



Sequences and Series

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

In this booster

24.1 Arithmetic Sequences

24.2 Geometric Sequences

24.3 Series and Sigma Notation

24.4 Infinite Geometric Series and Applications

Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

24.1 Arithmetic Sequences

An arithmetic sequence changes by the same amount each step. Learn to find the common difference, write recursive and explicit formulas, and locate any term.

Why This Matters

A sequence is an ordered list of numbers, and an **arithmetic sequence** is the simplest and most useful kind: it moves by the same amount every step. Seats in a stadium that grow by a fixed number per row, a savings plan that adds the same deposit each week, and the mileage on a trip at constant speed are all arithmetic. The same idea sits behind linear functions, so mastering it prepares you for series, calculus, and finance.

Students get stuck on three things. They mix up the term number n with the value of the term a_n . They write $a_1 + nd$ when the formula needs $(n - 1)d$. And they find the common difference by subtracting in the wrong order, which flips its sign. We will fix each of these with a clear method and a simple picture: an arithmetic sequence is a **line** hiding inside a list.

Core Ideas

A **sequence** is a list of numbers in order: $a_1, a_2, a_3, \dots$. The small number is the position, called the **index**, and a_n is the term at position n . A sequence is **arithmetic** if every term is found by adding the same number to the one before it.

Arithmetic Sequence

A sequence is arithmetic if $a_{n+1} - a_n = d$ is the same number for every n . The number d is the **common difference**.

Recursive form: a_1 is given, and $a_n = a_{n-1} + d$ for $n \geq 2$.

Explicit form:

$$a_n = a_1 + (n - 1)d.$$

Why $(n - 1)$? To get from the first term to the n th term, you take $n - 1$ steps. Standing on step 1, you need only one jump to reach step 2, two jumps to reach step 3, and so on. The explicit formula is faster than the recursive one because it jumps straight to any term. For example, in the list 7, 12, 17, 22, $\dots$ we have $a_1 = 7$ and $d = 5$, so $a_n = 7 + 5(n - 1) = 5n + 2$.

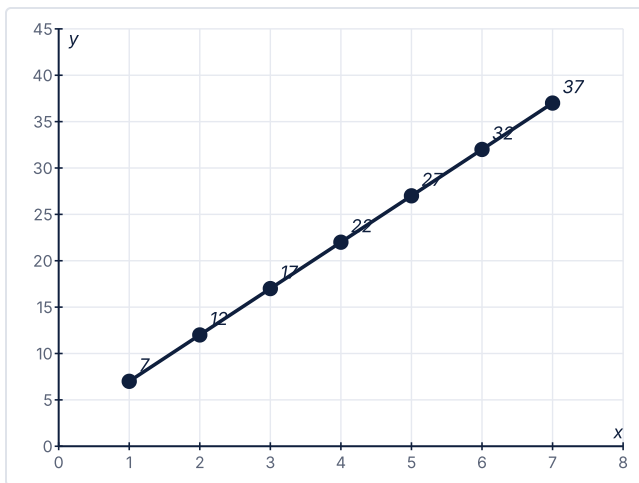
n	1	2	3	4	5	6
$a_n = 5n + 2$	7	12	17	22	27	32

Look at the graph of these terms. Plotting the points (n, a_n) gives dots on a straight line. The common difference d is the **slope** and a_n is a linear function of n (with the domain limited to whole numbers). In $a_n = 5n + 2$ the slope is 5, matching $d = 5$.

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The terms 7, 12, 17, ... lie on a line with slope 5.

Finding the Pieces

Common difference: $d = a_{n+1} - a_n$ (later term minus earlier term).

Two known terms: $d = \frac{a_m - a_k}{m - k}$ for positions $m \neq k$.

Number of terms from first value a_1 to last value L :

$$n = \frac{L - a_1}{d} + 1.$$

A positive d means the sequence climbs, a negative d means it falls, and $d = 0$ gives a constant list. Whenever you see a pattern like 'increases by 3 each time', 'drops 4 per step', or 'each row has 2 more seats', you are looking at an arithmetic sequence.

Worked Examples

EXAMPLE 1: IS IT ARITHMETIC?

Decide whether each sequence is arithmetic. If yes, give d . (a) 7, 12, 17, ... (b) 20, 14, 8, 2, ... (c) 2, 4, 8, 16, ...

SOLUTION

(a) Differences: $12 - 7 = 5$ and $17 - 12 = 5$. Arithmetic, $d = 5$.

(b) Differences: $14 - 20 = -6$, $8 - 14 = -6$, $2 - 8 = -6$. Arithmetic, $d = -6$.

(c) Differences: $4 - 2 = 2$ but $8 - 4 = 4$. The differences change, so it is not arithmetic. (It multiplies by 2, so it is geometric.)

EXAMPLE 2: EXPLICIT FORMULA AND A DISTANT TERM

For 7, 12, 17, ... write the explicit formula and find a_{25} .

SOLUTION

$a_1 = 7$ and $d = 5$.

$a_n = 7 + (n - 1)5 = 7 + 5n - 5 = 5n + 2$.

$a_{25} = 5(25) + 2 = 127$.

Check against the pattern: terms rise by 5 about 24 times from 7, giving $7 + 120 = 127$.

EXAMPLE 3: WHICH TERM IS 302?

In the sequence $a_n = 5n + 2$, is 302 a term? Is 100 a term? If so, which one?

SOLUTION

Set $5n + 2 = 302$. Then $5n = 300$ and $n = 60$. Yes, 302 is the 60th term.

Set $5n + 2 = 100$. Then $5n = 98$ and $n = 19.6$.

An index must be a whole number, so 100 is **not** a term.

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EXAMPLE 4: TWO TERMS GIVEN

An arithmetic sequence has $a_4 = 19$ and $a_9 = 44$. Find a_1 and the explicit formula.

SOLUTION

Find d : from term 4 to term 9 is 5 steps and the value rises by $44 - 19 = 25$. So $d = \frac{25}{5} = 5$.

Go back 3 steps from a_4 to a_1 : $a_1 = 19 - 3(5) = 4$.

Formula: $a_n = 4 + 5(n - 1) = 5n - 1$.

Check: $a_4 = 20 - 1 = 19$ and $a_9 = 45 - 1 = 44$.

EXAMPLE 5: RECURSIVE TO EXPLICIT

A sequence is defined by $a_1 = 30$ and $a_n = a_{n-1} - 4$. Write an explicit formula and find the first negative term.

SOLUTION

The recursive rule adds -4 each time, so $d = -4$.

$a_n = 30 + (n - 1)(-4) = 34 - 4n$.

Negative means $34 - 4n < 0$, so $n > 8.5$. The first whole number is $n = 9$.

Check: $a_8 = 2$ and $a_9 = -2$. The first negative term is $a_9 = -2$.

EXAMPLE 6: COUNTING TERMS AND ARITHMETIC MEANS

(a) How many terms are in $11, 17, 23, \dots, 203$? (b) Insert three numbers between 5 and 29 so the five numbers form an arithmetic sequence.

SOLUTION

(a) $d = 6$, so $n = \frac{203 - 11}{6} + 1 = 32 + 1 = 33$ terms.

(b) The new list has 5 terms, with $a_1 = 5$ and $a_5 = 29$. So $4d = 24$ and $d = 6$.

The numbers are 5, 11, 17, 23, 29. The three inserted numbers, called arithmetic means, are 11, 17, 23.

Coach's Playbook

- 1 Find d first.** Subtract any term from the one after it. Check with a second pair.
 - 2 Write the formula.** $a_n = a_1 + (n - 1)d$. Simplify it to the form $dn + c$ if you like.
 - 3 Substitute the index you want.** The index n is a whole number $1, 2, 3, \dots$
 - 4 Solving for n ?** Set the formula equal to the value and solve. If n is not a whole number, the value is not in the sequence.
 - 5 Test your formula.** Plug in $n = 1$ and $n = 2$. You should get back the first two terms.
- ◆ Two known terms: divide the difference in values by the difference in positions. This is the slope formula in disguise.
 - ◆ The simplified form $a_n = dn + (a_1 - d)$ shows that the number in front of n is d . For $a_n = 5n - 1$, the common difference is immediately 5.
 - ◆ For counting terms, remember to add 1: the number of terms is the number of steps plus one, like fence posts and gaps.

Common traps

1. Using $a_1 + nd$ instead of $a_1 + (n - 1)d$, which gives the term one step too far. 2. Finding d in the wrong direction, so a falling sequence gets a positive d . 3. Forgetting to add 1 when counting terms from a_1 to L . 4. Accepting a non-integer n as an answer when asked which term a value is.

Mistake Clinic

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Spot the error 1

A student finds the 12th term of $6, 3, 0, -3, \dots$ and writes $a_{12} = 6 + 12(-3) = -30$. The student used nd instead of $(n - 1)d$. Correct: $a_{12} = 6 + 11(-3) = 6 - 33 = -27$. Test the formula at $n = 1$: the faulty version gives $6 - 3 = 3$, which is not the first term.

Spot the error 2

A student counts the terms of $4, 9, 14, \dots, 129$ as $\frac{129 - 4}{5} = 25$. The student counted steps, not terms. Always add 1: there are $25 + 1 = 26$ terms. Check with a short list, $4, 9, 14$: there are 2 steps but 3 terms.

Spot the error 3

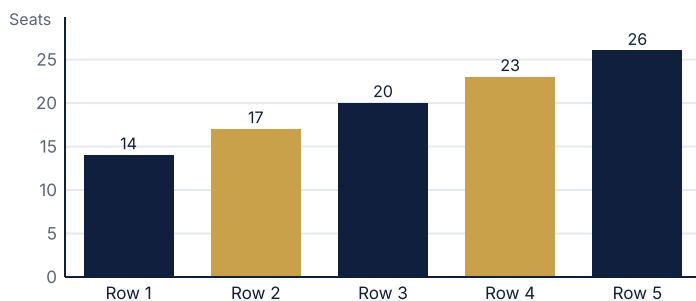
A student says that $3, 6, 12, 24, \dots$ is arithmetic with $d = 3$ because the first jump is $+3$. The student checked only one pair. The next jumps are $+6$ and $+12$, so the sequence is not arithmetic. A sequence is arithmetic only if **every** consecutive difference is the same.

Real-World Connection

A theater has 14 seats in the first row, and each row after has 3 more seats. Row n has $a_n = 14 + 3(n - 1) = 3n + 11$ seats. Row 20 has 71 seats. The row with 59 seats satisfies $3n + 11 = 59$, so $n = 16$.

Challenge Corner

Three numbers in an arithmetic sequence add to 36 and have product 1428. Find them. Solution: call them $12 - d$, 12 , $12 + d$ (the middle is the average, $36 \div 3 = 12$). The product is $12(144 - d^2) = 1428$, so $144 - d^2 = 119$ and $d^2 = 25$. Then $d = 5$ or -5 , and the numbers are 7, 12, 17 (or in the reverse order). Check: $7 \cdot 12 \cdot 17 = 1428$.



Seats per row in the theater: each bar is 3 taller than the last.

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Practice

PRACTICE Arithmetic Sequences

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Which sequence is arithmetic?
 (A) 3, 6, 12, 24, ... (B) 10, 7, 4, 1, ... (C) 2, 5, 9, 14, ...
 (D) 1, 4, 9, 16, ...
- What is the common difference of $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \dots$?
 (A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) 2 (D) 3
- An arithmetic sequence has $a_1 = 6$ and $d = -3$. What is a_{12} ?
 (A) -30 (B) -33 (C) -27 (D) 27
- Which is an explicit formula for 9, 13, 17, 21, ...?
 (A) $a_n = 4n + 9$ (B) $a_n = 9n + 4$ (C) $a_n = 4n + 13$
 (D) $a_n = 4n + 5$
- A sequence has $a_1 = 5$ and $a_n = a_{n-1} + 7$. What is a_5 ?
 (A) 40 (B) 33 (C) 26 (D) 35
- Which term of 8, 15, 22, ... equals 197?
 (A) the 27th term (B) the 29th term (C) the 28th term
 (D) the 26th term
- An arithmetic sequence has $a_3 = 11$ and $a_8 = 36$. What is a_{20} ?
 (A) 100 (B) 101 (C) 91 (D) 96
- How many terms are in 4, 9, 14, ..., 129?
 (A) 26 (B) 25 (C) 27 (D) 24
- For $a_n = 3n - 7$, what is the smallest n with $a_n > 100$?
 (A) 35 (B) 36 (C) 37 (D) 34
- Find the 15th term of 22, 19, 16, ...
- An arithmetic sequence has $a_1 = -5$ and $d = 4$. Write the explicit formula and find a_{30} .
- An arithmetic sequence has $a_6 = 3$ and $a_{11} = -17$. Write the explicit formula.
- How many terms are in 2.5, 4, 5.5, ..., 40?
- Insert four numbers between 3 and 28 so that all six numbers form an arithmetic sequence. List the four numbers.
- A student deposits 40 dollars in week 1 and raises the weekly deposit by 15 dollars each week. In which week does the deposit first exceed 500 dollars, and how much is it?

24.2 Geometric Sequences

A geometric sequence multiplies by the same ratio each step. Learn to find the common ratio, write recursive and explicit formulas, and model growth and decay.

Why This Matters

Some things grow by adding, and some grow by multiplying. A **geometric sequence** is the multiplying kind: each term is the previous term times a fixed number. Money earning interest, a video being shared by more and more people, a phone losing value, a bouncing ball losing height, and a drug leaving the body are all geometric patterns. Because multiplying repeatedly is so powerful, geometric patterns can grow explosively or fade quickly, and understanding them is key to finance, biology, and computer science.

Students get stuck in familiar places. They add when they should multiply and treat a sequence like 2, 6, 18 as arithmetic. They write the exponent as n instead of $n - 1$. They confuse a percent decrease of 15 percent with a ratio of 0.15 when it is 0.85. This section shows you how to avoid each of these, with one idea: find the **ratio**.

Core Ideas

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Geometric Sequence

A sequence is geometric if $\frac{a_{n+1}}{a_n} = r$ is the same number for every n . The number r is the **common ratio**.

Recursive form: a_1 is given, and $a_n = r \cdot a_{n-1}$ for $n \geq 2$.

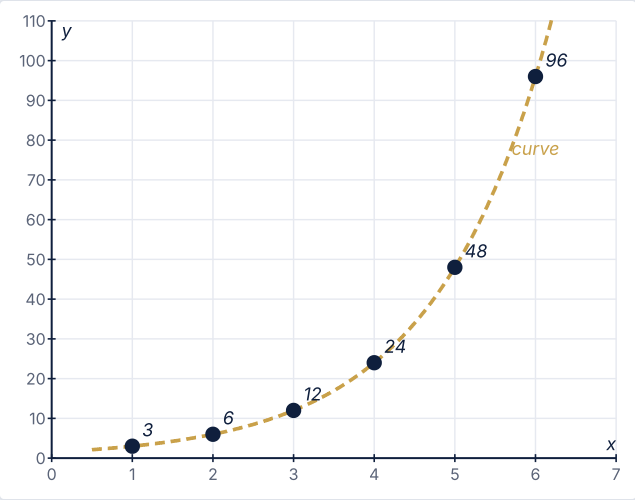
Explicit form:

$$a_n = a_1 r^{n-1}.$$

The explicit formula works like the arithmetic one, but multiplication replaces addition. To get from term 1 to term n you multiply by r exactly $n - 1$ times. For the list 3, 6, 12, 24, . . . the first term is 3 and the ratio is 2, so $a_n = 3 \cdot 2^{n-1}$. For example $a_5 = 3 \cdot 2^4 = 48$.

n	1	2	3	4	5	6
$a_n = 3 \cdot 2^{n-1}$	3	6	12	24	48	96

Notice what the picture below shows. The points do not lie on a line. They lie on an **exponential curve**, which rises faster and faster. A geometric sequence is an exponential function whose domain is the whole numbers. The ratio r tells you how steep the curve is.



The terms 3, 6, 12, 24, 48, 96 sit on an exponential curve.

What the Ratio Tells You

- $r > 1$: the terms grow (exponential growth). If $a_1 > 0$ they increase.
- $0 < r < 1$: the terms shrink toward 0 (exponential decay).
- $r < 0$: the terms **alternate** in sign, positive then negative.
- $r = 1$ gives a constant list. A percent change of p percent per step has $r = 1 \pm \frac{p}{100}$.

Situation	Ratio r
Grows by 20 percent each year	$1 + 0.20 = 1.2$
Loses 15 percent each year	$1 - 0.15 = 0.85$
Doubles each step	2
Halves each step	0.5
Flips sign and triples	-3

To find r , divide a term by the one before it, **later over earlier**. To check that a sequence is geometric you must test several pairs, not only the first. Two sequences with the same first term can look very different. Starting at 5, the ratio 3 gives 5, 15, 45, but the ratio -3 gives 5, -15, 45.

Worked Examples

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EXAMPLE 1: FIND THE RATIO

Is each sequence geometric? If so, find r . (a) 3, 6, 12, ... (b) 80, 40, 20, ... (c) 5, -10, 20, ... (d) 2, 4, 6, 8, ...

SOLUTION

(a) $6 \div 3 = 2$ and $12 \div 6 = 2$. Geometric, $r = 2$.

(b) $40 \div 80 = \frac{1}{2}$ and $20 \div 40 = \frac{1}{2}$. Geometric, $r = \frac{1}{2}$.

(c) $-10 \div 5 = -2$ and $20 \div (-10) = -2$. Geometric, $r = -2$, with alternating signs.

(d) $4 \div 2 = 2$ but $6 \div 4 = 1.5$. Not geometric (it is arithmetic with $d = 2$).

EXAMPLE 2: EXPLICIT FORMULA AND A LATER TERM

Write the explicit formula for 3, 6, 12, ... and find a_8 .

SOLUTION

$a_1 = 3$ and $r = 2$, so $a_n = 3 \cdot 2^{n-1}$.

$a_8 = 3 \cdot 2^7 = 3 \cdot 128 = 384$.

Check the pattern: the terms double each step, and $3 \rightarrow 6 \rightarrow 12 \rightarrow 24 \rightarrow 48 \rightarrow 96 \rightarrow 192 \rightarrow 384$ is seven doublings.

EXAMPLE 3: A FRACTIONAL RATIO

Find the sixth term of 162, 54, 18, ...

SOLUTION

$$r = \frac{54}{162} = \frac{1}{3}.$$

$$a_6 = 162 \left(\frac{1}{3}\right)^5 = \frac{162}{243}.$$

Divide top and bottom by 81: $\frac{2}{3}$.

Check by listing: 162, 54, 18, 6, 2, $\frac{2}{3}$. The sixth term is $\frac{2}{3}$.

EXAMPLE 4: TWO TERMS GIVEN

(a) A geometric sequence has $a_2 = 12$ and $a_5 = 324$. Find a_1 and r . (b) Another has $a_3 = 20$ and $a_5 = 80$. Find all possible values of r .

SOLUTION

(a) From term 2 to term 5 is 3 steps, so $a_5 = a_2 \cdot r^3$: $324 = 12r^3$, so $r^3 = 27$ and $r = 3$. Then $a_1 = \frac{12}{3} = 4$.

(b) Two steps from term 3 to term 5: $80 = 20r^2$, so $r^2 = 4$ and $r = 2$ or $r = -2$.

Both work. With $a_1 = 5$: 5, 10, 20, 40, 80 and 5, -10, 20, -40, 80 each have $a_3 = 20$ and $a_5 = 80$.

EXAMPLE 5: WHICH TERM IS IT?

In the sequence 3, 6, 12, ..., which term equals 1536?

SOLUTION

Set $3 \cdot 2^{n-1} = 1536$.

Divide by 3: $2^{n-1} = 512$.

Since $512 = 2^9$, we get $n - 1 = 9$, so $n = 10$.

Check: $a_{10} = 3 \cdot 2^9 = 3 \cdot 512 = 1536$. The 10th term.

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EXAMPLE 6: DECAY WITH PERCENT

A car is worth 24000 dollars and loses 15 percent of its value each year. (a) Write a model for the value after n years. (b) Find the value after 4 years. (c) After how many whole years is it first worth less than 10000 dollars?

SOLUTION

(a) The ratio is $1 - 0.15 = 0.85$, so $V_n = 24000(0.85)^n$ (here $n = 0$ is today).

(b) $V_4 = 24000(0.85)^4 \approx 24000(0.5220) \approx 12528.15$ dollars.

(c) We need $24000(0.85)^n < 10000$, so $0.85^n < 0.41667$. Try values: $V_5 \approx 10648.93$ is still above 10000, but $V_6 \approx 9051.59$ is below.

So after 6 years the car is first worth less than 10000 dollars.

Coach's Playbook

- 1 Divide, do not subtract.** Compute later term divided by earlier term. If the quotient repeats, it is geometric.
 - 2 Write the formula.** $a_n = a_1 r^{n-1}$. Keep the exponent as $n - 1$.
 - 3 Two terms given?** Divide them to get $r^{(\text{number of steps})}$, then take the root. Remember that an even root gives $\pm$.
 - 4 Percent problems.** Turn the percent into a ratio: 1.07 for 7 percent growth, 0.93 for a 7 percent drop.
 - 5 Check the size.** If $r > 1$, terms should increase in size. If $0 < r < 1$, they should shrink.
- ◆ To solve for the index, divide to isolate the power. If both sides are powers of the same base, equate the exponents. Otherwise use logs.
 - ◆ Compare types: arithmetic adds the same amount (a line). Geometric multiplies by the same amount (a curve).
 - ◆ A table of the first four terms is a fast way to detect the pattern, and to sanity-check a formula.
 - ◆ Do not round r in the middle of a problem. Keep fractions like $\frac{1}{3}$ exact.

Common traps

1. Using $a_1 r^n$ instead of $a_1 r^{n-1}$. 2. Computing r by subtracting terms (that gives d for an arithmetic sequence). 3. Using $r = 0.15$ for a 15 percent loss. A 15 percent loss leaves 85 percent, so $r = 0.85$. 4. Forgetting the negative root: $r^2 = 4$ gives $r = 2$ or $r = -2$.

Mistake Clinic**Spot the error 1**

A student says 2, 6, 18, 54, ... has $a_n = 2 + 4(n - 1)$ because the first jump is +4. The jumps are +4, +12, +36, so it is not arithmetic. The ratio is $6 \div 2 = 3$ each time. Correct formula: $a_n = 2 \cdot 3^{n-1}$. Check: $a_3 = 2 \cdot 9 = 18$.

Spot the error 2

A student computes the 6th term of 5, 15, 45, ... as $5 \cdot 3^6 = 3645$. The exponent should be $n - 1 = 5$. Correct: $a_6 = 5 \cdot 3^5 = 5 \cdot 243 = 1215$. Listing helps: 5, 15, 45, 135, 405, 1215 is six terms, and the 3 was multiplied only five times.

Spot the error 3

A student writes that a population which falls by 30 percent each year has ratio $r = 0.30$ and gets $P_n = 800(0.3)^n$. A 30 percent fall leaves 70 percent, so $r = 0.7$. The correct model is $P_n = 800(0.7)^n$. After one year the population is 560, not 240.

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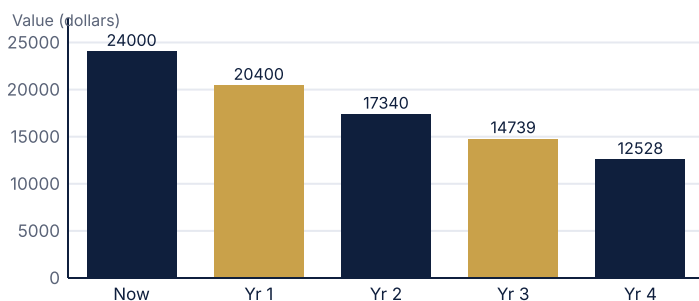
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Real-World Connection

A social post is seen by 4 people on the first day, and each day every viewer shares it with 3 new people, so the number of new viewers triples daily: 4, 12, 36, 108, ... On day n the count is $4 \cdot 3^{n-1}$. Day 7 gives $4 \cdot 729 = 2916$ new viewers. This is why content can go viral so fast.

Challenge Corner

Three positive numbers form a geometric sequence. Their sum is 52 and their product is 1728. Find them. Solution: write them as $\frac{a}{r}, a, ar$. The product is $a^3 = 1728$, so $a = 12$. The sum is $12\left(\frac{1}{r} + 1 + r\right) = 52$, so $r + \frac{1}{r} = \frac{10}{3}$. Multiply by $3r$: $3r^2 - 10r + 3 = 0$, so $(3r - 1)(r - 3) = 0$ and $r = 3$ or $r = \frac{1}{3}$. The numbers are 4, 12, 36 (or in reverse order).



Value of the car, rounded to the nearest dollar: each bar is 85 percent of the one before it.

Practice

PRACTICE Geometric Sequences

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Which sequence is geometric?
 (A) 2, 4, 6, 8, ... (B) 1, 4, 9, 16, ... (C) 3, 9, 27, 81, ...
 (D) 5, 10, 15, 20, ...
- What is the common ratio of 81, 27, 9, ...?
 (A) $\frac{1}{3}$ (B) 3 (C) -54 (D) $\frac{1}{9}$
- A geometric sequence has $a_1 = 5$ and $r = 3$. What is a_6 ?
 (A) 3645 (B) 90 (C) 405 (D) 1215
- Which is the explicit formula for 4, 12, 36, ...?
 (A) $a_n = 4 \cdot 3^n$ (B) $a_n = 4 \cdot 3^{n-1}$ (C) $a_n = 3 \cdot 4^{n-1}$
 (D) $a_n = 8n - 4$
- A sequence has $a_1 = 2$ and $a_n = -3a_{n-1}$. What is a_4 ?
 (A) 54 (B) -18 (C) -54 (D) 81
- Which term of 7, 14, 28, ... equals 448?
 (A) the 7th term (B) the 6th term (C) the 8th term
 (D) the 64th term
- A population of 500 grows by 20 percent each year. About how many are there after 3 years?
 (A) 800 (B) 1728 (C) 600 (D) 864
- A geometric sequence with positive ratio has $a_2 = 6$ and $a_4 = 54$. What is a_6 ?
 (A) 162 (B) 486 (C) 1458 (D) 324
- A geometric sequence has $a_1 = 81$ and $r = \frac{1}{3}$. Which is the first term less than 1?
 (A) $a_6 = \frac{1}{3}$ (B) $a_5 = 1$ (C) $a_4 = 3$ (D) $a_7 = \frac{1}{9}$
- Find the 7th term of 2, 6, 18, ...

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11. Write the explicit formula for 625, 125, 25, ... and find a_5 .
12. A geometric sequence has $a_3 = 20$ and $a_5 = 80$. Find every possible value of r and the first term.
13. How many terms are in the geometric sequence 3, 6, 12, ..., 768?
14. A phone costs 900 dollars and loses 20 percent of its value each year. (a) What is it worth after 3 years? (b) After how many whole years is it first worth less than 300 dollars?
15. Insert two numbers between 8 and 27 so that all four numbers form a geometric sequence. List the two numbers.

24.3 Series and Sigma Notation

A series adds up the terms of a sequence. Learn sigma notation, partial sums, and the formulas for finite arithmetic and geometric series.

Why This Matters

A **series** is what you get when you add up the terms of a sequence. Totals matter in real life: the total money saved after 52 weekly deposits, the total distance of a bouncing ball, the total seats in a stadium, the total payments on a loan. Rather than adding terms one at a time, you will learn short formulas that find the total of 20, 200, or 2000 terms in seconds. Sigma notation, the symbol $\sum$, is the compact language mathematicians use to write such sums, and it returns in statistics and calculus.

Students get stuck in three places. The notation $\sum_{k=1}^n$ looks intimidating, even though it only means 'add these up'. They use the wrong formula for the type of sequence. And they miscount the number of terms, especially when the lower limit is not 1. We will turn each of these into a routine.

Core Ideas

Sigma Notation

$$\sum_{k=m}^n f(k) = f(m) + f(m+1) + \cdots + f(n).$$

The letter k is the **index**. It starts at the **lower limit** m and goes up by 1 until it reaches the **upper limit** n . The number of terms is $n - m + 1$.

To expand a sum, plug in each value of k and add. For example, $\sum_{k=1}^5 (3k + 2) = 5 + 8 + 11 + 14 + 17 = 55$. A **partial sum** S_n is the sum of the first n terms: $S_n = a_1 + a_2 + \cdots + a_n$.

Now the clever part. The story goes that a young student was asked to add $1 + 2 + \cdots + 10$. Instead of adding in order, the student **paired** the first and last numbers: $1 + 10$, $2 + 9$, $3 + 8$, and so on. Every pair gives 11 and there are 5 pairs, so the total is $5 \cdot 11 = 55$. That pairing trick is the proof of the arithmetic sum formula.

Pair	$1 + 10$	$2 + 9$	$3 + 8$	$4 + 7$	$5 + 6$
Sum	11	11	11	11	11

Sum of an Arithmetic Series

$$S_n = \frac{n}{2} (a_1 + a_n) = \frac{n}{2} (2a_1 + (n-1)d).$$

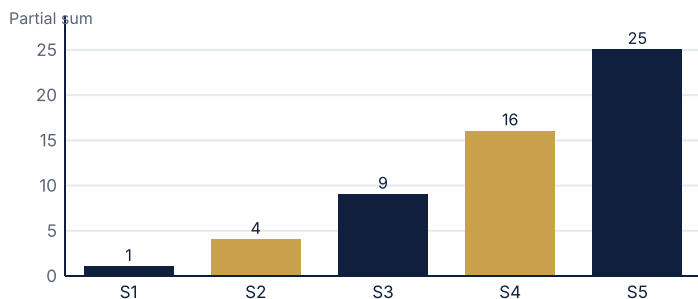
In words: the number of terms times the average of the first and last term.

The picture below shows partial sums of the odd numbers 1, 3, 5, 7, 9. The sums are 1, 4, 9, 16, 25, the perfect squares. This is the formula at work: the sum of the first n odd numbers is $\frac{n}{2}(1 + (2n-1)) = n^2$.

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Partial sums of 1, 3, 5, 7, 9 are 1, 4, 9, 16, 25.

Sum of a Finite Geometric Series

For $r \neq 1$:

$$S_n = a_1 + a_1r + \cdots + a_1r^{n-1} = a_1 \frac{1 - r^n}{1 - r}.$$

If $r = 1$, then $S_n = na_1$.

Where does the geometric formula come from? Write $S_n = a_1 + a_1r + \cdots + a_1r^{n-1}$ and multiply by r . Subtracting the two lines cancels almost everything: $S_n - rS_n = a_1 - a_1r^n$. Factor and divide by $1 - r$, and the formula appears. A useful link between partial sums and terms: $a_n = S_n - S_{n-1}$ for $n \geq 2$.

Series type	Quick test	Formula
Arithmetic	constant difference d	$S_n = \frac{n}{2}(a_1 + a_n)$
Geometric	constant ratio $r \neq 1$	$S_n = a_1 \frac{1 - r^n}{1 - r}$
Constant	each term the same c	$S_n = nc$

Worked Examples

EXAMPLE 1: EXPAND AND EVALUATE SIGMA NOTATION

Evaluate $\sum_{k=1}^5 (3k + 2)$.

SOLUTION

Plug in $k = 1, 2, 3, 4, 5$: 5, 8, 11, 14, 17.

Add: $5 + 8 + 11 + 14 + 17 = 55$.

Check with the formula: $\frac{5}{2}(5 + 17) = \frac{5}{2}(22) = 55$.

EXAMPLE 2: AN ARITHMETIC SERIES

Find $4 + 7 + 10 + \cdots + 97$.

SOLUTION

This is arithmetic with $a_1 = 4$ and $d = 3$.

Count the terms: $n = \frac{97 - 4}{3} + 1 = 32$.

$$S_{32} = \frac{32}{2}(4 + 97) = 16(101) = 1616.$$

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EXAMPLE 3: WRITE IN SIGMA NOTATION

Write each series in sigma notation and evaluate. (a) $6 + 10 + 14 + \dots$ (8 terms) (b) $3 + 6 + 12 + 24 + 48$

SOLUTION

(a) Arithmetic with $a_1 = 6$, $d = 4$. The k th term is $6 + 4(k - 1) = 4k + 2$. So $\sum_{k=1}^8 (4k + 2)$. The last term is 34, so the sum is $\frac{8}{2}(6 + 34) = 160$.

(b) Geometric with $a_1 = 3$, $r = 2$: $\sum_{k=1}^5 3 \cdot 2^{k-1}$.

Sum: $3 \cdot \frac{1 - 2^5}{1 - 2} = 3 \cdot 31 = 93$. Check: $3 + 6 + 12 + 24 + 48 = 93$.

EXAMPLE 4: GEOMETRIC SERIES

Find (a) the sum of the first 10 terms of $5, 10, 20, \dots$ and (b) $\sum_{k=1}^6 81 \left(\frac{1}{3}\right)^{k-1}$.

SOLUTION

(a) $a_1 = 5$, $r = 2$, $n = 10$: $S_{10} = 5 \cdot \frac{1 - 2^{10}}{1 - 2} = 5(1023) = 5115$.

(b) $a_1 = 81$, $r = \frac{1}{3}$, $n = 6$: $S_6 = 81 \cdot \frac{1 - (1/3)^6}{1 - 1/3} = 81 \cdot \frac{728/729}{2/3}$.

Simplify: $81 \cdot \frac{728}{729} \cdot \frac{3}{2} = \frac{364}{3}$, or about 121.33.

Check by listing: $81 + 27 + 9 + 3 + 1 + \frac{1}{3} = 121\frac{1}{3}$.

EXAMPLE 5: FROM PARTIAL SUMS TO A TERM

A sequence has partial sums $S_n = 3n^2 + n$. Find a_1 and a_6 .

SOLUTION

$a_1 = S_1 = 3 + 1 = 4$.

$a_6 = S_6 - S_5$. Compute $S_6 = 3(36) + 6 = 114$ and $S_5 = 3(25) + 5 = 80$.

$a_6 = 114 - 80 = 34$.

The idea: the sum through term 6 minus the sum through term 5 leaves only term 6.

EXAMPLE 6: HOW MANY TERMS?

The series $5 + 8 + 11 + \dots$ has sum 440. How many terms were added?

SOLUTION

Arithmetic with $a_1 = 5$, $d = 3$: $S_n = \frac{n}{2}(10 + 3(n - 1)) = \frac{n(3n + 7)}{2}$.

Set equal to 440: $n(3n + 7) = 880$, so $3n^2 + 7n - 880 = 0$.

Factor or use the quadratic formula: $n = \frac{-7 \pm 103}{6}$. The positive solution is $n = 16$.

Check: $a_{16} = 50$ and $\frac{16}{2}(5 + 50) = 440$. Reject the negative root.

Coach's Playbook

- 1 Classify.** Find the difference or the ratio. Arithmetic uses the average formula, geometric uses the ratio formula.
- 2 Count the terms.** For sigma notation use upper minus lower plus 1. For a list use $\frac{L - a_1}{d} + 1$.
- 3 Choose the formula.** If you know the last term, use $\frac{n}{2}(a_1 + a_n)$. If not, use $\frac{n}{2}(2a_1 + (n - 1)d)$.
- 4 Substitute carefully.** Keep parentheses around negative values and fractions.
- 5 Check with a small case.** Add the first 3 or 4 terms by hand to see that your formula behaves.

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- ◆ Constants pull out of a sum: $\sum 5k = 5 \sum k$. A sum of a constant c over n terms is nc , not c .
- ◆ Useful totals: $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ and the first n odd numbers add to n^2 .
- ◆ The index letter does not matter: $\sum_{k=1}^4 k$ and $\sum_{j=1}^4 j$ are the same sum.
- ◆ If the sum starts at $k = 0$, there is one more term than the upper limit.

Common traps

1. Miscounting terms when the lower limit is not 1, such as using 9 instead of $11 - 3 + 1$. 2. Dropping the $\frac{1}{2}$ in the arithmetic formula. 3. Using r^{n-1} instead of r^n in the geometric sum formula. 4. Splitting a sum with a constant incorrectly: $\sum_{k=1}^4 (2k + 1) \neq 2 \sum k + 1$ (the 1 is added four times).

Mistake Clinic

Spot the error 1

A student finds $\sum_{k=1}^4 (2k + 1)$ as $2(1 + 2 + 3 + 4) + 1 = 21$. The constant 1 is added once for each term, which is 4 times. Correct: $2(10) + 4(1) = 24$. Or list the terms: $3 + 5 + 7 + 9 = 24$.

Spot the error 2

A student adds 20 terms of 3, 7, 11, ... and writes $S_{20} = 20(3 + 79) = 1640$. The $\frac{1}{2}$ is missing. Pairing counts each pair of numbers once, so there are only 10 pairs of 82. Correct: $\frac{20}{2}(3 + 79) = 820$.

Spot the error 3

A student says $\sum_{k=3}^{11} k^2$ has 8 terms because $11 - 3 = 8$. The ends are both included. The count is $11 - 3 + 1 = 9$ terms: $k = 3, 4, 5, 6, 7, 8, 9, 10, 11$. Counting fence posts, not gaps, is the habit to build.

Real-World Connection

A person saves 5 dollars in week 1 and increases the deposit by 3 dollars each week. The deposits are 5, 8, 11, ... After 16 weeks the total saved is $\frac{16}{2}(5 + 50) = 440$ dollars, just as in Example 6. The formula gave the answer without adding 16 numbers.

Challenge Corner

An arithmetic series has $S_{10} = 210$ and $S_{20} = 820$. Find a_1 and d . Solution: $S_{10} = 5(2a_1 + 9d) = 210$ gives $2a_1 + 9d = 42$. $S_{20} = 10(2a_1 + 19d) = 820$ gives $2a_1 + 19d = 82$. Subtract: $10d = 40$, so $d = 4$. Then $2a_1 = 42 - 36 = 6$, so $a_1 = 3$. Check: $S_{10} = 5(6 + 36) = 210$.

Practice

PRACTICE Series and Sigma Notation

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is $\sum_{k=1}^4 (2k + 1)$?
 (A) 21 (B) 20 (C) 24 (D) 36
- An arithmetic sequence has $a_1 = 3$ and $d = 4$. What is the sum of the first 20 terms?
 (A) 820 (B) 1640 (C) 79 (D) 780
- What is $\sum_{k=1}^6 2^k$?
 (A) 63 (B) 127 (C) 128 (D) 126

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4. What is the sum of the first 8 terms of 3, 6, 12, ...?
 (A) 765 (B) 768 (C) 1533 (D) 381
5. Which sigma expression equals $5 + 9 + 13 + 17 + 21$?
 (A) $\sum_{k=1}^5 (4k + 5)$ (B) $\sum_{k=1}^5 (4k + 1)$ (C) $\sum_{k=1}^5 5k$
 (D) $\sum_{k=1}^5 5 \cdot 4^{k-1}$
6. How many terms are in $\sum_{k=3}^{11} k^2$?
 (A) 9 (B) 8 (C) 11 (D) 10
7. A sequence has partial sums $S_n = n^2 + 2n$. What is a_5 ?
 (A) 35 (B) 13 (C) 11 (D) 24
8. What is $100 + 50 + 25 + 12.5 + 6.25$?
 (A) 200 (B) 187.5 (C) 387.5 (D) 193.75
9. The sum $2 + 4 + 6 + \cdots + 2n$ equals 156. What is n ?
 (A) 13 (B) 12 (C) 78 (D) 11
10. Evaluate $\sum_{k=1}^6 (5k - 3)$.
11. Find $12 + 19 + 26 + \cdots + 138$.
12. Evaluate $\sum_{k=1}^7 4 \cdot 3^{k-1}$.
13. Write $2 + 5 + 8 + \cdots + 29$ in sigma notation and find its value.
14. Evaluate $\sum_{k=1}^8 64 \left(\frac{1}{2}\right)^{k-1}$.
15. An arithmetic series has $S_{10} = 210$ and $S_{20} = 820$. Find a_1 and d .

24.4 Infinite Geometric Series and Applications

Add infinitely many terms and get a finite answer. Learn when an infinite geometric series converges, how to find its sum, and how to use it for repeating decimals, bouncing balls, and money problems.

Why This Matters

Can you add up infinitely many numbers and get a finite answer? It sounds impossible, yet it happens all the time. A ball that bounces forever travels a finite distance. A repeating decimal such as $0.333\ldots$ is an infinite sum that equals exactly $\frac{1}{3}$. A payment plan with payments that shrink each year can have a finite total value. Infinite geometric series are the first step toward the central idea of calculus, the **limit**.

Students find this topic puzzling for good reason. They wonder how adding forever can stop, they forget to check whether the series converges before using the formula, and they set up bouncing-ball problems without counting both the way up and the way down. This section gives you a one-line test, one formula, and a clear method for every type of application.

Core Ideas

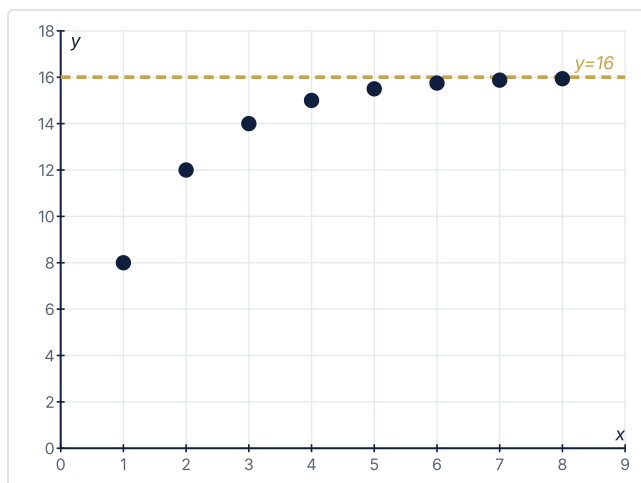
Look at the series $8 + 4 + 2 + 1 + \frac{1}{2} + \cdots$. The partial sums are 8, 12, 14, 15, 15.5, 15.75, ... Each new term is smaller, and the partial sums creep toward 16 without ever passing it. We say the series **converges** to 16. If the partial sums grow without bound, or jump around, the series **diverges**.

n	1	2	3	4	5	6	7	8
S_n	8	12	14	15	15.5	15.75	15.875	15.9375

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The partial sums of $8 + 4 + 2 + \dots$ climb toward the line $y = 16$ and never pass it.

Convergence Test

An infinite geometric series $a_1 + a_1r + a_1r^2 + \dots$ **converges** if and only if $|r| < 1$, that is, $-1 < r < 1$.

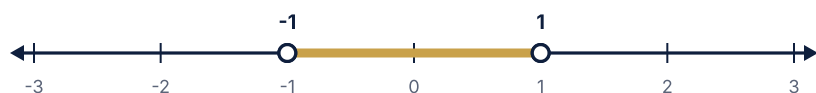
If $|r| \geq 1$ (and $a_1 \neq 0$), the series **diverges**: it has no finite sum.

Sum of an Infinite Geometric Series

If $|r| < 1$:

$$S = \sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1-r}.$$

Here a_1 is the **first** term of the series. Why it works: in $S_n = a_1 \frac{1-r^n}{1-r}$ the power r^n shrinks to 0 when $|r| < 1$.



The ratio must lie strictly between -1 and 1 for the series to converge.

Repeating decimals are infinite geometric series. For instance, $0.\overline{36} = 0.36 + 0.0036 + 0.000036 + \dots$ has $a_1 = 0.36$ and $r = 0.01$. Its sum is $\frac{0.36}{1-0.01} = \frac{0.36}{0.99} = \frac{36}{99} = \frac{4}{11}$. A shortcut: a block of m repeating digits over m nines gives the fraction.

Bouncing Ball

A ball is dropped from height h and rebounds to the fraction r of its previous height each time ($0 < r < 1$).

Total distance traveled:

$$D = h + 2 \cdot \frac{hr}{1-r} = h \frac{1+r}{1-r}.$$

The first drop counts once. Every later rebound counts twice, once up and once down.

Series	a_1	r	Converges?	Sum
$9 + 3 + 1 + \dots$	9	$\frac{1}{3}$	yes	$\frac{9}{2/3} = 13.5$
$2 + 4 + 8 + \dots$	2	2	no	none
$6 - 3 + 1.5 - \dots$	6	$-\frac{1}{2}$	yes	$\frac{6}{1.5} = 4$
$5 + 5 + 5 + \dots$	5	1	no	none

Worked Examples

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EXAMPLE 1: CONVERGE OR DIVERGE?

Decide whether each series converges. If so, find the sum. (a) $8 + 4 + 2 + \dots$ (b) $3 + 6 + 12 + \dots$ (c) $10 - 5 + 2.5 - \dots$

SOLUTION

(a) $r = \frac{1}{2}$, and $|r| < 1$. Converges. $S = \frac{8}{1 - 1/2} = 16$.

(b) $r = 2$, and $|r| \geq 1$. Diverges, so there is no sum.

(c) $r = -\frac{1}{2}$, and $|r| < 1$. Converges. $S = \frac{10}{1 - (-1/2)} = \frac{10}{3/2} = \frac{20}{3}$.

EXAMPLE 2: SIGMA NOTATION WITH AN INFINITE TOP

Evaluate (a) $\sum_{k=1}^{\infty} 5 \left(\frac{2}{3}\right)^{k-1}$ and (b) $\sum_{k=0}^{\infty} 4(0.3)^k$.

SOLUTION

(a) $a_1 = 5$ and $r = \frac{2}{3}$. $S = \frac{5}{1 - 2/3} = \frac{5}{1/3} = 15$.

(b) The sum starts at $k = 0$, so the first term is $4(0.3)^0 = 4$. Then $a_1 = 4$ and $r = 0.3$.

$S = \frac{4}{1 - 0.3} = \frac{4}{0.7} = \frac{40}{7} \approx 5.71$.

Always find the first term by plugging in the starting index.

EXAMPLE 3: REPEATING DECIMALS

Write each as a fraction in lowest terms. (a) $0.\overline{36}$ (b) $0.2\overline{7}$

SOLUTION

(a) $a_1 = 0.36$, $r = 0.01$. $S = \frac{0.36}{0.99} = \frac{36}{99} = \frac{4}{11}$.

(b) Split off the part that does not repeat: $0.2\overline{7} = 0.2 + 0.07 + 0.007 + 0.0007 + \dots$

The tail has $a_1 = 0.07$ and $r = 0.1$, so it equals $\frac{0.07}{0.9} = \frac{7}{90}$.

Total: $\frac{1}{5} + \frac{7}{90} = \frac{18}{90} + \frac{7}{90} = \frac{25}{90} = \frac{5}{18}$.

Check: $\frac{5}{18} = 0.2777\dots$

EXAMPLE 4: BOUNCING BALL

A ball is dropped from 10 meters. After each bounce it rises to 60 percent of the height it fell. Find the total distance traveled before it comes to rest.

SOLUTION

Drop: 10 m. The rebound heights are 6, 3.6, 2.16, ... with $a_1 = 6$ and $r = 0.6$.

Sum of the rebound heights: $\frac{6}{1 - 0.6} = \frac{6}{0.4} = 15$ m.

Each rebound is traveled up and down, so double: $2 \cdot 15 = 30$ m.

Total: $10 + 30 = 40$ m.

Check with the formula: $10 \cdot \frac{1.6}{0.4} = 40$.

EXAMPLE 5: WORK BACKWARD

(a) An infinite geometric series has $a_1 = 6$ and sum 24. Find r . (b) Another has $r = -\frac{1}{4}$ and sum 20. Find a_1 .

SOLUTION

(a) Set $\frac{6}{1 - r} = 24$. Then $1 - r = \frac{6}{24} = \frac{1}{4}$, so $r = \frac{3}{4}$. Since $|r| < 1$, this is valid.

(b) Set $\frac{a_1}{1 - (-1/4)} = 20$. Then $\frac{a_1}{5/4} = 20$, so $a_1 = 20 \cdot \frac{5}{4} = 25$.

Check (b): $25 - 6.25 + 1.5625 - \dots$ and $\frac{25}{1.25} = 20$.

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EXAMPLE 6: A FINANCIAL APPLICATION

A trust pays 1000 dollars this year. Each later year it pays 90 percent of the previous year's payment, forever. What is the total of all payments?

SOLUTION

The payments are 1000, 900, 810, ... with $a_1 = 1000$ and $r = 0.9$.

Since $|r| < 1$, the total is finite: $S = \frac{1000}{1 - 0.9} = \frac{1000}{0.1} = 10000$ dollars.

Though the payments never stop, their total never exceeds 10000 dollars.

For perspective, the first 10 payments total $1000 \cdot \frac{1 - 0.9^{10}}{0.1} \approx 6513$ dollars.

Coach's Playbook

- 1 Find r first.** Divide a term by the previous one. Check a second pair.
 - 2 Test $|r| < 1$.** If it fails, stop: the series diverges.
 - 3 Identify the first term.** Plug in the starting index of the sum if it is in sigma form.
 - 4 Apply $\frac{a_1}{1-r}$.** Be careful with negative r : $1 - (-\frac{1}{2}) = 1.5$, not 0.5.
 - 5 Sanity check.** The sum should be near a_1 when r is small, and much larger than a_1 when r is close to 1.
- ◆ Repeating decimal shortcut: $0.\overline{ab} = \frac{ab}{99}$ and $0.\overline{abc} = \frac{abc}{999}$. Then reduce.
 - ◆ Series with a first term of a_1 and r negative alternate around the sum: the partial sums go above, then below.
 - ◆ For bouncing balls draw a quick picture: drop, then pairs of (up, down).
 - ◆ Converting a percent loss: 25 percent lost per bounce means $r = 0.75$.

Common traps

1. Using the formula without checking $|r| < 1$. 2. Using the wrong first term, especially when the sum starts at $k = 0$. 3. Counting each bounce once instead of twice (up and down) in distance problems. 4. Sign mistakes with a negative ratio in $1 - r$.

Mistake Clinic**Spot the error 1**

A student writes $2 + 4 + 8 + \cdots = \frac{2}{1-2} = -2$. The ratio is $r = 2$, which breaks the condition $|r| < 1$, so the formula does not apply. The series diverges: the partial sums 2, 6, 14, 30, ... grow without bound, so no finite sum exists. A negative answer from adding positive numbers is the warning sign.

Spot the error 2

A student finds the total distance for a ball dropped from 20 feet that rebounds to $\frac{3}{4}$ of its height as $\frac{20}{1 - 3/4} = 80$ feet. This counts every height once, as if the ball were only falling. The first drop is 20 feet. The rebounds are 15, 11.25, ... with sum $\frac{15}{1/4} = 60$, counted twice: 120. Total: $20 + 120 = 140$ feet.

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Spot the error 3

A student converts $0.\overline{45}$ to $\frac{45}{999}$. The number of nines must match the number of repeating digits: two digits, so two nines. $0.\overline{45} = \frac{45}{99} = \frac{5}{11}$. Check: $5 \div 11 = 0.4545\dots$

Real-World Connection

A patient takes a 100 milligram dose each morning, and 20 percent of the drug remains in the body from one dose to the next day. Just after a dose, the amount is $100 + 20 + 4 + 0.8 + \dots = \frac{100}{1 - 0.2} = 125$ milligrams in the long run. Doctors use such sums to make sure the drug level settles at a safe value.

Challenge Corner

An infinite geometric series has sum 12, and the sum of its first two terms is 9. Find a_1 and r . Solution: $\frac{a_1}{1 - r} = 12$ gives $a_1 = 12(1 - r)$. The first two terms are $a_1 + a_1r = a_1(1 + r) = 12(1 - r)(1 + r) = 9$. So $1 - r^2 = \frac{3}{4}$, $r^2 = \frac{1}{4}$ and $r = \pm\frac{1}{2}$. Both satisfy $|r| < 1$. If $r = \frac{1}{2}$ then $a_1 = 6$ ($6 + 3 + \dots$). If $r = -\frac{1}{2}$ then $a_1 = 18$ ($18 - 9 + \dots$).

Practice**PRACTICE Infinite Geometric Series and Applications**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Which infinite series converges?
 (A) $2 + 4 + 8 + 16 + \dots$ (B) $3 + 3 + 3 + 3 + \dots$ (C) $9 + 3 + 1 + \frac{1}{3} + \dots$
 (D) $1 - 2 + 4 - 8 + \dots$
- What is the sum of $12 + 6 + 3 + \frac{3}{2} + \dots$?
 (A) 24 (B) 18 (C) 48 (D) 12
- What is $\sum_{k=1}^{\infty} 5\left(\frac{2}{3}\right)^{k-1}$?
 (A) 7.5 (B) 10 (C) 25 (D) 15
- What fraction equals $0.\overline{45}$?
 (A) $\frac{9}{20}$ (B) $\frac{5}{11}$ (C) $\frac{5}{111}$ (D) $\frac{4}{9}$
- A ball is dropped from 20 feet and rebounds to $\frac{3}{4}$ of its previous height each time. What is the total distance it travels?
 (A) 140 feet (B) 80 feet (C) 60 feet (D) 120 feet
- An infinite geometric series has $a_1 = 6$ and sum 24. What is r ?
 (A) $\frac{1}{4}$ (B) $\frac{4}{3}$ (C) $\frac{3}{4}$ (D) 18
- What is $\sum_{k=0}^{\infty} 4(0.3)^k$?
 (A) $\frac{40}{3}$ (B) $\frac{12}{7}$ (C) $\frac{40}{13}$ (D) $\frac{40}{7}$
- For which values of x does $\sum_{k=0}^{\infty} \left(\frac{x}{5}\right)^k$ converge?
 (A) $-5 < x < 5$ (B) $x < 5$ (C) $-1 < x < 1$
 (D) $|x| > 5$
- What is $5 - \frac{5}{2} + \frac{5}{4} - \dots$?
 (A) 10 (B) $\frac{10}{3}$ (C) $\frac{5}{2}$ (D) 15
- Find the sum of $27 + 18 + 12 + 8 + \dots$.
- Write $0.\overline{27}$ as a fraction in lowest terms.
- Write $0.8\overline{3}$ as a fraction in lowest terms.
- A ball is dropped from 12 feet and rebounds to $\frac{2}{3}$ of its previous height each time. Find the total distance traveled.
- A fund pays 500 dollars in year 1, and each later year pays 80 percent of the previous payment, forever. (a) Find the total of all payments. (b) What is the fewest number of payments whose total is at least 2400 dollars?

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15. For which values of x does $\sum_{k=0}^{\infty} \left(\frac{x-1}{4}\right)^k$ converge? Find the sum in terms of x .

Answers**Answers and Explanations**

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

24.1 Arithmetic Sequences**PRACTICE · ARITHMETIC SEQUENCES**

1. B. Each term is 3 less than the previous one. The others have changing differences.
2. A. $\frac{3}{4} - \frac{1}{4} = \frac{1}{2}$. Choice B uses the first term, not a difference.
3. C. $a_{12} = 6 + 11(-3) = -27$. Choice A uses $12d$ instead of $11d$.
4. D. $a_n = 9 + 4(n-1) = 4n + 5$. Choice A forgets to subtract the extra 4.
5. B. Four additions of 7: $5 + 28 = 33$. Choice A uses five additions, one step too many.
6. C. $a_n = 7n + 1$, so $7n + 1 = 197$ gives $n = 28$. Choice A computes $\frac{197-8}{7} = 27$ and forgets to add 1.
7. D. $d = \frac{36-11}{5} = 5$ and $a_1 = 1$, so $a_{20} = 1 + 19(5) = 96$. Choice A is $20d$.
8. A. $n = \frac{129-4}{5} + 1 = 26$. Choice B forgets the final $+1$.
9. B. $3n > 107$ gives $n > 35.67$, so $n = 36$. Here $a_{35} = 98$ and $a_{36} = 101$. Choice A rounds down.
10. -20. $d = -3$, so $a_{15} = 22 + 14(-3) = 22 - 42 = -20$.
11. $a_n = 4n - 9$ and $a_{30} = 111$. $a_n = -5 + 4(n-1) = 4n - 9$, and $4(30) - 9 = 111$.
12. $a_n = 27 - 4n$. $d = \frac{-17-3}{5} = -4$. Then $a_1 = 3 + 5 = 23$, so $a_n = 23 - 4(n-1) = 27 - 4n$.
13. 26. $d = 1.5$, so $n = \frac{40-2.5}{1.5} + 1 = 25 + 1 = 26$.
14. 8, 13, 18, 23. There are 5 steps from 3 to 28, so $d = \frac{25}{5} = 5$.
15. Week 32, with 505 dollars. $40 + 15(n-1) > 500$ gives $n-1 > 30.67$, so $n = 32$. Check: week 31 gives 490, week 32 gives 505.

24.2 Geometric Sequences**PRACTICE · GEOMETRIC SEQUENCES**

1. C. Each term is 3 times the previous one. The others have a constant difference (A, D) or changing ratios and differences.
2. A. $27 \div 81 = \frac{1}{3}$. Choice B divides in the wrong order, and choice C subtracts.
3. D. $a_6 = 5 \cdot 3^5 = 1215$. Choice A uses the exponent 6 instead of 5.
4. B. The first term is 4 and the ratio is 3. Choice D treats the sequence as arithmetic, and choice C swaps 3 and 4.
5. C. $a_2 = -6$, $a_3 = 18$, $a_4 = -54$. Choice A loses the sign: an odd number of multiplications by -3 is negative.
6. A. $7 \cdot 2^{n-1} = 448$ gives $2^{n-1} = 64 = 2^6$, so $n = 7$. Choice B forgets to add 1 to the exponent.
7. D. The ratio is 1.2, so $500(1.2)^3 = 864$. Choice A adds 20 percent of 500 three times, which ignores compounding.
8. B. $r^2 = 9$, so $r = 3$ (positive). Then $a_6 = 54 \cdot 9 = 486$. Choice A is a_5 .
9. A. $a_5 = 81 \cdot \frac{1}{81} = 1$, which is not less than 1. The next term, $a_6 = \frac{1}{3}$, is the first. Choice B is equal to 1.
10. 1458. $r = 3$, so $a_7 = 2 \cdot 3^6 = 2 \cdot 729 = 1458$.
11. $a_n = 625 \left(\frac{1}{5}\right)^{n-1}$ and $a_5 = 1$. $r = \frac{125}{625} = \frac{1}{5}$, and $625 \cdot \frac{1}{625} = 1$.
12. $r = 2$ or $r = -2$, with $a_1 = 5$ in both cases. $80 = 20r^2$ gives $r^2 = 4$. Then $a_1 = \frac{20}{r^2} = 5$.
13. 9. $3 \cdot 2^{n-1} = 768$ gives $2^{n-1} = 256 = 2^8$, so $n = 9$.
14. (a) 460.80 dollars. (b) 5 years. $900(0.8)^3 = 460.80$. Then $900(0.8)^4 = 368.64$ is still above 300, but $900(0.8)^5 = 294.91$ is below.
15. 12 and 18. There are 3 steps, so $r^3 = \frac{27}{8}$ and $r = \frac{3}{2}$. Then $8 \cdot \frac{3}{2} = 12$ and $12 \cdot \frac{3}{2} = 18$.

24.3 Series and Sigma Notation**PRACTICE · SERIES AND SIGMA NOTATION**

1. C. Terms: $3 + 5 + 7 + 9 = 24$. Choice A adds the constant 1 only once.
2. A. $a_{20} = 79$, so $S_{20} = \frac{20}{2}(3 + 79) = 820$. Choice B forgets the $\frac{1}{2}$, and choice D uses $a_{19} = 75$.
3. D. $2 + 4 + 8 + 16 + 32 + 64 = 126$. Choice B starts at $k = 0$, which adds an extra term, 1.
4. A. $S_8 = 3 \cdot \frac{2^8 - 1}{2 - 1} = 765$. Choice C is the sum of 9 terms and choice D is the sum of 7 terms.
5. B. The k th term is $5 + 4(k-1) = 4k + 1$. Choice A starts at 9, and choice D is geometric.
6. A. The count is $11 - 3 + 1 = 9$. Choice B subtracts and forgets the $+1$.
7. C. $a_5 = S_5 - S_4 = 35 - 24 = 11$. Choice A is the whole sum S_5 , and choice B is a_6 .
8. D. $S_5 = 100 \cdot \frac{1 - (1/2)^5}{1 - 1/2} = 193.75$. Choice A uses the infinite-sum idea $\frac{100}{1-0.5}$.
9. B. The sum is $n(n+1)$, so $n(n+1) = 156 = 12 \cdot 13$ and $n = 12$. Choice A is the larger factor.
10. 87. Terms: 2, 7, 12, 17, 22, 27. Sum: $\frac{6}{2}(2 + 27) = 87$.

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11. $1425. d = 7, n = \frac{138-12}{7} + 1 = 19, S = \frac{19}{2}(12 + 138) = 1425.$
12. $4372. S_7 = 4 \cdot \frac{3^7 - 1}{3 - 1} = 4 \cdot 1093 = 4372.$
13. $\sum_{k=1}^{10} (3k - 1) = 155.$ The k th term is $3k - 1$, and $3k - 1 = 29$ gives $k = 10.$ The sum is $\frac{10}{2}(2 + 29) = 155.$
14. $127.5. S_8 = 64 \cdot \frac{1 - (1/2)^8}{1/2} = 128 \cdot \frac{255}{256} = 127.5.$
15. $a_1 = 3$ and $d = 4.$ From the sums: $2a_1 + 9d = 42$ and $2a_1 + 19d = 82.$ Subtract to get $d = 4,$ then $a_1 = 3.$

24.4 Infinite Geometric Series and Applications

PRACTICE · INFINITE GEOMETRIC SERIES AND APPLICATIONS

1. C. The ratio is $\frac{1}{3},$ which satisfies $|r| < 1.$ The others have ratios 2, 1, and $-2.$
2. A. $r = \frac{1}{2},$ so $S = \frac{12}{1/2} = 24.$ Choice B is only the first two terms added.
3. D. $S = \frac{5}{1-2/3} = 15.$ Choice A divides by r instead of $1 - r.$
4. B. $0.\overline{45} = \frac{45}{99} = \frac{5}{11}.$ Choice C uses three nines for two digits.
5. A. Drop 20, plus twice the rebound sum $2 \cdot \frac{15}{1/4} = 120,$ total 140. Choice D forgets the first drop.
6. C. $\frac{6}{1-r} = 24$ gives $1 - r = \frac{1}{4},$ so $r = \frac{3}{4}.$ Choice A stops at $1 - r.$
7. D. The first term is $4(0.3)^0 = 4,$ so $S = \frac{4}{0.7} = \frac{40}{7}.$ Choice C uses $1 + r$ in the denominator.
8. A. Need $|\frac{x}{5}| < 1,$ so $|x| < 5.$ Choice B ignores very negative values of $x.$
9. B. $r = -\frac{1}{2},$ so $S = \frac{5}{1+1/2} = \frac{10}{3}.$ Choice A treats r as $+\frac{1}{2}.$
10. 81. $r = \frac{18}{27} = \frac{2}{3},$ so $S = \frac{27}{1/3} = 81.$
11. $\frac{3}{11} \cdot \frac{27}{99} = \frac{3}{11}.$
12. $\frac{5}{6}. 0.8 + 0.03 + 0.003 + \dots = \frac{4}{5} + \frac{0.03}{0.9} = \frac{4}{5} + \frac{1}{30} = \frac{25}{30} = \frac{5}{6}.$
13. 60 feet. Rebound sum $\frac{8}{1/3} = 24,$ doubled is 48. Add the first drop: $12 + 48 = 60.$
14. (a) 2500 dollars. (b) 15 payments. $S = \frac{500}{0.2} = 2500.$ Partial sums are $2500(1 - 0.8^n) \geq 2400,$ so $0.8^n \leq 0.04$ and $n \geq 14.4,$ so $n = 15.$
15. Converges for $-3 < x < 5,$ with sum $\frac{4}{5-x}.$ Need $|\frac{x-1}{4}| < 1,$ so $-4 < x - 1 < 4.$ The sum is $\frac{1}{1 - \frac{x-1}{4}} = \frac{4}{5-x}.$

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