in any denominator.
 - 4 Combine, then simplify.** Put everything over one denominator, expand the top, factor it, and cancel if possible.
- ◆ For subtraction, put the second numerator in parentheses: $\frac{x}{D} - \frac{x-1}{D} = \frac{x-(x-1)}{D}$.
 - ◆ If two denominators are opposites, such as $x-2$ and $2-x$, rewrite one of them with a negative sign so they match.
 - ◆ Test your final answer with one number. The original and the answer must agree.
 - ◆ In a complex fraction, find the LCD of the small fractions only, not of the big fraction.

Common traps

1. Adding denominators, as in $\frac{1}{x} + \frac{1}{3} = \frac{2}{x+3}$. 2. Not distributing the minus sign in a subtraction. 3. Flipping the first fraction instead of the second when dividing. 4. Cancelling terms across a sum, like crossing out x in $\frac{x+2}{x}$.

Mistake Clinic

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Spot the error

A student adds $\frac{2}{x} + \frac{3}{x+1}$ and writes $\frac{5}{2x+1}$. **Where it goes wrong:** Adding numerators and denominators separately is not a rule for adding fractions. Test $x = 1$: the correct sum is $2 + \frac{3}{2} = 3.5$, but $\frac{5}{3} \approx 1.67$. **Correct solution:** LCD is $x(x+1)$: $\frac{2(x+1) + 3x}{x(x+1)} = \frac{5x+2}{x(x+1)}$. At $x = 1$ this is $\frac{7}{2} = 3.5$.

Spot the error

A student subtracts $\frac{5x}{x+4} - \frac{x-3}{x+4}$ and writes $\frac{4x-3}{x+4}$. **Where it goes wrong:** The minus applies to the whole second numerator. The student computed $5x - x - 3$, which leaves the 3 with the wrong sign. **Correct solution:** $\frac{5x - (x-3)}{x+4} = \frac{5x - x + 3}{x+4} = \frac{4x+3}{x+4}$.

Spot the error

A student divides $\frac{6}{x} \div \frac{2}{x^2}$ and writes $\frac{6}{x} \cdot \frac{2}{x^2} = \frac{12}{x^3}$. **Where it goes wrong:** Division means multiply by the reciprocal of the second fraction. The student forgot to flip. **Correct solution:** $\frac{6}{x} \cdot \frac{x^2}{2} = \frac{6x^2}{2x} = 3x$ for $x \neq 0$. Check with $x = 2$: $3 \div 0.5 = 6$ and $3(2) = 6$.

Real-World Connection

You bike 12 miles at x miles per hour, then 12 more miles at $x + 4$ miles per hour. The total time is $\frac{12}{x} + \frac{12}{x+4} = \frac{24(x+2)}{x(x+4)}$ hours. At $x = 8$ this is $\frac{24 \cdot 10}{8 \cdot 12} = 2.5$ hours, matching $1.5 + 1 = 2.5$.

Challenge Corner

Simplify $\frac{\frac{1}{x-1} - \frac{1}{x+1}}{\frac{1}{x^2-1}}$. **Solution.** The LCD of the small fractions is $(x-1)(x+1) = x^2 - 1$. Multiply the top and bottom of the big fraction by it. The top becomes $(x+1) - (x-1) = 2$, and the bottom becomes 1. The answer is the constant 2, for $x \neq 1, -1$. Check at $x = 3$: top = $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, bottom = $\frac{1}{8}$, and $\frac{1/4}{1/8} = 2$.

Practice**PRACTICE** Multiplying, Dividing, Adding, and Subtracting Rational Expressions

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

1. Multiply: $\frac{2}{x} \cdot \frac{x^2}{6}$.

(A) $\frac{x}{3}$

(B) $3x$

(C) $\frac{x^3}{3}$

(D) $\frac{1}{3x}$

2. Divide: $\frac{x}{4} \div \frac{x}{8}$.

(A) $\frac{1}{2}$

(B) 2

(C) 32

(D) $\frac{x^2}{32}$

3. Add: $\frac{3}{x} + \frac{2}{x}$.

(A) $\frac{5}{2x}$

(B) $\frac{5}{x^2}$

(C) $\frac{5}{x}$

(D) $\frac{6}{x}$

4. Add: $\frac{1}{2} + \frac{1}{x}$.

(A) $\frac{2}{x+2}$

(B) $\frac{1}{2x}$

(C) $\frac{x+1}{2x}$

(D) $\frac{x+2}{2x}$

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5. Subtract: $\frac{x+5}{x-1} - \frac{2}{x-1}$.

(A) $\frac{x+7}{x-1}$

(B) $\frac{x+3}{x-1}$

(C) $x+3$

(D) $\frac{x+3}{2x-2}$

6. Multiply and simplify: $\frac{x^2-1}{x} \cdot \frac{3x}{x+1}$.

(A) $3(x+1)$

(B) $3x-1$

(C) $3(x-1)$

(D) $3(x^2-1)$

7. Divide: $\frac{x^2-4}{x+1} \div \frac{x-2}{x+1}$.

(A) $x+2$

(B) $x-2$

(C) $\frac{(x+2)(x-2)^2}{(x+1)^2}$

(D) $\frac{1}{x+2}$

8. Simplify $\frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{2}}$.

(A) $\frac{x+2}{2}$

(B) $\frac{2}{x+2}$

(C) $\frac{1}{x+2}$

(D) $\frac{2x}{x+2}$

9. Add: $\frac{4}{x^2-9} + \frac{2}{x+3}$.

(A) $\frac{2x+10}{(x-3)(x+3)}$

(B) $\frac{6}{x^2+x-6}$

(C) $\frac{2x-2}{x+3}$

(D) $\frac{2x-2}{(x-3)(x+3)}$

10. Multiply and simplify: $\frac{3x^2}{5y} \cdot \frac{10y^2}{9x}$.

11. Divide and simplify: $\frac{x^2-16}{x+2} \div \frac{x+4}{x^2-4}$.

12. Add: $\frac{4}{x-2} + \frac{3}{x+1}$.

13. Subtract and simplify: $\frac{x}{x^2-1} - \frac{1}{x+1}$.

14. Simplify the complex fraction $\frac{1-\frac{4}{x^2}}{1-\frac{2}{x}}$ and state the restrictions.

15. A cyclist rides 15 km at x km/h and then 15 km at $(x+5)$ km/h. Write the total time as one fraction, then find the time in hours when $x=10$.

15.4 Rational and Radical Equations

Solve equations with fractions and square roots, check for extraneous solutions, and set up work, direct variation, and inverse variation problems. You will learn why checking answers is essential.

Why This Matters

Equations with fractions and square roots model real situations: how long two people take to finish a job together, how the time for a trip changes with speed, how a falling object's speed relates to its height, and how the brightness of a lamp fades with distance. Solving them uses everything from the last three sections: factoring, common denominators, and radicals.

The big idea is that the steps used to solve these equations (multiplying by a variable expression, squaring both sides) can create answers that do not actually work. Those are called **extraneous solutions**. Students get stuck because they solve correctly and then forget to check. Other common struggles are setting up work problems (adding times instead of rates) and mixing up direct and inverse variation. This section gives you clear rules for each.

Core Ideas

To solve a **rational equation**, clear the fractions by multiplying both sides by the LCD of all the denominators. If the equation is just one fraction equal to another, cross-multiplying does the same job: $\frac{a}{b} = \frac{c}{d}$ means $ad = bc$. Then solve the resulting equation, which is often linear or quadratic.

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Solving Rational Equations

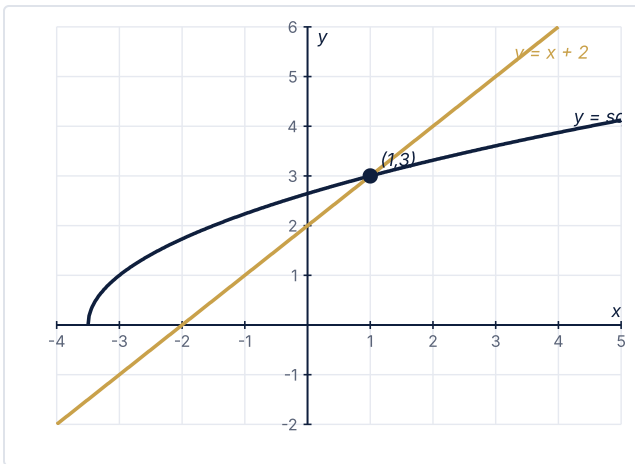
1. Factor the denominators and state the values of the variable that make any denominator zero.
2. Multiply both sides by the LCD.
3. Solve the resulting equation.
4. **Check** every answer. Reject any solution that makes an original denominator zero.

To solve a **radical equation**, first get the radical alone on one side. Then square both sides to remove the square root. Squaring can add a false solution, because $(-3)^2 = 3^2$ even though $-3 \neq 3$. So you must check each answer in the original equation.

Extraneous Solutions

An **extraneous solution** is a value that comes out of the algebra but fails in the original equation. Rational equations lose answers that make a denominator zero. Radical equations lose answers where the square root would equal a negative number, since $\sqrt{}$ is never negative.

If every answer fails, the equation has **no solution**.



Solving $\sqrt{2x+7} = x + 2$. The graphs cross only at $x = 1$. The algebra also produces $x = -3$, but the line is below the axis there while the root is positive, so -3 is extraneous.

Work problems use rates. If a job takes you 6 hours, your rate is $\frac{1}{6}$ of the job per hour. When people work together, their rates add. So if one worker takes a hours and another takes b hours, the time t together satisfies $\frac{1}{a} + \frac{1}{b} = \frac{1}{t}$. Together is always faster than the faster person alone.

Variation

Direct variation: $y = kx$. The ratio $\frac{y}{x}$ is constant, so doubling x doubles y .

Inverse variation: $y = \frac{k}{x}$, or $xy = k$. The product is constant, so doubling x halves y .

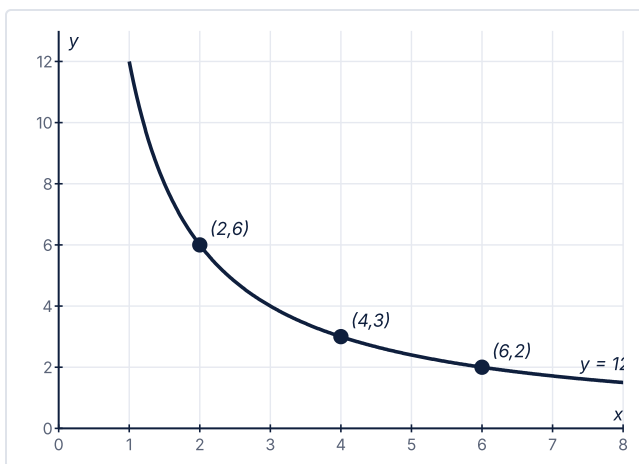
In both cases, use one known pair to find the constant k , then use k to answer the question.

Type	Equation	Graph	Example
Direct	$y = kx$	line through the origin	pay vs. hours worked
Inverse	$y = \frac{k}{x}$	curve that falls toward the axes	time vs. speed for a fixed trip

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Inverse variation with $k = 12$. The product of x and y is always 12, and the graph falls as x grows.

Worked Examples

EXAMPLE 1

Solve $\frac{5}{x-2} = \frac{3}{x}$.

SOLUTION

Restrictions: $x \neq 2$ and $x \neq 0$.

Cross-multiply: $5x = 3(x-2)$, so $5x = 3x - 6$.

Subtract $3x$: $2x = -6$, so $x = -3$.

Check: $\frac{5}{-5} = -1$ and $\frac{3}{-3} = -1$. The solution is $x = -3$.

EXAMPLE 2

Solve $\frac{x}{x-3} = \frac{3}{x-3} + 2$.

SOLUTION

Restriction: $x \neq 3$. Multiply every term by $x-3$: $x = 3 + 2(x-3)$.

Simplify: $x = 3 + 2x - 6 = 2x - 3$, so $x = 3$.

But $x = 3$ makes the denominators zero, so it is extraneous.

The equation has **no solution**.

EXAMPLE 3

Solve $\sqrt{2x+7} = x+2$.

SOLUTION

The radical is already alone. Square both sides: $2x+7 = x^2+4x+4$.

Move everything to one side: $x^2+2x-3 = 0$, so $(x+3)(x-1) = 0$ and $x = -3$ or $x = 1$.

Check $x = 1$: $\sqrt{9} = 3$ and $1+2 = 3$. It works.

Check $x = -3$: $\sqrt{1} = 1$ but $-3+2 = -1$. It fails.

The only solution is $x = 1$.

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EXAMPLE 4

Ana can paint a fence in 6 hours. Ben can paint it in 3 hours. How long will they take working together?

SOLUTION

Rates: Ana does $\frac{1}{6}$ of the fence per hour, Ben does $\frac{1}{3}$.

Together: $\frac{1}{6} + \frac{1}{3} = \frac{1}{t}$. Multiply by $6t$: $t + 2t = 6$, so $3t = 6$ and $t = 2$.

They need **2 hours**, which is less than Ben's 3 hours alone, as expected.

EXAMPLE 5

The cost of a bulk order varies directly with the weight. A 4 kilogram order costs 12 dollars. How much does a 10 kilogram order cost?

SOLUTION

Direct variation: $c = kw$. Use $12 = k \cdot 4$, so $k = 3$ dollars per kilogram.

Then $c = 3 \cdot 10 = 30$.

A 10 kilogram order costs **30 dollars**.

EXAMPLE 6

The time t for a trip varies inversely with the speed s . At 60 miles per hour it takes 3 hours. How long at 45 miles per hour?

SOLUTION

Inverse variation: $ts = k$. So $k = 3 \cdot 60 = 180$ miles.

Then $t = \frac{180}{45} = 4$.

At 45 miles per hour the trip takes **4 hours**. Slower speed, longer time: the answer makes sense.

Coach's Playbook

- 1 List the forbidden values.** Before solving a rational equation, write down what makes each denominator zero.
- 2 Clear fractions or isolate the radical.** Multiply by the LCD, or get the square root alone on one side.
- 3 Solve the resulting equation.** It is often linear or quadratic. Factor or rearrange as needed.
- 4 Always check.** Substitute into the original equation, not an intermediate line.

- ◆ For work problems, think "rate times time equals job done". Add rates, never times.
- ◆ In a variation problem, find k first and write the full equation before using it.
- ◆ A negative solution for a time, a length, or a count is almost always rejected.
- ◆ If the squared radical equation has two answers, expect that one of them may be extraneous.

Common traps

1. Skipping the check and keeping an answer that makes a denominator zero. 2. Squaring one side term by term, such as $(x + 2)^2 = x^2 + 4$. 3. In a work problem, adding times: 6 hours plus 3 hours is not the combined time. 4. Using $y = kx$ when the situation is inverse.

Mistake Clinic

Spot the error

A student solves $\sqrt{x+6} = x$ by squaring: $x+6 = x^2$, so $(x-3)(x+2) = 0$, and answers $x = 3$ and $x = -2$. **Where it goes wrong:** The algebra is right, but the student never checked. At $x = -2$ the left side is $\sqrt{4} = 2$, while the right side is -2 . **Correct solution:** $x = 3$ works since $\sqrt{9} = 3$. The only solution is $x = 3$, and $x = -2$ is extraneous.

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Spot the error

A student solves $\frac{1}{x-2} + \frac{1}{x+2} = \frac{4}{x^2-4}$ and writes $x = 2$ as the answer. **Where it goes wrong:** Multiplying by $x^2 - 4$ gives $(x+2) + (x-2) = 4$, so $x = 2$. That value makes $x - 2 = 0$ in the denominators, so it is not allowed. **Correct solution:** The equation has **no solution**, because the only candidate is extraneous.

Spot the error

A student says that two people who finish a job in 4 hours and 12 hours alone need $4 + 12 = 16$ hours together. **Where it goes wrong:** Working together should make the job faster, never slower than the faster worker. Times do not add. Rates add. **Correct solution:** $\frac{1}{4} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$ job per hour, so $t = 3$ hours.

Real-World Connection

A lamp's brightness varies inversely with the square of the distance: $I = \frac{k}{d^2}$. If a reading lamp gives a brightness of 36 units at 2 meters, then $k = 36 \cdot 4 = 144$. At 6 meters, $I = \frac{144}{36} = 4$ units. Tripling the distance cuts the brightness to one ninth.

Challenge Corner

Pipe A fills a tank in x hours. Pipe B takes 3 hours longer. Together they fill the tank in 2 hours. Find x . **Solution.** $\frac{1}{x} + \frac{1}{x+3} = \frac{1}{2}$. Multiply by $2x(x+3)$: $2(x+3) + 2x = x(x+3)$, so $4x + 6 = x^2 + 3x$. Then $x^2 - x - 6 = 0$, which factors as $(x-3)(x+2) = 0$. Time cannot be negative, so $x = 3$ is the only valid answer. Pipe A takes 3 hours and Pipe B takes 6 hours. Check: $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$.

Practice**PRACTICE Rational and Radical Equations**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Solve $\frac{12}{x} = 3$.
 (A) $x = 4$ (B) $x = 36$ (C) $x = \frac{1}{4}$ (D) $x = 9$
- Solve $\frac{2}{x+1} = \frac{1}{3}$.
 (A) $x = 7$ (B) $x = 5$ (C) $x = 1$ (D) $x = -5$
- Solve $\sqrt{x+4} = 5$.
 (A) $x = 9$ (B) $x = 29$ (C) $x = 21$ (D) $x = 25$
- How many solutions does $\frac{x}{x-2} = \frac{2}{x-2}$ have?
 (A) 1 (B) 2 (C) Infinitely many
 (D) 0
- y varies directly with x , and $y = 20$ when $x = 5$. What is y when $x = 8$?
 (A) 32 (B) 12.5 (C) 23 (D) 160
- y varies inversely with x , and $y = 6$ when $x = 4$. What is y when $x = 3$?
 (A) 4.5 (B) 8 (C) 18 (D) 7
- Solve $\sqrt{3x+1} = x-1$. Which is the solution set?
 (A) $x = 0$ only (B) $x = 0$ and $x = 5$ (C) $x = 5$ only
 (D) $x = -5$
- Pump A fills a tank in 4 hours and Pump B fills it in 12 hours. How long do they take together?
 (A) 3 hours (B) 8 hours (C) 16 hours (D) 4 hours

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9. Solve $x + \frac{6}{x} = 5$.
 (A) $x = 2$ only (B) $x = -2$ and $x = -3$ (C) $x = 5$ and $x = 6$
 (D) $x = 2$ and $x = 3$
10. Solve $\frac{9}{x-1} = \frac{6}{x}$.
11. Solve $\sqrt{4x+1} + 2 = 7$.
12. Solve $\frac{1}{x-2} + \frac{1}{x+2} = \frac{4}{x^2-4}$.
13. Solve $\sqrt{x+6} = x$.
14. Maya can mow a lawn in 5 hours. Leo can mow it in 4 hours. How long will they take together? Give an exact fraction of an hour.
15. A lamp's brightness I varies inversely with the square of the distance d . If $I = 36$ when $d = 2$ meters, find I when $d = 6$ meters.

Answers and Explanations

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

15.1 Simplifying Radicals and Radical Operations

PRACTICE · SIMPLIFYING RADICALS AND RADICAL OPERATIONS

- B. $98 = 49 \cdot 2$, so $\sqrt{98} = 7\sqrt{2}$. Choice A forgets to take the root of 49. Choice D is not simplified.
- B. Multiply under the root: $\sqrt{100} = 10$. Choice D forgets the root, and choice C adds 20 and 5.
- A. Like radicals combine like terms: $(2+5)\sqrt{3}$. The radicand stays 3, so choice B is wrong, and the coefficients are added, not multiplied (choice C).
- D. Combine with the quotient rule: $\sqrt{27/3} = \sqrt{9} = 3$. Choice C subtracts instead of dividing.
- C. Multiply top and bottom by $\sqrt{5}$: $\frac{10\sqrt{5}}{5} = 2\sqrt{5}$. Choice B forgets to divide by 5.
- A. This is a difference of squares: $3 - 4 = -1$. Choice B flips the order of subtraction.
- B. $\sqrt{75} = 5\sqrt{3}$ and $\sqrt{12} = 2\sqrt{3}$, so the difference is $3\sqrt{3}$. Choice A subtracts radicands, which is not allowed.
- C. $48a^7 = 16 \cdot 3 \cdot a^6 \cdot a$, so the result is $4a^3\sqrt{3a}$. Choice D drops the leftover a , and choice A forgets to root 16.
- D. Multiply by the conjugate: the denominator is $6 - 4 = 2$, and the numerator is $2(\sqrt{6} + 2)$, so the result is $\sqrt{6} + 2$. Choice B forgets to divide by 2.
- $6\sqrt{5}$. $180 = 36 \cdot 5$, so $\sqrt{180} = 6\sqrt{5}$.
- $24\sqrt{3}$. Multiply outside and inside: $12\sqrt{12} = 12 \cdot 2\sqrt{3} = 24\sqrt{3}$.
- $7 + 2\sqrt{10}$. $5 + 2\sqrt{10} + 2 = 7 + 2\sqrt{10}$.
- $3\sqrt{7} + 3\sqrt{3}$. Multiply by the conjugate. The denominator is $7 - 3 = 4$, so $\frac{12(\sqrt{7}+\sqrt{3})}{4} = 3\sqrt{7} + 3\sqrt{3}$.
- $40\sqrt{2}$ meters. Side = $\sqrt{200} = 10\sqrt{2}$. Perimeter = $4 \cdot 10\sqrt{2} = 40\sqrt{2}$.
- $\sqrt{5}$. Common denominator $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3}) = 2$. Numerator: $(\sqrt{5} + \sqrt{3}) + (\sqrt{5} - \sqrt{3}) = 2\sqrt{5}$. So the sum is $\frac{2\sqrt{5}}{2} = \sqrt{5}$.

15.2 Simplifying Rational Expressions

PRACTICE · SIMPLIFYING RATIONAL EXPRESSIONS

- C. The denominator is zero when $x - 7 = 0$, so $x = 7$. Choice A changes the sign, and the numerator never matters.
- A. Reduce $\frac{12}{18} = \frac{2}{3}$ and $x^3/x = x^2$. Choice B forgets to subtract 1 from the exponent.
- B. The bottom is $(x-4)(x+4)$; cancel $x+4$ to leave $\frac{1}{x-4}$. Choice A keeps the wrong factor.
- D. Restrictions come from the original denominator, so both 3 and -1 are forbidden. Choice A uses the simplified expression $\frac{1}{x+1}$.
- A. The top is $(x-5)(x+5)$ and the bottom is $(x+5)^2$. Cancel one $x+5$. Choice B inverts the result.
- C. Factor: $\frac{(x+2)(x+3)}{(x+1)(x+2)}$. Cancel $x+2$. Choice A cancels terms, and choice D cancels the wrong factor.
- B. $6 - x = -(x-6)$, so the ratio is -1 . Choice A ignores that the numerator is the opposite.
- D. $3x^2 - 12 = 3(x-2)(x+2)$ and $x^2 + x - 6 = (x+3)(x-2)$. Cancel $x-2$. Choice A drops the 3.
- C. $f(x) = x + 3$ except at $x = 3$, where the line has height 6. Choice A uses the zero of the numerator, which is not the hole.
- $\frac{3a}{5b^2}$. Reduce $\frac{15}{25} = \frac{3}{5}$, $a^2/a = a$, and $b/b^3 = 1/b^2$.
- $\frac{x-3}{x+4}$, with $x \neq 4$ and $x \neq -4$. Top: $(x-3)(x-4)$. Bottom: $(x-4)(x+4)$. Cancel $x-4$.
- $\frac{x-3}{x}$, with $x \neq 0$ and $x \neq -\frac{1}{2}$. Top: $(2x+1)(x-3)$. Bottom: $x(2x+1)$. Cancel $2x+1$.
- $x-5$ (with $x \neq 2, -2$). Group the top: $x^2(x-5) - 4(x-5) = (x-5)(x-2)(x+2)$. The bottom is $(x-2)(x+2)$.
- $x+4$. Length is area divided by width: $\frac{(x+2)(x+4)}{x+2} = x+4$.
- Undefined at $x = 2$ and $x = -1$. The simplified form is $\frac{2(x+2)}{x+1}$ and at $x = 3$ it equals $\frac{10}{4} = \frac{5}{2}$. Work: top = $2(x-2)(x+2)$, bottom = $(x-2)(x+1)$.

15.3 Multiplying, Dividing, Adding, and Subtracting Rational Expressions

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PRACTICE · MULTIPLYING, DIVIDING, ADDING, AND SUBTRACTING RATIONAL EXPRESSIONS

1. A. Multiply tops and bottoms: $\frac{2x^2}{6x} = \frac{x}{3}$. Choice B inverts the result.
2. B. Flip the second fraction: $\frac{x}{4} \cdot \frac{8}{x} = 2$. Choice D multiplies without flipping.
3. C. The denominators already match, so only the numerators add. Choices A and B wrongly change the denominator.
4. D. The LCD is $2x$: $\frac{x}{2x} + \frac{2}{2x} = \frac{x+2}{2x}$. Choice A adds the denominators.
5. B. Same denominator, so subtract the numerators: $x + 5 - 2 = x + 3$. Choice A adds instead.
6. C. Factor $x^2 - 1 = (x - 1)(x + 1)$. Cancel x and $x + 1$. Choice B forgets to multiply the -1 by 3.
7. A. Flip the second fraction: the factors $x + 1$ and $x - 2$ cancel, leaving $x + 2$. Choice B multiplies without flipping.
8. B. Multiply top and bottom by $2x$: top = 2, bottom = $2 + x$. Choice D multiplies the top and the bottom by different amounts.
9. D. The LCD is $(x - 3)(x + 3)$. The second fraction becomes $\frac{2(x-3)}{(x-3)(x+3)}$, so the top is $4 + 2x - 6 = 2x - 2$. Choice A uses $x + 3$ instead of $x - 3$.
10. $\frac{2xy}{3}$. Numbers: $\frac{30}{45} = \frac{2}{3}$. Variables: $x^2/x = x$ and $y^2/y = y$.
11. $(x - 4)(x - 2)$. Flip: $\frac{(x-4)(x+4)}{x+2} \cdot \frac{(x-2)(x+2)}{x+4}$. Cancel $x + 2$ and $x + 4$.
12. $\frac{7x - 2}{(x - 2)(x + 1)}$. Top: $4(x + 1) + 3(x - 2) = 4x + 4 + 3x - 6 = 7x - 2$.
13. $\frac{1}{(x - 1)(x + 1)}$. LCD is $(x - 1)(x + 1)$. Top: $x - (x - 1) = 1$.
14. $\frac{x + 2}{x}$, with $x \neq 0$ and $x \neq 2$. Multiply top and bottom by x^2 : $\frac{x^2 - 4}{x^2 - 2x} = \frac{(x-2)(x+2)}{x(x-2)}$.
15. $\frac{15(2x + 5)}{x(x + 5)}$ hours; at $x = 10$ it is $\frac{15 \cdot 25}{10 \cdot 15} = 2.5$ hours. Check: $1.5 + 1 = 2.5$.

15.4 Rational and Radical Equations**PRACTICE · RATIONAL AND RADICAL EQUATIONS**

1. A. Multiply both sides by x : $12 = 3x$, so $x = 4$. Choice B multiplies instead of dividing.
2. B. Cross-multiply: $6 = x + 1$, so $x = 5$. Choice A adds 1 instead of subtracting it.
3. C. Square both sides: $x + 4 = 25$, so $x = 21$. Choice D forgets the 4, and choice B adds it.
4. D. Multiplying by $x - 2$ gives $x = 2$, but $x = 2$ makes the denominator zero. It is extraneous, so there is no solution.
5. A. $k = 20/5 = 4$, so $y = 4 \cdot 8 = 32$. Choice B uses an inverse relationship.
6. B. $xy = 24$, so $y = 24/3 = 8$. Choice A treats the relationship as direct.
7. C. Squaring gives $x^2 - 5x = 0$, so $x = 0$ or 5. At $x = 0$: $\sqrt{1} = 1$ but $x - 1 = -1$, so it is extraneous. Choice B does not check.
8. A. Rates add: $\frac{1}{4} + \frac{1}{12} = \frac{1}{3}$. Choice B averages the times, and choice C adds them.
9. D. Multiply by x : $x^2 + 6 = 5x$, so $x^2 - 5x + 6 = 0$ and $(x - 2)(x - 3) = 0$. Both work when checked.
10. $x = -2$. Cross-multiply: $9x = 6x - 6$, so $3x = -6$. Check: $\frac{9}{-3} = -3$ and $\frac{6}{-2} = -3$.
11. $x = 6$. Isolate the root: $\sqrt{4x + 1} = 5$. Square: $4x + 1 = 25$. Check: $\sqrt{25} + 2 = 7$.
12. No solution. Multiplying by $x^2 - 4$ gives $2x = 4$, so $x = 2$, which makes a denominator zero. It is extraneous.
13. $x = 3$. Squaring gives $x^2 - x - 6 = 0$, so $x = 3$ or $x = -2$. Only $x = 3$ checks, since $\sqrt{4} = 2 \neq -2$.
14. $\frac{20}{9}$ hours (about 2.22 hours). $\frac{1}{5} + \frac{1}{4} = \frac{9}{20}$ of the lawn per hour, and time is the reciprocal.
15. 4. $I = \frac{k}{d^2}$ and $k = 36 \cdot 4 = 144$. Then $I = \frac{144}{36} = 4$.

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