



Rational Functions and Asymptotes

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

In this booster

22.1 Graphs of Rational Functions

22.2 Vertical, Horizontal, and Slant Asymptotes

22.3 Solving Rational Equations and Inequalities

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Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

22.1 Graphs of Rational Functions

Start from the parent graph of $1/x$ and learn how shifts, stretches, and flips move its asymptotes. Find the domain, range, and intercepts of rational functions and rewrite them in graphing form.

Why This Matters

A rational function is a fraction whose top and bottom are polynomials. Rational functions describe situations where one quantity shrinks as another grows: the time to drive a trip falls as speed rises, the cost per item falls as a factory makes more items, and the strength of a juice mixture changes as water is added. The simplest one, $y = \frac{1}{x}$, is the building block, and every other graph in this section is that one moved, stretched, or flipped.

Students usually get stuck on four things. First, **why the graph has a break**, since you can never divide by zero. Second, **which direction to shift**: $x - 2$ moves right, not left. Third, **mixing up the two asymptotes**, which are lines the graph approaches but never touches. Fourth, **finding intercepts**, where you must set the numerator equal to zero (not the whole fraction). You will learn each with a clear method.

Core Ideas

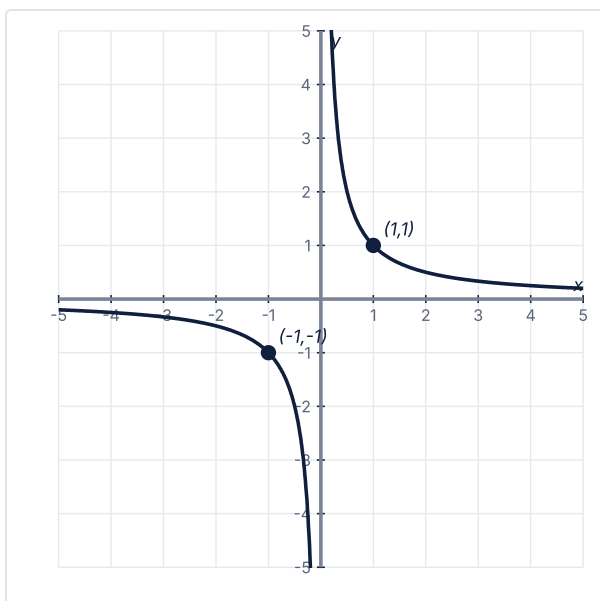
The **parent rational function** is $f(x) = \frac{1}{x}$. It is defined for every real number except $x = 0$, so its **domain** is all $x \neq 0$. Its graph has two separate pieces, one in the first quadrant and one in the third. As x gets very large, $\frac{1}{x}$ gets close to 0; as x gets close to 0 from either side, $\frac{1}{x}$ gets huge in size. The graph has a **vertical asymptote** at $x = 0$ (the y-axis) and a **horizontal asymptote** at $y = 0$ (the x-axis).

x	-4	-2	-1	-0.5	0.5	1	2	4
$\frac{1}{x}$	-0.25	-0.5	-1	-2	2	1	0.5	0.25

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The parent function $y = 1/x$. The dashed-style gray lines are the asymptotes $x = 0$ and $y = 0$. The graph has two branches and never crosses either asymptote.

Transformations of $1/x$

The function $f(x) = \frac{a}{x-h} + k$ is the parent $\frac{1}{x}$ with: a vertical stretch by $|a|$ (and a flip over the x-axis if $a < 0$), a shift **right** h units, and a shift **up** k units.

Vertical asymptote: $x = h$. **Horizontal asymptote:** $y = k$. **Domain:** all $x \neq h$. **Range:** all $y \neq k$.

The center of the graph is the point (h, k) where the asymptotes cross. If $a > 0$ the branches lie in the upper right and lower left of the center; if $a < 0$ they lie in the upper left and lower right.

Intercepts of a Rational Function

y-intercept: evaluate $f(0)$, as long as 0 is in the domain.

x-intercepts: set the **numerator** equal to 0 and solve (any solution that also makes the denominator 0 is not an intercept).

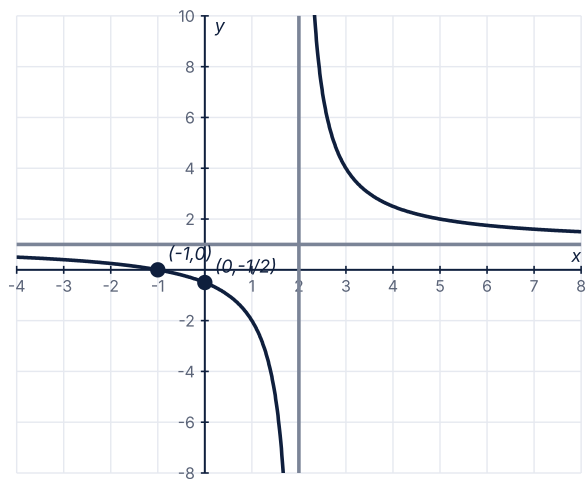
A function written as $\frac{a}{x-h} + k$ has an x-intercept when $\frac{a}{x-h} = -k$, which needs $k \neq 0$.

A rational function like $\frac{2x-6}{x+4}$ does not look like $\frac{a}{x-h} + k$, but long division (or rewriting the numerator) turns it into that form: $2x - 6 = 2(x + 4) - 14$, so the function equals $2 - \frac{14}{x+4}$. Now the center, stretch, and flip are all visible. Rewriting like this is the most useful trick of the section.

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The graph of $y = 3/(x-2) + 1$. The vertical asymptote is $x = 2$ and the horizontal asymptote is $y = 1$. The graph crosses the x -axis at $x = -1$ and the y -axis at $y = -1/2$.

Worked Examples

EXAMPLE 1

Describe the graph of $f(x) = \frac{1}{x}$: domain, range, asymptotes, and the sign of f in each quadrant.

SOLUTION

Domain: all real numbers except 0. Range: all real numbers except 0, because $\frac{1}{x}$ can never equal 0.

Asymptotes: vertical $x = 0$, horizontal $y = 0$.

For $x > 0$, $f(x) > 0$ (first quadrant). For $x < 0$, $f(x) < 0$ (third quadrant). The table of values shows $f(2) = 0.5$ and $f(-2) = -0.5$.

EXAMPLE 2

For $f(x) = \frac{3}{x-2} + 1$, find the asymptotes, domain, range, and both intercepts.

SOLUTION

Here $a = 3$, $h = 2$, $k = 1$. Vertical asymptote $x = 2$, horizontal asymptote $y = 1$.

Domain: $x \neq 2$. Range: $y \neq 1$.

y -intercept: $f(0) = \frac{3}{-2} + 1 = -\frac{1}{2}$.

x -intercept: $\frac{3}{x-2} = -1$, so $x - 2 = -3$ and $x = -1$.

Check: $f(-1) = \frac{3}{-3} + 1 = 0$. The graph appears above.

EXAMPLE 3

Describe $g(x) = -\frac{2}{x+3} - 4$ as a transformation of $\frac{1}{x}$ and find its intercepts.

SOLUTION

Rewrite $x + 3 = x - (-3)$, so $h = -3$. Also $a = -2$ and $k = -4$.

Transformation: stretch by 2, reflect over the x -axis, shift left 3 and down 4. Asymptotes: $x = -3$ and $y = -4$.

y -intercept: $g(0) = -\frac{2}{3} - 4 = -\frac{14}{3}$.

x -intercept: $-\frac{2}{x+3} = 4$, so $x + 3 = -\frac{1}{2}$ and $x = -\frac{7}{2}$.

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EXAMPLE 4

Write an equation of the form $y = \frac{a}{x-h} + k$ whose graph has vertical asymptote $x = 1$, horizontal asymptote $y = -2$, and passes through $(3, 0)$.

SOLUTION

The asymptotes give $h = 1$ and $k = -2$, so $y = \frac{a}{x-1} - 2$.

Substitute $(3, 0)$: $0 = \frac{a}{2} - 2$, so $a = 4$.

Equation: $y = \frac{4}{x-1} - 2$.

Check with the y-intercept: $y(0) = \frac{4}{-1} - 2 = -6$, which is reasonable for a branch below the asymptote $y = -2$ at the left of $x = 1$.

EXAMPLE 5

Rewrite $f(x) = \frac{2x-6}{x+4}$ in the form $\frac{a}{x-h} + k$, then find the asymptotes and intercepts.

SOLUTION

Write the numerator in terms of the denominator: $2x - 6 = 2(x + 4) - 14$.

So $f(x) = \frac{2(x+4) - 14}{x+4} = 2 - \frac{14}{x+4}$.

Asymptotes: $x = -4$ and $y = 2$. Domain $x \neq -4$.

x-intercept: numerator $2x - 6 = 0$ gives $x = 3$. y-intercept: $f(0) = \frac{-6}{4} = -\frac{3}{2}$.

EXAMPLE 6

For $f(x) = \frac{3x+5}{x-1}$, rewrite the function, identify the center of the graph, and find the input where $f(x) = 5$.

SOLUTION

$3x + 5 = 3(x - 1) + 8$, so $f(x) = 3 + \frac{8}{x-1}$.

Center: $(1, 3)$. Since $a = 8 > 0$, the branches are in the upper right and lower left of the center.

Solve $f(x) = 5$: $\frac{3x+5}{x-1} = 5$ gives $3x + 5 = 5x - 5$, so $x = 5$.

Check: $f(5) = \frac{20}{4} = 5$.

Coch's Playbook

- 1 **Rewrite first.** Put the function into the form $\frac{a}{x-h} + k$. Everything else reads off from it.
- 2 **Mark the center (h, k) and draw the asymptotes.** Use light dashed lines. They guide the graph.
- 3 **Plot two or three points.** Use the intercepts, or inputs one and two units from h .
- 4 **Choose branches by the sign of a .** Positive a : upper right and lower left. Negative a : upper left and lower right.
- 5 **State domain and range last.** Domain excludes h . Range excludes k .

Think of the asymptotes as invisible walls. The vertical wall is where the denominator is zero, because you cannot divide by zero. The horizontal wall is the height the graph settles toward far to the left and far to the right. The curve hugs both walls but never touches them, like a car that gets closer and closer to a stop sign without ever reaching it.

Common traps

- 1) Shifting the wrong way. The factor $x - 2$ means shift **right** 2; $x + 3$ means shift **left** 3. 2) Swapping the asymptotes. The vertical line comes from the denominator, the horizontal from the added constant k . 3) Setting the whole fraction equal to 0 for the y-intercept instead of evaluating $f(0)$. 4) Forgetting that the range also has a hole: $y \neq k$.

Mistake Clinic

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Spot the error 1

Problem: Find the vertical asymptote of $y = \frac{1}{x+3}$. Wrong work: The denominator is $x+3$, so the asymptote is $x=3$.
 What went wrong: Solving $x+3=0$ gives $x=-3$, not 3. The student dropped the sign when moving the 3. Right solution: $x=-3$. The graph is $\frac{1}{x}$ shifted 3 units left.

Spot the error 2

Problem: Find the x-intercept of $y = \frac{3}{x-2} - 1$. Wrong work: $\frac{3}{x-2} = 1$, so $x = 3 - 2 = 1$. What went wrong: To undo the division, multiply both sides by $x-2$: $3 = x - 2$. The student subtracted instead. Right solution: $3 = x - 2$ gives $x = 5$. Check: $\frac{3}{5} - 1 = 0$.

Spot the error 3

Problem: State the range of $y = \frac{2}{x} + 5$. Wrong work: The function is defined for every $x \neq 0$, so the range is all $y \neq 0$.
 What went wrong: The student confused domain with range. The graph never reaches the height of its horizontal asymptote, $y = 5$. Right solution: Domain: $x \neq 0$. Range: $y \neq 5$.

Real-World Connection

A family drives 240 miles. The trip time is $T(s) = \frac{240}{s}$ hours at speed s miles per hour. At 40 mph the trip takes 6 hours, at 60 mph it takes 4 hours, and at 120 mph it takes 2 hours. The graph is the right half of a $\frac{1}{x}$ curve. The time gets smaller as speed increases but never reaches 0, which is exactly what a horizontal asymptote at $T = 0$ means.

Challenge Corner

Problem: Find where the graph of $y = \frac{1}{x-1} + 1$ meets the line $y = x$. Solution: Set $\frac{1}{x-1} + 1 = x$ and multiply by $x-1$ (so $x \neq 1$): $1 + (x-1) = x(x-1)$, which gives $x = x^2 - x$, so $x^2 - 2x = 0$ and $x = 0$ or $x = 2$. Check: at $x = 0$, $y = \frac{1}{-1} + 1 = 0$. At $x = 2$, $y = 1 + 1 = 2$. The intersection points are $(0, 0)$ and $(2, 2)$.

Practice**PRACTICE** Graphs of Rational Functions

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is the domain of $f(x) = \frac{5}{x+2}$?
 (A) all real numbers except 2 (B) all real numbers except -2
 (C) all real numbers except 0 (D) all real numbers
- What is the vertical asymptote of $y = \frac{3}{x-4} + 7$?
 (A) $x = 4$ (B) $x = -4$ (C) $x = 7$ (D) $y = 7$
- What is the horizontal asymptote of $y = -\frac{2}{x+1} + 5$?
 (A) $y = -2$ (B) $y = -1$ (C) $y = 5$ (D) $y = 1$
- What is the x-intercept of $y = \frac{6}{x-2} - 3$?
 (A) $x = 0$ (B) $x = -1$ (C) $x = 2$ (D) $x = 4$
- What is the y-intercept of $y = \frac{4}{x+2} + 1$?
 (A) 3 (B) 1 (C) 2 (D) 5

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6. Which equation has vertical asymptote $x = -3$ and horizontal asymptote $y = 2$?

(A) $y = \frac{1}{x-3} + 2$ (B) $y = \frac{1}{x+3} + 2$ (C) $y = \frac{1}{x+2} + 3$ (D) $y = \frac{1}{x+3} - 2$

7. How does the graph of $y = -\frac{1}{x+2} - 3$ compare with $y = \frac{1}{x}$?

- (A) Reflected over the x-axis, shifted right 2 and down 3
 (B) Not reflected, shifted left 2 and down 3
 (C) Reflected over the x-axis, shifted left 2 and down 3
 (D) Reflected over the y-axis, shifted left 3 and down 2

8. What is the horizontal asymptote of $f(x) = \frac{4x+1}{x+2}$?

(A) $y = -2$ (B) $y = \frac{1}{2}$ (C) $x = -2$ (D) $y = 4$

9. What is the range of $y = \frac{2}{x-1} + 3$?

- (A) all $y \neq 3$ (B) all $y \neq 1$ (C) all $y \neq 0$
 (D) all real numbers

10. Find the y-intercept of $y = -\frac{4}{x+1} + 2$.

11. Find the x-intercept of $y = -\frac{4}{x+1} + 2$.

12. Write an equation of the form $y = \frac{a}{x-h} + k$ with vertical asymptote $x = 2$, horizontal asymptote $y = -1$, and passing through $(3, 1)$.

13. Rewrite $\frac{5x-3}{x+2}$ in the form $k + \frac{a}{x-h}$.

14. Solve $\frac{6}{x-1} + 2 = 5$.

15. State the domain and range of $f(x) = \frac{2x+7}{x-3}$.

22.2 Vertical, Horizontal, and Slant Asymptotes

Find vertical, horizontal, and slant asymptotes of rational functions, tell holes from asymptotes by factoring and cancelling, and read end behavior from degrees and leading coefficients.

Why This Matters

Asymptotes tell you what a rational function does at its extremes: what happens when the input gets huge, and what happens when the input gets close to a forbidden value. That information answers practical questions. How low can a cost per item go as production grows? What is the long-term level of a drug in the bloodstream? Where will a model break down? Rational functions model all of these, and asymptotes are the answer to what happens in the long run.

Students usually mix up three kinds of trouble spots. A **vertical asymptote** is a wall the graph never crosses. A **hole** is a single missing point where the graph is otherwise perfectly normal. A **horizontal or slant asymptote** describes end behavior far out to the left and right. The most common errors are calling a hole an asymptote, comparing the wrong parts of a fraction when finding the horizontal asymptote, and forgetting to use long division for the slant case. This section gives you a clean rule for each.

Core Ideas

Write the function as a fraction of polynomials, $f(x) = \frac{N(x)}{D(x)}$, and **factor both** top and bottom completely. Cancel any common factors, but remember what you cancelled. The cancelled factors will tell you about holes, and what remains tells you about asymptotes.

Vertical Asymptotes and Holes

If $(x - c)$ is a factor of the denominator and **remains after cancelling**, then $x = c$ is a **vertical asymptote**.

If $(x - c)$ is a factor of both numerator and denominator and cancels completely, there is a **hole** at $x = c$. Find its height by substituting c into the simplified function.

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Horizontal and Slant Asymptotes

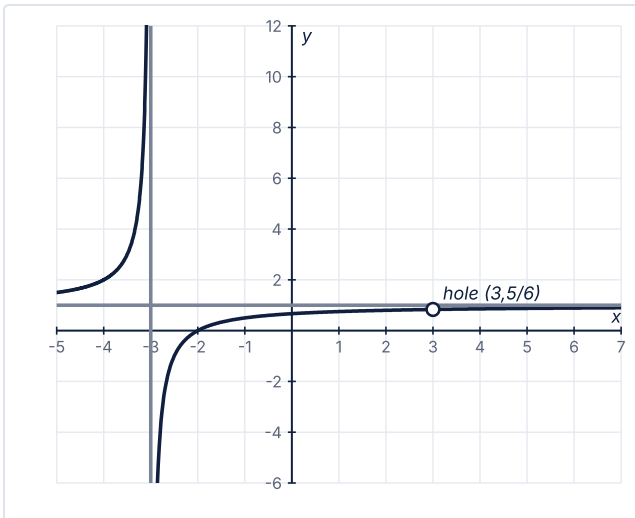
Let N have degree n and D have degree m .

If $n < m$: horizontal asymptote $y = 0$. If $n = m$: horizontal asymptote $y = \frac{\text{leading coefficient of } N}{\text{leading coefficient of } D}$.

If $n = m + 1$: no horizontal asymptote, but a **slant (oblique) asymptote** $y =$ the quotient from long division, ignoring the remainder. If $n > m + 1$: no horizontal or slant asymptote.

Degrees	Example	Asymptote
$n < m$	$\frac{4x + 7}{x^2 + 1}$	$y = 0$
$n = m$	$\frac{6x^2 - x}{2x^2 + 5}$	$y = 3$
$n = m + 1$	$\frac{2x^2 - 3x + 5}{x - 1}$	slant $y = 2x - 1$
$n \geq m + 2$	$\frac{x^4 + 1}{x}$	none (grows like x^3)

Why does the rule for $n = m$ use leading coefficients? When x is very large, only the highest powers matter, so $\frac{6x^2 - x}{2x^2 + 5}$ behaves like $\frac{6x^2}{2x^2} = 3$. For the slant case, long division splits the fraction into a line plus a leftover piece that shrinks to zero, so the graph hugs that line.

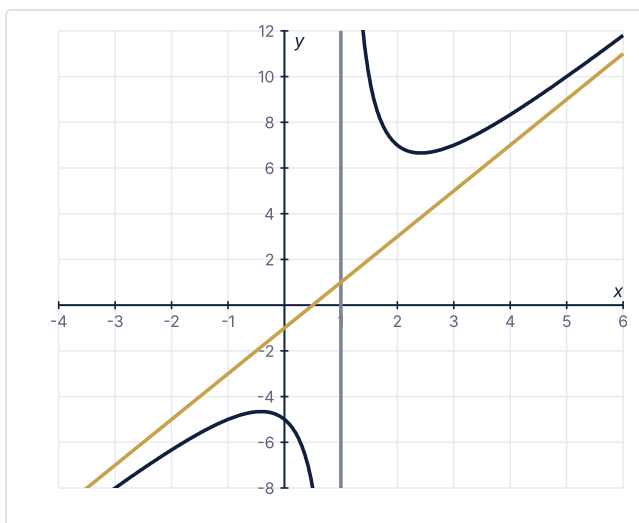


The graph of $y = \frac{x+2}{x+3}$, which is the simplified form of $\frac{x^2-x-6}{x^2-9}$. There is a vertical asymptote at $x = -3$, a horizontal asymptote at $y = 1$, and an open circle (a hole) at $x = 3$, height $5/6$.

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The graph of $y = (2x^2 - 3x + 5)/(x - 1)$. The vertical asymptote is $x = 1$ and the slant asymptote is the line $y = 2x - 1$. The branches hug the line far from the center.

Worked Examples

EXAMPLE 1

Find the vertical and horizontal asymptotes of $f(x) = \frac{3x + 1}{x - 2}$.

SOLUTION

Denominator zero: $x - 2 = 0$, so the vertical asymptote is $x = 2$ (the numerator is $7 \neq 0$ there).

Degrees are equal (1 and 1), so the horizontal asymptote is the ratio of leading coefficients: $y = \frac{3}{1} = 3$.

EXAMPLE 2

Find the horizontal asymptote, if any: (a) $\frac{4x + 7}{x^2 + 1}$, (b) $\frac{6x^2 - x}{2x^2 + 5}$, (c) $\frac{x^3 - 2}{x^2 + 4}$.

SOLUTION

(a) Degree 1 is less than degree 2, so $y = 0$.

(b) Equal degrees 2 and 2, so $y = \frac{6}{2} = 3$.

(c) Degree 3 is greater than degree 2. There is no horizontal asymptote. Because $3 = 2 + 1$ there is a slant asymptote (found by long division).

EXAMPLE 3

Find all holes and asymptotes of $f(x) = \frac{x^2 - x - 6}{x^2 - 9}$.

SOLUTION

$$\text{Factor: } f(x) = \frac{(x - 3)(x + 2)}{(x - 3)(x + 3)}.$$

The factor $x - 3$ cancels, so there is a **hole** at $x = 3$. The simplified function is $\frac{x + 2}{x + 3}$, and its height at 3 is $\frac{5}{6}$. Hole: $(3, \frac{5}{6})$.

The factor $x + 3$ remains, so the vertical asymptote is $x = -3$.

Equal degrees with leading coefficients 1 and 1 give the horizontal asymptote $y = 1$.

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EXAMPLE 4

Find the slant asymptote of $f(x) = \frac{2x^2 - 3x + 5}{x - 1}$.

SOLUTION

Divide $2x^2 - 3x + 5$ by $x - 1$. Using synthetic division with $c = 1$ on 2, -3, 5: bring down 2; $-3 + 2 = -1$; $5 - 1 = 4$.

Quotient $2x - 1$, remainder 4. So $f(x) = 2x - 1 + \frac{4}{x - 1}$.

The leftover $\frac{4}{x-1}$ shrinks to 0 for large $|x|$, so the slant asymptote is $y = 2x - 1$.

The vertical asymptote is $x = 1$.

EXAMPLE 5

Fully analyze $f(x) = \frac{x^2 - 4}{x^2 - x - 2}$: holes, asymptotes, and intercepts.

SOLUTION

Factor: $f(x) = \frac{(x - 2)(x + 2)}{(x - 2)(x + 1)}$.

Hole at $x = 2$; the simplified function is $\frac{x+2}{x+1}$, which gives $\frac{4}{3}$ at $x = 2$. Hole: $(2, \frac{4}{3})$.

Vertical asymptote: $x = -1$. Horizontal asymptote: $y = 1$.

x-intercept: $x + 2 = 0$ gives $x = -2$ (the zero $x = 2$ is the hole, not an intercept). y-intercept: $f(0) = \frac{-4}{-2} = 2$.

EXAMPLE 6

Write a rational function with vertical asymptotes $x = -2$ and $x = 3$, horizontal asymptote $y = 2$, and an x-intercept only at $x = 1$.

SOLUTION

The denominator needs the factors $(x + 2)(x - 3)$.

For a horizontal asymptote of $y = 2$ the degrees must match, so the numerator has degree 2. To have only one x-intercept use $(x - 1)^2$, and the leading coefficient must be 2.

$$f(x) = \frac{2(x - 1)^2}{(x + 2)(x - 3)}.$$

Check: leading coefficients give $\frac{2}{1} = 2$, and at $x = 1$ the numerator is 0 while the denominator is $(3)(-2) \neq 0$. The graph touches the axis there.

Coach's Playbook

- 1 Factor everything.** Top and bottom. Pull out common factors first, then factor trinomials and differences of squares.
- 2 Cancel and record.** Write down every cancelled factor $x - c$. Each one is a hole at $x = c$.
- 3 Read the vertical asymptotes.** Set each remaining denominator factor equal to zero.
- 4 Compare degrees.** Smaller on top: $y = 0$. Equal: ratio of leading coefficients. One bigger on top: divide to get the slant line.
- 5 Find intercepts last.** Use the simplified numerator for x-intercepts and $f(0)$ for the y-intercept.

A good way to remember holes versus walls: a hole is a point the graph skips, so a single point is missing. A vertical asymptote is a wall, and the graph shoots off toward positive or negative infinity next to it. If a factor cancels completely, nothing blows up, so it must be a hole. One quick check on a calculator table is that near a hole the outputs settle toward a number, while near a vertical asymptote they grow without bound.

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Common traps

1) Calling a cancelled factor a vertical asymptote. It is a hole. 2) Using the coefficients of the wrong terms. The horizontal asymptote uses the highest-power terms only. 3) Writing a slant asymptote without doing the division, or including the remainder in the line. 4) Believing a graph can never cross a horizontal asymptote. It can cross it near the center; the asymptote only describes the far ends.

Mistake Clinic**Spot the error 1**

Problem: Find the horizontal asymptote of $f(x) = \frac{3x^2 + 1}{x^3 + 2}$. Wrong work: The leading coefficients are 3 and 1, so $y = \frac{3}{1} = 3$. What went wrong: That rule is only for equal degrees. Here the numerator has degree 2 and the denominator has degree 3. Right solution: Since $2 < 3$, the horizontal asymptote is $y = 0$.

Spot the error 2

Problem: Find the vertical asymptotes of $f(x) = \frac{x^2 - 4}{x - 2}$. Wrong work: The denominator is zero at $x = 2$, so there is a vertical asymptote at $x = 2$. What went wrong: The numerator also has the factor $x - 2$, and $x^2 - 4 = (x - 2)(x + 2)$ cancels it, so nothing blows up at 2. Right solution: $f(x) = x + 2$ for $x \neq 2$. There is a hole at $(2, 4)$ and no vertical asymptote.

Spot the error 3

Problem: Find the slant asymptote of $f(x) = \frac{x^2 + 1}{x - 1}$. Wrong work: $x^2 \div x = x$, so the slant asymptote is $y = x$. What went wrong: Dividing only the leading terms gives the first term of the quotient. The full division continues: $x^2 + 1 = (x - 1)(x + 1) + 2$. Right solution: The quotient is $x + 1$, so the slant asymptote is $y = x + 1$.

Real-World Connection

A drug's concentration in the blood is modeled by $c(t) = \frac{5t}{t^2 + 1}$ milligrams per liter, t hours after a dose. The numerator has degree 1 and the denominator has degree 2, so the horizontal asymptote is $y = 0$. That matches reality: far in the future the drug has left the body and the concentration approaches zero. The graph rises first (the drug is absorbed) and then slowly falls toward the asymptote.

Challenge Corner

Problem: Find all holes and asymptotes of $f(x) = \frac{x^3 - x^2 - 4x + 4}{x^2 - 1}$. Solution: Factor the numerator by grouping: $x^2(x - 1) - 4(x - 1) = (x - 1)(x - 2)(x + 2)$. The denominator is $(x - 1)(x + 1)$. Cancel $x - 1$: there is a hole at $x = 1$, and the simplified function is $\frac{x^2 - 4}{x + 1}$, so the hole is at $(1, -\frac{3}{2})$. The vertical asymptote is $x = -1$. The degrees are 2 over 1, so there is a slant asymptote: $\frac{x^2 - 4}{x + 1} = x - 1 - \frac{3}{x + 1}$, so $y = x - 1$. The x-intercepts are $x = 2$ and $x = -2$.

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Practice

PRACTICE Vertical, Horizontal, and Slant Asymptotes

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is the vertical asymptote of $f(x) = \frac{x+1}{x-5}$?
 (A) $x = -1$ (B) $x = 1$ (C) $x = 5$ (D) $y = 1$
- What is the horizontal asymptote of $f(x) = \frac{6x-1}{2x+9}$?
 (A) $y = 3$ (B) $y = -\frac{1}{9}$ (C) $y = 6$ (D) $y = 0$
- What is the horizontal asymptote of $f(x) = \frac{x^2+3}{x^3-1}$?
 (A) $y = 1$ (B) $y = 0$ (C) $y = 3$
 (D) There is none
- Which statement describes the graph of $f(x) = \frac{x^2-16}{x-4}$?
 (A) A vertical asymptote at $x = 4$
 (B) A hole at $(4, 0)$ and a vertical asymptote at $x = 4$
 (C) A hole at $(-4, 0)$
 (D) A hole at $(4, 8)$ and no vertical asymptote
- What is the slant asymptote of $f(x) = \frac{x^2+3x-1}{x+1}$?
 (A) $y = x + 2$ (B) $y = x + 3$ (C) $y = x - 1$ (D) $y = x + 1$
- For $f(x) = \frac{x-3}{(x-3)(x+4)}$, which statement is true?
 (A) The vertical asymptotes are $x = 3$ and $x = -4$
 (B) The only vertical asymptote is $x = 3$
 (C) The only vertical asymptote is $x = -4$, and there is a hole at $x = 3$
 (D) There are no vertical asymptotes
- How many vertical asymptotes does $f(x) = \frac{x+2}{x^2-x-6}$ have?
 (A) 0 (B) 2 (C) 3 (D) 1
- As x grows without bound, what value does $f(x) = \frac{2x^2+1}{x^2-5}$ approach?
 (A) 0 (B) 2 (C) $\frac{1}{5}$ (D) -5
- Which function has horizontal asymptote $y = -\frac{3}{2}$?
 (A) $f(x) = \frac{3x-1}{2x+5}$ (B) $f(x) = \frac{3x^3}{2x^2-1}$ (C) $f(x) = \frac{-3x^2+7}{2x^2+x}$
 (D) $f(x) = \frac{2x^2}{-3x^2+1}$
- Find the vertical asymptote and the horizontal asymptote of $f(x) = \frac{4x-3}{2x+6}$.
- Find the slant asymptote of $f(x) = \frac{3x^2+x-2}{x-2}$.
- Find the coordinates of the hole in the graph of $f(x) = \frac{x^2+x-12}{x-3}$.
- Find the x-intercept and y-intercept of $f(x) = \frac{2x-8}{x+2}$.
- Write a rational function with vertical asymptote $x = 4$, horizontal asymptote $y = 3$, and x-intercept at $x = 1$.
- For $f(x) = \frac{x^2-9}{x^2-x-6}$, state the hole, the vertical asymptote, and the horizontal asymptote.

22.3 Solving Rational Equations and Inequalities

Solve equations with variable denominators by clearing fractions, screen out extraneous solutions with restrictions, and solve rational inequalities using sign charts and number lines.

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Why This Matters

Whenever two quantities are linked by a ratio, a rational equation appears. Two pipes filling a tank, two printers sharing a job, a lens that focuses light, and a boat traveling with and against a current all lead to equations with variables in the denominators. Rational inequalities answer the practical follow-up: for which inputs is the cost low enough, the speed high enough, or the mixture strong enough?

Students usually get stuck in three places. First, **clearing denominators** correctly, especially multiplying every term by the least common denominator. Second, **extraneous solutions**: a root you find may make a denominator zero, and then it is not allowed. Third, **inequalities**, where the temptation is to multiply both sides by an expression that may be negative, which flips the inequality in some regions and not in others. A sign chart avoids that problem completely.

Core Ideas

Solving a Rational Equation

- 1) Factor each denominator and write down the **restrictions**: every value that makes a denominator 0 is excluded.
- 2) Multiply **every term** on both sides by the least common denominator (LCD).
- 3) Solve the resulting polynomial equation.
- 4) Reject any solution that is a restricted value. A rejected value is an **extraneous solution**. If every solution is rejected, the equation has **no solution**.

Extraneous solutions appear because multiplying by an expression that equals zero at some x creates a new equation that is true there even though the original was undefined. That is why the restriction list comes first, not last. Checking in the original equation is the final safety net.

Solving a Rational Inequality

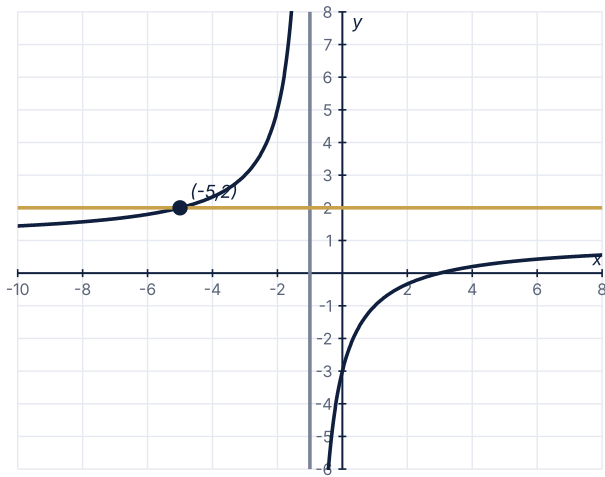
- 1) Move everything to one side so the other side is 0. Never multiply by a variable expression.
- 2) Combine into a single fraction and factor top and bottom.
- 3) The **critical values** are the zeros of the numerator and the zeros of the denominator. They split the number line into intervals.
- 4) Test one number per interval (a sign chart). Include numerator zeros only when the inequality allows equality. **Never** include denominator zeros.

Interval	Test x	$x + 5$	$x + 1$	$\frac{x + 5}{x + 1}$
$x < -5$	-6	-	-	+
$-5 < x < -1$	-3	+	-	-
$x > -1$	0	+	+	+

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Solving $(x-3)/(x+1) \leq 2$. The gold line is $y = 2$ and the navy curve is $y = (x-3)/(x+1)$. The curve is on or below the line for $x \leq -5$ and for $x > -1$. They meet at $x = -5$.



Solution of $(x-3)/(x+1) \leq 2$: $x \leq -5$ or $x > -1$. The open circle at -1 shows the excluded value; the closed circle at -5 is included.

Worked Examples

EXAMPLE 1

Solve $\frac{3}{x-2} = \frac{5}{x+4}$.

SOLUTION

Restrictions: $x \neq 2$ and $x \neq -4$.

Cross multiply (this is multiplying by the LCD): $3(x+4) = 5(x-2)$.

Distribute: $3x + 12 = 5x - 10$, so $22 = 2x$ and $x = 11$.

11 is not a restricted value. Check: $\frac{3}{9} = \frac{1}{3}$ and $\frac{5}{15} = \frac{1}{3}$. Solution: $x = 11$.

EXAMPLE 2

Solve $\frac{x}{x-3} = 2 + \frac{3}{x-3}$.

SOLUTION

Restriction: $x \neq 3$. The LCD is $x - 3$.

Multiply every term by $x - 3$: $x = 2(x - 3) + 3$.

Simplify: $x = 2x - 3$, so $x = 3$.

But $x = 3$ is restricted, so it is extraneous. The equation has **no solution**.

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EXAMPLE 3

Solve $\frac{1}{x} + \frac{1}{x+2} = \frac{3}{4}$.

SOLUTIONRestrictions: $x \neq 0$, $x \neq -2$. The LCD is $4x(x+2)$.Multiply every term: $4(x+2) + 4x = 3x(x+2)$.Simplify: $8x + 8 = 3x^2 + 6x$, so $3x^2 - 2x - 8 = 0$.Factor: $(3x+4)(x-2) = 0$, so $x = 2$ or $x = -\frac{4}{3}$.Neither is restricted. Check $x = 2$: $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. Check $x = -\frac{4}{3}$: $-\frac{3}{4} + \frac{3}{2} = \frac{3}{4}$. Both work.**EXAMPLE 4**

Solve $\frac{x+1}{x-2} > 0$.

SOLUTIONThe critical values are $x = -1$ (numerator zero) and $x = 2$ (denominator zero).Test $x = -2$: $\frac{-1}{-4} > 0$ (positive). Test $x = 0$: $\frac{1}{-2} < 0$ (negative). Test $x = 3$: $\frac{4}{1} > 0$ (positive).

The inequality is strict, so exclude both critical values.

Solution: $x < -1$ or $x > 2$.**EXAMPLE 5**

Solve $\frac{x-3}{x+1} \leq 2$.

SOLUTIONDo not multiply by $x+1$. Move the 2: $\frac{x-3}{x+1} - 2 \leq 0$.Combine: $\frac{x-3-2(x+1)}{x+1} = \frac{-x-5}{x+1} \leq 0$. Multiply by -1 (which flips the sign): $\frac{x+5}{x+1} \geq 0$.Critical values: -5 and -1 . The sign chart (above) shows positive for $x < -5$ and $x > -1$. The numerator zero -5 is included; the denominator zero -1 is not.Solution: $x \leq -5$ or $x > -1$.Check $x = 0$: $\frac{-3}{1} = -3 \leq 2$ true. Check $x = -3$: $\frac{-6}{-2} = 3 \leq 2$ false, as expected.**EXAMPLE 6**

Solve $\frac{x}{x-2} + \frac{2}{x+2} = \frac{8}{x^2-4}$.

SOLUTIONFactor $x^2 - 4 = (x-2)(x+2)$. Restrictions: $x \neq 2$, $x \neq -2$. LCD: $(x-2)(x+2)$.Multiply every term: $x(x+2) + 2(x-2) = 8$.Expand: $x^2 + 2x + 2x - 4 = 8$, so $x^2 + 4x - 12 = 0$, which factors as $(x+6)(x-2) = 0$.The candidates are $x = -6$ and $x = 2$. But $x = 2$ is restricted, so reject it.Check $x = -6$: left side is $\frac{-6}{-8} + \frac{2}{-4} = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ and the right side is $\frac{8}{32} = \frac{1}{4}$. Solution: $x = -6$ only.**Coach's Playbook**

- 1 Restrictions first.** Factor the denominators and write down every excluded value before doing anything else.
- 2 Clear fractions with the LCD.** Multiply every term, including whole numbers, by the LCD.
- 3 Solve the simpler equation.** Usually linear or quadratic.
- 4 Reject extraneous solutions.** Cross out any answer on the restriction list and substitute the rest to check.

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5 For inequalities, build the sign chart. Get 0 on one side, combine into one fraction, list critical values, and test an interval each.

A helpful picture is the graph: the solution of an equation is where two curves meet, and the solution of an inequality is where one curve lies above or below the other. If your algebra says the curves meet at a place where one of them has a vertical asymptote, something went wrong. Another quick check is the answer count. After clearing denominators, a quadratic gives at most two candidates, and some of them may be thrown out.

Common traps

1) Forgetting to multiply the terms without fractions (like the 2 in $\frac{x}{x-3} = 2 + \dots$) by the LCD. 2) Keeping an extraneous solution because you never wrote the restrictions. 3) Multiplying an inequality by a variable expression without considering its sign. 4) Including a denominator zero in an inequality's solution, or leaving out a numerator zero when the inequality is $\leq$ or $\geq$.

Mistake Clinic

Spot the error 1

Problem: Solve $\frac{x+1}{x-2} > 3$. Wrong work: Multiply by $x-2$: $x+1 > 3(x-2)$, so $7 > 2x$ and $x < 3.5$. What went wrong: The sign of $x-2$ is unknown, so you cannot multiply by it without splitting into cases. The student's answer includes $x=1$, but $\frac{2}{-1} = -2$ is not greater than 3. Right solution: $\frac{x+1-3(x-2)}{x-2} = \frac{7-2x}{x-2} > 0$. The critical values are 2 and 3.5. Testing gives positive only for $2 < x < 3.5$.

Spot the error 2

Problem: Solve $\frac{x}{x-2} = \frac{2}{x-2}$. Wrong work: Multiply by $x-2$ to get $x=2$. The solution is $x=2$. What went wrong: The value $x=2$ makes both denominators zero, so it is not allowed. It is extraneous. Right solution: There is no solution.

Spot the error 3

Problem: Solve $\frac{1}{x} + \frac{1}{x+2} = \frac{3}{4}$. Wrong work: Add the fractions: $\frac{1+1}{x+x+2} = \frac{2}{2x+2} = \frac{3}{4}$. What went wrong: You cannot add fractions by adding numerators and denominators separately. Use a common denominator. Right solution: $\frac{(x+2)+x}{x(x+2)} = \frac{3}{4}$ leads to $3x^2 - 2x - 8 = 0$, so $x=2$ or $x=-\frac{4}{3}$.

Real-World Connection

Two taps fill a tank. The second tap alone takes 3 hours longer than the first. Together they fill it in 2 hours. If the first tap takes x hours alone, then $\frac{1}{x} + \frac{1}{x+3} = \frac{1}{2}$. Multiplying by $2x(x+3)$ gives $2(x+3) + 2x = x(x+3)$, which simplifies to $x^2 - x - 6 = 0$, so $x=3$ or $x=-2$. A negative time makes no sense, so the first tap takes 3 hours and the second takes 6 hours. The restriction from the context rejects the other root.

Challenge Corner

Problem: Solve $\frac{x+1}{x-1} \geq \frac{x-1}{x+1}$. Solution: Move everything to one side and combine: $\frac{(x+1)^2 - (x-1)^2}{(x-1)(x+1)} = \frac{4x}{(x-1)(x+1)} \geq 0$. The critical values are -1 , 0 , and 1 . Test: $x=-2$ gives $\frac{-8}{3} < 0$. $x=-0.5$ gives $\frac{-2}{-0.75} > 0$. $x=0.5$ gives $\frac{2}{-0.75} < 0$. $x=2$ gives $\frac{8}{3} > 0$. Include $x=0$ (numerator zero) but exclude ± 1 . Solution: $-1 < x \leq 0$ or $x > 1$.

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Practice

PRACTICE Solving Rational Equations and Inequalities

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Solve $\frac{6}{x+1} = 2$.
 (A) $x = 2$ (B) $x = 3$ (C) $x = -2$ (D) $x = 1$
- Solve $\frac{x}{x-5} = \frac{5}{x-5} + 2$.
 (A) $x = 5$ (B) no solution (C) $x = 0$
 (D) $x = -5$
- Solve $\frac{2}{x} + 3 = \frac{5}{x}$.
 (A) $x = 0$ (B) $x = 3$ (C) $x = 1$ (D) $x = \frac{1}{3}$
- Which is the solution set of $\frac{x+3}{x-1} < 0$?
 (A) $x < -3$ or $x > 1$ (B) $x < 1$ (C) $-3 \leq x < 1$
 (D) $-3 < x < 1$
- Which is the solution set of $\frac{1}{x-2} \geq 1$?
 (A) $x \leq 3$ (B) $2 < x \leq 3$ (C) $x \geq 3$ (D) $2 \leq x \leq 3$
- What are all solutions of $x - \frac{6}{x} = 1$?
 (A) $x = 3$ and $x = -2$ (B) $x = 3$ and $x = 2$ (C) $x = -3$ and $x = 2$
 (D) $x = 3$ only
- Which values must be excluded from the domain of $\frac{1}{x-2} + \frac{3}{x^2-9}$?
 (A) $x = 2$ only (B) $x = 3$ and -3 only (C) $x = 2, 3, -3$
 (D) $x = -2, 3, -3$
- What is the least common denominator of $\frac{1}{x^2-4}$ and $\frac{1}{x^2-2x}$?
 (A) $(x-2)(x+2)$ (B) $x(x-2)(x+2)$ (C) $x(x-2)^2(x+2)$
 (D) $x^2(x^2-4)$
- Which is the solution set of $\frac{x-4}{x+2} \geq 0$?
 (A) $x \leq -2$ or $x \geq 4$ (B) $-2 < x \leq 4$ (C) $x \geq 4$
 (D) $x < -2$ or $x \geq 4$
- Solve $\frac{5}{x-3} = \frac{2}{x+1}$.
- Solve $\frac{3}{x} + \frac{2}{x+1} = 2$. Give exact answers.
- Solve $\frac{x^2}{x-3} = \frac{9}{x-3}$.
- Solve $\frac{2x+1}{x-4} \leq 3$.
- Solve $\frac{1}{x} < \frac{1}{3}$.
- Solve $\frac{2}{x-1} + \frac{3}{x+1} = \frac{12}{x^2-1}$.

22.4 Variation and Applications

Model situations with direct, inverse, joint, and combined variation, then apply rational functions to mixtures, rates, and average cost, including the long-run behavior shown by asymptotes.

Why This Matters

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Variation describes how two or more quantities change together. Double the number of hours you work and your pay doubles. Double the number of people sharing a job and the time per person is cut in half. These relationships, direct, inverse, and joint, are the simplest rational and polynomial models, and they show up in physics formulas, cooking, finance, and engineering. Once you can spot the type, a single constant of variation k solves almost every problem.

Students usually get stuck in three places. First, **deciding the type**: does the second quantity grow or shrink when the first one grows? Second, **setting up the equation** from words like 'varies jointly' or 'inversely as the square'. Third, the rational applications of mixtures, rates, and average cost, where the equation has a variable in a denominator and the answer must make sense in context. This section shows a repeatable method that works for every type.

Core Ideas

Direct, Inverse, and Joint Variation

Direct variation: $y = kx$. The ratio $\frac{y}{x} = k$ is constant. When x doubles, y doubles.

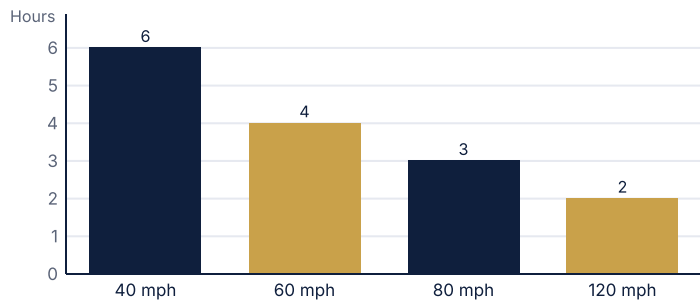
Inverse variation: $y = \frac{k}{x}$. The product $xy = k$ is constant. When x doubles, y is cut in half.

Joint variation: $z = kxy$. The quantity z varies directly with each of x and y together. **Combined variation** mixes the types, for example $y = \frac{kx}{z^2}$ means y varies directly with x and inversely with the square of z .

The Four-Step Method

- 1) Write the general equation with k from the wording.
- 2) Substitute the given values and solve for k .
- 3) Write the specific equation with the number k .
- 4) Substitute the new value and solve for the unknown.

Phrase	Equation	Example
y varies directly as x	$y = kx$	Pay = hourly rate $\times$ hours
y varies inversely as x	$y = \frac{k}{x}$	Time = $\frac{\text{distance}}{\text{speed}}$
z varies jointly as x and y	$z = kxy$	Area of a rectangle = length $\times$ width
y varies directly as x and inversely as z^2	$y = \frac{kx}{z^2}$	Gravity force between two masses



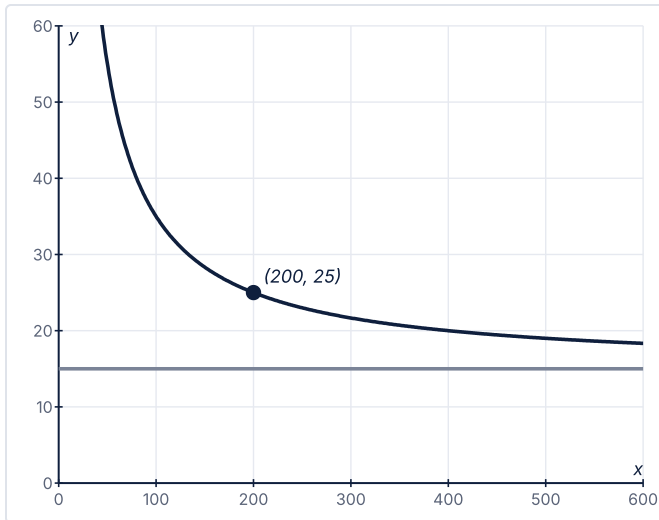
A 240-mile trip: travel time is inversely proportional to speed. Doubling the speed from 60 to 120 mph cuts the time from 4 hours to 2 hours.

The rational applications use the same idea. In a **mixture** problem the concentration is the amount of the substance divided by the total amount, so adding material changes both parts of the fraction. In an **average cost** problem the average cost per item is total cost divided by the number of items, $A(n) = \frac{C(n)}{n}$, and the horizontal asymptote tells you the cost per item in the long run. In a **rate** problem, rates add: if one machine does $\frac{1}{x}$ of a job per hour and another does $\frac{1}{y}$, together they do $\frac{1}{x} + \frac{1}{y}$ per hour.

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Average cost $A(n) = (2000 + 15n)/n$ for a company with fixed cost 2000 dollars and cost 15 dollars per item. The average cost falls as n grows and approaches the horizontal asymptote $y = 15$, the cost per item.

Worked Examples

EXAMPLE 1

The distance d a cyclist travels varies directly with the time t . If $d = 24$ km when $t = 8$ hours, find d when $t = 15$ hours.

SOLUTION

General equation: $d = kt$.

Substitute: $24 = k \cdot 8$, so $k = 3$.

Specific equation: $d = 3t$.

At $t = 15$: $d = 3(15) = 45$ km.

EXAMPLE 2

The time T to complete a trip varies inversely with the speed s . At 60 mph the trip takes 4 hours. How long does it take at 80 mph?

SOLUTION

General equation: $T = \frac{k}{s}$.

Substitute: $4 = \frac{k}{60}$, so $k = 240$ (the trip length in miles).

At 80 mph: $T = \frac{240}{80} = 3$ hours.

Check: faster speed gives less time, which makes sense for inverse variation.

EXAMPLE 3

The quantity z varies jointly with x and y . If $z = 60$ when $x = 3$ and $y = 5$, find z when $x = 6$ and $y = 2$.

SOLUTION

General equation: $z = kxy$.

Substitute: $60 = k(3)(5) = 15k$, so $k = 4$.

Specific equation: $z = 4xy$.

At $x = 6$, $y = 2$: $z = 4(6)(2) = 48$.

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EXAMPLE 4

The quantity y varies directly with x and inversely with the square of z . If $y = 12$ when $x = 6$ and $z = 2$, find y when $x = 9$ and $z = 3$.

SOLUTION

General equation: $y = \frac{kx}{z^2}$.

Substitute: $12 = \frac{k \cdot 6}{4} = \frac{3k}{2}$, so $k = 8$.

Specific equation: $y = \frac{8x}{z^2}$.

At $x = 9$, $z = 3$: $y = \frac{8 \cdot 9}{9} = 8$.

EXAMPLE 5

A 40 liter tank holds a solution containing 8 liters of acid. How many liters of pure acid must be added to make the solution 50% acid?

SOLUTION

After adding x liters of pure acid, the acid is $8 + x$ liters and the total is $40 + x$ liters.

Concentration: $C(x) = \frac{8+x}{40+x}$. Set $C(x) = 0.5$.

Multiply by $40 + x$: $8 + x = 0.5(40 + x) = 20 + 0.5x$.

So $0.5x = 12$ and $x = 24$ liters.

Check: $\frac{32}{64} = 0.5$. Also $C(x)$ can never reach 100%, since its horizontal asymptote is $y = 1$.

EXAMPLE 6

A company has a fixed cost of 2000 dollars and spends 15 dollars to make each item. Write the average cost function $A(n)$, find how many items give an average cost of 25 dollars, and describe the long-run average cost.

SOLUTION

Total cost: $C(n) = 2000 + 15n$. Average cost: $A(n) = \frac{2000 + 15n}{n} = \frac{2000}{n} + 15$.

Set $A(n) = 25$: $2000 + 15n = 25n$, so $10n = 2000$ and $n = 200$ items.

Long run: as n grows, $\frac{2000}{n} \rightarrow 0$, so $A(n) \rightarrow 15$. The horizontal asymptote $y = 15$ means the average cost per item approaches 15 dollars but never reaches it.

Check: $A(200) = \frac{2000}{200} + 15 = 25$. The graph above shows this point.

Coach's Playbook

- 1 Name the type.** Direct if both grow together, inverse if one grows as the other shrinks, joint if there are two variables multiplied.
- 2 Write the general equation.** Use k . Powers apply only to the variable named in the words (for example 'the square of z ').
- 3 Find k with the first data set.** Do this before using the new numbers.
- 4 Answer the question asked.** Substitute the new values, solve, and attach units.
- 5 Check reasonableness.** Inverse: if the input goes up, the output should go down. Mixtures: concentration must stay between 0 and 1.

A fast mental model for inverse variation: the product stays fixed. If 6 workers need 10 days, the job is 60 worker-days. With 4 workers it takes $60 \div 4 = 15$ days. The same product idea works for speed and time, and for pressure and volume in science.

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Common traps

1) Treating an inverse variation as direct, so the output moves the wrong way. 2) Forgetting to square (or cube) the variable named in the phrase, such as 'inversely as the square of d '. 3) In mixture problems, adding the new amount to the numerator but not to the denominator. 4) Accepting a solution that does not fit the context, like a negative number of items or a concentration above 100%.

Mistake Clinic**Spot the error 1**

Problem: y varies inversely with x , and $y = 8$ when $x = 3$. Find y when $x = 6$. Wrong work: $\frac{8}{3} = \frac{y}{6}$, so $y = 16$. What went wrong: That setup is for direct variation. For inverse variation xy stays constant, so doubling x should cut y in half. Right solution: $k = xy = 24$, so $y = \frac{24}{6} = 4$.

Spot the error 2

Problem: A 40 L solution has 8 L acid. How much pure acid must be added to make it 50% acid? Wrong work: $\frac{8+x}{40} = 0.5$, so $8+x = 20$ and $x = 12$. What went wrong: Adding acid also increases the total volume, so the denominator must be $40+x$. Right solution: $\frac{8+x}{40+x} = 0.5$ gives $x = 24$ liters.

Spot the error 3

Problem: y varies directly with x and inversely with the square of z . $y = 12$ when $x = 6$ and $z = 2$. Find the equation. Wrong work: $y = kxz^2$, so $12 = k(6)(4)$ and $k = 0.5$. What went wrong: 'Inversely' means z^2 belongs in the denominator. Right solution: $y = \frac{kx}{z^2}$, so $12 = \frac{6k}{4}$ and $k = 8$. The equation is $y = \frac{8x}{z^2}$.

Real-World Connection

A school club is planning a field trip with a bus that costs a fixed 400 dollars to rent, plus 5 dollars of snacks per student. The cost per student is $A(n) = \frac{400+5n}{n} = \frac{400}{n} + 5$. With 20 students it is 25 dollars each, with 40 students it is 15 dollars each, and with 80 students it is 10 dollars each. As more students join, the price per student falls toward the 5 dollar snack cost, which is the horizontal asymptote.

Challenge Corner

Problem: A boat travels 24 km upstream and 24 km back downstream in a total of 5 hours. The current flows at 2 km/h. Find the boat's speed in still water. Solution: Let v be the speed in still water. Time upstream is $\frac{24}{v-2}$ and downstream is $\frac{24}{v+2}$. Then $\frac{24}{v-2} + \frac{24}{v+2} = 5$. Multiply by $(v-2)(v+2)$: $24(v+2) + 24(v-2) = 5(v^2-4)$, so $48v = 5v^2 - 20$ and $5v^2 - 48v - 20 = 0$. Factor: $(5v+2)(v-10) = 0$, so $v = 10$ or $v = -\frac{2}{5}$. A speed cannot be negative, so $v = 10$ km/h. Check: $\frac{24}{8} + \frac{24}{12} = 3 + 2 = 5$.

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Practice

PRACTICE Variation and Applications

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- The value y varies directly with x . If $y = 12$ when $x = 4$, what is y when $x = 10$?
(A) 40 (B) 30 (C) 4.8 (D) 120
- The value y varies inversely with x . If $y = 6$ when $x = 10$, what is y when $x = 4$?
(A) 2.4 (B) 6 (C) 15 (D) 24
- Which equation says that z varies jointly with x and y ?
(A) $z = kxy$ (B) $z = \frac{kx}{y}$ (C) $z = k(x + y)$ (D) $z = \frac{k}{xy}$
- The value y varies inversely with the square of x . If $y = 5$ when $x = 2$, what is y when $x = 5$?
(A) 2 (B) $\frac{1}{5}$ (C) 20 (D) $\frac{4}{5}$
- In the inverse variation $y = \frac{k}{x}$, what happens to y when x doubles?
(A) It doubles (B) It is cut in half
(C) It stays the same (D) It becomes four times as large
- A 10 liter solution is 30% salt (3 liters of salt). How many liters of pure water must be added to make it 10% salt?
(A) 10 L (B) 30 L (C) 7 L (D) 20 L
- A company's total cost is $C(n) = 500 + 8n$ dollars for n items. What is the average cost per item when $n = 50$?
(A) 18 dollars (B) 10 dollars (C) 8 dollars (D) 900 dollars
- One pipe fills a tank in 6 hours and a second pipe fills it in 3 hours. How long do they take together?
(A) 4.5 hours (B) 9 hours (C) 2 hours (D) 3 hours
- What is the horizontal asymptote of the average cost function $A(n) = \frac{1200 + 9n}{n}$?
(A) $y = 9$ (B) $y = 0$ (C) $y = 1200$ (D) $y = 1209$
- The distance d varies directly with time t . If $d = 150$ when $t = 2.5$, find d when $t = 4$.
- The number of days d to finish a job varies inversely with the number of workers w . If 6 workers take 10 days, how many days will 4 workers take?
- The value z varies jointly with x and y . If $z = 42$ when $x = 2$ and $y = 7$, find z when $x = 5$ and $y = 4$.
- The value F varies directly with m and inversely with the square of d . If $F = 18$ when $m = 3$ and $d = 2$, find F when $m = 5$ and $d = 4$.
- A 60 liter mixture is 25% juice (15 liters of juice). How many liters of pure juice must be added to make the mixture 40% juice?
- The average cost per item is $A(n) = \frac{1800 + 12n}{n}$ dollars. For what number of items is the average cost 20 dollars?

Answers Answers and Explanations

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

22.1 Graphs of Rational Functions**PRACTICE · GRAPHS OF RATIONAL FUNCTIONS**

- B. B. The denominator is zero when $x = -2$. Choice A has the wrong sign.
- A. A. The denominator $x - 4$ is zero at $x = 4$. The line $y = 7$ is the horizontal asymptote, which Choice D describes by mistake.
- C. C. The constant added outside the fraction is $k = 5$. Choice B is the x -value of the vertical asymptote, $x = -1$, mixed up with the horizontal one.
- D. D. Set $\frac{6}{x-2} = 3$, so $x - 2 = 2$ and $x = 4$. Choice C comes from the sign slip $x - 2 = -3$.
- A. A. $y(0) = \frac{4}{2} + 1 = 3$. Choice B drops the $+1$.
- B. B. $x + 3 = 0$ gives $x = -3$ and $+2$ gives $y = 2$. Choice A has the wrong sign in the shift, and Choice C swaps the roles of 2 and 3.
- C. C. The minus sign in front flips it over the x -axis, $x + 2$ means left 2, and -3 means down 3. Choice A shifts the wrong horizontal way.
- D. D. Rewrite: $4x + 1 = 4(x + 2) - 7$, so $f(x) = 4 - \frac{7}{x+2}$ and the horizontal asymptote is $y = 4$. Choice C, $x = -2$, is the vertical asymptote, not the horizontal one.
- A. A. The graph never reaches its horizontal asymptote $y = 3$. Choice B gives the excluded x -value instead.
- 2. $y(0) = -\frac{4}{1} + 2 = -2$.

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11. $x = 1$. Solve $-\frac{4}{x+1} = -2$, so $x + 1 = 2$ and $x = 1$.
12. $y = \frac{2}{x-2} - 1$. With $h = 2$, $k = -1$: $1 = \frac{a}{1} - 1$, so $a = 2$.
13. $5 - \frac{13}{x+2}$. $5x - 3 = 5(x+2) - 13$.
14. $x = 3$. $\frac{6}{x-1} = 3$, so $x - 1 = 2$ and $x = 3$.
15. Domain: $x \neq 3$. Range: $y \neq 2$. Rewrite $f(x) = 2 + \frac{13}{x-3}$.

22.2 Vertical, Horizontal, and Slant Asymptotes

PRACTICE · VERTICAL, HORIZONTAL, AND SLANT ASYMPTOTES

1. C. C. The denominator is zero at $x = 5$. Choice A is the zero of the numerator, which is an x-intercept, not an asymptote.
2. A. A. Equal degrees, so divide leading coefficients: $\frac{6}{2} = 3$. Choice D (0) applies only when the numerator's degree is smaller.
3. B. B. The numerator has the smaller degree, so $y = 0$. Choice A uses the ratio of leading coefficients, which applies only to equal degrees.
4. D. D. $f(x) = \frac{(x-4)(x+4)}{x-4} = x + 4$ for $x \neq 4$, so there is a hole at (4, 8). Choice A is the common wrong answer.
5. A. A. Division gives $x^2 + 3x - 1 = (x+1)(x+2) - 3$, so the quotient is $x + 2$. Choice B uses the coefficient 3 without dividing.
6. C. C. The factor $x - 3$ cancels, leaving a hole there. Choice A calls the hole an asymptote.
7. D. D. The denominator factors as $(x-3)(x+2)$. The factor $x + 2$ cancels (a hole at $x = -2$), so the only vertical asymptote is $x = 3$. Choice B counts both denominator zeros.
8. B. B. Equal degrees, so the end value is $\frac{2}{1} = 2$. Choice A would be right only if the numerator had the smaller degree.
9. C. C. Equal degrees with leading coefficients -3 and 2 give $-\frac{3}{2}$. Choice A gives $+\frac{3}{2}$, and Choice D gives $-\frac{2}{3}$ (the ratio upside down).
10. Vertical: $x = -3$. Horizontal: $y = 2$. The denominator is zero at $x = -3$, and $\frac{4}{2} = 2$.
11. $y = 3x + 7$. Synthetic division with $c = 2$ on $3, 1, -2$ gives $3, 7$ with remainder 12 .
12. (3, 7). Factor: $\frac{(x-4)(x-3)}{x-3} = x + 4$, and $3 + 4 = 7$.
13. x-intercept (4, 0) and y-intercept (0, -4). Set $2x - 8 = 0$ for the first; compute $f(0) = \frac{-8}{2}$ for the second.
14. $f(x) = \frac{3(x-1)}{x-4}$, or $\frac{3x-3}{x-4}$. The degrees are equal with leading coefficients 3 and 1.
15. Hole at $(3, \frac{6}{5})$, vertical asymptote $x = -2$, horizontal asymptote $y = 1$. Factor: $\frac{(x-3)(x+3)}{(x-3)(x+2)} = \frac{x+3}{x+2}$.

22.3 Solving Rational Equations and Inequalities

PRACTICE · SOLVING RATIONAL EQUATIONS AND INEQUALITIES

1. A. A. Multiply by $x + 1$: $6 = 2(x + 1)$, so $x = 2$. Choice B subtracts incorrectly ($6/2 = 3$ is the value of $x + 1$, not x).
2. B. B. Multiplying by $x - 5$ gives $x = 5 + 2(x - 5)$, so $x = 5$. That value makes the denominators zero, so it is extraneous. Choice A is the extraneous value.
3. C. C. Multiply by x : $2 + 3x = 5$, so $x = 1$. Choice B subtracts $5 - 2 = 3$ and stops before dividing by the coefficient.
4. D. D. The critical values are -3 and 1 , and the fraction is negative between them. Choice C wrongly includes -3 , but the inequality is strict.
5. B. B. $\frac{1}{x-2} - 1 = \frac{1-x}{x-2} \geq 0$ is true for $2 < x \leq 3$. Choice A comes from multiplying by $x - 2$ without checking its sign.
6. A. A. Multiply by x : $x^2 - 6 = x$, so $x^2 - x - 6 = 0$ and $(x-3)(x+2) = 0$. Choice B has a sign error in one root.
7. C. C. $x - 2 = 0$ gives 2 , and $x^2 - 9 = 0$ gives ± 3 . Choice D has the wrong sign on the first restriction.
8. B. B. Factor: $x^2 - 4 = (x-2)(x+2)$ and $x^2 - 2x = x(x-2)$. The LCD uses each factor once: $x(x-2)(x+2)$. Choice C repeats the factor $x - 2$.
9. D. D. The critical values are -2 and 4 . The fraction is positive for $x < -2$ and $x > 4$, and zero at $x = 4$, which is included. The value $x = -2$ makes the denominator zero, so Choice A is wrong to include it.
10. $x = -\frac{11}{3}$. Cross multiply: $5(x + 1) = 2(x - 3)$, so $5x + 5 = 2x - 6$ and $3x = -11$.
11. $x = \frac{3 \pm \sqrt{33}}{4}$. Multiply by $x(x + 1)$: $3(x + 1) + 2x = 2x(x + 1)$, so $2x^2 - 3x - 3 = 0$. The quadratic formula gives $\frac{3 \pm \sqrt{9 + 24}}{4}$.
12. $x = -3$. Multiply by $x - 3$: $x^2 = 9$, so $x = \pm 3$. The value $x = 3$ is restricted, so only $x = -3$ remains.
13. $x < 4$ or $x \geq 13$. Rewrite as $\frac{13-x}{x-4} \leq 0$, i.e. $\frac{x-13}{x-4} \geq 0$. The value $x = 4$ is excluded and $x = 13$ is included.
14. $x < 0$ or $x > 3$. Rewrite as $\frac{3-x}{3x} < 0$. The critical values are 0 and 3 , and testing gives negative for $x < 0$ and for $x > 3$.
15. $x = \frac{13}{5}$. Multiply by $(x-1)(x+1)$: $2(x+1) + 3(x-1) = 12$, so $5x - 1 = 12$.

22.4 Variation and Applications

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PRACTICE · VARIATION AND APPLICATIONS

1. B. B. $k = \frac{12}{4} = 3$, so $y = 3(10) = 30$. Choice A multiplies 4 by 10 and ignores the given pair.
2. C. C. $k = xy = 60$, so $y = \frac{60}{4} = 15$. Choice A treats it as direct variation ($6 \cdot \frac{4}{10}$).
3. A. A. Joint variation means z is a constant times the product xy . Choice B adds instead of multiplying.
4. D. D. $k = yx^2 = 5 \cdot 4 = 20$, so $y = \frac{20}{25} = \frac{4}{5}$. Choice A uses x instead of x^2 .
5. B. B. Since $xy = k$ is fixed, doubling x halves y . Choice A describes direct variation.
6. D. D. $\frac{3}{10+x} = 0.1$ gives $10 + x = 30$, so $x = 20$. Choice A subtracts the percentages and does not use the equation; Choice B forgets to subtract the original 10 L.
7. A. A. $C(50) = 900$ and $\frac{900}{50} = 18$. Choice D is the total cost, not the average cost.
8. C. C. Rates add: $\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ of the tank per hour, so 2 hours. Choice A averages the two times.
9. A. A. Rewrite $A(n) = \frac{1200}{n} + 9$. For large n , $A(n) \rightarrow 9$, the cost per item. Choice B would be correct only if there were no per-item cost.
10. 240. $k = \frac{150}{2.5} = 60$, so $d = 60(4) = 240$.
11. 15 days. $wd = 60$ is constant, so $d = \frac{60}{4} = 15$.
12. 60. $k = \frac{42}{14} = 3$, so $z = 3(5)(4) = 60$.
13. $F = 7.5$. $F = \frac{km}{d^2}$, so $18 = \frac{3k}{4}$ and $k = 24$. Then $F = \frac{24 \cdot 5}{16} = 7.5$.
14. 15 liters. Solve $\frac{15+x}{60+x} = 0.4$: $15 + x = 24 + 0.4x$, so $0.6x = 9$ and $x = 15$.
15. $n = 225$. Solve $1800 + 12n = 20n$, so $8n = 1800$ and $n = 225$.

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