



Logarithms and Exponential Equations

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

In this booster

23.1 Logarithms as Inverses of Exponentials

23.2 Properties of Logarithms

23.3 Solving Exponential and Logarithmic Equations

23.4 Applications of Logs and e

Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

23.1 Logarithms as Inverses of Exponentials

A logarithm answers the question: what exponent do I need? Learn to read, evaluate, and convert logs, find their domains, and sketch their graphs as mirror images of exponentials.

Why This Matters

Exponents tell you what happens when you multiply the same number again and again. Logarithms answer the opposite question: how many times did we multiply? If a video gets doubled in views every day, an exponent tells you the views after 10 days. A logarithm tells you how many days it takes to reach one million views. Scientists, engineers, and bankers ask this second kind of question all the time. Students get stuck on logs for three main reasons. First, the notation $\log_b x$ looks strange and nobody tells them it is just a **question in disguise**. Second, they mix up which number is the base, which is the exponent, and which is the result. Third, they forget that a log can only be taken of a positive number. This section fixes all three. By the end you will read $\log_2 8$ as the question "2 to what power gives 8?" and answer it in seconds.

Core Ideas

Start with something you already know: $2^3 = 8$. This sentence has three parts: the **base** 2, the **exponent** 3, and the **result** 8. A logarithm lets you ask for the exponent when you know the base and the result. We write $\log_2 8 = 3$ and read it as "the log base 2 of 8 is 3." Nothing new has happened. It is the same fact written in a different form.

Definition of a Logarithm

For $b > 0$, $b \neq 1$, and $x > 0$:

$$\log_b x = y \text{ means exactly the same as } b^y = x.$$

In words: $\log_b x$ is the **exponent** you put on b to get x .

The memory trick is to read the log as a question and answer it with the exponent. The base of the log stays the base of the power. The answer to the log is the exponent. The number inside the log is the result. This is why logs are called the **inverse** of exponentials: the log undoes the exponential, the same way subtraction undoes addition.

Exponential form	Logarithmic form	Read as
$3^4 = 81$	$\log_3 81 = 4$	3 to the 4 is 81
$10^2 = 100$	$\log_{10} 100 = 2$	10 to the 2 is 100
$2^{-3} = \frac{1}{8}$	$\log_2 \frac{1}{8} = -3$	a negative exponent gives a fraction
$25^{1/2} = 5$	$\log_{25} 5 = \frac{1}{2}$	a root is a fractional exponent

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Two logs are so common that they have their own names. The **common logarithm** has base 10 and is written $\log x$ with no base shown. The **natural logarithm** has base $e \approx 2.718$ and is written $\ln x$. Your calculator has buttons for both. Everything we learn about $\log_b$ works for these two as well.

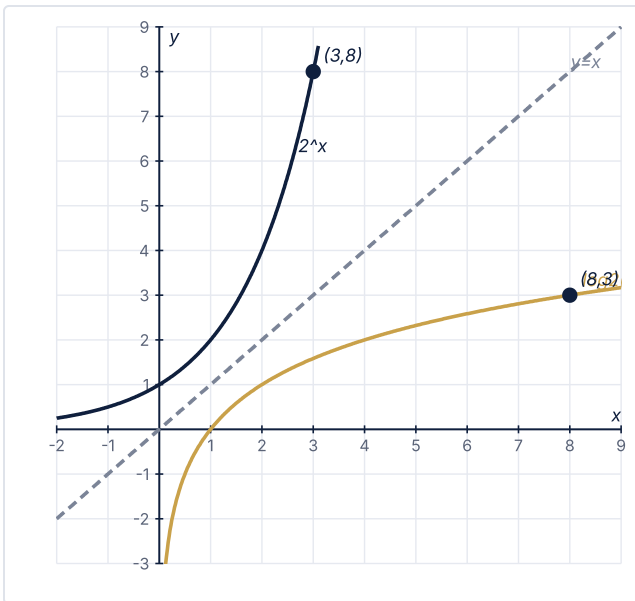
Facts That Follow From the Definition

$\log_b 1 = 0$ because $b^0 = 1$. $\log_b b = 1$ because $b^1 = b$.

$\log_b(b^x) = x$ and $b^{\log_b x} = x$. The log and the power cancel each other.

You can only take the log of a **positive** number. $\log_b 0$ and $\log_b(-5)$ do not exist, because no power of a positive base is zero or negative.

Because the log is the inverse of the exponential, their graphs are mirror images across the line $y = x$. Every point (a, c) on $y = 2^x$ becomes the point (c, a) on $y = \log_2 x$. For example, $(3, 8)$ is on the exponential graph, so $(8, 3)$ is on the log graph. Look at the picture and find these pairs.



The graphs of $y = 2^x$ (navy) and $y = \log_2 x$ (gold) are mirror images across the dashed line $y = x$.

Graph of $y = \log_b x$ (with $b > 1$)

Domain: $x > 0$. Range: all real numbers.

It passes through $(1, 0)$ and $(b, 1)$. It rises slowly and never touches the **vertical asymptote** $x = 0$.

For $y = \log_b(x - h) + k$, the asymptote moves to $x = h$ and the point $(1, 0)$ moves to $(h + 1, k)$.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2 x$	-2	-1	0	1	2	3

Notice how slowly the log grows. To go from output 2 to output 3 the input must double from 4 to 8, and the next step needs 16. Exponentials explode, so their inverses creep.

Worked Examples

EXAMPLE 1: EVALUATE A LOGARITHM

Evaluate (a) $\log_3 81$, (b) $\log_5 \frac{1}{25}$, (c) $\log_4 2$.

SOLUTION

Ask: base to what power gives the number inside?

(a) $3^? = 81$. Since $3^4 = 81$, the answer is 4.

(b) $5^? = \frac{1}{25}$. Since $5^2 = 25$ and a negative exponent flips the fraction, $5^{-2} = \frac{1}{25}$. The answer is -2.

(c) $4^? = 2$. Since $\sqrt{4} = 2$ and a square root is the power $\frac{1}{2}$, the answer is $\frac{1}{2}$.

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EXAMPLE 2: CONVERT BETWEEN FORMS

(a) Write $\log_7 343 = 3$ in exponential form. (b) Write $2^{-3} = \frac{1}{8}$ in logarithmic form.

SOLUTION

(a) The base is 7, the log value 3 is the exponent, and 343 is the result: $7^3 = 343$.

(b) The base is 2, the exponent is -3 , and the result is $\frac{1}{8}$: $\log_2 \frac{1}{8} = -3$.

Check (a): $7 \cdot 7 \cdot 7 = 343$. Good.

EXAMPLE 3: SOLVE FOR AN UNKNOWN

Solve (a) $\log_x 64 = 3$ and (b) $\log_2 x = -4$.

SOLUTION

(a) Convert to exponential form: $x^3 = 64$. The base must be positive, so $x = 4$.

(b) Convert: $2^{-4} = x$. So $x = \frac{1}{2^4} = \frac{1}{16}$.

Notice that in (a) the unknown is the base, and in (b) it is the result. Converting to exponential form handles both.

EXAMPLE 4: EVALUATE WHEN THE ANSWER IS A FRACTION

Evaluate $\log_8 32$.

SOLUTION

Let $y = \log_8 32$, so $8^y = 32$.

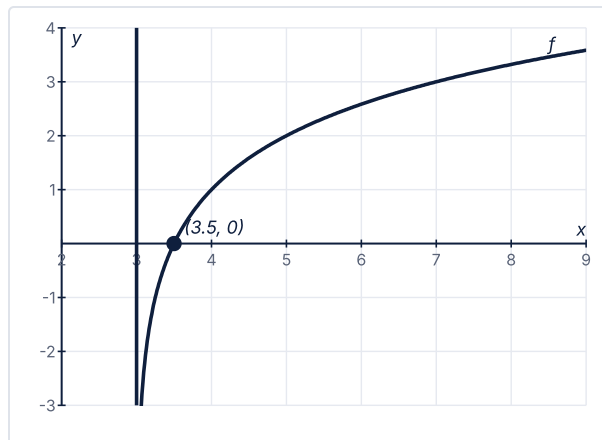
Write both sides with base 2: $8 = 2^3$ and $32 = 2^5$. Then $(2^3)^y = 2^5$, so $2^{3y} = 2^5$.

Equal bases mean equal exponents: $3y = 5$, so $y = \frac{5}{3}$.

Check: $8^{5/3} = (\sqrt[3]{8})^5 = 2^5 = 32$.

EXAMPLE 5: DOMAIN, ASYMPTOTE, AND INTERCEPT

For $f(x) = \log_2(x - 3) + 1$ find the domain, the vertical asymptote, and the x -intercept.



The vertical line $x = 3$ is the asymptote.

SOLUTION

The inside must be positive: $x - 3 > 0$, so the domain is $x > 3$.

The asymptote is where the inside equals zero: $x = 3$.

For the x -intercept set $f(x) = 0$: $\log_2(x - 3) = -1$. Convert: $x - 3 = 2^{-1} = \frac{1}{2}$, so $x = 3.5$.

The graph below confirms that the curve crosses the x -axis at 3.5.

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EXAMPLE 6: FIND AN INVERSE FUNCTION

Find the inverse of $f(x) = 4^{x-1} + 3$, then check with $f(2)$.

SOLUTION

Write $y = 4^{x-1} + 3$ and swap x and y : $x = 4^{y-1} + 3$.

Isolate the power: $x - 3 = 4^{y-1}$.

Convert to a log: $y - 1 = \log_4(x - 3)$, so $f^{-1}(x) = \log_4(x - 3) + 1$.

Check: $f(2) = 4^1 + 3 = 7$. Then $f^{-1}(7) = \log_4 4 + 1 = 1 + 1 = 2$. The inverse sends 7 back to 2.

Coach's Playbook

- 1 Read the log as a question.** For $\log_b x$ say to yourself: "b to what power is x?"
 - 2 Rewrite as an exponent.** If you are stuck, convert to $b^y = x$ and solve with what you know about powers.
 - 3 Use common bases.** Write numbers as powers of 2, 3, 5, or 10 so you can match exponents, as in $8 = 2^3$.
 - 4 Check by going backward.** Raise the base to your answer. If you do not land on the number inside the log, something is off.
 - 5 Check the domain.** For any log of an expression, set the inside greater than zero before you trust an answer.
- ◆ Memorize the powers of 2 up to 2^{10} , powers of 3 up to 3^5 , and powers of 5 up to 5^4 . Most logs you will see use these.
 - ◆ Anchors: $\log_b 1 = 0$ and $\log_b b = 1$ for every valid base. These two give instant points on any log graph.
 - ◆ If the number inside is bigger than the base, the log is greater than 1. If it is between 0 and 1, the log is negative.

Common traps

1. Swapping the roles of base and exponent, for example writing $\log_2 8 = 2^3$ instead of $2^3 = 8$. 2. Thinking $\log_b 0 = 0$. It does not exist. 3. Forgetting that the base of a log must be positive and not equal to 1, and that the inside must be positive. 4. Treating $\log_2 32$ as $32 \div 2 = 16$. The log is an exponent, not a quotient.

Mistake Clinic**Spot the error 1**

A student writes: " $\log_4 16 = 4$ because $16 \div 4 = 4$." Where did this go wrong? The student divided instead of asking for an exponent. The correct question is "4 to what power is 16?" Since $4^2 = 16$, the answer is $\log_4 16 = 2$. Coincidences like $16 \div 4 = 4$ can fool you, so always check by raising the base: $4^4 = 256$, not 16.

Spot the error 2

A student solves $\log_x 25 = 2$ and writes $x = 2^{25}$. The student turned the problem into a power of the wrong number. The base of the log is x , so the exponential form is $x^2 = 25$. Then $x = 5$ (the base must be positive, so -5 is rejected). Rule: the base of the log stays the base of the power.

Spot the error 3

A student says the domain of $f(x) = \log_3(x + 6)$ is $x > 6$. The student ignored the sign. The inside must be positive: $x + 6 > 0$, so $x > -6$. A quick test shows the error: at $x = 0$ we get $\log_3 6$, which exists, but $0 > 6$ is false. Always solve the inequality, not just copy the number.

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Real-World Connection

A music app notes that each octave doubles a pitch's frequency. Starting at a note of 110 hertz, how many octaves up is a note of 880 hertz? The number of octaves is $\log_2 \frac{880}{110} = \log_2 8 = 3$ octaves. Musicians use a log scale without ever saying the word.

Challenge Corner

Solve $\log_3(x^2 - 2x) = 1$ and check both answers. Solution: convert to exponential form, $x^2 - 2x = 3^1 = 3$. Then $x^2 - 2x - 3 = 0$, so $(x - 3)(x + 1) = 0$ and $x = 3$ or $x = -1$. Check the inside: for $x = 3$, $9 - 6 = 3 > 0$. For $x = -1$, $1 + 2 = 3 > 0$. Both are valid. The solutions are $x = 3$ and $x = -1$.

Practice**PRACTICE Logarithms as Inverses of Exponentials**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Which equation is the exponential form of $\log_5 125 = 3$?
 (A) $3^5 = 125$ (B) $5^3 = 125$ (C) $125^5 = 3$ (D) $5^{125} = 3$
- What is the value of $\log_2 32$?
 (A) 16 (B) 10 (C) 5 (D) 4
- What is the value of $\log_9 3$?
 (A) 2 (B) 3 (C) 27 (D) $\frac{1}{2}$
- What is the value of $\log_{1/2} 8$?
 (A) -3 (B) 3 (C) $-\frac{1}{3}$ (D) $\frac{1}{3}$
- If $\log_x 49 = 2$ and $x > 0$, what is x ?
 (A) 98 (B) 7 (C) 24.5 (D) 2^{49}
- What is the domain of $f(x) = \log_3(5 - x)$?
 (A) $x > 5$ (B) $x > -5$ (C) $x < 5$ (D) $x \neq 5$
- What is the vertical asymptote of $g(x) = \log_2(x + 4) - 3$?
 (A) $x = -4$ (B) $x = 4$ (C) $x = -3$ (D) $x = 3$
- Which point lies on the graph of $y = \log_4 x$?
 (A) (2, 16) (B) (4, 0) (C) (0, 1) (D) (16, 2)
- What is the value of $\ln e^3 + \log 100$?
 (A) 13 (B) 5 (C) 2 (D) 6
- Evaluate $\log_3 \frac{1}{27}$.
- Solve $\log_2(x + 1) = 5$.
- Evaluate $\log_{16} 8$. Give a fraction.
- Find the inverse function of $f(x) = 2^x - 5$.
- For $f(x) = \log_3(x - 2) + 4$, find the exact x -intercept.
- Solve $\log_5(x^2 - 24) = 2$. List every solution.

23.2 Properties of Logarithms

Every log property is an exponent rule in disguise. Learn the product, quotient, and power rules, change of base, and how to expand and condense log expressions without the usual traps.

Why This Matters

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Logs turn hard operations into easy ones. Before calculators existed, scientists multiplied huge numbers by adding their logarithms, because adding is far easier than multiplying. That same idea powers slide rules, decibel scales, and the way computers measure how fast an algorithm grows. In algebra, the properties of logs are the tools that let you solve equations where the unknown is stuck in an exponent.

Students struggle here because the three rules look alike and are easy to blur. The most common slips are writing $\log(a + b) = \log a + \log b$, dropping a power instead of moving it in front, and forgetting that the rules only work when everything inside is positive. In this section you will learn each rule from the exponent rule behind it, so you do not have to memorize blindly.

Core Ideas

Every log property is an exponent rule in disguise. Remember that a log is an exponent. When you multiply powers of the same base, you **add** the exponents: $2^3 \cdot 2^2 = 2^5$. In log language, that says $\log_2 8 + \log_2 4 = \log_2 32$, since $3 + 2 = 5$. Notice that the numbers inside the logs were multiplied: $8 \cdot 4 = 32$.

The Three Log Properties

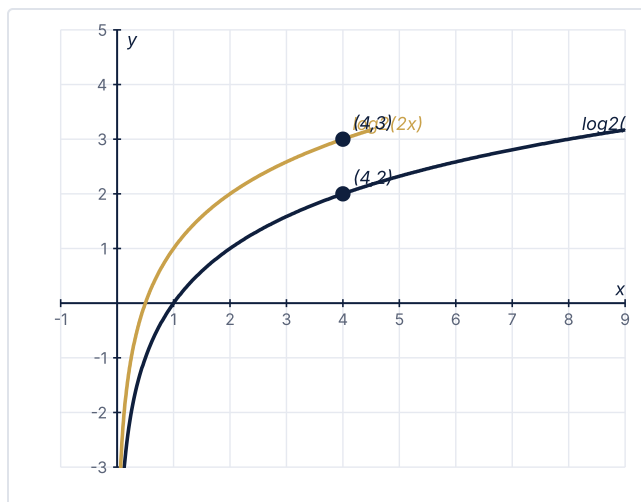
Let $b > 0$, $b \neq 1$, and let $M, N > 0$. Then:

Product rule: $\log_b(MN) = \log_b M + \log_b N$

Quotient rule: $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$

Power rule: $\log_b(M^p) = p \log_b M$ for any real number p .

The quotient rule comes from dividing powers, where exponents subtract. The power rule comes from raising a power to a power, where exponents multiply: $(2^3)^4 = 2^{12}$. A picture helps with the product rule. The graph of $y = \log_2(2x)$ is the graph of $y = \log_2 x$ moved up 1 unit, because $\log_2(2x) = \log_2 2 + \log_2 x = 1 + \log_2 x$.



At $x = 4$ the gold curve is exactly 1 unit above the navy curve, matching $\log_2(2x) = 1 + \log_2(x)$.

Statement	Is it true?	Why
$\log_b(MN) = \log_b M + \log_b N$	Yes	product rule
$\log_b(M + N) = \log_b M + \log_b N$	No	there is no rule for a sum inside
$\log_b(M^p) = p \log_b M$	Yes	power rule
$(\log_b M)^p = p \log_b M$	No	the power is on the whole log, not the inside
$\frac{\log_b M}{\log_b N} = \log_b M - \log_b N$	No	a quotient of logs is not a log of a quotient

Two more facts are used constantly. The first is the pair $\log_b b^x = x$ and $b^{\log_b x} = x$ from the last section. The second is the **change of base** formula. Your calculator only has $\log$ (base 10) and $\ln$ (base e), so to evaluate a log with any other base you convert.

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Change of Base Formula

For any valid bases:

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

Usually $a = 10$ or $a = e$, so $\log_b x = \frac{\ln x}{\ln b}$.

A useful special case: $\log_b a = \frac{1}{\log_a b}$.

Expanding means using the rules to break one log of a product, quotient, or power into several simple logs. **Condensing** means the reverse: combine several logs into a single log. When you expand, powers come down. When you condense, coefficients go back up as powers.

Task	Before	After
Expand	$\log_3 \frac{x^2 y}{5}$	$2 \log_3 x + \log_3 y - \log_3 5$
Condense	$\log_2 6 + 3 \log_2 x$	$\log_2 (6x^3)$
Condense	$2 \log 5 - \log 5$	$\log 5$

Worked Examples

EXAMPLE 1: EXPAND A LOGARITHM

Expand $\log_2 \frac{8x^3}{y}$ as completely as possible.

SOLUTION

Quotient rule: $\log_2 (8x^3) - \log_2 y$.

Product rule on the top: $\log_2 8 + \log_2 x^3 - \log_2 y$.

Power rule: $\log_2 x^3 = 3 \log_2 x$. Also $\log_2 8 = 3$.

Result: $3 + 3 \log_2 x - \log_2 y$.

EXAMPLE 2: CONDENSE INTO A SINGLE LOGARITHM

Write $3 \log_5 2 + \log_5 x - 2 \log_5 y$ as one logarithm.

SOLUTION

Move coefficients up as powers: $\log_5 2^3 + \log_5 x - \log_5 y^2$.

Simplify $2^3 = 8$: $\log_5 8 + \log_5 x - \log_5 y^2$.

Added logs multiply and subtracted logs divide: $\log_5 \frac{8x}{y^2}$.

EXAMPLE 3: EVALUATE USING THE RULES, NO CALCULATOR

Evaluate (a) $\log_6 4 + \log_6 9$ and (b) $\log_2 40 - \log_2 5$.

SOLUTION

(a) Add logs means multiply inside: $\log_6 (4 \cdot 9) = \log_6 36$. Since $6^2 = 36$, the value is 2.

(b) Subtract logs means divide inside: $\log_2 \frac{40}{5} = \log_2 8 = 3$.

Check (b): $2^3 = 8$. Good.

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EXAMPLE 4: USE GIVEN VALUES

Suppose $\log_b 2 = 0.35$ and $\log_b 3 = 0.55$. Find (a) $\log_b 12$ and (b) $\log_b \frac{3}{8}$.

SOLUTION

(a) Break 12 into primes: $12 = 2^2 \cdot 3$. So $\log_b 12 = 2(0.35) + 0.55 = 1.25$.

(b) $\frac{3}{8} = \frac{3}{2^3}$. So $\log_b \frac{3}{8} = \log_b 3 - 3 \log_b 2 = 0.55 - 1.05 = -0.5$.

The negative result makes sense: $\frac{3}{8}$ is less than 1.

EXAMPLE 5: CHANGE OF BASE

(a) Evaluate $\log_4 32$ exactly. (b) Approximate $\log_5 40$ to three decimal places.

SOLUTION

(a) Convert to base 2: $\log_4 32 = \frac{\log_2 32}{\log_2 4} = \frac{5}{2}$.

(b) Use natural logs: $\log_5 40 = \frac{\ln 40}{\ln 5} \approx \frac{3.6889}{1.6094} \approx 2.292$.

Sense check: $5^2 = 25$ and $5^3 = 125$, so the answer should lie between 2 and 3. It does.

EXAMPLE 6: CONDENSE AND WATCH THE DOMAIN

Simplify $\log_2(x^2 - 9) - \log_2(x + 3)$ and state for which x the original expression is defined.

SOLUTION

Quotient rule: $\log_2 \frac{x^2 - 9}{x + 3}$.

Factor: $x^2 - 9 = (x - 3)(x + 3)$, so the fraction reduces to $x - 3$. The result is $\log_2(x - 3)$.

Domain of the original: both $x^2 - 9 > 0$ and $x + 3 > 0$. The second gives $x > -3$. The first gives $x < -3$ or $x > 3$. Together: $x > 3$.

So the simplification is valid for $x > 3$.

Coach's Playbook

- 1 Name the operation inside.** Product, quotient, or power? Pick the rule that matches.
 - 2 Expand: top to bottom.** Split the fraction first, then split products, then bring down powers.
 - 3 Condense: coefficients first.** Turn every coefficient into a power before you combine anything.
 - 4 Group adds and subtracts.** Positive logs go in the numerator, subtracted logs go in the denominator.
 - 5 Check with a number.** Plug in a simple value, such as $x = 2$, into both forms to see that they match.
- ◆ Write numbers as prime factors. If you know $\log 2$ and $\log 3$, you can build $\log 6$, $\log 12$, $\log 18$, $\log 1.5$ and many more.
 - ◆ A log of a root is a fractional power: $\log_b \sqrt{x} = \frac{1}{2} \log_b x$.
 - ◆ Change of base is the answer whenever your calculator lacks the base. Remember to divide the log of the number by the log of the base, in that order.

Common traps

1. Writing $\log(a + b) = \log a + \log b$. Sums inside a log do not split.
2. Writing $\log_b M \div \log_b N = \log_b(M - N)$. Dividing two logs is not the same as subtracting them.
3. Leaving a coefficient in front when condensing, for example $2 \log x + \log y = \log(2xy)$. The 2 must become a power on x .
4. Flipping the change-of-base fraction, giving $\log_b x = \ln b \div \ln x$.

Mistake Clinic

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Spot the error 1

A student simplifies $\log_3 9 + \log_3 27$ as $\log_3(9 + 27) = \log_3 36$. Where is the error? Adding logs means multiplying the insides, not adding them. Correct work: $\log_3(9 \cdot 27) = \log_3 243 = 5$. Quick check: $\log_3 9 = 2$ and $\log_3 27 = 3$, and $2 + 3 = 5$.

Spot the error 2

A student expands $\log_2(x^3y)$ and writes $\log_2 x^3 \cdot \log_2 y$. The student multiplied logs instead of adding them. The product rule says the log of a product is a **sum**: $\log_2(x^3y) = 3\log_2 x + \log_2 y$. A product of logs is a completely different quantity.

Spot the error 3

A student computes $\log_3 20$ on a calculator as $\ln 3 \div \ln 20 \approx 0.367$. The fraction is upside down. The number inside goes on top: $\log_3 20 = \frac{\ln 20}{\ln 3} \approx \frac{2.996}{1.099} \approx 2.727$. Check: $3^2 = 9$ and $3^3 = 27$, so the answer must be between 2 and 3, not below 1.

Real-World Connection

A sound technician knows that a signal gets 10 times stronger in the first stage of an amplifier and 100 times stronger in the second. Because $\log 10 = 1$ and $\log 100 = 2$, the combined log gain is $1 + 2 = 3$, which is $\log 1000$. Engineers add gains in this log form rather than multiplying huge factors.

Challenge Corner

Show that $\log_a b \cdot \log_b a = 1$ and use it to evaluate $\log_8 2 \cdot \log_2 8$. Solution: by change of base, $\log_a b = \frac{\ln b}{\ln a}$ and $\log_b a = \frac{\ln a}{\ln b}$. Multiplying, the logs cancel, giving 1. Directly: $\log_2 8 = 3$ and $\log_8 2 = \frac{1}{3}$, and $\frac{1}{3} \cdot 3 = 1$.

Practice**PRACTICE Properties of Logarithms**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is the value of $\log_3 4 + \log_3 \frac{9}{4}$?
(A) 2 (B) 9 (C) $\frac{25}{4}$ (D) 3
- Which expression equals $\log_5(x^3y)$ for $x, y > 0$?
(A) $3 \log_5 x \cdot \log_5 y$ (B) $3 \log_5 (x + y)$ (C) $3 \log_5 x + \log_5 y$
(D) $\log_5 x + 3 \log_5 y$
- What is the value of $\log_2 20 - \log_2 5$?
(A) 4 (B) 2 (C) $\log_2 15$ (D) 3
- Which is equivalent to $2\log_a 5 + \log_a 3$?
(A) $\log_a 13$ (B) $\log_a 30$ (C) $\log_a 225$ (D) $\log_a 75$
- If $\log_b 2 = p$ and $\log_b 5 = q$, which expression equals $\log_b 20$?
(A) $2p + q$ (B) $2pq$ (C) $p + 2q$ (D) $2(p + q)$
- Which expression is equal to $\log_7 30$?
(A) $\frac{\ln 7}{\ln 30}$ (B) $\frac{\ln 30}{\ln 7}$ (C) $\ln 30 - \ln 7$ (D) $\ln 30 \cdot \ln 7$
- What is the exact value of $\log_9 27$?
(A) 3 (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ (D) 18
- For $x > 0$, which expression is equal to $\log_2(x^4)$?
(A) $4 \log_2 x$ (B) $(\log_2 x)^4$ (C) $x \log_2 4$ (D) $\log_2 x + 4$

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9. Given $\log 2 \approx 0.301$ and $\log 3 \approx 0.477$, what is $\log 12$ to three decimal places?
 (A) 0.778 (B) 1.255 (C) 0.144 (D) 1.079
10. Evaluate $\log_6 9 + \log_6 4$.
11. Expand completely: $\log_2 \frac{16\sqrt{x}}{y^3}$. Assume $x, y > 0$.
12. Write as a single logarithm and simplify: $4\log_3 2 - 2\log_3 6 + \log_3 9$.
13. Evaluate $\log_{125} 25$ as a fraction.
14. Suppose $\log_b 3 = 0.5$ and $\log_b 5 = 0.8$. Find $\log_b \frac{25}{\sqrt{3}}$.
15. If $\log_2 x = 9$, find $\log_8 x$.

23.3 Solving Exponential and Logarithmic Equations

Get the unknown out of the exponent or out of the log. Learn the same-base method, taking logs of both sides, quadratic-form equations, and how to catch extraneous solutions.

Why This Matters

Every time you ask "how long until my savings reach a goal?" or "when will this population hit a million?" you are solving an equation with the unknown in an exponent. Radioactive dating, loan payoffs, spread of news online, and the cooling of a hot drink are all modeled this way. The skill you build here is the bridge from simple formulas to real forecasting.

Students get stuck for two reasons. First, they try to solve $3^x = 20$ with ordinary algebra and divide or subtract the base. Second, with log equations they forget to check the answer, and a solution that looks perfect turns out to be impossible. These impossible answers are called **extraneous solutions**. Learning to expect and test for them is what separates careful solvers from careless ones.

Core Ideas

There are two big families of equations. In an **exponential equation** the unknown is in an exponent, such as $2^{x+1} = 32$. In a **logarithmic equation** the unknown is inside a log, such as $\log_3(x+4) = 2$. The strategy for each is to get the unknown out of the exponent or out of the log, using the fact that exponentials and logs undo each other.

Method 1: Same Base

If $b^m = b^n$ with $b > 0$ and $b \neq 1$, then $m = n$.

If $\log_b M = \log_b N$ with $M, N > 0$, then $M = N$.

Use this whenever both sides can be written with the same base, such as $9 = 3^2$ and $27 = 3^3$.

Method 2: Take a Logarithm of Both Sides

If the bases cannot be matched, isolate the power and take $\ln$ (or $\log$) of both sides. Then use the power rule:

$$b^x = c \Rightarrow x \ln b = \ln c \Rightarrow x = \frac{\ln c}{\ln b}.$$

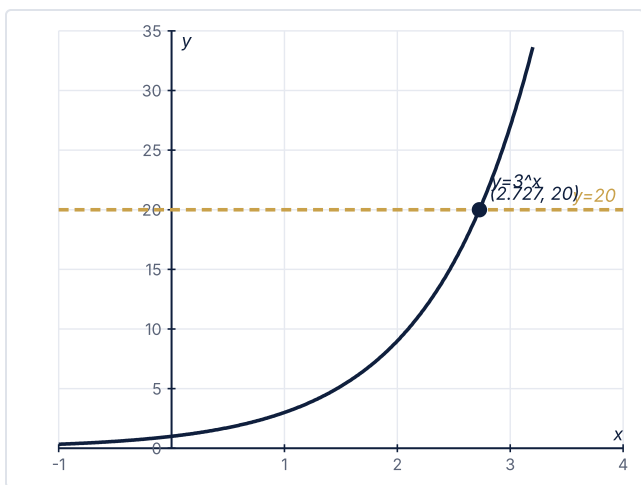
Isolate the exponential part **before** taking logs. For $a \cdot b^x = c$, first divide by a .

Here is a picture of Method 2. Solving $3^x = 20$ means finding where the curve $y = 3^x$ reaches the height 20. Since $3^2 = 9$ and $3^3 = 27$, the answer must lie between 2 and 3. The formula gives $x = \frac{\ln 20}{\ln 3} \approx 2.727$, and the graph agrees.

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The curve $y = 3^x$ meets the line $y = 20$ at $x = 2.727$.

Method 3: Solving Log Equations

Combine logs into one log on each side using the product, quotient, and power rules.

Then either set the insides equal (if both sides have a log with the same base) or convert to exponential form (if one side is a number).

Always check each answer in the original equation. Every log argument must be positive.

Why do extraneous solutions appear? The rules like $\log M + \log N = \log(MN)$ work only when M and N are positive. After combining, the equation $x(x - 2) = 8$ allows negative x values that the original equation never permitted. Algebra gave a candidate, but the original log cannot accept it. Rejecting such a value is not a mistake. It is a required step.

Equation type	First move	Then
$b^{f(x)} = b^{g(x)}$	set exponents equal	solve $f(x) = g(x)$
$a \cdot b^x = c$	divide by a	take $\ln$ of both sides
$\log_b f(x) = k$	write $f(x) = b^k$	solve, then check $f(x) > 0$
$\log_b f + \log_b g = k$	combine to $\log_b(fg)$	convert, solve, check both insides
$b^{2x} + p b^x + q = 0$	let $u = b^x$	solve quadratic, then $x = \log_b u$ (need $u > 0$)

Worked Examples

EXAMPLE 1: SAME-BASE METHOD

Solve (a) $2^{3x-1} = 32$ and (b) $9^x = 27^{x-1}$.

SOLUTION

(a) Write $32 = 2^5$. Then $3x - 1 = 5$, so $3x = 6$ and $x = 2$.

(b) Write both sides as powers of 3: $(3^2)^x = (3^3)^{x-1}$, which is $3^{2x} = 3^{3x-3}$.

Set exponents equal: $2x = 3x - 3$, so $x = 3$.

Check (b): $9^3 = 729$ and $27^2 = 729$. Good.

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EXAMPLE 2: TAKE A LOGARITHM

Solve $3^x = 20$ and give the answer to three decimal places.

SOLUTION

The bases cannot be matched, so take $\ln$ of both sides: $\ln 3^x = \ln 20$.

Power rule: $x \ln 3 = \ln 20$.

Divide: $x = \frac{\ln 20}{\ln 3} \approx \frac{2.9957}{1.0986} \approx 2.727$.

This matches the graph above, where the curve meets $y = 20$.

EXAMPLE 3: ISOLATE FIRST

Solve $3(1.5)^x + 4 = 40$ to three decimal places.

SOLUTION

Subtract 4: $3(1.5)^x = 36$.

Divide by 3: $(1.5)^x = 12$.

Take $\ln$: $x \ln 1.5 = \ln 12$, so $x = \frac{\ln 12}{\ln 1.5} \approx \frac{2.4849}{0.4055} \approx 6.129$.

Check: $1.5^{6.129} \approx 12$, and $3(12) + 4 = 40$. Good.

EXAMPLE 4: QUADRATIC FORM

Solve $4^x - 6 \cdot 2^x + 8 = 0$.

SOLUTION

Since $4^x = (2^x)^2$, let $u = 2^x$. The equation becomes $u^2 - 6u + 8 = 0$.

Factor: $(u - 2)(u - 4) = 0$, so $u = 2$ or $u = 4$.

Replace u : $2^x = 2$ gives $x = 1$, and $2^x = 4$ gives $x = 2$.

Check $x = 2$: $16 - 24 + 8 = 0$. Good. Both solutions work, because 2^x is always positive.

EXAMPLE 5: SIMPLE LOG EQUATIONS

Solve (a) $\log_3(2x + 1) = 2$ and (b) $\log_4(x + 3) = \log_4(2x - 5)$.

SOLUTION

(a) Convert: $2x + 1 = 3^2 = 9$, so $x = 4$. Check: $2(4) + 1 = 9 > 0$.

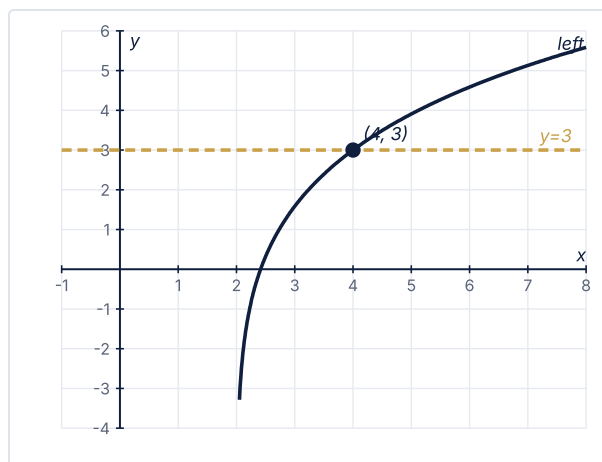
(b) Same base on both sides, so the insides are equal: $x + 3 = 2x - 5$, giving $x = 8$.

Check (b): $x + 3 = 11 > 0$ and $2x - 5 = 11 > 0$. The solution is valid.

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EXAMPLE 6: AN EXTRANEOUS SOLUTIONSolve $\log_2 x + \log_2(x - 2) = 3$.The left side exists only for $x > 2$ and equals 3 at $x = 4$.**SOLUTION**Combine: $\log_2[x(x - 2)] = 3$.Convert: $x(x - 2) = 2^3 = 8$, so $x^2 - 2x - 8 = 0$.Factor: $(x - 4)(x + 2) = 0$, so the candidates are $x = 4$ and $x = -2$.Check $x = -2$: $\log_2(-2)$ does not exist. Reject it.Check $x = 4$: $\log_2 4 + \log_2 2 = 2 + 1 = 3$. Valid.The only solution is $x = 4$. The graph shows that the left side reaches 3 just once.**Coach's Playbook**

- 1 Identify the type.** Is the unknown in an exponent or inside a log?
- 2 Isolate.** Get the exponential alone (divide, subtract) or the log alone before you convert.
- 3 Choose the tool.** Same base: equate exponents. Different bases: take logs. Log equation: convert or equate insides.
- 4 Solve the resulting equation.** It may be linear or quadratic.
- 5 Check every candidate.** Put it back into the original equation. Reject anything that makes a log argument zero or negative.

- ◆ When you take logs, use the log that is most convenient. $\ln$ works for every base, and it makes answers like $x = \frac{\ln 20}{\ln 3}$ easy to type.
- ◆ Keep exact answers (with $\ln$) until the end. Round only the final step.
- ◆ An exponential expression such as 2^x is **never** zero or negative. If you get $2^x = -4$, the equation has no solution.
- ◆ Estimate first: bracket the answer between two integers using nearby powers.

Common traps

1. Dividing the base: from $5 \cdot 2^x = 40$ writing $10^x = 40$. The 5 multiplies the power and must be divided out of it. 2. Taking the log of one term of a sum, such as $\ln(2^x + 3) = x \ln 2 + \ln 3$. That is not valid. 3. Skipping the domain check on log equations. 4. Rounding early. Using $\ln 3 \approx 1.1$ in the middle of a problem gives a wrong second decimal.

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Spot the error 1

A student solves $5 \cdot 2^x = 40$ and writes $10^x = 40$, so $x = \log 40 \approx 1.602$. The 5 and the 2 cannot be multiplied, because 5 multiplies the whole power and 2 is the base. Correct: divide by 5 first, $2^x = 8$, so $x = 3$. Check: $5 \cdot 8 = 40$.

Spot the error 2

A student solves $\log_3 x + \log_3(x + 6) = 3$ and finds $x^2 + 6x - 27 = 0$, then answers " $x = 3$ or $x = -9$." The student never checked the domain. For $x = -9$ both $\log_3(-9)$ and $\log_3(-3)$ are undefined, so -9 is extraneous. Only $x = 3$ works: $\log_3 3 + \log_3 9 = 1 + 2 = 3$.

Spot the error 3

A student solves $\ln(x + 5) = \ln x + \ln 2$ by writing $x + 5 = x + 2$ and concludes there is no solution. The student wrongly split an equal sign across a sum. The right side combines to $\ln(2x)$, so $x + 5 = 2x$ and $x = 5$. Check: $\ln 10 = \ln 5 + \ln 2$. True. Use the product rule on the right side before comparing insides.

Real-World Connection

A savings account grows by 5 percent a year. How many years until a deposit doubles? Solve $1.05^t = 2$. Then $t = \frac{\ln 2}{\ln 1.05} \approx \frac{0.6931}{0.0488} \approx 14.2$ years. Banks and investors use exactly this log step to turn a growth rate into a waiting time.

Challenge Corner

Solve $5^{x+1} = 3^{2x}$. Give the exact answer and a decimal to three places. Solution: take $\ln$ of both sides: $(x + 1) \ln 5 = 2x \ln 3$. Expand: $x \ln 5 + \ln 5 = 2x \ln 3$. Gather the x terms: $\ln 5 = x(2 \ln 3 - \ln 5)$. So $x = \frac{\ln 5}{2 \ln 3 - \ln 5} = \frac{\ln 5}{\ln(9/5)} \approx \frac{1.6094}{0.5878} \approx 2.738$.

Practice**PRACTICE Solving Exponential and Logarithmic Equations**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Solve $2^{x+3} = 64$.
(A) $x = 3$ (B) $x = 6$ (C) $x = 61$ (D) $x = 32$
- Solve $9^x = 27$.
(A) $x = 3$ (B) $x = \frac{2}{3}$ (C) $x = \frac{3}{2}$ (D) $x = 18$
- Solve $5^x = 40$. Round to three decimal places.
(A) $x = 8$ (B) $x \approx 0.436$ (C) $x = 35$ (D) $x \approx 2.292$
- Which is the exact solution of $2e^x = 14$?
(A) $x = \ln 7$ (B) $x = \ln 14$ (C) $x = \ln 12$ (D) $x = 7$
- Solve $\log_2(x - 1) = 4$.
(A) $x = 5$ (B) $x = 17$ (C) $x = 15$ (D) $x = 16$
- Solve $\log_4(x + 2) = \log_4(3x - 8)$.
(A) $x = -5$ (B) $x = 3$ (C) $x = 5$ (D) $x = \frac{5}{2}$
- Solve $\log_3 x + \log_3(x + 6) = 3$. Which statement is correct?
(A) $x = -9$ only (B) $x = 3$ or $x = -9$ (C) $x = 9$
(D) $x = 3$ only

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8. Solve $9^x - 4 \cdot 3^x + 3 = 0$.

(A) $x = 0$ or $x = 1$

(B) $x = 1$ or $x = 3$

(C) $x = 0$ only

(D) $x = 1$ only

9. Solve $4^x = 8^{x-1}$.

(A) $x = -3$

(B) $x = 3$

(C) $x = \frac{3}{2}$

(D) $x = 1$

10. Solve $3^{2x-1} = 81$.

11. Solve $4(1.2)^x = 50$. Round to three decimal places.

12. Solve $4^x - 10 \cdot 2^x + 16 = 0$.

13. Solve $\log_2(x+3) + \log_2(x-3) = 4$.

14. Find the exact solution of $\ln(3x+1) = 2$.

15. Solve $5^{x+1} = 3^{2x}$. Give the exact form and a decimal to three places.

23.4 Applications of Logs and e

Use logarithms to measure sound, acidity, and earthquakes, and use the number e for continuous growth, decay, doubling time, and half-life. Learn to pull the time out of the exponent.

Why This Matters

Exponential growth and decay are everywhere: money earning interest, a viral post spreading, medicine leaving your bloodstream, a hot drink cooling down. Logarithms are the other half of the story. They turn enormous ranges into small, readable numbers. A whisper and a jet engine differ by a factor of a trillion in energy, yet on the decibel scale they are just 20 and 140. That is the power of a log scale.

Students struggle with applications for two reasons. They are not sure which formula to use or what each letter means, and they get lost when the unknown is the time, which is stuck in an exponent. The cure is simple: **label the quantities, write the model, and use a log to pull the time down**. In this section you will do this for decibels, pH, earthquakes, continuous growth and decay, and doubling time.

Core Ideas

Some quantities cover such a huge range that we measure them with a **logarithmic scale**. On a log scale, adding a fixed amount means multiplying by a fixed factor. The three famous ones follow.

Three Log Scales

Sound level (decibels): $L = 10 \log \left(\frac{I}{I_0} \right)$, where I is intensity and $I_0 = 10^{-12}$ is the quietest sound we can hear.

Acidity (pH): $\text{pH} = -\log[H^+]$, where $[H^+]$ is the concentration of hydrogen ions.

Earthquake size: each step of 1 on the magnitude scale means the shaking amplitude is 10 times as large. A difference of d steps gives an amplitude ratio of 10^d .

Scale	If the number goes ...	Then the quantity ...
Decibels	up by 10	multiplies by 10
Decibels	up by 20	multiplies by 100
pH	down by 1	hydrogen ions multiply by 10
Magnitude	up by 2	amplitude multiplies by 100

Now for growth and decay. When growth happens in tiny steps all the time, the natural base is the number $e \approx 2.71828$. It appears when you compound more and more often. Look at what happens to $(1 + \frac{1}{n})^n$ as n grows. The values settle down toward e .

n (times compounded)	1	2	4	12	365
$(1 + \frac{1}{n})^n$	2.0000	2.2500	2.4414	2.6130	2.7146

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Continuous Growth and Decay

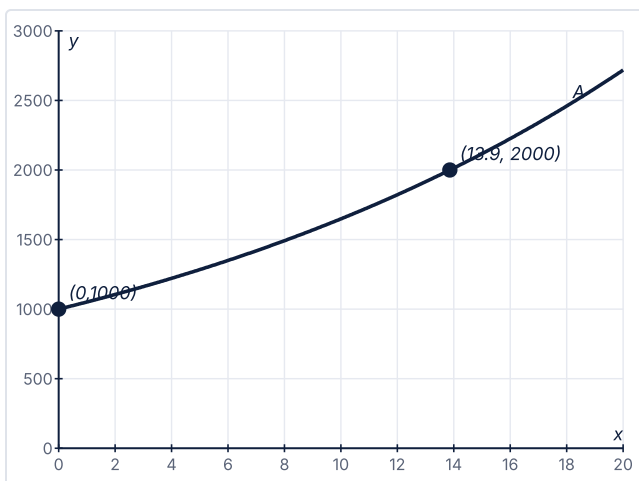
Continuous model: $A = A_0 e^{rt}$. Here A_0 is the starting amount, r is the continuous rate (positive for growth, negative for decay), and t is time.

Doubling time: growth at rate r doubles in $t = \frac{\ln 2}{r}$.

Half-life: decay at rate k (writing $A = A_0 e^{-kt}$) halves in $t = \frac{\ln 2}{k}$.

Compound interest for comparison: $A = P \left(1 + \frac{r}{n}\right)^{nt}$, which approaches Pe^{rt} as n grows.

The doubling-time rule comes from solving $2A_0 = A_0 e^{rt}$. Divide by A_0 to get $e^{rt} = 2$, then apply $\ln$: $rt = \ln 2$. The starting amount cancels, which is why doubling time does not depend on how much you start with. The graph shows an amount of 1000 growing at 5 percent per year. It doubles at $t = \ln 2 \div 0.05 \approx 13.9$ years.



Continuous growth at 5 percent: the amount doubles after about 13.9 years.

Worked Examples

EXAMPLE 1: DECIBELS

Use $L = 10 \log \left(\frac{I}{I_0} \right)$ with $I_0 = 10^{-12}$. (a) Find the level of a sound with intensity 10^{-5} . (b) A sound is 30 decibels louder than another. How many times as intense is it?

SOLUTION

(a) $\frac{I}{I_0} = \frac{10^{-5}}{10^{-12}} = 10^7$. Then $L = 10 \log 10^7 = 10 \cdot 7 = 70$ decibels.

(b) The difference in levels is $30 = 10 \log(\text{ratio})$, so $\log(\text{ratio}) = 3$ and the ratio is $10^3 = 1000$.

A 30-decibel jump means 1000 times the intensity.

EXAMPLE 2: PH

(a) Find the pH when $[H^+] = 3.2 \times 10^{-4}$. (b) Find $[H^+]$ when the pH is 8.2.

SOLUTION

(a) $\text{pH} = -\log(3.2 \times 10^{-4}) = -(\log 3.2 + \log 10^{-4}) = -(0.505 - 4) \approx 3.49$.

(b) Solve $8.2 = -\log[H^+]$. Then $\log[H^+] = -8.2$, so $[H^+] = 10^{-8.2} \approx 6.31 \times 10^{-9}$.

Check the direction: a pH below 7 is acid, and 3.49 is fairly acidic. A pH of 8.2 is slightly basic, with very few hydrogen ions.

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EXAMPLE 3: COMPARING EARTHQUAKES

How many times as large is the shaking amplitude of a magnitude 6.8 earthquake compared to a magnitude 5.6 earthquake?

SOLUTION

The difference in magnitude is $6.8 - 5.6 = 1.2$.

The amplitude ratio is $10^{1.2}$.

Calculate: $10^{1.2} \approx 15.85$.

The larger quake shakes the ground about 15.8 times as much. Since $10^1 = 10$ and $10^2 = 100$, a result between 10 and 100 makes sense.

EXAMPLE 4: CONTINUOUS COMPOUNDING

An account holds 2500 dollars at 4 percent interest compounded continuously. Find the balance after 10 years.

SOLUTION

Use $A = Pe^{rt}$ with $P = 2500$, $r = 0.04$, $t = 10$.

$rt = 0.04 \cdot 10 = 0.4$.

$A = 2500e^{0.4} \approx 2500(1.4918) \approx 3729.56$ dollars.

Compare with yearly compounding: $2500(1.04)^{10} \approx 3700.61$. Continuous is slightly higher.

EXAMPLE 5: TIME AND DOUBLING TIME

A bacteria culture follows $N = 500e^{0.35t}$ with t in hours. (a) When does it reach 5000? (b) What is its doubling time?

SOLUTION

(a) Set $500e^{0.35t} = 5000$. Divide: $e^{0.35t} = 10$.

Take $\ln$: $0.35t = \ln 10$, so $t = \frac{\ln 10}{0.35} \approx 6.58$ hours.

(b) Doubling time is $\frac{\ln 2}{0.35} \approx 1.98$ hours.

Sense check: in about 6.6 hours there are about 3.3 doublings, and $2^{3.3} \approx 10$. It matches.

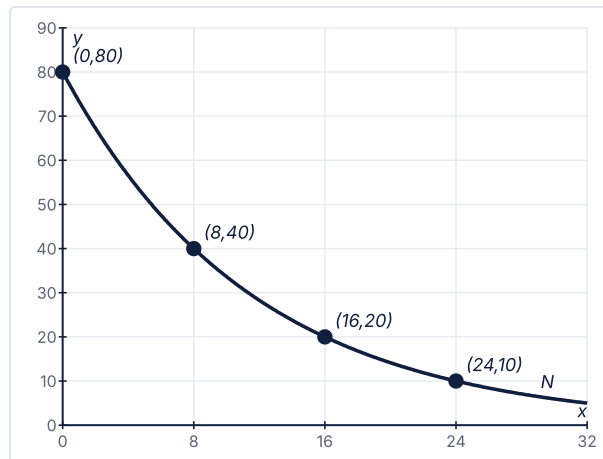
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EXAMPLE 6: HALF-LIFE

A medicine has a half-life of 8 days. A sample starts at 80 grams. (a) Find the decay constant k in $N = 80e^{-kt}$. (b) How much remains after 20 days? (c) When will 10 grams remain?



Each 8 days the amount is cut in half.

SOLUTION

(a) Half-life gives $k = \frac{\ln 2}{8} \approx 0.0866$ per day.

(b) $N = 80e^{-0.0866 \cdot 20} = 80e^{-1.733} \approx 14.14$ grams.

(c) 80 to 40 to 20 to 10 takes 3 half-lives, so $3 \cdot 8 = 24$ days.

The graph shows the points (0, 80), (8, 40), (16, 20), and (24, 10).

Coach's Playbook

- 1 Label every quantity.** Write what each letter means and what units you have. Identify the unknown.
 - 2 Pick the model.** Scale problem: decibels, pH, or magnitude. Growth or decay: A_0e^{rt} . Interest with n periods: $P(1 + r/n)^{nt}$.
 - 3 Substitute, then isolate.** Plug in numbers, then get the exponential alone.
 - 4 Take ln.** Use $\ln e^x = x$ to bring the time down.
 - 5 Interpret and check.** Does the answer make sense? Include units, and check it against a half-life or doubling estimate.
- ◆ For comparing two log-scale readings, subtract them first. The difference tells you the ratio of the actual quantities.
 - ◆ The **Rule of 70** estimates doubling time: years $\approx 70 \div$ (percent rate). At 5 percent: about 14 years, close to the exact 13.9.
 - ◆ Percent rates must become decimals before they go in: 4 percent is 0.04.
 - ◆ The base-10 log of a power of 10 is just the exponent: $\log 10^{-9} = -9$.

Common traps

1. Using the rate as a percent: putting $r = 4$ instead of $r = 0.04$.
2. Forgetting the negative sign in pH, so acids appear to have negative pH.
3. Treating a 10-decibel increase as "10 percent louder" when it means 10 times the intensity.
4. Rounding $e^{0.4}$ too early, which changes the answer's second decimal.

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Spot the error 1

A student says that a magnitude 7 earthquake is "two times as strong" as a magnitude 5 earthquake because $7 - 5 = 2$. The scale is logarithmic, so the difference is an exponent. The amplitude ratio is $10^{7-5} = 10^2 = 100$. The magnitude 7 quake shakes the ground 100 times as much.

Spot the error 2

A student solves $1000e^{0.05t} = 2000$ by writing $e^{0.05t} = 1000$, after subtracting 1000 from both sides. The student subtracted, but the 1000 is a multiplier. Correct: divide by 1000, so $e^{0.05t} = 2$, then $0.05t = \ln 2$ and $t \approx 13.86$ years.

Spot the error 3

A student computes the pH of $[H^+] = 10^{-3}$ as $\log 10^{-3} = -3$ and says "the pH is negative three." The definition has a minus sign: $\text{pH} = -\log[H^+] = -(-3) = 3$. Nearly every solution you will see has a pH between 0 and 14.

Real-World Connection

A phone speaker plays at 60 decibels, and a second speaker plays at 80 decibels. The 20-decibel difference means the second has $10^2 = 100$ times the sound intensity, yet your ears perceive it as only a few times louder. The log scale matches how hearing actually works, which is why audio engineers use it.

Challenge Corner

A town of 12000 people shrinks continuously. After 4 years the population is 9000. Find the continuous rate k in $P = 12000e^{kt}$, then the time when the population first falls below 6000. Solution: $9000 = 12000e^{4k}$, so $e^{4k} = 0.75$ and $k = \frac{\ln 0.75}{4} \approx -0.0719$. For 6000: $e^{kt} = 0.5$, so $t = \frac{\ln 0.5}{k} = \frac{-0.6931}{-0.0719} \approx 9.64$ years.

Practice**PRACTICE Applications of Logs and e**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- A sound has intensity $I = 10^{-4}$. Using $L = 10 \log \left(\frac{I}{10^{-12}} \right)$, what is its level in decibels?
 (A) 8 (B) 80 (C) 40 (D) 120
- Solution X has pH 3 and solution Y has pH 5. How many times as many hydrogen ions does X have?
 (A) 100 (B) 2 (C) 20 (D) 1000
- How many times as large is the shaking amplitude of a magnitude 6 earthquake as that of a magnitude 4.5 earthquake?
 (A) 1.5 (B) 15 (C) about 31.6 (D) 316
- An account has 1000 dollars at 6 percent compounded continuously. What is the balance after 5 years?
 (A) 1338.23 (B) 1300.00 (C) 1060.00 (D) about 1349.86
- A quantity grows continuously at 3 percent per year. About how long until it doubles?
 (A) about 23.1 years (B) 33.3 years (C) 66.7 years (D) 12.0 years
- A substance decays by $N = N_0 e^{-0.2t}$, with t in years. What is its half-life?
 (A) 5 years (B) about 3.47 years (C) 2.5 years (D) 0.139 years

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7. A population of 400 triples every 4 years. Written as $N = 400e^{kt}$, what is k per year?
 (A) $\frac{3}{4} = 0.75$ (B) $4 \ln 3 \approx 4.39$ (C) $\frac{\ln 3}{4} \approx 0.275$
 (D) $\frac{1}{4} = 0.25$
8. A culture follows $N = 200e^{0.4t}$. When does it reach 1000?
 (A) $t \approx 4.02$ (B) $t = 12.5$ (C) $t \approx 16.7$ (D) $t \approx 17.3$
9. What is $[H^+]$ for a solution with pH 6.5?
 (A) about 6.5×10^{-7} (B) about 3.16×10^{-7} (C) about 3.16×10^6
 (D) about 6.5×10^{-1}
10. Using $L = 10 \log \left(\frac{I}{10^{-12}} \right)$, find the sound level of a sound with $I = 2 \times 10^{-6}$. Round to the nearest tenth.
11. Find the pH of a solution with $[H^+] = 2.5 \times 10^{-5}$. Round to two decimal places.
12. An investment of 4200 dollars earns 3.5 percent compounded continuously. Find the balance after 12 years, to the nearest cent.
13. How many years for 5000 dollars at 4.5 percent compounded continuously to grow to 8000 dollars? Round to the nearest hundredth.
14. A 120 milligram sample has a half-life of 6 hours. How much remains after 15 hours? Round to the nearest hundredth.
15. A population falls from 900 to 600 in 3 years, following $N = 900e^{kt}$. Find k to four decimal places, then the time for the population to reach 300, to the nearest hundredth.

Answers**Answers and Explanations**

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

23.1 Logarithms as Inverses of Exponentials**PRACTICE · LOGARITHMS AS INVERSES OF EXPONENTIALS**

- B. The base 5 gets the exponent 3 and the result is 125. Choice A swaps base and exponent.
- C. $2^5 = 32$. Choice A divides 32 by 2, and 4 is one power short.
- D. $9^{1/2} = 3$, since the square root of 9 is 3. Choice A is the reciprocal, which is the log of 9 base 3.
- A. $(\frac{1}{2})^{-3} = 2^3 = 8$. A positive 3 would give $\frac{1}{8}$, not 8.
- B. Convert to $x^2 = 49$, so $x = 7$. Choice A multiplies 49 by 2 and C halves it, neither is an exponent step.
- C. The inside must be positive: $5 - x > 0$ gives $x < 5$. Choice A flips the inequality.
- A. The asymptote is where $x + 4 = 0$, so $x = -4$. The -3 moves the graph down but does not move the asymptote.
- D. $\log_4 16 = 2$ because $4^2 = 16$. Choice A has the coordinates reversed, and $\log_4 4 = 1$, not 0.
- B. $\ln e^3 = 3$ and $\log 100 = 2$ because $10^2 = 100$. The sum is 5. Choice D multiplies instead of adding.
- 3. Since $3^3 = 27$, a negative exponent gives $3^{-3} = \frac{1}{27}$.
- $x = 31$. Convert: $x + 1 = 2^5 = 32$, so $x = 31$. The inside is $32 > 0$.
- $\frac{3}{4}$. Let $16^y = 8$. Then $2^{4y} = 2^3$, so $4y = 3$ and $y = \frac{3}{4}$.
- $f^{-1}(x) = \log_2(x + 5)$. Swap: $x = 2^y - 5$. Then $x + 5 = 2^y$, so $y = \log_2(x + 5)$.
- $x = \frac{163}{81}$. Set $f(x) = 0$: $\log_3(x - 2) = -4$. Then $x - 2 = 3^{-4} = \frac{1}{81}$, so $x = 2 + \frac{1}{81} = \frac{163}{81}$.
- $x = 7$ or $x = -7$. Convert: $x^2 - 24 = 25$, so $x^2 = 49$. Check the inside: $49 - 24 = 25 > 0$ for both values, so both work.

23.2 Properties of Logarithms**PRACTICE · PROPERTIES OF LOGARITHMS**

- A. Added logs multiply the insides: $\log_3(4 \cdot \frac{9}{4}) = \log_3 9 = 2$. Choice C adds the insides, which is not the rule.
- C. The 3 is on x only, and a product becomes a sum. Choice D puts the 3 on the wrong variable.
- B. Subtracted logs divide the insides: $\log_2 4 = 2$. Choice A divides but forgets to take the log.
- D. $2 \log_a 5 = \log_a 25$, then $\log_a 25 + \log_a 3 = \log_a 75$. Choice B leaves the 2 as a multiplier instead of a power.
- A. $20 = 2^2 \cdot 5$, so $\log_b 20 = 2p + q$. Choice D doubles both logs, which would be $\log_b 100$.
- B. Change of base puts the number on top and the base on the bottom. Choice C is the log of a quotient, $\ln \frac{30}{7}$, which is different.
- C. $\log_9 27 = \frac{\log_3 27}{\log_3 9} = \frac{3}{2}$. Choice B is the flipped fraction.
- A. The power rule brings the exponent in front as a multiplier. Choice B puts the power on the log itself.
- D. $12 = 2^2 \cdot 3$, so $\log 12 = 2(0.301) + 0.477 = 1.079$. Choice B is $\log 18$ and choice A is $\log 6$.
2. Multiply inside: $\log_6 36 = 2$ because $6^2 = 36$.
- $4 + \frac{1}{2} \log_2 x - 3 \log_2 y$. Split: $\log_2 16 + \log_2 x^{1/2} - \log_2 y^3$. Since $\log_2 16 = 4$, apply the power rule.

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12. $\log_3 4$. Powers: $\log_3 16 - \log_3 36 + \log_3 9 = \log_3 \frac{16 \cdot 9}{36} = \log_3 4$.
13. $\frac{2}{3}$. Both are powers of 5: $\frac{\log_5 25}{\log_5 125} = \frac{2}{3}$.
14. 1.35. $\log_b 25 = 2(0.8) = 1.6$ and $\log_b \sqrt{3} = \frac{1}{2}(0.5) = 0.25$. Subtract: $1.6 - 0.25 = 1.35$.
15. 3. By change of base, $\log_8 x = \frac{\log_2 x}{\log_2 8} = \frac{9}{3} = 3$.

23.3 Solving Exponential and Logarithmic Equations

PRACTICE · SOLVING EXPONENTIAL AND LOGARITHMIC EQUATIONS

- A. Write $64 = 2^6$, so $x + 3 = 6$ and $x = 3$. Choice B forgets to subtract 3.
- C. Write $3^{2x} = 3^3$, so $2x = 3$ and $x = \frac{3}{2}$. Choice B inverts the fraction.
- D. $x = \frac{\ln 40}{\ln 5} \approx 2.292$. Choice A divides 40 by 5, and choice C subtracts.
- A. Divide by 2: $e^x = 7$, so $x = \ln 7$. Choice B forgets to divide by 2.
- B. Convert: $x - 1 = 2^4 = 16$, so $x = 17$. Choice D forgets to add 1, and choice A adds 4 and 1.
- C. Set the insides equal: $x + 2 = 3x - 8$, so $2x = 10$ and $x = 5$. Check: both insides equal 7.
- D. Combine: $x(x + 6) = 27$, so $(x + 9)(x - 3) = 0$. The value -9 makes the logs undefined, so only $x = 3$ works. Choice B ignores the check.
- A. Let $u = 3^x$: $u^2 - 4u + 3 = 0$, so $u = 1$ or $u = 3$. Then $x = 0$ or $x = 1$. Choice B forgets the last step, from u back to x .
- B. Write $2^{2x} = 2^{3x-3}$, so $2x = 3x - 3$ and $x = 3$. Choice A has a sign slip.
- $x = \frac{5}{2}$. Since $81 = 3^4$, $2x - 1 = 4$, so $x = \frac{5}{2}$.
- $x \approx 13.853$. Divide: $1.2^x = 12.5$. Then $x = \frac{\ln 12.5}{\ln 1.2} \approx 13.853$.
- $x = 1$ or $x = 3$. Let $u = 2^x$: $u^2 - 10u + 16 = 0$, so $u = 2$ or $u = 8$. Then $x = 1$ or $x = 3$.
- $x = 5$. Combine: $x^2 - 9 = 16$, so $x = 5$ or $x = -5$. Reject -5 because $\log_2(-2)$ does not exist.
- $x = \frac{e^2 - 1}{3}$. Convert: $3x + 1 = e^2$. Subtract 1 and divide by 3. This is about 2.130.
- $x = \frac{\ln 5}{2 \ln 3 - \ln 5} \approx 2.738$. Take $\ln$: $(x + 1) \ln 5 = 2x \ln 3$. Gather x terms: $\ln 5 = x(2 \ln 3 - \ln 5)$.

23.4 Applications of Logs and e

PRACTICE · APPLICATIONS OF LOGS AND E

- B. $\frac{I}{I_0} = 10^8$, so $L = 10 \cdot 8 = 80$. Choice A drops the factor of 10, and choice D uses only the exponent 12 of the denominator.
- A. Each pH step is a factor of 10, so two steps give $10^2 = 100$. Choice B reports the pH difference, 2, and ignores that the scale is logarithmic.
- C. The difference is 1.5, so the ratio is $10^{1.5} \approx 31.6$. Choice A reports the difference, not the ratio.
- D. $A = 1000e^{0.06 \cdot 5} = 1000e^{0.3} \approx 1349.86$. Choice A is yearly compounding, which gives a slightly smaller balance.
- A. $t = \frac{\ln 2}{0.03} \approx 23.1$. Choice B uses $1 \div 0.03$, forgetting the $\ln 2$.
- B. Half-life is $\frac{\ln 2}{0.2} \approx 3.47$. Choice A uses $1 \div 0.2$, and choice D multiplies 0.2 by $\ln 2$ instead of dividing.
- C. At $t = 4$ we need $e^{4k} = 3$, so $4k = \ln 3$ and $k = \frac{\ln 3}{4}$. Choice D ignores the factor of 3.
- A. Divide by 200: $e^{0.4t} = 5$, so $t = \frac{\ln 5}{0.4} \approx 4.02$. Choice C subtracts 200 first, and choice D forgets to divide by 200.
- B. $[H^+] = 10^{-6.5} \approx 3.16 \times 10^{-7}$. Choice A misreads the decimal exponent as digits, and choice C drops the minus sign.
- 63.0 decibels. $\frac{I}{I_0} = 2 \times 10^6$, so $L = 10(\log 2 + 6) = 10(6.301) \approx 63.0$.
- 4.60. $\text{pH} = -\log(2.5 \times 10^{-5}) = -(0.398 - 5) = 4.602$.
- 6392.24 dollars. $A = 4200e^{0.035 \cdot 12} = 4200e^{0.42} \approx 6392.24$.
- 10.45 years. Solve $5000e^{0.045t} = 8000$: $e^{0.045t} = 1.6$, so $t = \frac{\ln 1.6}{0.045} \approx 10.45$.
- 21.21 milligrams. 15 hours is 2.5 half-lives, so $120 \cdot (\frac{1}{2})^{2.5} = 120 \cdot 0.17678 \approx 21.21$.
- $k \approx -0.1352$ and $t \approx 8.13$ years. From $600 = 900e^{3k}$: $k = \frac{\ln(2/3)}{3}$. For 300 : $e^{kt} = \frac{1}{3}$, so $t = \frac{\ln(1/3)}{k} \approx 8.13$.

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