



Complex Numbers and Polynomial Functions

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

In this booster

21.1 Imaginary and Complex Numbers

21.2 Polynomial Functions and Their Graphs

21.3 Zeros of Polynomials

21.4 Modeling with Polynomials

Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

21.1 Imaginary and Complex Numbers

Meet the imaginary unit i , learn its repeating powers, and master adding, multiplying, dividing, and conjugating complex numbers. Then use them to solve quadratics that have no real solutions.

Why This Matters

Try to solve $x^2 = -1$ using only the real numbers and you hit a wall: no real number squared gives a negative. Mathematicians fixed that by inventing one new number, i , whose square is -1 . That single idea opens a whole number system called the **complex numbers**, and it is not just a classroom curiosity. Electrical engineers use complex numbers to describe alternating current, audio engineers use them to study sound waves, and computer graphics, signal processing, and quantum physics are built on them.

Students usually get stuck in four places. First, the **sign of i^2** : it is -1 , not $+1$, and this one fact drives every simplification. Second, **powers of i** , where students try to multiply i forty times instead of spotting the pattern. Third, **division**, because you cannot leave i in a denominator and must use the conjugate. Fourth, **quadratics with negative discriminants**, where students stop at $\sqrt{-16}$ instead of writing $4i$. This section turns each of these into a short routine you can run without thinking.

Core Ideas

The **imaginary unit** is defined by $i = \sqrt{-1}$, which means $i^2 = -1$. Any square root of a negative number can be pulled apart using i . A **complex number** is a number of the form $a + bi$ where a and b are real numbers. We call a the **real part** and b the **imaginary part**. If $b = 0$ the number is real, so every real number is also a complex number. If $a = 0$ and $b \neq 0$ the number is **pure imaginary**.

Definition and Square Roots of Negatives

$i = \sqrt{-1}$ and $i^2 = -1$.

For $c > 0$: $\sqrt{-c} = i\sqrt{c}$. Example: $\sqrt{-49} = 7i$ and $\sqrt{-12} = 2i\sqrt{3}$.

Convert to i form **before** you multiply radicals. The rule $\sqrt{a}\sqrt{b} = \sqrt{ab}$ is only safe when a and b are not both negative.

Powers of i repeat in a cycle of length four. After $i^4 = 1$ the pattern starts over, so you only need the remainder when the exponent is divided by 4.

Power	i^1	i^2	i^3	i^4	i^5	i^6	i^7	i^8
Value	i	-1	$-i$	1	i	-1	$-i$	1

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Powers of i

Divide the exponent n by 4 and look at the remainder r : remainder 0 gives 1, remainder 1 gives i , remainder 2 gives -1 , remainder 3 gives $-i$.

Example: i^{58} : $58 = 4(14) + 2$, so $i^{58} = i^2 = -1$.

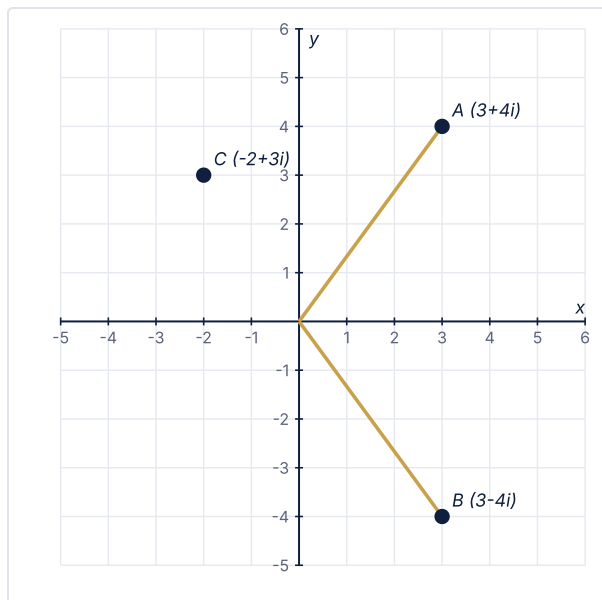
To **add or subtract** complex numbers, combine real parts with real parts and imaginary parts with imaginary parts, just like combining like terms. To **multiply**, use the distributive property (FOIL) and then replace every i^2 with -1 . The result should always be written in **standard form** $a + bi$.

The Conjugate

The **conjugate** of $a + bi$ is $a - bi$. Change only the sign of the imaginary part.

Key product: $(a + bi)(a - bi) = a^2 + b^2$, which is always a real, nonnegative number.

To divide, multiply the top and bottom by the conjugate of the **denominator**: $\frac{p + qi}{a + bi} = \frac{(p + qi)(a - bi)}{a^2 + b^2}$.



The complex plane: the horizontal axis holds the real part and the vertical axis holds the imaginary part. The point A is $3+4i$ and its conjugate B is $3-4i$, a reflection across the real axis.

Every complex number $a + bi$ can be drawn as the point (a, b) in the **complex plane**. The conjugate is the mirror image across the real (horizontal) axis. The distance from the origin to $a + bi$ is $\sqrt{a^2 + b^2}$, called the **modulus**. For $3 + 4i$ that distance is 5.

Complex Solutions of Quadratics

For $ax^2 + bx + c = 0$ with real coefficients, the quadratic formula gives $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

If the discriminant $b^2 - 4ac < 0$, there are **no real solutions** but exactly **two complex solutions that are conjugates** of each other:

$$x = \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} i.$$

Discriminant $b^2 - 4ac$	Number and type of solutions	Graph of $y = ax^2 + bx + c$
Positive	Two different real	Crosses the x-axis twice
Zero	One repeated real	Touches the x-axis once
Negative	Two complex conjugates	Never touches the x-axis

Worked Examples

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EXAMPLE 1Simplify i^{47} and i^{102} .**SOLUTION**

Divide each exponent by 4 and keep the remainder.

$$47 = 4(11) + 3, \text{ so } i^{47} = i^3 = -i.$$

$$102 = 4(25) + 2, \text{ so } i^{102} = i^2 = -1.$$

EXAMPLE 2Write $\sqrt{-48}$ in terms of i , and then simplify $\sqrt{-4} \cdot \sqrt{-9}$.**SOLUTION**

$$\sqrt{-48} = i\sqrt{48} = i\sqrt{16 \cdot 3} = 4i\sqrt{3}.$$

$$\text{Convert first: } \sqrt{-4} = 2i \text{ and } \sqrt{-9} = 3i.$$

$$\text{Multiply: } (2i)(3i) = 6i^2 = -6.$$

Note that $\sqrt{36} = 6$ would be wrong, because the radical rule fails when both numbers are negative.**EXAMPLE 3**Multiply $(3 + 4i)(2 - 5i)$ and write the answer in standard form.**SOLUTION**

$$\text{FOIL: } 3 \cdot 2 = 6, 3 \cdot (-5i) = -15i, 4i \cdot 2 = 8i, 4i \cdot (-5i) = -20i^2.$$

$$\text{Combine: } 6 - 15i + 8i - 20i^2.$$

$$\text{Replace } i^2 = -1: -20i^2 = +20.$$

$$\text{Total: } 6 + 20 + (-15i + 8i) = 26 - 7i.$$

EXAMPLE 4Divide: $\frac{5+i}{2-3i}$.**SOLUTION**The denominator's conjugate is $2 + 3i$. Multiply top and bottom by it.

$$\text{Numerator: } (5+i)(2+3i) = 10 + 15i + 2i + 3i^2 = 10 + 17i - 3 = 7 + 17i.$$

$$\text{Denominator: } (2-3i)(2+3i) = 4 + 9 = 13.$$

$$\text{Answer: } \frac{7+17i}{13} = \frac{7}{13} + \frac{17}{13}i.$$

EXAMPLE 5Solve $x^2 - 6x + 25 = 0$.**SOLUTION**Here $a = 1$, $b = -6$, $c = 25$. Discriminant: $36 - 100 = -64$.

$$x = \frac{6 \pm \sqrt{-64}}{2} = \frac{6 \pm 8i}{2}.$$

$$x = 3 + 4i \text{ or } x = 3 - 4i.$$

Check: the sum of the roots is $6 = -b/a$ and the product is $9 + 16 = 25 = c/a$.Want a plan built around **your** student?

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EXAMPLE 6

Solve $2x^2 + 4x + 7 = 0$, and state the real part and imaginary part of each solution.

SOLUTION

Discriminant: $16 - 4(2)(7) = 16 - 56 = -40$.

$$\sqrt{-40} = i\sqrt{40} = 2i\sqrt{10}.$$

$$x = \frac{-4 \pm 2i\sqrt{10}}{4} = -1 \pm \frac{\sqrt{10}}{2}i.$$

Real part -1 ; imaginary parts $\pm \frac{\sqrt{10}}{2}$. Always divide **both** terms of the numerator by the denominator.

Coach's Playbook

- 1 Reduce before you work.** Turn every $\sqrt{-c}$ into $i\sqrt{c}$ and every high power of i into one of $1, i, -1, -i$ before doing anything else.
- 2 Multiply like binomials.** Use FOIL, then immediately replace i^2 with -1 . Writing $i^2 = -1$ beside your work prevents sign slips.
- 3 Divide with the conjugate.** Multiply top and bottom by the conjugate of the denominator. The new denominator is $a^2 + b^2$, a plain positive number.
- 4 Finish in standard form.** Write the answer as $a + bi$. Never leave i in a denominator or leave an i^2 anywhere.
- 5 Check with a sum and product.** For a quadratic, the two complex roots must add to $-b/a$ and multiply to c/a . This catches arithmetic errors in seconds.

Here is a quick way to see why the conjugate trick works. Any product of the form $(a + bi)(a - bi)$ expands to $a^2 - b^2i^2$, and because $i^2 = -1$ the two middle terms cancel and the last term becomes $+b^2$. The imaginary part vanishes, which is exactly what you want in a denominator. It is the same idea as multiplying by $\sqrt{2} + 1$ to rationalize a denominator containing $\sqrt{2} - 1$.

A good mental model: treat i like a variable x during algebra, with one extra rule that every x^2 becomes -1 . Another useful habit is to estimate the discriminant first. If it is negative, you already know to expect a conjugate pair, so a lone complex root is a sign of a mistake.

Common traps

1) Writing $\sqrt{-4}\sqrt{-9} = \sqrt{36} = 6$. The correct value is -6 . 2) Writing $i^2 = 1$ or forgetting the sign when i^2 appears from FOIL. 3) Using the wrong number to reduce i^n , for example dividing by 2 instead of 4. 4) Changing the sign of the real part when taking a conjugate. Only the imaginary part flips.

Mistake Clinic**Spot the error 1**

Problem: Simplify $(2 + 3i)^2$. Wrong work: $(2 + 3i)^2 = 4 + 9i^2 = 4 - 9 = -5$. What went wrong: Squaring a binomial is not the same as squaring each term. The middle term $2 \cdot 2 \cdot 3i = 12i$ was dropped, so the answer lost its imaginary part. Right solution: $(2 + 3i)^2 = 4 + 12i + 9i^2 = 4 + 12i - 9 = -5 + 12i$.

Spot the error 2

Problem: Divide $\frac{1 + 3i}{2 - i}$. Wrong work: Multiply top and bottom by $2 - i$ to get $\frac{(1 + 3i)(2 - i)}{(2 - i)^2}$. What went wrong: The student multiplied by the denominator itself instead of its conjugate, so the denominator still contains i . Right solution: Multiply by $2 + i$: numerator $(1 + 3i)(2 + i) = 2 + i + 6i + 3i^2 = -1 + 7i$, denominator $4 + 1 = 5$. Answer: $-\frac{1}{5} + \frac{7}{5}i$.

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Spot the error 3

Problem: Simplify $\sqrt{-4} \cdot \sqrt{-9}$. Wrong work: $\sqrt{-4} \cdot \sqrt{-9} = \sqrt{(-4)(-9)} = \sqrt{36} = 6$. What went wrong: The product rule for square roots does not hold when both radicands are negative. Rewrite with i first. Right solution: $(2i)(3i) = 6i^2 = -6$.

Real-World Connection

In an electrical circuit, a resistor and a coil can be combined into one number such as $3 + 4i$ ohms, where the real part measures resistance and the imaginary part measures how the coil resists changing current. Adding a second part $1 - 2i$ ohms in series is just complex addition: $(3 + 4i) + (1 - 2i) = 4 + 2i$. Engineers rely on the rules in this section every day.

Challenge Corner

Problem: Compute $(1 + i)^8$ without expanding eight times. Solution: First square: $(1 + i)^2 = 1 + 2i + i^2 = 2i$. Then $(1 + i)^8 = ((1 + i)^2)^4 = (2i)^4 = 16i^4 = 16$. Check: the modulus of $1 + i$ is $\sqrt{2}$, and $(\sqrt{2})^8 = 16$, which matches.

Practice**PRACTICE Imaginary and Complex Numbers**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is the value of i^{23} ?
(A) i (B) $-i$ (C) 1 (D) -1
- Which expression equals $(4 - 3i) + (2 + 7i)$?
(A) $6 + 4i$ (B) $6 - 4i$ (C) $2 - 10i$ (D) $6 + 10i$
- Which is equal to $\sqrt{-75}$?
(A) $5\sqrt{3}$ (B) $-5i\sqrt{3}$ (C) $5i\sqrt{3}$ (D) $25i\sqrt{3}$
- What is the product $(2 + 3i)(2 - 3i)$?
(A) -5 (B) $4 + 9i$ (C) $4 - 9i$ (D) 13
- What is the conjugate of $-5 + 2i$?
(A) $5 - 2i$ (B) $-5 - 2i$ (C) $5 + 2i$ (D) $-5 + 2i$
- Which is equal to $(3 + 2i)^2$?
(A) $9 + 4i$ (B) $13 + 12i$ (C) $5 + 12i$ (D) $5 + 6i$
- Which expression equals $\frac{6 + 2i}{1 + i}$?
(A) $4 + 2i$ (B) $3 + i$ (C) $8 - 4i$ (D) $4 - 2i$
- What are the solutions of $x^2 + 4x + 13 = 0$?
(A) $-2 \pm 3i$ (B) $2 \pm 3i$ (C) $-2 \pm 9i$ (D) $-4 \pm 3i$
- For which values of k does $x^2 + 6x + k = 0$ have two nonreal solutions?
(A) $k < 9$ (B) $k > 36$ (C) $k > 9$ (D) $k < -9$
- Simplify i^{2026} .
- Multiply and write in standard form: $(7 - 2i)(3 + 4i)$.
- Write $\frac{4 + 9i}{1 - 2i}$ in the form $a + bi$.
- Solve $3x^2 - 2x + 5 = 0$ over the complex numbers.
- Find real numbers a and b so that $(2 + ai)(3 - i) = b + 7i$.
- A quadratic with real coefficients and leading coefficient 1 has the root $2 - 5i$. Write the quadratic in the form $x^2 + bx + c$.

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21.2 Polynomial Functions and Their Graphs

Learn to read a polynomial's degree and leading coefficient, predict end behavior, find zeros and multiplicities, count turning points, and sketch accurate graphs from factored form.

Why This Matters

Polynomial functions are the workhorses of algebra. Engineers model the curve of a roller coaster with them, economists use them for cost and profit, and scientists fit them to data. Everything you learned about lines and parabolas is a special case: a line is a polynomial of degree 1 and a parabola is a polynomial of degree 2. This section shows how to look at a polynomial's formula and predict the shape of its graph without plotting a single point.

Students usually struggle with three ideas. First, **end behavior**, because it depends on two things at once (the degree being even or odd, and the leading coefficient being positive or negative). Second, **multiplicity**, where a repeated factor makes the graph bounce off the axis instead of crossing it. Third, confusing **zeros, x-intercepts, and turning points**. By the end of this section you will keep these straight with a simple checklist.

Core Ideas

A **polynomial function** has the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where the exponents are whole numbers and the coefficients are real numbers. The **degree** is the largest exponent n and the **leading coefficient** is a_n , the coefficient of that highest power. Expressions with negative or fractional exponents, or with x under a root or in a denominator, are **not** polynomials. A polynomial in factored form can be read at a glance: the degree is the total count of factors, so $(2x - 1)^2(x + 3)$ has degree 3.

End Behavior

The ends of the graph depend only on the **degree** and the **sign of the leading coefficient**.

Even degree, positive leading coefficient: both ends rise (up, up). Even degree, negative: both ends fall (down, down).
Odd degree, positive: falls on the left, rises on the right. Odd degree, negative: rises on the left, falls on the right.

Degree	Leading coeff.	As $x \rightarrow -\infty$	As $x \rightarrow +\infty$	Example
Even	Positive	$f \rightarrow +\infty$	$f \rightarrow +\infty$	$x^4 - 3x$
Even	Negative	$f \rightarrow -\infty$	$f \rightarrow -\infty$	$-2x^2 + 5$
Odd	Positive	$f \rightarrow -\infty$	$f \rightarrow +\infty$	$x^3 - x$
Odd	Negative	$f \rightarrow +\infty$	$f \rightarrow -\infty$	$-x^5 + 2x$

Why does only the leading term matter? When x is huge, say $x = 1000$, the term x^3 equals a billion while x^2 is only a million. The highest power overwhelms everything else, so far from the origin the graph looks like its leading term.

Zeros, Multiplicity, and Turning Points

A **zero** of f is an input c with $f(c) = 0$. If $(x - c)^m$ is a factor and $(x - c)^{m+1}$ is not, then c is a zero of **multiplicity** m .

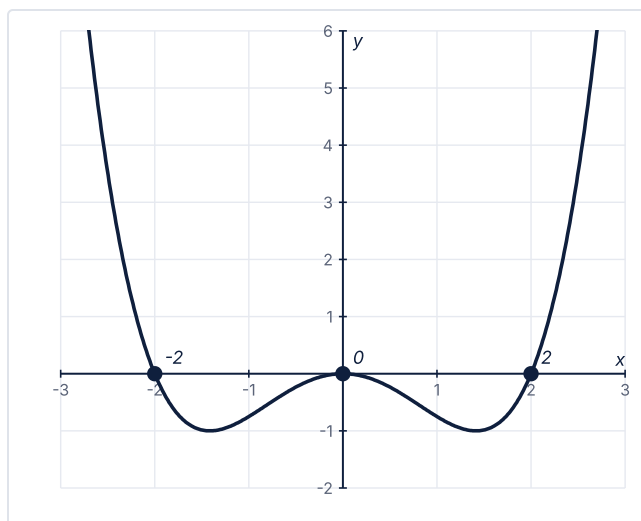
Odd multiplicity: the graph **crosses** the x-axis at c (and flattens out when $m \geq 3$). Even multiplicity: the graph **touches** the axis and turns back without crossing.

A polynomial of degree n has **at most** n **real zeros** and **at most** $n - 1$ **turning points**. Counting multiplicity, it has exactly n complex zeros.

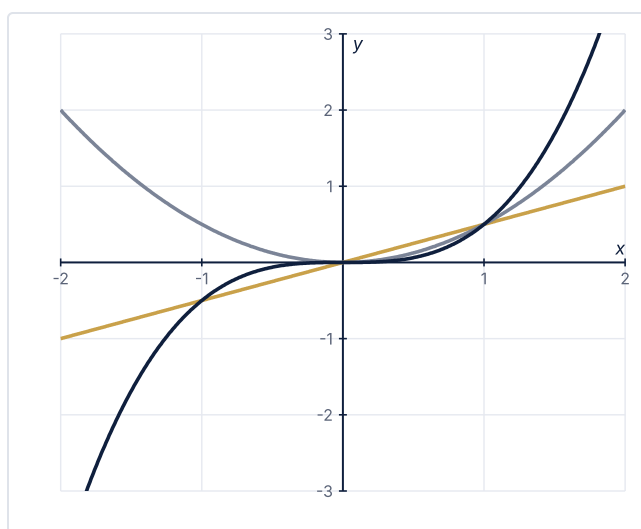
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The graph of $y = (x+2)x^2(x-2)/4$ crosses the axis at $x = -2$ and $x = 2$ and touches it at $x = 0$, where the multiplicity is 2.



Multiplicity at one zero. Gold: $y = x/2$ (multiplicity 1, crosses straight). Gray: $y = x^2/2$ (multiplicity 2, touches and bounces). Navy: $y = x^3/2$ (multiplicity 3, crosses while flattening).

To sketch a polynomial from factored form, follow four moves: mark the zeros on the x-axis, decide cross or touch at each one from its multiplicity, find the y-intercept by computing $f(0)$, and use end behavior to decide where the left and right tails go. Then connect the pieces with a smooth, unbroken curve. Polynomial graphs never have gaps, corners, or sharp points.

Worked Examples

EXAMPLE 1

Describe the end behavior of $f(x) = -2x^4 + 5x^3 - x + 7$ and of $g(x) = 3x^5 - x^2$.

SOLUTION

For f : degree 4 (even), leading coefficient -2 (negative). Both ends fall.

So $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ and as $x \rightarrow +\infty$.

For g : degree 5 (odd), leading coefficient 3 (positive). The left end falls and the right end rises.

So $g(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ and $g(x) \rightarrow +\infty$ as $x \rightarrow +\infty$.

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EXAMPLE 2

A polynomial has degree 6. State the maximum number of real zeros and the maximum number of turning points. Then state the least number of turning points a polynomial of degree 5 can have.

SOLUTION

A degree-6 polynomial has at most 6 real zeros and at most $6 - 1 = 5$ turning points.

A degree-5 polynomial has odd degree, so its ends go in opposite directions and it must cross the x-axis at least once.

It can be as simple as x^5 , which has **no** turning points. So the least number is 0 (the number of turning points is always one less than the degree, or fewer by an even amount).

EXAMPLE 3

Find the zeros of $f(x) = x^2(x - 3)^3(x + 2)$, state each multiplicity, say whether the graph crosses or touches, and give the degree and end behavior.

SOLUTION

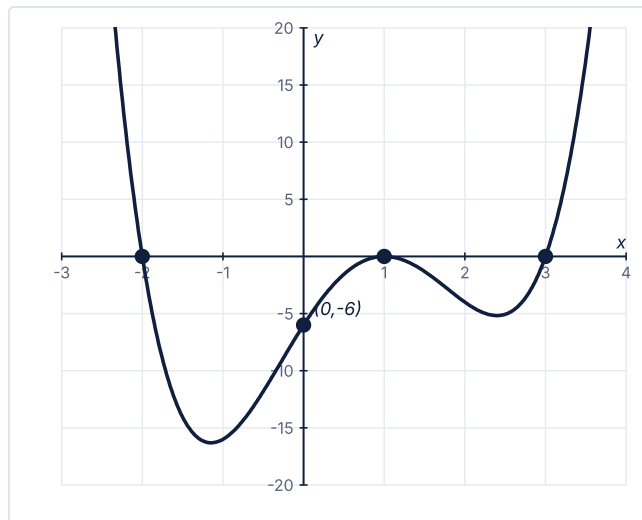
Zeros: $x = 0$ with multiplicity 2 (touches), $x = 3$ with multiplicity 3 (crosses and flattens), $x = -2$ with multiplicity 1 (crosses).

Degree: $2 + 3 + 1 = 6$ (even). The leading coefficient is $+1$ because every factor has leading coefficient $+1$.

End behavior: both ends rise.

EXAMPLE 4

Sketch $f(x) = (x + 2)(x - 1)^2(x - 3)$. Find the zeros, the y-intercept, and the end behavior.



Graph of $f(x) = (x+2)(x-1)^2(x-3)$. The graph touches the axis at $x = 1$ and crosses at $x = -2$ and $x = 3$.

SOLUTION

Zeros: -2 (multiplicity 1, crosses), 1 (multiplicity 2, touches), 3 (multiplicity 1, crosses).

y-intercept: $f(0) = (2)(1)(-3) = -6$.

Degree 4 with positive leading coefficient: both ends rise.

Sign check: for $x < -2$ the graph is positive, between -2 and 1 it is negative, it touches at $x = 1$ and stays negative until 3 , then it is positive for $x > 3$.

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EXAMPLE 5

The graph of a polynomial crosses the x -axis at $x = -3$, touches it at $x = 0$, and crosses it at $x = 2$ while flattening there (like x^3). What is the least possible degree?

SOLUTION

Crossing straight means multiplicity 1 at $x = -3$.

Touching means an even multiplicity, at least 2, at $x = 0$.

Crossing and flattening means an odd multiplicity of at least 3 at $x = 2$.

Least degree: $1 + 2 + 3 = 6$.

EXAMPLE 6

Show that $f(x) = x^3 - 4x + 1$ has three real zeros and say roughly where they are.

x	-3	-2	-1	0	1	2
$f(x)$	-14	1	4	1	-2	1

SOLUTION

Evaluate the function at several integers (see the table).

The sign changes between -3 and -2 , between 0 and 1 , and between 1 and 2 . Since f is continuous, the Intermediate Value Theorem guarantees a zero in each interval.

A cubic has at most 3 real zeros, so these are all of them.

Coach's Playbook

- 1 Read the degree and leading coefficient first.** They settle the end behavior before you do any other work.
- 2 Factor completely.** Zeros and multiplicities live in the factors. Pull out any common factor of x first.
- 3 Mark cross or touch.** Odd exponent on a factor means cross; even exponent means touch (bounce).
- 4 Compute $f(0)$.** That is the y -intercept and it anchors the graph.
- 5 Check the count.** Turning points are at most $n - 1$. If your sketch has more, something is wrong.

A helpful mental picture: think of the graph as a rope that is pinned on the x -axis at each zero. At an odd-multiplicity pin the rope passes through. At an even-multiplicity pin the rope is only hooked on the axis and bounces back. Between pins the rope can wander up or down, but it can never jump.

Common traps

1) Reading end behavior from the sign of the first term written instead of the term with the highest power. 2) Saying a zero of multiplicity 2 crosses the axis. Even multiplicity touches and turns. 3) Counting turning points as equal to the degree. The maximum is $n - 1$. 4) Forgetting to expand the degree from factors, for example calling $(x - 1)^2(x + 4)$ degree 2.

Mistake Clinic**Spot the error 1**

Problem: Find the end behavior of $f(x) = 7x - x^3 + 2$. Wrong work: The leading coefficient is 7, which is positive, so the graph falls left and rises right. What went wrong: The leading term is the one with the highest power, $-x^3$, not the first term written. Its coefficient is -1 . Right solution: Rewrite as $-x^3 + 7x + 2$. Odd degree with a negative leading coefficient: the graph rises on the left and falls on the right.

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Spot the error 2

Problem: Describe the graph of $f(x) = (x+1)^2(x-2)$ at its zeros. Wrong work: Both zeros cross the axis because the graph is a cubic. What went wrong: Whether the graph crosses depends on the multiplicity of each zero, not the total degree. The factor $(x+1)^2$ has even multiplicity. Right solution: At $x = -1$ (multiplicity 2) the graph touches and turns. At $x = 2$ (multiplicity 1) it crosses. Also $f(0) = (1)(-2) = -2$.

Spot the error 3

Problem: What is the degree of $f(x) = (x-1)^2(x+4)(x^2+3)$? Wrong work: There are three factors, so the degree is 3. What went wrong: The degree is the sum of the exponents after combining all factors, not the count of factors. Right solution: $2 + 1 + 2 = 5$, so the degree is 5.

Real-World Connection

A company's profit over its first several years sometimes follows a cubic, rising, dipping during a costly expansion, and then rising again. The two turning points of the cubic mark the best year before the dip and the worst year of the dip. Knowing that a cubic has at most 2 turning points tells an analyst that a model with more wiggles than that cannot be a cubic.

Challenge Corner

Problem: The cubic $f(x) = x^3 - 3x$ has two turning points. Use a table to locate the local maximum, then use symmetry to find the local minimum. Solution: Factor: $f(x) = x(x - \sqrt{3})(x + \sqrt{3})$, so the zeros are $-\sqrt{3} \approx -1.73$, 0, and $\sqrt{3} \approx 1.73$. A turn must occur between each pair of neighboring zeros. Table: $f(-1.5) = 1.125$, $f(-1) = 2$, $f(-0.5) = 1.375$. The values rise to 2 at $x = -1$ and then fall, so the local maximum is $(-1, 2)$. Because $f(-x) = -f(x)$, the graph is symmetric about the origin, so the local minimum is $(1, -2)$.

Practice**PRACTICE Polynomial Functions and Their Graphs**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is the end behavior of $f(x) = -3x^5 + 2x^2 - 1$?
 (A) $f \rightarrow -\infty$ as $x \rightarrow -\infty$ and $f \rightarrow +\infty$ as $x \rightarrow +\infty$
 (B) $f \rightarrow +\infty$ at both ends
 (C) $f \rightarrow +\infty$ as $x \rightarrow -\infty$ and $f \rightarrow -\infty$ as $x \rightarrow +\infty$
 (D) $f \rightarrow -\infty$ at both ends
- What is the degree of $f(x) = (2x-1)^2(x+3)(x^2+1)$?
 (A) 5 (B) 4 (C) 6 (D) 7
- What is the greatest number of turning points a degree-7 polynomial can have?
 (A) 7 (B) 6 (C) 8 (D) 3
- For $f(x) = x(x-4)^2(x+1)^3$, at which zero does the graph cross the x-axis while flattening?
 (A) $x = 0$ (B) $x = 4$ (C) $x = 1$ (D) $x = -1$
- A polynomial graph falls on the left and rises on the right. Which could be its equation?
 (A) $f(x) = -2x^3 + x$ (B) $f(x) = 2x^3 - x$ (C) $f(x) = 2x^4 - x$
 (D) $f(x) = -x^4 + 3$
- Which of the following is NOT a polynomial function?
 (A) $f(x) = 3x^4 - 2x$ (B) $f(x) = \sqrt{2}x^2 + 5$ (C) $f(x) = x^{-2} + x$
 (D) $f(x) = (x-1)(x+4)$

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7. What is the y-intercept of $f(x) = (x - 2)^2(x + 3)(x - 5)$?
 (A) -60 (B) 60 (C) -30 (D) 30
8. Which interval must contain a real zero of $g(x) = x^3 + x - 9$?
 (A) $(0, 1)$ (B) $(2, 3)$ (C) $(-2, -1)$ (D) $(1, 2)$
9. How many x-intercepts does the graph of $f(x) = x^2(x - 1)^3(x + 2)^2$ have?
 (A) 7 (B) 2 (C) 3 (D) 5
10. A polynomial graph crosses the x-axis at $x = -4$, touches it at $x = -1$, and crosses it while flattening at $x = 3$ (multiplicity 3). What is the least possible degree?
11. Factor $f(x) = x^3 - 6x^2 + 9x$ completely and list each zero with its multiplicity.
12. What is the greatest number of turning points of $f(x) = (x^2 - 1)^2(x + 3)$?
13. For which values of x is $f(x) = (x + 4)(x - 1)^2(x - 6)$ negative?
14. The polynomial $f(x) = x^3 + kx^2 - 4$ has a zero at $x = 2$. Find k .
15. The graph of $g(x) = a(x + 2)(x - 1)^2$ passes through $(0, -6)$. Find a and then describe the end behavior of g .

21.3 Zeros of Polynomials

Use the Rational Root Theorem, the Remainder and Factor Theorems, and synthetic division to find every zero of a polynomial, including the complex ones that never appear on a graph.

Why This Matters

Finding the zeros of a polynomial is one of the oldest problems in mathematics. For a quadratic you have the quadratic formula, but for a cubic or quartic there is no short formula that students can use by hand. Instead, algebra gives you a toolkit: the Rational Root Theorem to list candidate zeros, the Remainder and Factor Theorems to test them, synthetic division to shrink the polynomial, and the Fundamental Theorem of Algebra to tell you when you are done. The same toolkit appears in precalculus, calculus, engineering design, and computer algebra.

Students typically struggle with four things. First, the **sign of the test value** in synthetic division: to test $x + 3$ you use -3 . Second, the **candidate list**, where they forget that the numerator comes from the constant term and the denominator from the leading coefficient. Third, **knowing when to stop**: a cubic has exactly three zeros, counting complex ones. Fourth, **complex zeros** that never show up on a graph. This section gives you a clear routine for each.

Core Ideas

Dividing a polynomial $f(x)$ by $x - c$ gives a quotient $q(x)$ and a constant remainder r , so that $f(x) = (x - c)q(x) + r$. **Synthetic division** is a shortcut for this: you write only the coefficients (use 0 for any missing power), bring down the first one, then repeatedly multiply by c and add.

Remainder Theorem and Factor Theorem

Remainder Theorem: When $f(x)$ is divided by $x - c$, the remainder is $f(c)$.

Factor Theorem: $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$, that is, if and only if c is a zero of f .

To test $x + 3$, use $c = -3$. The sign always flips.

Rational Root Theorem

Let $f(x) = a_nx^n + \cdots + a_1x + a_0$ have **integer** coefficients with $a_0 \neq 0$. Every rational zero in lowest terms $\frac{p}{q}$ satisfies: p divides the constant term a_0 , and q divides the leading coefficient a_n .

So the candidates are $\pm \frac{\text{factors of } a_0}{\text{factors of } a_n}$. The theorem only lists only possible rational zeros. You must test each one.

Polynomial	Constant term	Leading coeff.	Possible rational zeros
$x^3 - 2x^2 - 5x + 6$	6	1	$\pm 1, \pm 2, \pm 3, \pm 6$
$2x^3 + x^2 - 13x + 6$	6	2	$\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$
$3x^2 - x - 4$	-4	3	$\pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$

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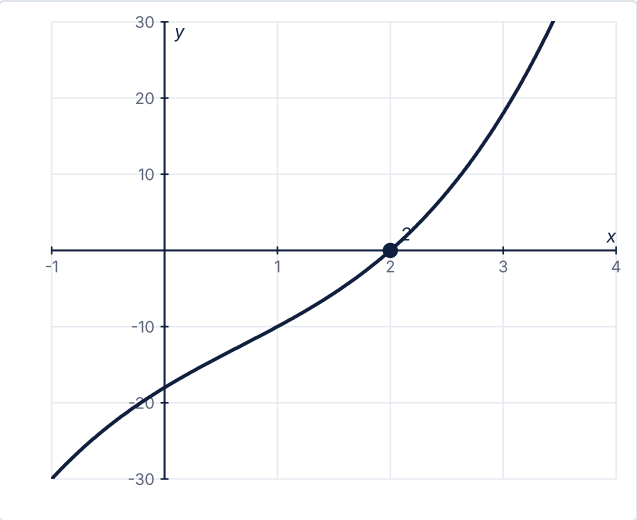
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Fundamental Theorem and Conjugate Pairs

A polynomial of degree $n \geq 1$ has exactly n complex zeros, counting multiplicity.
If the coefficients are **real** and $a + bi$ is a zero, then its conjugate $a - bi$ is also a zero. Complex zeros of real polynomials come in conjugate pairs. (Likewise, if the coefficients are rational and $a + \sqrt{b}$ is a zero, so is $a - \sqrt{b}$.)

The routine for finding all zeros is therefore: list the candidates, test them with synthetic division until one works, then repeat on the smaller quotient. When you reach a quadratic, factor it or use the quadratic formula, which will produce complex zeros if the discriminant is negative.



The graph of $y = x^3 - 2x^2 + 9x - 18$ crosses the x-axis only at $x = 2$. Its other two zeros, $3i$ and $-3i$, are complex, so they do not appear on the graph. The graph shows only real zeros.

Worked Examples

EXAMPLE 1

Use synthetic division to divide $f(x) = 2x^3 - 5x^2 + 3x - 4$ by $x - 3$. State the quotient and the remainder.

3	2	-5	3	-4
		6	3	18
	2	1	6	14

SOLUTION

Use $c = 3$ and the coefficients 2, -5 , 3, -4 . Bring down 2. Then $2 \cdot 3 = 6$ and $-5 + 6 = 1$. Then $1 \cdot 3 = 3$ and $3 + 3 = 6$. Then $6 \cdot 3 = 18$ and $-4 + 18 = 14$.
Quotient: $2x^2 + x + 6$. Remainder: 14.
By the Remainder Theorem this means $f(3) = 14$. Check: $2(27) - 5(9) + 9 - 4 = 54 - 45 + 5 = 14$.

EXAMPLE 2

Find the remainder when $f(x) = x^4 - 3x^2 + 2x - 5$ is divided by $x + 2$, without dividing.

SOLUTION

$x + 2 = x - (-2)$, so $c = -2$.
By the Remainder Theorem the remainder is $f(-2) = 16 - 3(4) + 2(-2) - 5 = 16 - 12 - 4 - 5 = -5$.

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EXAMPLE 3

Show that $x - 2$ is a factor of $f(x) = x^3 - 3x^2 - 4x + 12$ and then find all zeros.

SOLUTION

$f(2) = 8 - 12 - 8 + 12 = 0$, so by the Factor Theorem $x - 2$ is a factor.

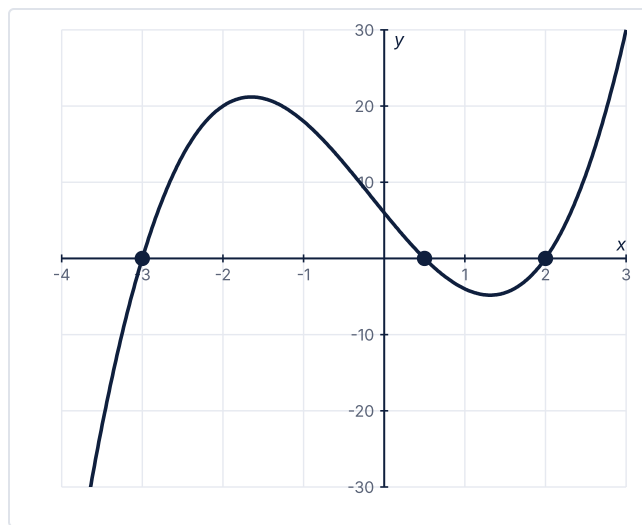
Synthetic division by 2 on 1, -3, -4, 12 gives 1, -1, -6 with remainder 0. The quotient is $x^2 - x - 6$.

Factor: $x^2 - x - 6 = (x - 3)(x + 2)$.

So $f(x) = (x - 2)(x - 3)(x + 2)$ and the zeros are 2, 3, and -2.

EXAMPLE 4

Find all zeros of $f(x) = 2x^3 + x^2 - 13x + 6$.



Graph of $y = 2x^3 + x^2 - 13x + 6$, crossing the axis at $x = -3$, $x = 1/2$, and $x = 2$.

SOLUTION

Possible rational zeros: $p \in \{\pm 1, \pm 2, \pm 3, \pm 6\}$ and $q \in \{1, 2\}$, giving $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$.

Test $x = 2$: $16 + 4 - 26 + 6 = 0$. It works.

Synthetic division by 2 on 2, 1, -13, 6 gives 2, 5, -3. The quotient is $2x^2 + 5x - 3 = (2x - 1)(x + 3)$.

Zeros: $x = 2$, $x = \frac{1}{2}$, $x = -3$.

EXAMPLE 5

Find all complex zeros of $f(x) = x^3 - 2x^2 + 9x - 18$.

SOLUTION

Candidates divide 18: $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$. Test $x = 2$: $8 - 8 + 18 - 18 = 0$.

Synthetic division by 2 on 1, -2, 9, -18 gives 1, 0, 9, so the quotient is $x^2 + 9$.

Solve $x^2 + 9 = 0$: $x = \pm 3i$.

All three zeros: $x = 2$, $x = 3i$, $x = -3i$. The graph (above) shows only $x = 2$, because the other two are not real.

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EXAMPLE 6

A polynomial with real coefficients is $f(x) = x^4 - 3x^3 + 7x^2 + 21x - 26$. Given that $2 + 3i$ is a zero, find all four zeros.

SOLUTION

The coefficients are real, so the conjugate $2 - 3i$ is also a zero.

Build the quadratic factor with these two zeros: sum = 4, product = $4 + 9 = 13$, so the factor is $x^2 - 4x + 13$.

Divide $f(x)$ by $x^2 - 4x + 13$. The quotient is $x^2 + x - 2$ with remainder 0.

Factor: $x^2 + x - 2 = (x + 2)(x - 1)$.

The four zeros are $2 + 3i$, $2 - 3i$, -2 , and 1 .

Coach's Playbook

- 1 List candidates.** Write the factors of the constant term over the factors of the leading coefficient. Cross out repeats.
- 2 Test the easy ones first.** Try 1, -1 , 2, -2 before fractions. Use synthetic division, and if the remainder is 0 you have found a zero.
- 3 Reduce.** Replace the polynomial with the quotient. Its degree is one smaller, and its zeros are the remaining zeros.
- 4 Finish with a quadratic.** Once you have degree 2, factor or use the quadratic formula. A negative discriminant gives a conjugate pair.
- 5 Count to check.** The number of zeros, with multiplicity, must equal the degree.

A fast screening trick: the sum of the coefficients equals $f(1)$. If it is zero, then 1 is a zero. And $f(-1)$ is the sum of the even-power coefficients minus the sum of the odd-power coefficients. These two checks take five seconds and often save a long search. Also remember that a graphing tool can point you to a real zero; then you confirm it exactly with synthetic division.

Common traps

1) Using $c = 3$ to test the factor $x + 3$. The test value is -3 . 2) Forgetting to insert a 0 for a missing power, for example dividing $x^3 - 7x + 6$ using the coefficients 1, $-7, 6$ instead of 1, 0, $-7, 6$. 3) Building the candidate list from only the leading coefficient or only the constant term. 4) Stopping after finding the real zeros and ignoring that a degree- n polynomial has n zeros in all.

Mistake Clinic**Spot the error 1**

Problem: Find the remainder when $f(x) = x^3 + 2x^2 - 4x + 5$ is divided by $x + 3$. Wrong work: Use $c = 3$: $f(3) = 27 + 18 - 12 + 5 = 38$. What went wrong: The divisor is $x + 3 = x - (-3)$, so $c = -3$, not 3. Right solution: $f(-3) = -27 + 18 + 12 + 5 = 8$. The remainder is 8.

Spot the error 2

Problem: List the possible rational zeros of $2x^3 + 3x^2 - 8x - 12$. Wrong work: The factors of the leading coefficient 2 are $\pm 1, \pm 2$, so the candidates are ± 1 and ± 2 . What went wrong: The numerators come from the constant term -12 and the denominators from the leading coefficient 2. The student used only the leading coefficient. Right solution: $p \in \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$ and $q \in \{1, 2\}$. The candidates include $\pm \frac{1}{2}, \pm \frac{3}{2}$, and so on. In fact $f(x) = (x + 2)(x - 2)(2x + 3)$, so $-\frac{3}{2}$ is a zero that the wrong list missed.

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Spot the error 3

Problem: A cubic with real coefficients has zeros 5 and $2i$. Write it in factored form with leading coefficient 1. Wrong work: $f(x) = (x - 5)(x - 2i)$. What went wrong: That has degree 2, and the polynomial has real coefficients, so the conjugate $-2i$ must also be a zero. Right solution: $f(x) = (x - 5)(x - 2i)(x + 2i) = (x - 5)(x^2 + 4) = x^3 - 5x^2 + 4x - 20$.

Real-World Connection

A designer models a hill profile on a trail with $h(x) = x^3 - 7x + 6$, where x is measured in hundreds of meters from the trailhead and h is height above a flat reference level. The reference level is crossed where $h(x) = 0$. Candidates are divisors of 6, and $h(1) = 1 - 7 + 6 = 0$. Dividing gives $x^2 + x - 6 = (x + 3)(x - 2)$, so the crossings are at $x = 1$, $x = 2$, and $x = -3$. Finding exact crossing points this way is faster and more reliable than zooming in on a graph.

Challenge Corner

Problem: Use the Rational Root Theorem to prove that $\sqrt{3}$ is irrational. Solution: The number $\sqrt{3}$ is a zero of $f(x) = x^2 - 3$. If it were rational, the Rational Root Theorem says it must be one of ± 1 or ± 3 (divisors of -3 over divisors of 1). But $f(1) = -2$, $f(-1) = -2$, $f(3) = 6$, and $f(-3) = 6$, none of which is zero. So f has no rational zero, and therefore $\sqrt{3}$ is irrational.

Practice**PRACTICE Zeros of Polynomials**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is the remainder when $x^3 + 2x^2 - 5x + 1$ is divided by $x - 2$?
(A) 1 (B) -7 (C) 7 (D) 9
- Which of these is a factor of $x^3 - x^2 - 8x + 12$?
(A) $x - 2$ (B) $x + 2$ (C) $x - 3$ (D) $x + 1$
- Which of the following is NOT a possible rational zero of $3x^3 - x^2 + 5x - 4$?
(A) $\frac{4}{3}$ (B) $\frac{2}{3}$ (C) $\frac{1}{3}$ (D) $\frac{3}{4}$
- A polynomial with real coefficients has zeros 3 and $1 - 2i$. What is the least possible degree?
(A) 2 (B) 3 (C) 4 (D) 5
- What is the quotient when $x^3 - 4x^2 + 5x - 2$ is divided by $x - 1$?
(A) $x^2 + 3x + 2$ (B) $x^2 - 3x - 2$ (C) $x^2 - 3x + 2$ (D) $x^2 - 5x + 10$
- Which list gives all zeros of $x^3 - 3x^2 + 4x - 12$?
(A) 3, 2, -2 (B) -3, 2i, -2i (C) 3, 4i, -4i (D) 3, 2i, -2i
- A polynomial f satisfies $f(-3) = 0$. Which statement must be true?
(A) $x - 3$ is a factor of f (B) $x + 3$ is a factor of f
(C) The remainder when f is divided by $x + 3$ is -3 (D) $f(3) = 0$
- How many nonreal zeros does $f(x) = x^4 - 1$ have?
(A) 2 (B) 0 (C) 3 (D) 4
- A degree-5 polynomial has real coefficients. Which cannot be its number of nonreal zeros?
(A) 0 (B) 2 (C) 3 (D) 4
- Use the Remainder Theorem to find the remainder when $2x^4 - x^3 + 3x - 7$ is divided by $x + 1$.
- Find all zeros of $x^3 - 2x^2 - 5x + 6$.
- Find all complex zeros of $x^3 + x^2 + 9x + 9$.

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13. For what value of k is $x - 3$ a factor of $2x^3 + kx^2 - x - 6$?
14. A cubic with real coefficients is $x^3 - 8x^2 + 22x - 20$. Given that $3 - i$ is a zero, find all zeros.
15. Find all complex zeros of $2x^3 - 3x^2 + 8x - 12$.

21.4 Modeling with Polynomials

Build a polynomial from its zeros and one extra point, solve polynomial inequalities with sign charts, and use cubic models to find volumes and estimate maximums in box problems.

Why This Matters

Polynomials are not just things to factor on a worksheet. They are the language engineers use to describe the volume of a box, the height of a bridge arch, or the profit of a business. The two skills in this section run in opposite directions. **Building** a polynomial starts from a picture or a list of zeros and ends with a formula. **Analyzing** a polynomial starts from a formula and asks when it is positive, negative, or as large as possible. Together they turn the zeros and graphs you studied earlier into a problem-solving tool.

Students usually get stuck in three places. First, they forget the **stretch factor** a , so the polynomial has the right zeros but misses the given point. Second, they solve polynomial inequalities by treating them like equations or by dividing by a variable, which loses solutions. Third, in volume problems they forget the **domain**, accepting answers that give negative lengths. We will fix all three with a short routine each.

Core Ideas

Building a Polynomial from Its Zeros

If f has zeros $r_1, r_2, \dots, r_n$ then $f(x) = a(x - r_1)(x - r_2) \cdots (x - r_n)$ for some nonzero constant a .

A repeated zero gets a repeated factor. A complex zero $p + qi$ of a real polynomial brings its conjugate, and together they give the real quadratic factor $x^2 - 2px + (p^2 + q^2)$.

Find a by substituting one more known point, often the y-intercept $(0, f(0))$.

Different values of a stretch or flip the graph without moving the zeros. For example, $a = 2$ makes the graph twice as steep and $a = -1$ flips it upside down. That is why the zeros alone never fix a polynomial: you also need one extra point.

Solving a Polynomial Inequality

- 1) Move everything to one side so you have $f(x) > 0$, $f(x) < 0$, $f(x) \geq 0$, or $f(x) \leq 0$.
- 2) Factor and find the zeros. They split the number line into intervals.
- 3) Test one number in each interval (a **sign chart**). The sign cannot change inside an interval.
- 4) Choose the intervals that match. Use brackets for $\leq$ or $\geq$ (zeros are included) and parentheses for $<$ or $>$.

Interval	Test x	$(x + 3)$	$(x - 1)$	$(x - 4)$	Product
$x < -3$	-4	-	-	-	-
$-3 < x < 1$	0	+	-	-	+
$1 < x < 4$	2	+	+	-	-
$x > 4$	5	+	+	+	+



Solution of $(x+3)(x-1)(x-4) > 0$: the product is positive on $(-3, 1)$ and on $(4, \infty)$.

Volume Models

An open-top box made by cutting squares of side x from each corner of an L by W sheet and folding up the sides has dimensions x , $L - 2x$, and $W - 2x$.

Volume: $V(x) = x(L - 2x)(W - 2x)$ with **domain** $0 < x < \frac{W}{2}$ (using the shorter side W). Any answer outside this interval is rejected, because it would give a negative length.

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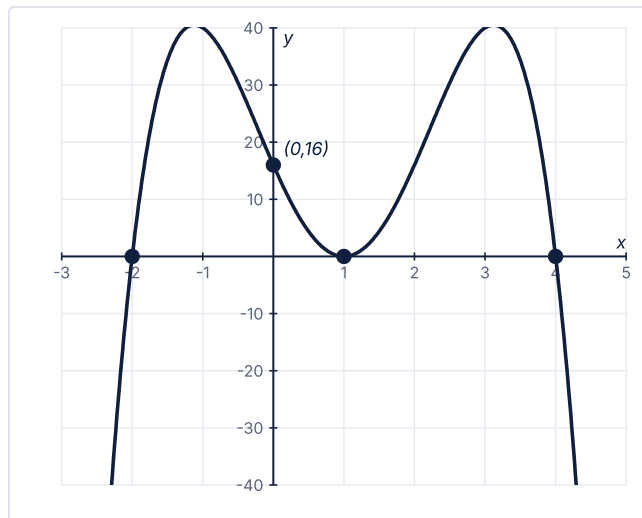
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To find a maximum when you have no calculus yet, use a table of values or the graphing tool, then zoom in. The highest point of the graph over the domain is the maximum. To hit a target volume exactly, set $V(x)$ equal to the target, move everything to one side, and use the zero-finding tools from the last section.

Worked Examples

EXAMPLE 1

Write a polynomial of least degree with zeros -2 , 1 (multiplicity 2), and 4 , whose graph passes through $(0, 16)$.



Graph of $f(x) = -2(x+2)(x-1)^2(x-4)$. It touches the axis at $x = 1$, crosses at $x = -2$ and $x = 4$, and passes through $(0, 16)$.

SOLUTION

Zeros with multiplicity give $f(x) = a(x+2)(x-1)^2(x-4)$.

Use $f(0) = 16$: $a(2)(1)(-4) = -8a = 16$, so $a = -2$.

$f(x) = -2(x+2)(x-1)^2(x-4)$. Expanded: $f(x) = -2x^4 + 8x^3 + 6x^2 - 28x + 16$.

Check: at $x = 2$, $f(2) = -2(4)(1)(-2) = 16$. The degree is 4 with a negative leading coefficient, so both ends fall.

EXAMPLE 2

A polynomial with real coefficients has zeros 3 and $1 + 2i$, and $f(0) = 30$. Find $f(x)$ in expanded form.

SOLUTION

The conjugate $1 - 2i$ is also a zero. The quadratic factor is $x^2 - 2x + 5$, since the sum of the zeros is 2 and the product is $1 + 4 = 5$.

$f(x) = a(x-3)(x^2 - 2x + 5)$. Then $f(0) = a(-3)(5) = -15a = 30$, so $a = -2$.

Expand: $(x-3)(x^2 - 2x + 5) = x^3 - 5x^2 + 11x - 15$.

$f(x) = -2x^3 + 10x^2 - 22x + 30$.

EXAMPLE 3

Solve $(x+3)(x-1)(x-4) > 0$.

SOLUTION

The zeros -3 , 1 , 4 split the line into four intervals. Test each (see the table and number line above).

At $x = -4$: $(-)(-)(-) = -$. At $x = 0$: $(+)(-)(-) = +$. At $x = 2$: $(+)(+)(-) = -$. At $x = 5$: $(+)(+)(+) = +$.

We need positive values and the inequality is strict, so exclude the zeros.

Solution: $-3 < x < 1$ or $x > 4$, that is $(-3, 1) \cup (4, \infty)$.

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EXAMPLE 4

Solve $x^2(x - 5)(x + 2) \geq 0$.

SOLUTION

Zeros: 0 (double), 5, and -2 . Test $x = -3$: $9 \cdot (-8) \cdot (-1) = 72 > 0$. Test $x = -1$: $1 \cdot (-6) \cdot 1 < 0$. Test $x = 1$: $1 \cdot (-4) \cdot 3 < 0$. Test $x = 6$: $36 \cdot 1 \cdot 8 > 0$.

The double zero at 0 does not change the sign, but because the inequality is $\geq$, the value $x = 0$ is a solution (the product equals 0 there).

Solution: $x \leq -2$, or $x = 0$, or $x \geq 5$, that is $(-\infty, -2] \cup \{0\} \cup [5, \infty)$.

EXAMPLE 5

A 20 cm by 30 cm sheet of cardboard has a square of side x cut from each corner, and the sides are folded up to make an open box. Write the volume $V(x)$ and its domain, then find x when the volume is 1008 cm^3 .

SOLUTION

Dimensions: x , $20 - 2x$, $30 - 2x$. So $V(x) = x(20 - 2x)(30 - 2x) = 4x^3 - 100x^2 + 600x$.

Domain: the shorter side gives $20 - 2x > 0$, so $0 < x < 10$.

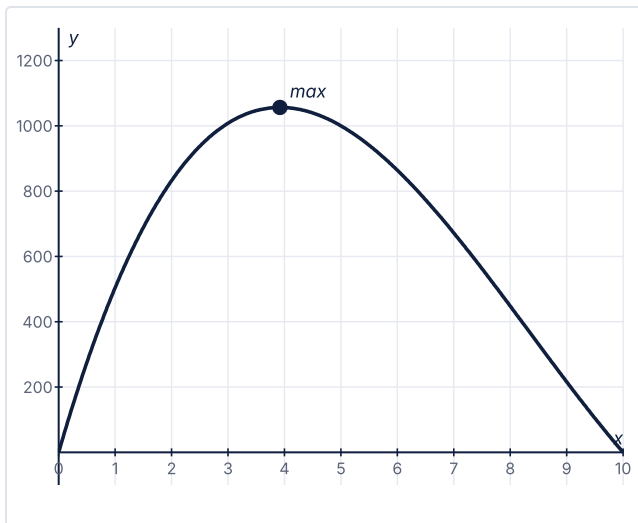
Set $V = 1008$: $4x^3 - 100x^2 + 600x - 1008 = 0$, or $x^3 - 25x^2 + 150x - 252 = 0$. Test $x = 3$: $27 - 225 + 450 - 252 = 0$. It works.

Divide: $x^2 - 22x + 84 = 0$, so $x = 11 \pm \sqrt{37}$, giving $x \approx 4.92$ or $x \approx 17.08$.

Only values in $0 < x < 10$ are allowed. So $x = 3 \text{ cm}$ or $x \approx 4.92 \text{ cm}$. The root 17.08 is rejected.

EXAMPLE 6

For the box in Example 5, estimate the cutout that gives the largest volume, and the maximum volume.



Volume $V(x) = x(20-2x)(30-2x)$ for the box. The highest point is near $x = 3.92$. The domain of the model is $0 < x < 10$.

x	3	3.5	4	4.5	5
$V(x)$	1008	1046.5	1056	1039.5	1000

SOLUTION

Compute a table of values over the domain.

The volume rises, peaks near $x \approx 3.9$, and then falls toward 0 at $x = 10$. Refining the table near the peak gives $x \approx 3.92$.

Maximum volume $\approx 1056.3 \text{ cm}^3$ at $x \approx 3.92 \text{ cm}$.

Check: $V(3) = 1008$ and $V(5) = 1000$, both smaller than the peak value, and the table agrees with the graph.

Coach's Playbook

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- 1 **Sketch before you solve.** A rough graph from the zeros and end behavior tells you what the answer should look like before any algebra.
- 2 **Never divide by a variable.** To solve $x^3 > 9x$, move everything to one side and factor. Dividing by x loses the case where x is negative.
- 3 **Make a sign chart.** Label each interval, test one point, and write plus or minus. Then read the answer straight from the chart.
- 4 **Write the domain first in a word problem.** Lengths must be positive. Write the allowed interval for x before you start solving.
- 5 **Check the answer against the context.** A box with cutout $x = 17$ cm from a 20 cm sheet is impossible, no matter what the equation says.

A useful mental model for inequalities: the zeros are the only places where a polynomial can change sign, like gates on a road. Between two gates the sign is locked. Even-multiplicity gates are 'fake gates', where the graph touches zero but does not switch sides, so the sign on both sides is the same.

Common traps

1) Leaving out the stretch factor a and writing only the product of the factors. 2) Dividing both sides of an inequality by x or by an expression that can be negative. 3) Forgetting that a double zero does not change the sign, or forgetting to include it when the inequality allows equality. 4) Reporting a volume solution outside the domain, such as a cutout longer than half the sheet.

Mistake Clinic

Spot the error 1

Problem: Solve $(x - 2)(x + 3) > 0$. Wrong work: $x - 2 > 0$ gives $x > 2$, and $x + 3 > 0$ gives $x > -3$, so the answer is $x > 2$. What went wrong: A product is positive when both factors are positive **or** both are negative. The student ignored the case where both are negative. Right solution: Test intervals: positive for $x < -3$, negative for $-3 < x < 2$, positive for $x > 2$. Answer: $x < -3$ or $x > 2$.

Spot the error 2

Problem: Solve $x^3 > 9x$. Wrong work: Divide by x to get $x^2 > 9$, so $x > 3$ or $x < -3$. What went wrong: Dividing by x is not allowed because x may be negative (which flips the inequality) or zero. The student lost solutions and gained wrong ones. Right solution: $x^3 - 9x = x(x - 3)(x + 3) > 0$. Test: $x = -4$ gives negative, $x = -1$ gives positive, $x = 1$ gives negative, $x = 4$ gives positive. Answer: $-3 < x < 0$ or $x > 3$.

Spot the error 3

Problem: A 10 cm by 16 cm sheet has corner squares of side x cut out to make a box with volume 144 cm^3 . Solve for x . Wrong work: $x(10 - 2x)(16 - 2x) = 144$ factors as $4(x - 2)^2(x - 9) = 0$, so $x = 2$ or $x = 9$. What went wrong: The algebra is right, but the domain is $0 < x < 5$ (half of the short side). A cutout of 9 cm would make the width $10 - 18 = -8$ cm, which is impossible. Right solution: Reject $x = 9$. The only valid cutout is $x = 2$ cm.

Real-World Connection

A packaging company wants a shipping box with the greatest volume that can be made from one flat sheet. The volume formula is a cubic in the cutout size, the domain comes from the sheet's shorter side, and the peak is found by table or graph. The same process applies to designing gutters, planters, and fish tanks, anywhere a fixed amount of material must hold as much as possible.

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Challenge Corner

Problem: A rectangular block has dimensions x , $x + 2$, and $x + 5$ centimeters and volume 120 cm^3 . Find x . Solution: $x(x + 2)(x + 5) = 120$ gives $x^3 + 7x^2 + 10x - 120 = 0$. The candidates divide 120, and $x = 3$ works: $27 + 63 + 30 - 120 = 0$. Dividing by $x - 3$ leaves $x^2 + 10x + 40$, whose discriminant is $100 - 160 = -60 < 0$, so the other two zeros are complex. The only real solution is $x = 3$, so the block is 3 cm by 5 cm by 8 cm. Check: $3 \cdot 5 \cdot 8 = 120$.

Practice**PRACTICE Modeling with Polynomials**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Which polynomial has zeros 1, -2 , and 3 and leading coefficient 1?
 (A) $(x + 1)(x - 2)(x + 3)$ (B) $(x - 1)(x - 2)(x - 3)$ (C) $(x - 1)(x + 2)(x - 3)$
 (D) $(x + 1)(x + 2)(x + 3)$
- The graph of $f(x) = a(x + 2)(x - 3)$ passes through $(0, 12)$. What is a ?
 (A) -2 (B) 2 (C) -12 (D) 12
- What is the solution set of $(x - 2)(x + 5) < 0$?
 (A) $x < -5$ or $x > 2$ (B) $x < 2$ (C) $x > -5$
 (D) $-5 < x < 2$
- What is the solution set of $x(x - 3)^2 \geq 0$?
 (A) $x \geq 3$ (B) $x \geq 0$ (C) $0 \leq x \leq 3$ (D) $x \leq 0$ or $x = 3$
- An open box is made from a 12 cm by 12 cm square by cutting out squares of side x cm from the corners. Its volume is $V(x) = x(12 - 2x)^2$. What is $V(2)$?
 (A) 128 cm^3 (B) 256 cm^3 (C) 64 cm^3 (D) 200 cm^3
- For the box in the previous question, which values of x are allowed in the model?
 (A) $0 < x < 12$ (B) $0 < x < 3$ (C) $0 < x < 6$ (D) $x > 0$
- A polynomial with real coefficients and leading coefficient 1 has the zeros 0 and $2 - i$ and least possible degree. What is it?
 (A) $x^3 + 4x^2 + 5x$ (B) $x^3 - 4x^2 - 5x$ (C) $x^3 - 2x^2 + x$
 (D) $x^3 - 4x^2 + 5x$
- What is the solution set of $x^3 \leq 4x$?
 (A) $x \leq -2$ or $0 \leq x \leq 2$ (B) $-2 \leq x \leq 2$ (C) $x \leq 2$
 (D) $x \geq 2$ or $x \leq -2$
- For which values of x is $(x + 1)^2(x - 3) > 0$?
 (A) $x < -1$ or $x > 3$ (B) $x > 3$ (C) $x < 3$
 (D) $x > -1$
- Write a polynomial of least degree in factored form with zeros 2, -3 , and 5 whose graph passes through $(0, -30)$.
- Solve $(x + 4)(x - 1)(x - 6) \leq 0$.
- A box is made from an 8 cm by 14 cm sheet by cutting squares of side x from the corners. Write $V(x)$, give the domain, and find $V(2)$.
- For the box in the previous problem, which whole-number cutout (1, 2, or 3 cm) gives the greatest volume, and what is that volume?
- A polynomial with real coefficients and leading coefficient 1 has the zeros 1 and $3 + 2i$ and least possible degree. Write it in expanded form.
- Solve $x^3 - 7x + 6 > 0$.

Answers**Answers and Explanations**

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

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21.1 Imaginary and Complex Numbers**PRACTICE · IMAGINARY AND COMPLEX NUMBERS**

1. B. $B. 23 = 4(5) + 3$, so $i^{23} = i^3 = -i$. Choice A is what you get from remainder 1.
2. A. A. Add real parts: $4 + 2 = 6$. Add imaginary parts: $-3 + 7 = 4$. Choice B subtracts the imaginary parts by mistake.
3. C. $C. \sqrt{-75} = i\sqrt{25 \cdot 3} = 5i\sqrt{3}$. Choice D forgets to take the square root of 25.
4. D. D. A number times its conjugate is $a^2 + b^2 = 4 + 9 = 13$. Choice A comes from treating i^2 as $+1$, which gives $4 - 9 = -5$.
5. B. B. Only the sign of the imaginary part changes. Choice A flips the real part too.
6. C. $C. (3 + 2i)^2 = 9 + 12i + 4i^2 = 9 - 4 + 12i = 5 + 12i$. Choice B leaves $4i^2$ as $+4$.
7. D. D. Multiply by $1 - i$: numerator $6 - 6i + 2i - 2i^2 = 8 - 4i$, denominator 2, giving $4 - 2i$. Choice C forgets the $-2i^2$ term.
8. A. $A. x = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$. Choice D forgets to divide the real part by 2.
9. C. C. Need $36 - 4k < 0$, so $k > 9$. Choice A reverses the inequality.
10. -1. $2026 = 4(506) + 2$, so $i^{2026} = i^2 = -1$.
11. $29 + 22i$. FOIL: $21 + 28i - 6i - 8i^2 = 21 + 22i + 8 = 29 + 22i$.
12. $-\frac{14}{5} + \frac{17}{5}i$. Multiply by $1 + 2i$: numerator $4 + 8i + 9i + 18i^2 = -14 + 17i$; denominator $1 + 4 = 5$.
13. $x = \frac{1 \pm i\sqrt{14}}{3}$. Discriminant $4 - 60 = -56$, $\sqrt{-56} = 2i\sqrt{14}$, so $x = \frac{2 \pm 2i\sqrt{14}}{6} = \frac{1 \pm i\sqrt{14}}{3}$.
14. $a = 3$, $b = 9$. Expand: $6 - 2i + 3ai - ai^2 = (6 + a) + (3a - 2)i$. Matching imaginary parts: $3a - 2 = 7$, so $a = 3$. Then $b = 6 + 3 = 9$.
15. $x^2 - 4x + 29$. The other root is the conjugate $2 + 5i$. Sum = 4, product = $4 + 25 = 29$, so $x^2 - 4x + 29$.

21.2 Polynomial Functions and Their Graphs**PRACTICE · POLYNOMIAL FUNCTIONS AND THEIR GRAPHS**

1. C. C. Degree 5 is odd and the leading coefficient -3 is negative, so the left end rises and the right end falls. Choice A is the pattern for a positive leading coefficient.
2. A. A. Add the degrees of the factors: $2 + 1 + 2 = 5$. Counting factors instead of adding exponents gives the tempting wrong value 4.
3. B. B. At most $n - 1 = 6$. Choice A confuses turning points with the degree.
4. D. D. Only $x = -1$ has odd multiplicity 3, so the graph crosses and flattens there. The zero $x = 4$ has multiplicity 2 (touch) and $x = 0$ crosses straight.
5. B. B. Falling left and rising right means odd degree with positive leading coefficient. Choice A has a negative leading coefficient, and the even-degree choices have both ends the same.
6. C. C. A polynomial has only whole-number exponents. The exponent -2 breaks that rule. Choice B is fine because $\sqrt{2}$ is just a coefficient.
7. A. A. $f(0) = (-2)^2(3)(-5) = 4 \cdot 3 \cdot (-5) = -60$. Choice D comes from using -2 instead of $(-2)^2 = 4$, which gives $(-2)(3)(-5) = 30$.
8. D. D. $g(1) = -7$ and $g(2) = 1$ change sign, so a zero lies in $(1, 2)$. On the other intervals the values do not change sign: $g(0) = -9$, $g(3) = 21$.
9. C. C. The distinct zeros are 0, 1, and -2 . Choice A counts the degree, which counts multiplicities, but the graph has only one intercept per distinct zero.
10. 6. Multiplicities at least $1 + 2 + 3 = 6$.
11. $f(x) = x(x - 3)^2$. Zero $x = 0$ with multiplicity 1 and zero $x = 3$ with multiplicity 2. (Factor out x , then $x^2 - 6x + 9 = (x - 3)^2$.)
12. 4. The degree is $4 + 1 = 5$, so at most $5 - 1 = 4$ turning points.
13. $-4 < x < 6$ with $x \neq 1$. Positive for $x < -4$, negative on $(-4, 1)$, negative again on $(1, 6)$ (the double zero at 1 does not change the sign), positive for $x > 6$.
14. $k = -1$. $f(2) = 8 + 4k - 4 = 0$, so $4k = -4$ and $k = -1$.
15. $a = -3$. $g(0) = a(2)(1) = 2a = -6$. The degree is 3 and the leading coefficient is -3 , so $g \rightarrow +\infty$ as $x \rightarrow -\infty$ and $g \rightarrow -\infty$ as $x \rightarrow +\infty$.

21.3 Zeros of Polynomials**PRACTICE · ZEROS OF POLYNOMIALS**

1. C. C. The remainder is $f(2) = 8 + 8 - 10 + 1 = 7$. Choice B has a sign error with the middle terms.
2. A. A. $f(2) = 8 - 4 - 16 + 12 = 0$. The others fail: $f(-2) = 16$, $f(3) = 6$, and $f(-1) = 18$. Choice B comes from testing $c = 2$ with the wrong sign.
3. D. D. The numerator must divide 4 and the denominator must divide 3. The fraction $\frac{3}{4}$ has numerator 3 and denominator 4, which reverses the roles.
4. B. B. The conjugate $1 + 2i$ must also be a zero, so there are at least three zeros. Choice A forgets the conjugate.
5. C. C. Synthetic division with $c = 1$ gives 1, -3 , 2 with remainder 0. Choice A flips the sign of the second coefficient.
6. D. D. Factor by grouping: $x^2(x - 3) + 4(x - 3) = (x - 3)(x^2 + 4)$. Choice A solves $x^2 = 4$ instead of $x^2 = -4$.
7. B. B. By the Factor Theorem, $f(-3) = 0$ means $x - (-3) = x + 3$ is a factor. Choice A has the sign wrong.
8. A. A. $x^4 - 1 = (x^2 - 1)(x^2 + 1)$ with zeros 1, -1 , i , $-i$. Only i and $-i$ are nonreal.
9. C. C. Nonreal zeros come in conjugate pairs, so their number is even. Three is odd, which is impossible.

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10. -7 . $f(-1) = 2 + 1 - 3 - 7 = -7$.
11. 1, 3, -2 . Candidates divide 6. $f(1) = 0$, so divide to get $x^2 - x - 6 = (x - 3)(x + 2)$.
12. -1 , $3i$, $-3i$. Factor by grouping: $x^2(x + 1) + 9(x + 1) = (x + 1)(x^2 + 9)$.
13. $k = -5$. Need $f(3) = 54 + 9k - 3 - 6 = 45 + 9k = 0$, so $k = -5$.
14. 2, $3 + i$, $3 - i$. The conjugate $3 + i$ is also a zero. The factor $x^2 - 6x + 10$ divides the cubic to give $x - 2$.
15. $\frac{3}{2}$, $2i$, $-2i$. Group: $x^2(2x - 3) + 4(2x - 3) = (2x - 3)(x^2 + 4)$.

21.4 Modeling with Polynomials

PRACTICE · MODELING WITH POLYNOMIALS

- C. C. A zero r gives the factor $x - r$, so $1 \rightarrow (x - 1)$, $-2 \rightarrow (x + 2)$, $3 \rightarrow (x - 3)$. Choice A has every sign reversed.
- A. A. $f(0) = a(2)(-3) = -6a = 12$, so $a = -2$. Choice B ignores the sign of -6 .
- D. D. The product is negative between the zeros, so $-5 < x < 2$. Choice A is the answer to the opposite inequality > 0 .
- B. B. The double zero at 3 does not change the sign, and $x = 3$ is already in $x \geq 0$. The sign is positive for $x > 0$ and negative for $x < 0$. Choice A wrongly drops $0 \leq x < 3$.
- A. A. $V(2) = 2(12 - 4)^2 = 2 \cdot 64 = 128$. Choice B comes from squaring the product $2 \cdot 8$, and choice C forgets the factor $x = 2$.
- C. C. The width $12 - 2x$ must be positive, so $x < 6$. Choice A forgets that two cuts come from each side.
- D. D. The conjugate $2 + i$ must also be a zero. Then $x(x^2 - 4x + 5) = x^3 - 4x^2 + 5x$. Choice A has the sign of the middle term wrong.
- A. A. $x^3 - 4x = x(x - 2)(x + 2) \leq 0$. Test: negative for $x < -2$, positive on $(-2, 0)$, negative on $(0, 2)$, positive for $x > 2$. Choice B comes from dividing by x .
- B. B. The factor $(x + 1)^2$ is never negative, so the sign comes from $x - 3$. Choice A treats the double zero as a sign change.
- $f(x) = -(x - 2)(x + 3)(x - 5)$. $f(0) = a(-2)(3)(-5) = 30a = -30$, so $a = -1$.
- $x \leq -4$ or $1 \leq x \leq 6$. Test intervals: negative for $x < -4$, positive on $(-4, 1)$, negative on $(1, 6)$, positive for $x > 6$. Zeros are included.
- $V(x) = x(8 - 2x)(14 - 2x)$, domain $0 < x < 4$, $V(2) = 2(4)(10) = 80 \text{ cm}^3$.
- 2 cm, with volume 80 cubic cm. $V(1) = 1(6)(12) = 72$, $V(2) = 80$, $V(3) = 3(2)(8) = 48$.
- $x^3 - 7x^2 + 19x - 13$. The conjugate $3 - 2i$ is also a zero, giving $(x - 1)(x^2 - 6x + 13)$.
- $-3 < x < 1$ or $x > 2$. Factor: $(x - 1)(x - 2)(x + 3)$. Test: negative for $x < -3$, positive on $(-3, 1)$, negative on $(1, 2)$, positive for $x > 2$.

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