



Trigonometric Functions and the Unit Circle

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

In this booster

25.1 Radians, Degrees, and the Unit Circle

25.2 Graphs of Sine and Cosine

25.3 Tangent and the Reciprocal Functions

25.4 Modeling with Trigonometric Functions

Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

25.1 Radians, Degrees, and the Unit Circle

Learn to measure angles in degrees and radians, build the unit circle from two special triangles, and use reference angles and quadrant signs to find exact values of sine, cosine, and tangent without a calculator.

Why This Matters

Every wave, wheel, and orbit you will ever study is described with angles, and the unit circle is the single picture that organizes all of them. Sound, light, tides, alternating current, the motion of a Ferris wheel, and the rotation of a satellite dish are all modeled with sine and cosine. If you can read the unit circle fluently, graphing, identities, and equations in the next sections become routine. If you cannot, each new topic feels like a pile of unrelated formulas.

Students usually get stuck in four places. First, **radians feel unnatural** because we grow up with degrees, so $\frac{5\pi}{6}$ does not look like an angle. Second, **reference angles** get confused with coterminal angles. Third, students memorize a chart of exact values without understanding it, so one forgotten entry ruins a whole problem. Fourth, **signs** go wrong: a sine that should be negative comes out positive. This section builds the unit circle from the ground up so that you can rebuild any value in seconds instead of hoping you remember it.

Core Ideas

An angle in standard position has its vertex at the origin and its initial side along the positive x -axis. Positive angles turn **counterclockwise** and negative angles turn **clockwise**. The final position of the rotating side is the **terminal side**. One full turn is 360° , but mathematicians prefer a second unit built from the circle itself.

Radian Measure

One **radian** is the angle that cuts off an arc whose length equals the radius. In general, an angle of θ radians at the center of a circle of radius r cuts off an arc of length

$$s = r\theta.$$

A full turn is 2π radians, so $360^\circ = 2\pi$ and $180^\circ = \pi$ radians. Conversion: multiply degrees by $\frac{\pi}{180}$ to get radians, and multiply radians by $\frac{180}{\pi}$ to get degrees.

The formula $s = r\theta$ is why radians are so useful: there is no conversion constant. A sector of angle θ also has area $A = \frac{1}{2}r^2\theta$, and both formulas require θ in **radians**. The unit circle is the circle of radius 1 centered at the origin, so on it the arc length from $(1, 0)$ equals the angle itself.

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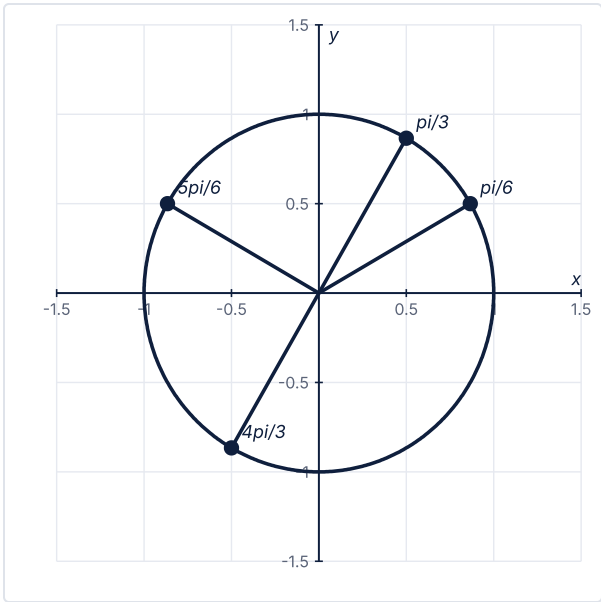
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The Unit Circle Definition

For any angle t in standard position, the terminal side meets the unit circle $x^2 + y^2 = 1$ at exactly one point. That point is $(\cos t, \sin t)$.

So $\cos t$ is the x -coordinate, $\sin t$ is the y -coordinate, and $\tan t = \frac{\sin t}{\cos t}$ whenever $\cos t \neq 0$. Because the point is on the circle, $\cos^2 t + \sin^2 t = 1$.



The unit circle with four marked angles. Each point has coordinates $(\cos t, \sin t)$.

You do not need to memorize the whole circle. You need only the first quadrant, and then symmetry does the rest. The three special first-quadrant angles come from two triangles: the 45° - 45° - 90° triangle (legs equal) and the 30° - 60° - 90° triangle (sides in ratio $1 : \sqrt{3} : 2$).

Degrees	Radians	$\cos t$	$\sin t$	$\tan t$
0°	0	1	0	0
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	0	1	undefined

A memory trick: the sines of $0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$ are $\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$, and cosines run the same list backward.

Reference Angles and Signs

The **reference angle** of t is the acute angle (between 0 and $\frac{\pi}{2}$) that the terminal side makes with the x -axis. In quadrant II it is $\pi - t$, in quadrant III it is $t - \pi$, and in quadrant IV it is $2\pi - t$.

Then $|\cos t|, |\sin t|, |\tan t|$ equal the values for the reference angle, and the sign comes from the quadrant: **I** all positive, **II** only sine positive, **III** only tangent positive, **IV** only cosine positive (the phrase All Students Take Calculus).

Two angles are **coterminal** if they share a terminal side. They differ by a whole number of full turns, so you find them by adding or subtracting 2π (or 360°). Coterminal angles have identical trigonometric values, which lets you reduce huge or negative angles to something on the first lap around the circle.

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Quadrant	Angle range	Reference angle	cos	sin	tan
I	$0 < t < \frac{\pi}{2}$	t	+	+	+
II	$\frac{\pi}{2} < t < \pi$	$\pi - t$	-	+	-
III	$\pi < t < \frac{3\pi}{2}$	$t - \pi$	-	-	+
IV	$\frac{3\pi}{2} < t < 2\pi$	$2\pi - t$	+	-	-

Worked Examples

EXAMPLE 1: CONVERTING BETWEEN UNITS

(a) Convert 150° to radians. (b) Convert $\frac{7\pi}{4}$ radians to degrees.

SOLUTION

(a) Multiply by $\frac{\pi}{180}$: $150 \cdot \frac{\pi}{180} = \frac{150\pi}{180} = \frac{5\pi}{6}$ (divide top and bottom by 30).

(b) Multiply by $\frac{180}{\pi}$: $\frac{7\pi}{4} \cdot \frac{180}{\pi} = \frac{7 \cdot 180}{4} = 7 \cdot 45 = 315^\circ$.

Check: 315° is 45° short of a full turn, and $\frac{7\pi}{4}$ is $\frac{\pi}{4}$ short of 2π . Both agree.

EXAMPLE 2: ARC LENGTH AND SECTOR AREA

A circular garden sector has radius 12 meters and central angle $\frac{5\pi}{6}$. Find the arc length along its curved edge. Then find the area of a sector with radius 10 meters and central angle 72° .

SOLUTION

Arc length: $s = r\theta = 12 \cdot \frac{5\pi}{6} = 10\pi$ meters, about 31.4 meters.

For the second sector, convert first: $72^\circ = 72 \cdot \frac{\pi}{180} = \frac{2\pi}{5}$.

Area: $A = \frac{1}{2}r^2\theta = \frac{1}{2} \cdot 100 \cdot \frac{2\pi}{5} = 20\pi$ square meters.

EXAMPLE 3: COTERMINAL ANGLES AND QUADRANTS

Find the angle between 0 and 2π that is coterminal with (a) $\frac{29\pi}{6}$ and (b) $-\frac{11\pi}{4}$, and name its quadrant.

SOLUTION

(a) Subtract full turns of $2\pi = \frac{12\pi}{6}$. Two turns remove $\frac{24\pi}{6}$: $\frac{29\pi}{6} - \frac{24\pi}{6} = \frac{5\pi}{6}$. Since $\frac{\pi}{2} < \frac{5\pi}{6} < \pi$, the angle is in quadrant II.

(b) The angle is negative, so add turns. $2\pi = \frac{8\pi}{4}$, and two turns add $\frac{16\pi}{4}$: $-\frac{11\pi}{4} + \frac{16\pi}{4} = \frac{5\pi}{4}$. Since $\pi < \frac{5\pi}{4} < \frac{3\pi}{2}$, it is in quadrant III.

EXAMPLE 4: REFERENCE ANGLES

Find the reference angle for (a) $\frac{7\pi}{6}$, (b) 315° , and (c) $\frac{5\pi}{3}$.

SOLUTION

(a) $\frac{7\pi}{6}$ is in quadrant III, so reference angle $= \frac{7\pi}{6} - \pi = \frac{\pi}{6}$.

(b) 315° is in quadrant IV, so reference angle $= 360^\circ - 315^\circ = 45^\circ$.

(c) $\frac{5\pi}{3}$ is in quadrant IV, so reference angle $= 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$.

In every case the reference angle is measured to the nearest part of the x -axis, never to the y -axis.

EXAMPLE 5: EXACT VALUES

Find the exact values of $\sin \frac{5\pi}{6}$, $\cos \frac{5\pi}{6}$, and $\tan \frac{5\pi}{6}$. Then find $\cos \frac{7\pi}{4}$ and $\sin(-\frac{2\pi}{3})$.

SOLUTION

$\frac{5\pi}{6}$ is in quadrant II with reference angle $\pi - \frac{5\pi}{6} = \frac{\pi}{6}$. Reference values: $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\sin \frac{\pi}{6} = \frac{1}{2}$.

Quadrant II: sine positive, cosine negative. So $\sin \frac{5\pi}{6} = \frac{1}{2}$, $\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$, and $\tan \frac{5\pi}{6} = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$.

$\frac{7\pi}{4}$: quadrant IV, reference $\frac{\pi}{4}$, cosine positive, so $\cos \frac{7\pi}{4} = \frac{\sqrt{2}}{2}$.

$-\frac{2\pi}{3}$ is coterminal with $\frac{4\pi}{3}$ (quadrant III, reference $\frac{\pi}{3}$). Sine is negative there: $\sin(-\frac{2\pi}{3}) = -\frac{\sqrt{3}}{2}$.

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EXAMPLE 6: USING THE PYTHAGOREAN RELATION AND A QUADRANT

Suppose $\cos t = -\frac{3}{5}$ and t is in quadrant III. Find $\sin t$ and $\tan t$. Then find every t in $[0, 2\pi)$ with $\sin t = -\frac{\sqrt{2}}{2}$.

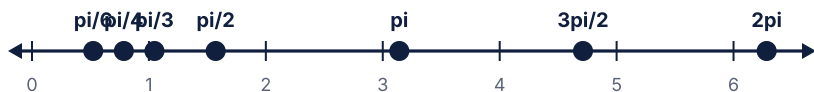
SOLUTION

From $\sin^2 t = 1 - \cos^2 t = 1 - \frac{9}{25} = \frac{16}{25}$ we get $\sin t = \pm \frac{4}{5}$. In quadrant III sine is negative, so $\sin t = -\frac{4}{5}$.

Then $\tan t = \frac{\sin t}{\cos t} = \frac{-4/5}{-3/5} = \frac{4}{3}$, positive as expected in quadrant III.

For $\sin t = -\frac{\sqrt{2}}{2}$: the reference angle is $\frac{\pi}{4}$, and sine is negative in quadrants III and IV. Quadrant III: $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$. Quadrant IV: $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

Answer: $t = \frac{5\pi}{4}$ or $t = \frac{7\pi}{4}$.



Landmark angles on a number line of radians. One lap is about 6.28 radians.

Coach's Playbook

- 1 Think in fractions of pi.** A half turn is π , so $\frac{\pi}{2}$ is a quarter turn, $\frac{\pi}{3}$ is a sixth of a full turn, and $\frac{\pi}{6}$ is a twelfth. Picture a clock: each hour mark is $\frac{\pi}{6}$.
 - 2 Find the quadrant first.** Before computing, decide which quadrant the angle is in. This fixes the signs and tells you which reference-angle formula to use.
 - 3 Reduce to the reference angle.** Use the first-quadrant table, then attach the sign from the quadrant.
 - 4 Reduce with common denominators.** To subtract full turns from a fraction like $\frac{29\pi}{6}$, write 2π with the same denominator ($\frac{12\pi}{6}$).
 - 5 Check with a sketch.** A quick circle with the angle drawn tells you if your answer has the right sign and size.
- ♦ **Hand trick for the table.** Hold your left hand up, palm facing you. Fingers from thumb to pinky stand for $90^\circ, 60^\circ, 45^\circ, 30^\circ, 0^\circ$. To get the sine, count fingers below and take $\frac{\sqrt{n}}{2}$; to get cosine, count the fingers above.
 - ♦ **Axis angles.** At $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ the point is $(1, 0), (0, 1), (-1, 0), (0, -1)$. Tangent is undefined where $\cos t = 0$.
 - ♦ **Degree-radian shortcut.** Think π means 180° and substitute: $\frac{2\pi}{3}$ becomes $\frac{2 \cdot 180}{3} = 120^\circ$.

Common traps

(1) Using the reference angle as the answer without fixing the sign. (2) Measuring a reference angle to the y -axis instead of the x -axis. (3) Forgetting that $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ need radians; with degrees the answer is wrong by a factor of about 57.3. (4) Swapping the values at $\frac{\pi}{6}$ and $\frac{\pi}{3}$: the smaller angle has the smaller sine ($\frac{1}{2}$) and the larger cosine.

Mistake Clinic**Spot the error**

Problem: Find $\cos \frac{5\pi}{6}$. **Student work:** The reference angle is $\frac{\pi}{6}$, so $\cos \frac{5\pi}{6} = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$. **Where it went wrong:** the student stopped at the reference angle and never applied the quadrant sign. $\frac{5\pi}{6}$ is in quadrant II, where the x -coordinate is negative. **Correct:** $\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$.

Spot the error

Problem: Find the reference angle of $\frac{4\pi}{3}$. **Student work:** $\frac{4\pi}{3}$ is in quadrant III, and $\pi - \frac{4\pi}{3} = -\frac{\pi}{3}$, so the reference angle is $-\frac{\pi}{3}$. **Where it went wrong:** the student used the quadrant II formula. A reference angle is always positive and acute. For quadrant III use $t - \pi$. **Correct:** $\frac{4\pi}{3} - \pi = \frac{\pi}{3}$.

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Spot the error

Problem: A wheel of radius 5 cm turns through 90° . Find the arc length. **Student work:** $s = r\theta = 5 \cdot 90 = 450$ cm. **Where it went wrong:** the formula $s = r\theta$ only works when θ is in radians. A 450 cm arc on a wheel of radius 5 cm is absurd (the whole circumference is only about 31.4 cm). **Correct:** $90^\circ = \frac{\pi}{2}$, so $s = 5 \cdot \frac{\pi}{2} = \frac{5\pi}{2} \approx 7.85$ cm.

Real-World Connection

A bicycle wheel has radius 0.35 m. Each revolution is 2π radians, so one revolution moves the bike $s = 0.35 \cdot 2\pi = 0.7\pi \approx 2.2$ meters. If the wheel turns 5 revolutions every 2 seconds, the angle turns at $\frac{10\pi}{2} = 5\pi$ radians per second, and the bike travels about $0.35 \cdot 5\pi \approx 5.5$ meters per second. This is how a speedometer converts spinning into speed.

Challenge Corner

Problem: A point P moves counterclockwise on the unit circle starting at $(1, 0)$. After traveling an arc length of $\frac{17\pi}{3}$, find the coordinates of P and the quadrant. **Solution:** Remove full laps: $\frac{17\pi}{3} - 4\pi = \frac{17\pi}{3} - \frac{12\pi}{3} = \frac{5\pi}{3}$ (two laps). The angle $\frac{5\pi}{3}$ is in quadrant IV with reference angle $\frac{\pi}{3}$. Cosine is positive and sine is negative, so $P = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ in quadrant IV.

Practice**PRACTICE Radians, Degrees, and the Unit Circle**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Convert 225° to radians.
(A) $\frac{3\pi}{4}$ (B) $\frac{5\pi}{4}$ (C) $\frac{5\pi}{6}$ (D) $\frac{9\pi}{4}$
- What is the reference angle for $\frac{11\pi}{6}$?
(A) $\frac{\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{11\pi}{6}$
- What is $\cos \frac{2\pi}{3}$?
(A) $\frac{1}{2}$ (B) $-\frac{\sqrt{3}}{2}$ (C) $-\frac{1}{2}$ (D) $\frac{\sqrt{3}}{2}$
- A circle has radius 8 cm. What is the length of the arc cut off by a central angle of $\frac{3\pi}{4}$?
(A) 24π cm (B) $\frac{3\pi}{32}$ cm (C) 1080π cm (D) 6π cm
- What is $\tan \frac{7\pi}{4}$?
(A) -1 (B) 1 (C) $-\frac{\sqrt{2}}{2}$ (D) 0
- Which angle is coterminal with $-\frac{\pi}{3}$?
(A) $\frac{2\pi}{3}$ (B) $\frac{5\pi}{3}$ (C) $\frac{4\pi}{3}$ (D) $\frac{7\pi}{3}$
- If $\sin t = \frac{3}{5}$ and t is in quadrant II, what is $\cos t$?
(A) $\frac{4}{5}$ (B) $-\frac{3}{4}$ (C) $-\frac{4}{5}$ (D) $\frac{4}{3}$
- The point $P\left(-\frac{5}{13}, y\right)$ lies on the unit circle with $y > 0$. What is y ?
(A) $\frac{12}{13}$ (B) $-\frac{12}{13}$ (C) $\frac{8}{13}$ (D) $\frac{5}{13}$
- A sector has radius 6 and central angle $\frac{2\pi}{3}$. What is its area?
(A) 24π (B) 4π (C) 8π (D) 12π
- Convert $\frac{7\pi}{12}$ radians to degrees.
- Find the exact value of $\cos\left(-\frac{5\pi}{3}\right) + \sin \frac{7\pi}{6}$.
- Find every t in $[0, 2\pi)$ with $\cos t = -\frac{\sqrt{2}}{2}$.
- The terminal side of angle t is in quadrant III and $\tan t = 2$. Find the exact value of $\cos t$.
- The tip of a fan blade is 0.4 m from the center, and the fan spins at 150 revolutions per minute. Give the exact speed of the tip in meters per minute.
- Angle t is in quadrant IV and $\sin t = -\frac{\sqrt{7}}{4}$. Find the exact value of $\tan t$.

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25.2 Graphs of Sine and Cosine

Master amplitude, period, phase shift, and midline for sine and cosine. Sketch any sinusoid with the five-point method, then work backward from a graph to write its equation.

Why This Matters

Sine and cosine are the mathematics of anything that repeats: a swinging pendulum, the hum of a speaker, the voltage in a wall outlet, daily temperature, the height of the tide, the position of a seat on a Ferris wheel. A graph of a repeating quantity is called a **sinusoid**, and once you can read its amplitude, period, shift, and midline, you can describe the whole pattern with one short equation and predict what it will do at any time.

Students usually struggle with four things. They forget to **factor out the coefficient of x** before reading the phase shift, so $\sin(2x - \pi)$ is misread as a shift of π instead of $\frac{\pi}{2}$. They confuse the **period** with the **frequency**, using B instead of $\frac{2\pi}{B}$. They mix up **amplitude** with the total height from bottom to top. And when **writing an equation from a graph**, they guess instead of following a method. This section gives you a reliable routine for each one.

Core Ideas

Start with the parent graphs. The graph of $y = \sin x$ starts at $(0, 0)$, rises to 1 at $\frac{\pi}{2}$, returns to 0 at π , falls to -1 at $\frac{3\pi}{2}$, and returns to 0 at 2π , where it repeats. The graph of $y = \cos x$ is the same wave shifted left by $\frac{\pi}{2}$: it starts at its maximum $(0, 1)$. Both are **periodic** with period 2π and both have range $[-1, 1]$ and domain all real numbers.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	-1	0
$\cos x$	1	0	-1	0	1

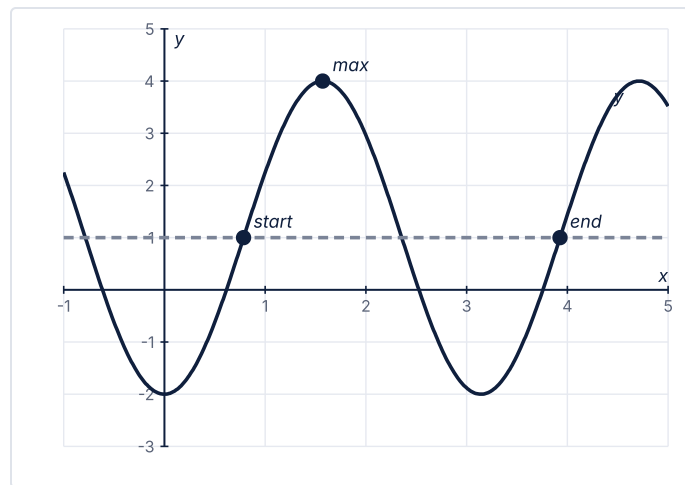
Every sinusoid is built from these five key points, stretched, flipped, and moved. The general form shows four independent controls.

General Sinusoid

$$y = A \sin(B(x - C)) + D \quad \text{or} \quad y = A \cos(B(x - C)) + D.$$

Amplitude = $|A|$, the distance from the midline to a peak. **Period** = $\frac{2\pi}{|B|}$, the length of one full cycle. **Phase shift** = C (right if $C > 0$, left if $C < 0$) once the argument is factored as $B(x - C)$. **Midline** $y = D$. **Maximum** = $D + |A|$ and **minimum** = $D - |A|$.

If $A < 0$, the graph is flipped upside down: a negative sine starts by going down, and a negative cosine starts at its minimum. If $B < 0$, use the identities $\sin(-u) = -\sin u$ and $\cos(-u) = \cos u$ to make B positive before reading the graph.



Graph of $y = 3 \sin(2(x - \pi/4)) + 1$. Midline $y = 1$ (dashed), amplitude 3, period π , shifted right $\pi/4$.

The Five-Point Method

Divide one period into four equal parts. The five points x_0, x_1, x_2, x_3, x_4 are the start, quarter, half, three-quarter, and end of the cycle. For $A \sin$: values are midline, max, midline, min, midline. For $A \cos$: max, midline, min, midline, max. (Reverse max and min when $A < 0$.) Starting x is C , and each step is $\frac{\text{period}}{4}$.

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This gives a fast way to sketch any sinusoid: find the midline and amplitude, compute the period, locate the start at $x = C$, then step by a quarter period, plotting five points and connecting them with a smooth curve. To read an equation off a graph, work the process in reverse: maximum and minimum give D and $|A|$, the distance between two consecutive peaks gives the period and so B , and a peak or midline crossing gives C .

Quantity	How to find it from a graph
Midline D	$\frac{\max + \min}{2}$
Amplitude $ A $	$\frac{\max - \min}{2}$
Period	distance between two consecutive maxima (or minima)
B	$\frac{2\pi}{\text{period}}$
Shift C	x -value of a max (cosine form) or an up-crossing of the midline (sine form)

Worked Examples

EXAMPLE 1: READING THE FOUR FEATURES

State the amplitude, period, phase shift, midline, and range of $y = -3 \sin\left(2x - \frac{\pi}{2}\right) + 1$.

SOLUTION

First factor the argument: $2x - \frac{\pi}{2} = 2\left(x - \frac{\pi}{4}\right)$. So $A = -3$, $B = 2$, $C = \frac{\pi}{4}$, $D = 1$.

Amplitude $|A| = 3$. Period $\frac{2\pi}{2} = \pi$. Phase shift $\frac{\pi}{4}$ to the right. Midline $y = 1$.

Maximum $= 1 + 3 = 4$, minimum $= 1 - 3 = -2$, so the range is $[-2, 4]$. Because $A < 0$, the graph starts at the midline at $x = \frac{\pi}{4}$ and goes **down** first.

EXAMPLE 2: SKETCHING A COSINE

Sketch one period of $y = 2 \cos\left(\frac{x}{2}\right)$ and list its five key points.

SOLUTION

Amplitude 2, midline $y = 0$, $B = \frac{1}{2}$, so the period is $\frac{2\pi}{1/2} = 4\pi$. There is no shift.

A quarter of the period is π , so the key x -values are $0, \pi, 2\pi, 3\pi, 4\pi$.

Cosine goes max, midline, min, midline, max: $(0, 2), (\pi, 0), (2\pi, -2), (3\pi, 0), (4\pi, 2)$. Connect with a smooth wave.

EXAMPLE 3: MAXIMUM, MINIMUM, AND WHERE THEY OCCUR

For $y = 4 \cos\left(3\left(x - \frac{\pi}{6}\right)\right) - 2$ find the range, the period, and the first $x \geq \frac{\pi}{6}$ where the minimum occurs.

SOLUTION

Midline -2 , amplitude 4: maximum 2, minimum -6 . Range $[-6, 2]$.

Period $= \frac{2\pi}{3}$. Cosine starts at its maximum at $x = C = \frac{\pi}{6}$.

The minimum is half a period later: $\frac{\pi}{6} + \frac{1}{2} \cdot \frac{2\pi}{3} = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$. So the first minimum is at $x = \frac{\pi}{2}$ with value -6 .

EXAMPLE 4: WRITING A COSINE EQUATION FROM A GRAPH

The graph shows the depth y (in feet) of water in a channel x hours after midnight. Write a cosine equation for it.

SOLUTION

The graph shows maximum 7 at $x = 1$ and minimum -1 at $x = 5$, so the half period is $5 - 1 = 4$ and the period is 8.

Midline $D = \frac{7+(-1)}{2} = 3$. Amplitude $A = \frac{7-(-1)}{2} = 4$. $B = \frac{2\pi}{8} = \frac{\pi}{4}$.

Cosine peaks at its start, and the first peak is at $x = 1$, so $C = 1$:

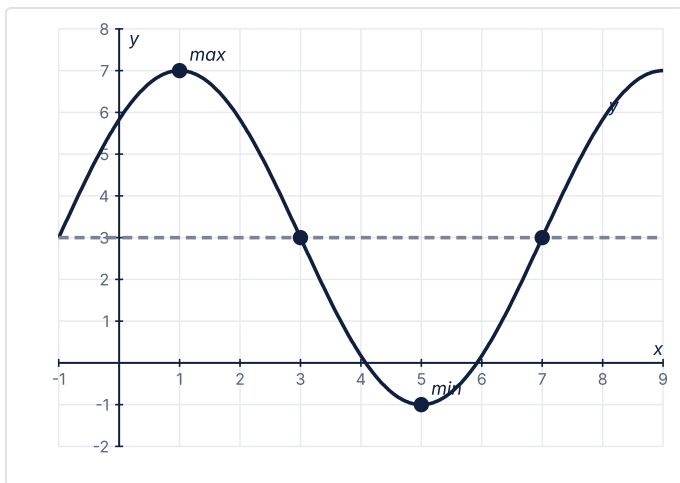
$$y = 4 \cos\left(\frac{\pi}{4}(x - 1)\right) + 3.$$

Check: at $x = 1$ the cosine is 1, giving $4 + 3 = 7$. At $x = 5$, $\frac{\pi}{4} \cdot 4 = \pi$, cosine is -1 , giving -1 . Both match.

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Graph for Example 4. Maximum 7 at $x = 1$, minimum -1 at $x = 5$.

EXAMPLE 5: THE SAME GRAPH AS A SINE

Rewrite the graph from Example 4 as a sine function.

SOLUTION

Sine starts on the midline going **up**. On this graph the curve crosses the midline going up one quarter period before the peak, at $x = 1 - 2 = -1$ (the period is 8, so a quarter is 2).

So $C = -1$, with the same $A = 4$, $B = \frac{\pi}{4}$, $D = 3$:

$$y = 4 \sin \left(\frac{\pi}{4}(x + 1) \right) + 3.$$

Check: $x = 1$ gives $\sin \frac{\pi}{2} = 1$ and $y = 7$. Alternatively, use $\cos u = \sin \left(u + \frac{\pi}{2} \right)$. Many equations describe the same graph; any correct shift plus a whole number of periods works.

EXAMPLE 6: SOLVING FROM A SINUSOID MODEL

Let $f(x) = -2 \cos \left(\frac{\pi x}{3} \right) + 1$. Find $f(1)$, then find all x in $[0, 6)$ with $f(x) = 3$ and all x in $[0, 6)$ with $f(x) = 0$.

SOLUTION

$$f(1) = -2 \cos \frac{\pi}{3} + 1 = -2 \cdot \frac{1}{2} + 1 = 0.$$

For $f(x) = 3$: $-2 \cos \frac{\pi x}{3} = 2$, so $\cos \frac{\pi x}{3} = -1$. Cosine is -1 only at π (within one period), so $\frac{\pi x}{3} = \pi$ and $x = 3$.

For $f(x) = 0$: $\cos \frac{\pi x}{3} = \frac{1}{2}$. On $[0, 2\pi)$ the angles are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ (the period of f is 6, so $\frac{\pi x}{3}$ runs from 0 to 2π). So $\frac{\pi x}{3} = \frac{\pi}{3}$ gives $x = 1$ and $\frac{\pi x}{3} = \frac{5\pi}{3}$ gives $x = 5$.

Answers: $f(1) = 0$; $f(x) = 3$ at $x = 3$; $f(x) = 0$ at $x = 1$ and $x = 5$.

Coach's Playbook

- 1 Factor first.** Rewrite the inside as $B(x - C)$ before reading any shift. $\sin(2x + \pi)$ is $\sin \left(2 \left(x + \frac{\pi}{2} \right) \right)$, a shift of $\frac{\pi}{2}$ to the left.
- 2 Midline and amplitude from the extremes.** Average the max and min for D , and halve their difference for $|A|$.
- 3 Period from the picture.** Measure peak to peak, then set $B = \frac{2\pi}{\text{period}}$. A trough to trough distance works too, and half the distance from max to min is a half period.
- 4 Choose the easiest anchor.** For a cosine use a maximum (or a minimum with $A < 0$). For a sine use an upward midline crossing.
- 5 Verify with two points.** Substitute a max and a min or midline point into your equation. If both give the right height, you are done.

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- ◆ **Quarter-period stepping.** Once you know the period, every key point is spaced by a quarter period. List them in a table and the whole sketch follows.
- ◆ **Solving $\sin(Bx) = c$.** First solve for the angle Bx on one lap, then divide by B . The number of solutions per period of x is the same as for the basic wave.
- ◆ **Number of cycles.** On an interval of length L , a sinusoid completes $\frac{L}{\text{period}}$ cycles.

Common traps

(1) Reading the shift from $\sin(2x - \pi)$ as π when it is $\frac{\pi}{2}$. (2) Using B itself as the period instead of $\frac{2\pi}{B}$. (3) Calling the amplitude the full height $\max - \min$; amplitude is half of that. (4) Forgetting that a negative A reflects the graph, so a negative cosine starts at its minimum.

Mistake Clinic

Spot the error

Problem: Find the period of $y = \sin\left(\frac{\pi x}{4}\right)$. **Student work:** The coefficient of x is $\frac{\pi}{4}$, so the period is $\frac{\pi}{4}$. **Where it went wrong:** the coefficient of x is B , and the period is $\frac{2\pi}{B}$, not B . **Correct:** period = $\frac{2\pi}{\pi/4} = 8$.

Spot the error

Problem: Find the phase shift of $y = 3 \cos(2x - \pi)$. **Student work:** The shift is π to the right. **Where it went wrong:** the argument must be factored as $B(x - C)$. Since $2x - \pi = 2\left(x - \frac{\pi}{2}\right)$, the shift is $\frac{\pi}{2}$. **Correct:** $\frac{\pi}{2}$ to the right. Check: the first maximum is at $x = \frac{\pi}{2}$, since $2 \cdot \frac{\pi}{2} - \pi = 0$ and $\cos 0 = 1$.

Spot the error

Problem: A sinusoid has maximum 9 and minimum 1. Find its amplitude and midline. **Student work:** Amplitude = $9 - 1 = 8$, midline = $\frac{9+1}{2} = 5$. **Where it went wrong:** amplitude is the distance from the midline to a peak, which is half the total height. **Correct:** amplitude = $\frac{9-1}{2} = 4$, midline $y = 5$. Check: $5 + 4 = 9$ and $5 - 4 = 1$.

Real-World Connection

The height of a seat on a Ferris wheel follows a sinusoid. Suppose the wheel has radius 20 m, its center is 22 m above the ground, and it makes one turn every 12 minutes. Starting at the lowest point, the height is $h = -20 \cos\left(\frac{\pi}{6}t\right) + 22$. At $t = 3$ minutes, $h = -20 \cos \frac{\pi}{2} + 22 = 22$ m, the level of the center. At $t = 6$ minutes, $h = 42$ m, the top.

Challenge Corner

Problem: A sinusoid has amplitude 3, period 6, midline $y = -1$, and its maximum occurs at $x = 2$. Write a sine equation for it. **Solution:** $B = \frac{2\pi}{6} = \frac{\pi}{3}$. A sine curve reaches its maximum a quarter period after an upward midline crossing, and a quarter period is 1.5. So $C = 2 - 1.5 = \frac{1}{2}$. The equation is $y = 3 \sin\left(\frac{\pi}{3}\left(x - \frac{1}{2}\right)\right) - 1$. Check: at $x = 2$, $\frac{\pi}{3} \cdot \frac{3}{2} = \frac{\pi}{2}$, so $y = 3 - 1 = 2$, the maximum $-1 + 3$.

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Practice

PRACTICE Graphs of Sine and Cosine

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is the amplitude of $y = -5 \cos(2x) + 3$?
(A) -5 (B) 5 (C) 3 (D) 8
- What is the period of $y = \sin\left(\frac{\pi x}{4}\right)$?
(A) 8 (B) 4 (C) 2π (D) $\frac{\pi}{4}$
- Which describes the phase shift of $y = \sin(2x + \pi)$?
(A) π to the left (B) $\frac{\pi}{2}$ to the right (C) $\frac{\pi}{2}$ to the left
(D) π to the right
- What is the range of $y = -2 \sin x + 4$?
(A) $[2, 6]$ (B) $[-2, 2]$ (C) $[-2, 4]$ (D) $[4, 6]$
- A sinusoid has maximum 9 at $x = 0$, minimum 1, and period 12. Which equation fits?
(A) $y = 4 \cos\left(\frac{\pi x}{12}\right) + 5$ (B) $y = 9 \cos\left(\frac{\pi x}{6}\right) + 1$ (C) $y = 4 \cos(12x) + 5$
(D) $y = 4 \cos\left(\frac{\pi x}{6}\right) + 5$
- What is the smallest positive x at which $y = 2 \sin(2x)$ reaches its maximum?
(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{8}$ (D) π
- Which function has the same graph as $y = \cos x$?
(A) $y = \sin\left(x - \frac{\pi}{2}\right)$ (B) $y = -\sin\left(x + \frac{\pi}{2}\right)$ (C) $y = \sin\left(x + \frac{\pi}{2}\right)$ (D) $y = \sin(x + \pi)$
- How many complete periods of $y = \cos(3x)$ fit in the interval $[0, 4\pi]$?
(A) 6 (B) 12 (C) 3 (D) 2
- If $y = 3 \cos(2x - \pi) + 2$, what is y when $x = \frac{\pi}{2}$?
(A) -1 (B) 2 (C) 3 (D) 5
- Give the amplitude and period of $y = -\frac{1}{2} \sin(4\pi x)$.
- A sinusoid has its minimum -2 at $x = 0$ and its next maximum 6 at $x = 5$. Write a cosine equation for it.
- For $y = 3 \sin\left(x - \frac{\pi}{6}\right) + 1$, find the maximum value, the minimum value, and the smallest positive x at which the maximum occurs.
- The temperature (in degrees) in a greenhouse is modeled by $T = 10 \cos\left(\frac{\pi}{6}(t - 3)\right) + 60$, where t is hours after 6 a.m. Find the lowest temperature and the first time $t > 0$ it occurs.
- Solve $2 \cos(3x) + 1 = 2$ for all x in $\left[0, \frac{2\pi}{3}\right]$.
- Write a sine function with amplitude 3, period 6, and midline $y = -1$ whose maximum occurs at $x = 2$.

25.3 Tangent and the Reciprocal Functions

Define tangent, cotangent, secant, and cosecant from sine and cosine. Graph them with asymptotes, find exact values, state domain and range, and write a tangent equation from a graph.

Why This Matters

Sine and cosine describe smooth waves, but a lot of the world is about **ratios and slopes**. The tangent of an angle is the slope of a line. It tells a surveyor how high a hillside rises over a given distance, tells a game programmer where a camera ray lands on a screen, and tells an engineer how steep a ramp is. The three reciprocal functions, cosecant, secant, and cotangent, appear in calculus (you will integrate $\sec^2 x$), in physics (refraction and optics), and in navigation.

Students usually get stuck in four places. They cannot remember **which function is the reciprocal of which**, mixing up secant with sine. They forget that the tangent graph has **vertical asymptotes** and instead draw a curve that connects through them. They use the wrong period, 2π instead of π . And they struggle with **domain and range**, especially for secant and cosecant, whose graphs have gaps. This section removes those problems by building every function from sine and cosine, so you only need one set of facts.

Core Ideas

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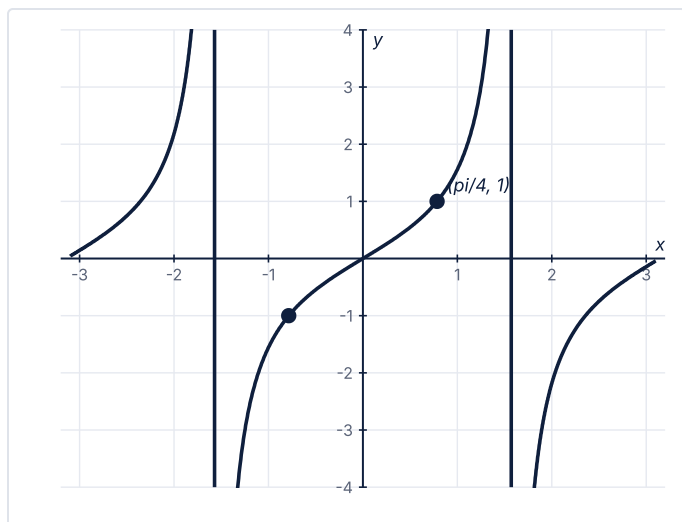
Definitions

$$\tan t = \frac{\sin t}{\cos t}, \quad \cot t = \frac{\cos t}{\sin t} = \frac{1}{\tan t}, \quad \sec t = \frac{1}{\cos t}, \quad \csc t = \frac{1}{\sin t}.$$

Each is undefined exactly where its denominator is 0. Memory aid: the names are crossed. **Secant** goes with **cosine**, and **cosecant** goes with **sine**.

Since the unit circle point is $(\cos t, \sin t)$, tangent is the slope of the terminal side, $\frac{y}{x}$. Where the terminal side is vertical ($x = 0$, at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$) the slope does not exist, so tangent has a **vertical asymptote** there. Its signs follow the quadrants: positive in I and III, negative in II and IV.

Angle t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan t$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined
$\csc t$	undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
$\sec t$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	undefined
$\cot t$	undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0



Graph of $y = \tan x$. Vertical asymptotes at $x = -\pi/2$ and $x = \pi/2$ (and every π after). Period π , zeros at $0, \pi$.

Tangent Graph Facts

$y = \tan x$ has **period** π , domain all $x \neq \frac{\pi}{2} + k\pi$, range all real numbers, zeros at $x = k\pi$, and increases on every interval between asymptotes. It is an odd function: $\tan(-x) = -\tan x$.

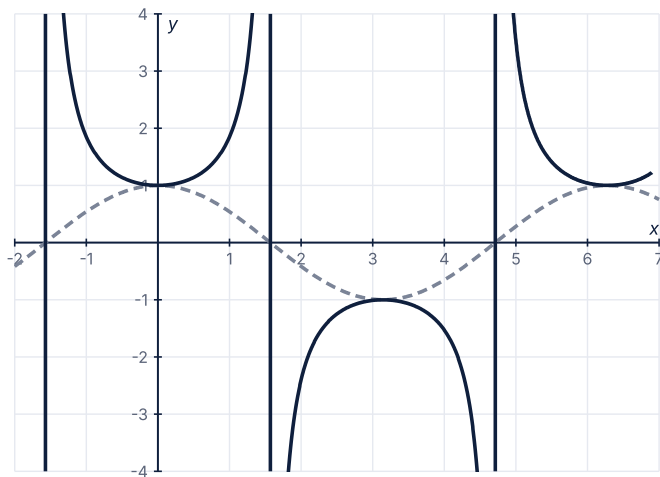
For $y = A \tan(B(x - C)) + D$: period = $\frac{\pi}{|B|}$, the center of each branch is $(C + \text{multiple of period}, D)$, and asymptotes are at $x = C + \frac{\pi/2 + k\pi}{B}$. A negative A flips the branch so it decreases.

Secant and cosecant are best sketched **from their partner waves**. Lightly sketch $y = \cos x$. Wherever cosine is 0, secant has an asymptote. Wherever cosine peaks at 1 or bottoms at -1 , secant touches the same point and the graph turns away from the axis, forming U-shaped branches that never enter the band $-1 < y < 1$. Cosecant works the same way from sine.

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$y = \sec x$ (solid) built from $y = \cos x$ (dashed). Secant has asymptotes where cosine is zero and branches that touch the peaks and valleys of cosine.

Secant, Cosecant, Cotangent

$y = \sec x$: period 2π , domain $x \neq \frac{\pi}{2} + k\pi$, range $(-\infty, -1] \cup [1, \infty)$. $y = \csc x$: period 2π , domain $x \neq k\pi$, same range. $y = \cot x$: period π , domain $x \neq k\pi$, range all reals, zeros at $x = \frac{\pi}{2} + k\pi$, decreasing on each branch.

Because $|\cos x| \leq 1$ and $|\sin x| \leq 1$, the reciprocals satisfy $|\sec x| \geq 1$ and $|\csc x| \geq 1$.

Worked Examples

EXAMPLE 1: EXACT VALUES OF RECIPROCAL FUNCTIONS

Find the exact value of (a) $\sec \frac{5\pi}{6}$, (b) $\csc \frac{7\pi}{4}$, (c) $\cot \frac{2\pi}{3}$.

SOLUTION

(a) $\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$, so $\sec \frac{5\pi}{6} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$.

(b) $\sin \frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$, so $\csc \frac{7\pi}{4} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$.

(c) $\cos \frac{2\pi}{3} = -\frac{1}{2}$ and $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, so $\cot \frac{2\pi}{3} = \frac{-1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$.

Always find the sine and cosine first, then flip or divide.

EXAMPLE 2: FINDING ALL SIX VALUES

Given $\sec t = -\frac{5}{3}$ and $\sin t > 0$, find the other five trigonometric values.

SOLUTION

$\cos t = \frac{1}{\sec t} = -\frac{3}{5}$. Cosine negative and sine positive puts t in quadrant II.

$\sin^2 t = 1 - \frac{9}{25} = \frac{16}{25}$, so $\sin t = \frac{4}{5}$ (positive).

$\tan t = \frac{4/5}{-3/5} = -\frac{4}{3}$, $\csc t = \frac{5}{4}$, $\cot t = -\frac{3}{4}$.

Check signs for quadrant II: only sine and cosecant are positive. They are.

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EXAMPLE 3: SKETCHING A STRETCHED TANGENT

Find the period, asymptotes, and three key points of one branch of $y = \tan(2x)$.

SOLUTION

$B = 2$, so the period is $\frac{\pi}{2}$. The branch around the origin runs between the asymptotes where $2x = \pm\frac{\pi}{2}$, that is $x = \pm\frac{\pi}{4}$. Others are at $x = \frac{\pi}{4} + k \cdot \frac{\pi}{2}$.

Zeros: $2x = k\pi$, so $x = \frac{k\pi}{2}$.

Key points on the central branch: $(-\frac{\pi}{8}, -1)$, $(0, 0)$, $(\frac{\pi}{8}, 1)$ because $\tan(2 \cdot \frac{\pi}{8}) = \tan \frac{\pi}{4} = 1$. Draw an increasing curve that approaches the asymptotes.

EXAMPLE 4: SHIFTED, FLIPPED TANGENT

Graph one branch of $y = -2 \tan(x - \frac{\pi}{3}) + 1$. Give the asymptotes, the center, and two other points.

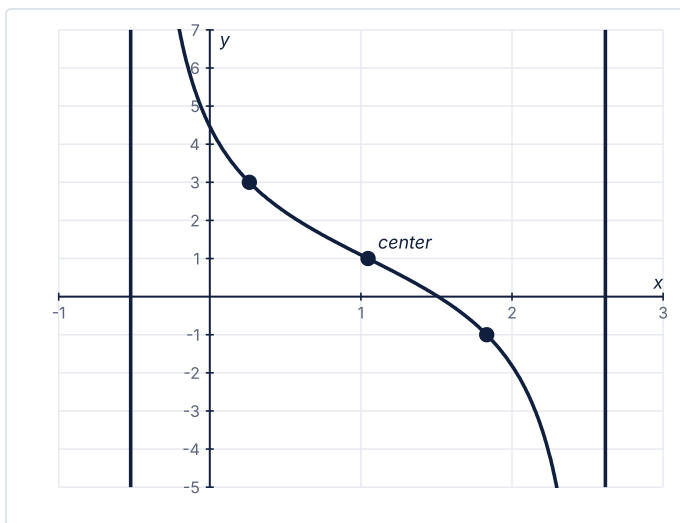
SOLUTION

Period π . The center of the branch is $(\frac{\pi}{3}, 1)$, where the inside equals 0.

Asymptotes where $x - \frac{\pi}{3} = \pm\frac{\pi}{2}$: $x = \frac{5\pi}{6}$ and $x = -\frac{\pi}{6}$.

At $x = \frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$: $\tan \frac{\pi}{4} = 1$, so $y = -2 + 1 = -1$. At $x = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$: $\tan(-\frac{\pi}{4}) = -1$, so $y = 2 + 1 = 3$.

Because $A = -2 < 0$, the branch **decreases** from top left to bottom right.



Graph for Example 4. The vertical lines are asymptotes at $x = -\pi/6$ and $x = 5\pi/6$.

EXAMPLE 5: DOMAIN AND RANGE OF A RECIPROCAL GRAPH

State the domain and range of $y = 2 \csc x - 1$.

SOLUTION

Cosecant is undefined where $\sin x = 0$, so the domain is all real x with $x \neq k\pi$.

Since $|\csc x| \geq 1$, we have $\csc x \geq 1$ or $\csc x \leq -1$. Multiply by 2: $2 \csc x \geq 2$ or $2 \csc x \leq -2$. Subtract 1: $y \geq 1$ or $y \leq -3$.

Range: $(-\infty, -3] \cup [1, \infty)$. The graph never visits the band $-3 < y < 1$.

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EXAMPLE 6: WRITING A TANGENT EQUATION FROM A GRAPH

A tangent graph has consecutive vertical asymptotes at $x = 1$ and $x = 5$, passes through its center at $(3, 2)$, and passes through $(4, 5)$. Write its equation.

SOLUTION

The distance between asymptotes is the period: $5 - 1 = 4$. So $B = \frac{\pi}{4}$.

The center is $(3, 2)$, so $C = 3$ and $D = 2$: $y = A \tan\left(\frac{\pi}{4}(x - 3)\right) + 2$.

Use $(4, 5)$: $5 = A \tan \frac{\pi}{4} + 2 = A + 2$, so $A = 3$.

Equation:

$$y = 3 \tan\left(\frac{\pi}{4}(x - 3)\right) + 2.$$

Check with $x = 2$: $3 \tan\left(-\frac{\pi}{4}\right) + 2 = -1$, which sits symmetrically below the center as it should.

Coach's Playbook

- 1 Rewrite in sine and cosine.** Any value of $\tan$, $\sec$, $\csc$, or $\cot$ is found by getting $\sin$ and $\cos$ first. Then divide or flip.
 - 2 Locate the asymptotes first.** For tangent, set the inside equal to $\frac{\pi}{2} + k\pi$. For secant, find the zeros of the cosine. For cosecant and cotangent, find the zeros of the sine.
 - 3 Mark the center or turning points.** One branch of tangent is pinned by its center and two points a quarter period away. A reciprocal branch is pinned by the peaks and valleys of its partner wave.
 - 4 Decide direction.** Tangent increases when $A > 0$ and decreases when $A < 0$. Cotangent decreases when $A > 0$.
 - 5 State domain and range carefully.** Write excluded values with a ' k ' for all integers, and keep the gap in the range of secant and cosecant.
- ◆ **Period check.** Tangent and cotangent repeat every π . Secant and cosecant repeat every 2π .
 - ◆ **Slope view.** $\tan t$ is the slope of the terminal side. At 45° the slope is 1, at 60° it is steeper ($\sqrt{3}$), and it grows without bound near 90° .
 - ◆ **Sign check.** A reciprocal always has the same sign as the original, so $\csc t$ has the sign of $\sin t$.

Common traps

(1) Pairing secant with sine instead of cosine. (2) Using $\frac{2\pi}{B}$ for the period of tangent; it is $\frac{\pi}{B}$. (3) Connecting the branches of tangent across an asymptote. (4) Saying the range of secant or cosecant is $[-1, 1]$ or all reals; it is $(-\infty, -1] \cup [1, \infty)$ before stretching.

Mistake Clinic**Spot the error**

Problem: Find the period of $y = \tan(2x)$. **Student work:** Period = $\frac{2\pi}{2} = \pi$. **Where it went wrong:** that formula is for sine and cosine. Tangent already repeats every π , so the period is $\frac{\pi}{2}$. **Correct:** $\frac{\pi}{2}$. Check: the asymptotes are at $\pm\frac{\pi}{4}$, a distance of $\frac{\pi}{2}$ apart.

Spot the error

Problem: Find $\sec \frac{\pi}{3}$. **Student work:** $\sec \frac{\pi}{3} = \frac{1}{\sin(\pi/3)} = \frac{2}{\sqrt{3}}$. **Where it went wrong:** secant is the reciprocal of **cosine**, not sine. **Correct:** $\sec \frac{\pi}{3} = \frac{1}{\cos(\pi/3)} = \frac{1}{1/2} = 2$.

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Spot the error

Problem: State the range of $y = \csc x$. **Student work:** Sine has range $[-1, 1]$, so cosecant has range $[-1, 1]$ too. **Where it went wrong:** taking a reciprocal of a number between 0 and 1 makes it larger than 1, not smaller. **Correct:** $|\csc x| \geq 1$, so the range is $(-\infty, -1] \cup [1, \infty)$. Example: $\sin x = \frac{1}{2}$ gives $\csc x = 2$.

Real-World Connection

A lighthouse beam rotates while the shoreline is a straight line 100 m away. When the beam makes angle θ with the line perpendicular to the shore, the spot on the shore is $d = 100 \tan \theta$ meters from the closest point. At $\theta = \frac{\pi}{3}$, $d = 100\sqrt{3} \approx 173$ m. As θ nears $\frac{\pi}{2}$ the spot races toward infinity: that is the vertical asymptote of the tangent graph in action.

Challenge Corner

Problem: How many vertical asymptotes does $y = \tan(3x)$ have on the interval $[0, 2\pi]$? **Solution:** Asymptotes occur where $3x = \frac{\pi}{2} + k\pi$, so $x = \frac{\pi}{6} + \frac{k\pi}{3}$. For $k = 0, 1, 2, 3, 4, 5$ we get $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$, all at most 2π . For $k = 6$ the value is $\frac{13\pi}{6} > 2\pi$. Answer: 6 asymptotes (the period is $\frac{\pi}{3}$ and $\frac{2\pi}{\pi/3} = 6$ cycles).

Practice**PRACTICE Tangent and the Reciprocal Functions**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is $\tan \frac{2\pi}{3}$?
(A) $\sqrt{3}$ (B) $-\frac{\sqrt{3}}{3}$ (C) $-\sqrt{3}$ (D) $-\frac{1}{2}$
- What is the period of $y = \tan(3x)$?
(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) 3π (D) π
- Which is a vertical asymptote of $y = \tan(\frac{x}{2})$?
(A) $x = \frac{\pi}{2}$ (B) $x = \frac{\pi}{4}$ (C) $x = 2\pi$ (D) $x = \pi$
- What is $\csc \frac{\pi}{6}$?
(A) $\frac{1}{2}$ (B) 2 (C) $\frac{2\sqrt{3}}{3}$ (D) $\sqrt{3}$
- What is the range of $y = \sec x$?
(A) $[-1, 1]$ (B) all real numbers (C) $(-\infty, -1) \cup (1, \infty)$
(D) $(-\infty, -1] \cup [1, \infty)$
- What is $\cot \frac{5\pi}{4}$?
(A) -1 (B) 0 (C) 1 (D) $-\sqrt{2}$
- If $\sec t = -2$ and t is in quadrant III, what is $\tan t$?
(A) $-\sqrt{3}$ (B) $\sqrt{3}$ (C) $\frac{\sqrt{3}}{3}$ (D) $-\frac{\sqrt{3}}{3}$
- Which describes the domain of $y = \cot x$?
(A) all real x except $x = k\pi$ (B) all real x except $x = \frac{\pi}{2} + k\pi$ (C) all real x
(D) all real x except $x = 2k\pi$
- Which is a vertical asymptote of $y = \tan(x - \frac{\pi}{4})$?
(A) $x = \frac{\pi}{4}$ (B) $x = \frac{\pi}{2}$ (C) $x = \frac{3\pi}{4}$ (D) $x = \frac{5\pi}{4}$
- Find the exact value of $\csc \frac{7\pi}{6} + \sec \frac{\pi}{3}$.
- List the vertical asymptotes of $y = \tan(2x)$ on the interval $(0, \pi)$.
- Given $\cot t = \frac{3}{4}$ and t in quadrant III, find $\csc t$.
- A tangent graph has asymptotes at $x = -1$ and $x = 3$, passes through $(2, 1)$ and $(0, -3)$. Write its equation.
- Solve $\csc x = -2$ for x in $[0, 2\pi)$.

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15. Find the domain and range of $y = 3 \sec x + 2$.

25.4 Modeling with Trigonometric Functions

Turn words, tables, and graphs into sinusoidal models. Handle Ferris wheels, tides, and temperature cycles, solve for times and windows, and read the output of a sinusoidal regression.

Why This Matters

Nature repeats itself. Tides rise and fall twice a day, temperatures swing with the seasons, daylight stretches and shrinks over the year, a spring bounces, a wheel turns, and wall outlets push current back and forth sixty times each second. Sinusoidal functions are the standard model for all of these. Once you can build a model, you can answer practical questions: when will the water be deep enough for a boat, how many hours will a rider spend above 30 meters, when will the greenhouse be too hot?

Students usually struggle with modeling in four ways. They cannot **translate words into the four parameters** A , B , C , D . They forget the **units of time**, so the period gets mixed up between minutes, hours, and months. They solve $\sin$ or $\cos$ equations and **find only one of the two solutions** per cycle. And they treat a calculator's regression output as a mystery instead of reading it as an equation. This section gives a four-step modeling routine and then practices it in context after context.

Core Ideas

A **sinusoidal model** has the form $f(t) = A \cos(B(t - C)) + D$ (or the same with sine). The variable t is usually time. The parameters carry real meaning: D is the average level, $|A|$ is the swing above and below average, $\frac{2\pi}{B}$ is how long one cycle takes, and C marks when the pattern is at a chosen starting point (a maximum, for cosine).

Four-Step Modeling Routine

1. **Midline:** $D = \frac{\max + \min}{2}$. 2. **Amplitude:** $|A| = \frac{\max - \min}{2}$. 3. **Period** P from context, then $B = \frac{2\pi}{P}$. 4. **Anchor:** pick C from a known maximum, minimum, or midline crossing, and choose the sign of A to match.
Always check the model by substituting two known points before using it.

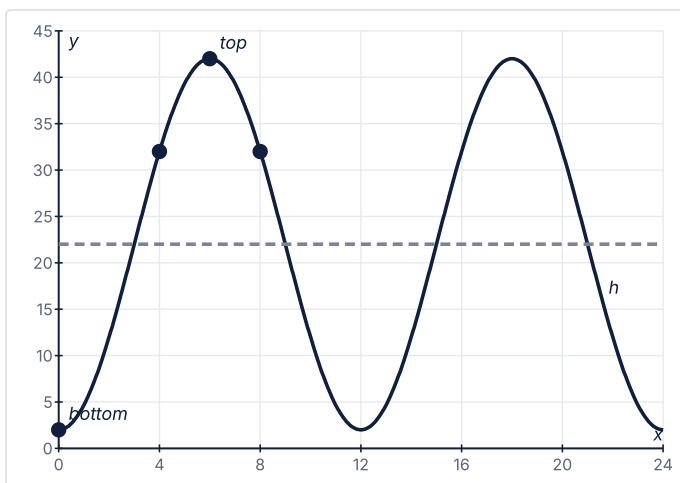
Choosing between sine and cosine is a matter of convenience, not correctness. A good rule: if the story begins at a **maximum**, use $A \cos$ with $C = 0$. If it begins at a **minimum**, use $-A \cos$. If it begins at the **midline going up**, use $A \sin$. Because the starting point is often a bottom (a rider boarding a Ferris wheel at the lowest point), the negative cosine form $-A \cos(Bt) + D$ appears constantly.

Context	Typical period	Typical B
Ferris wheel, 12 min per turn	12 minutes	$\frac{\pi}{6}$
Tides (two highs per day)	about 12.4 hours, often rounded to 12	$\frac{\pi}{6}$
Daily temperature	24 hours	$\frac{\pi}{12}$
Seasonal temperature (months)	12 months	$\frac{\pi}{6}$
Seasonal temperature (days)	365 days	$\frac{2\pi}{365}$
Spring that bobs twice per second	0.5 seconds	4π

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Height $h(t)$ of a rider on a wheel with radius 20 m and center 22 m high, one turn per 12 minutes. The dashed line is the midline.

Reading a Model

For $f(t) = A \cos(B(t - C)) + D$: the **maximum** is $D + |A|$, the **minimum** is $D - |A|$, the **period** is $\frac{2\pi}{|B|}$, and the **frequency** (cycles per unit time) is the reciprocal $\frac{|B|}{2\pi}$.

To find when $f(t) = k$: isolate the cosine (or sine), solve for the inside angle on one lap using the unit circle, then solve for t . There are usually **two** solutions per cycle, unless k is the maximum or minimum.

Sinusoidal regression is a technology tool: you enter a table of data and a calculator or software finds the sine curve $y = a \sin(bx + c) + d$ that fits best. The output is just an equation. To use it, convert: amplitude $|a|$, midline d , period $\frac{2\pi}{b}$, and phase shift $-\frac{c}{b}$ (the shift is $-\frac{c}{b}$ because $bx + c = b(x + \frac{c}{b})$). A good fit needs at least one full cycle of data, and you should check that the period is sensible. If the context says 12 months, a fitted period of 12.08 months is fine, but 20 months is a warning. No model predicts far outside the data it was built from.

Worked Examples

EXAMPLE 1: A FERRIS WHEEL

A Ferris wheel has radius 20 m, its center is 22 m above the ground, and it takes 12 minutes for one turn. A rider boards at the lowest point at $t = 0$. Write the height $h(t)$, then find the height at $t = 4$ and every time in the first turn when the height is 32 m.

SOLUTION

Midline $D = 22$, amplitude 20, period 12 so $B = \frac{2\pi}{12} = \frac{\pi}{6}$. The rider starts at a minimum, so use $-\cos$:

$$h(t) = -20 \cos\left(\frac{\pi t}{6}\right) + 22.$$

Check: $h(0) = -20 + 22 = 2$ m, the bottom.

$$h(4) = -20 \cos \frac{2\pi}{3} + 22 = -20 \left(-\frac{1}{2}\right) + 22 = 32 \text{ m.}$$

Solve $h(t) = 32$: $-20 \cos \frac{\pi t}{6} = 10$, so $\cos \frac{\pi t}{6} = -\frac{1}{2}$. On one lap the angles are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, so $\frac{\pi t}{6} = \frac{2\pi}{3}$ gives $t = 4$ and $\frac{\pi t}{6} = \frac{4\pi}{3}$ gives $t = 8$.

The rider is at 32 m twice, once going up (4 minutes) and once coming down (8 minutes).

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EXAMPLE 2: TIDES AND A TIME WINDOW

At a harbor, the depth is greatest, 5 ft, at $t = 3$ hours after midnight, and least, 1 ft, at $t = 9$. Model the depth with a cosine and find when a boat that needs at least 4 ft of water can enter during the first tide cycle.

SOLUTION

Midline $\frac{5+1}{2} = 3$, amplitude 2, half period $9 - 3 = 6$ so period 12 and $B = \frac{\pi}{6}$. A maximum at $t = 3$ gives $C = 3$:

$$d(t) = 2 \cos\left(\frac{\pi}{6}(t - 3)\right) + 3.$$

Need $d(t) \geq 4$: $2 \cos\left(\frac{\pi}{6}(t - 3)\right) \geq 1$, so $\cos\left(\frac{\pi}{6}(t - 3)\right) \geq \frac{1}{2}$. Cosine is at least $\frac{1}{2}$ when its angle is within $\frac{\pi}{3}$ of 0: $\left|\frac{\pi}{6}(t - 3)\right| \leq \frac{\pi}{3}$.

So $|t - 3| \leq 2$, giving $1 \leq t \leq 5$. The boat can enter from 1 a.m. to 5 a.m., a window of 4 hours. Check endpoints: $d(1) = 2 \cos\left(-\frac{\pi}{3}\right) + 3 = 4$ and $d(5) = 4$.



Hours after midnight when the depth is at least 4 ft: from $t = 1$ to $t = 5$.

EXAMPLE 3: A SEASONAL TEMPERATURE MODEL

The average monthly temperature in a town is lowest, 30°F , in January ($t = 1$) and highest, 84°F , in July ($t = 7$). Model it, find the temperature in April, and estimate the months when it is 70°F . Round to one decimal.

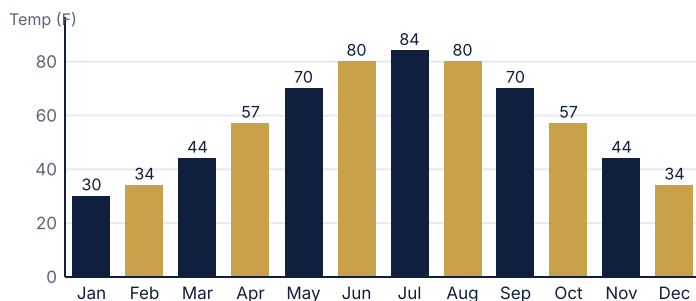
SOLUTION

Midline $\frac{84+30}{2} = 57$, amplitude 27, period 12, so $B = \frac{\pi}{6}$. Maximum at $t = 7$:

$$T(t) = 27 \cos\left(\frac{\pi}{6}(t - 7)\right) + 57.$$

April is $t = 4$: $T(4) = 27 \cos\left(-\frac{\pi}{2}\right) + 57 = 57^\circ\text{F}$, the midline.

For $T = 70$: $\cos\left(\frac{\pi}{6}(t - 7)\right) = \frac{13}{27}$. A calculator gives $\arccos \frac{13}{27} \approx 1.068$. So $\frac{\pi}{6}(t - 7) = \pm 1.068$ and $t - 7 = \pm \frac{6 \cdot 1.068}{\pi} \approx \pm 2.04$. $t \approx 4.96$ and $t \approx 9.04$: early May and early September.



Monthly averages from the model in Example 3. The pattern is symmetric around July.

EXAMPLE 4: BUILDING A MODEL FROM A DATA TABLE

The table gives the water level (in cm) in a tank that is filled and drained in a repeating cycle. Find a sine model.

t (hours)	0	3	6	9	12
L	8	13	8	3	8

SOLUTION

Max 13, min 3: midline 8, amplitude 5. The values return to 8 at $t = 12$ and repeat, so the period is 12 and $B = \frac{\pi}{6}$.

At $t = 0$ the level is on the midline and then rises to 13, which is exactly how a sine starts:

$$L(t) = 5 \sin\left(\frac{\pi t}{6}\right) + 8.$$

Check all five points: $t = 3$: $5 \cdot 1 + 8 = 13$. $t = 6$: $5 \cdot 0 + 8 = 8$. $t = 9$: $5(-1) + 8 = 3$. $t = 12$: 8. All match.

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EXAMPLE 5: A BOBBING SPRING

A weight on a spring has height $h(t) = 6 \cos(4\pi t) + 20$ centimeters at t seconds. Find the period, the frequency, the height at $t = 0.1$ (use a calculator, radian mode), and the first time the weight is at its lowest point.

SOLUTION

$B = 4\pi$, so the period is $\frac{2\pi}{4\pi} = 0.5$ seconds. Frequency is the reciprocal: 2 oscillations per second.

$h(0.1) = 6 \cos(0.4\pi) + 20 \approx 6(0.3090) + 20 \approx 21.85$ cm.

Lowest when $\cos(4\pi t) = -1$: $4\pi t = \pi$, so $t = \frac{1}{4}$ second, and the lowest height is $20 - 6 = 14$ cm.

EXAMPLE 6: INTERPRETING A REGRESSION EQUATION

Using monthly data ($x = 1$ for January), a calculator's sinusoidal regression gives hours of daylight as $y = 4.9 \sin(0.52x - 1.3) + 12.1$. Describe the fit and predict the maximum.

SOLUTION

Amplitude 4.9, midline 12.1 hours. Period $\frac{2\pi}{0.52} \approx 12.08$ months, close to 12, so the model makes sense for a yearly cycle (a modeler may round b to $\frac{\pi}{6} \approx 0.5236$).

Rewrite $0.52x - 1.3 = 0.52(x - 2.5)$: phase shift 2.5 months to the right.

Maximum = $12.1 + 4.9 = 17.0$ hours, minimum = $12.1 - 4.9 = 7.2$ hours. The maximum occurs when $0.52x - 1.3 = \frac{\pi}{2}$, so $x \approx 5.52$, in late May to June. That fits a Northern Hemisphere pattern, where daylight peaks around the June solstice.

Coach's Playbook

- 1 Name the variable and units.** Write what t and the output measure, with units, before writing anything else.
 - 2 Extract max, min, and period.** These three numbers give A , D , and B . Look for words like 'every 8 minutes' or 'twice a day'.
 - 3 Pick the starting point.** Bottom start: $-\cos$. Top start: $\cos$. Midline going up: $\sin$. Then choose C to match any given time.
 - 4 Check two points.** Plug in the starting point and one more. A model that fails at a given data point is wrong.
 - 5 Solve with the unit circle.** For $f(t) = k$ find all angles on one lap, then convert back to t and list every solution in the requested interval.
- ◆ **Sketch before solving.** A rough graph with the horizontal line $y = k$ shows how many solutions to expect.
 - ◆ **Inequalities.** To find when a model is above a level, solve the equation at the boundary times and then read the graph between them.
 - ◆ **Units.** If the period is 12 hours and t is in hours, $B = \frac{\pi}{6}$. If t is in minutes, convert the period too.

Common traps

(1) Forgetting the vertical shift, for example writing a Ferris wheel height as $-20 \cos(Bt)$ with no $+22$. (2) Using the period as B . (3) Finding only one of the two solutions per cycle. (4) Using degree mode for a model that is written in radians, or extending a model far past the data.

Mistake Clinic**Spot the error**

Problem: A wheel with radius 15 m has its lowest point 3 m above the ground and a period of 10 min. **Student work:**

$h = -15 \cos\left(\frac{\pi t}{5}\right)$. **Where it went wrong:** the student forgot the midline. The center is the bottom plus the radius, $3 + 15 = 18$ m. At $t = 0$ this model gives -15 m, below the ground. **Correct:** $h = -15 \cos\left(\frac{\pi t}{5}\right) + 18$, which gives 3 at $t = 0$.

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Spot the error

Problem: High tide is 6 m at $t = 2$, low tide is 2 m, and the period is 12 hours. **Student work:** $d = 2 \cos(12(t - 2)) + 4$.

Where it went wrong: the student put the period in the place of B . Since the period is $\frac{2\pi}{B}$, we need $B = \frac{2\pi}{12} = \frac{\pi}{6}$.

Correct: $d = 2 \cos\left(\frac{\pi}{6}(t - 2)\right) + 4$. Check: at $t = 2$ it gives 6.

Spot the error

Problem: For $h(t) = -20 \cos\left(\frac{\pi t}{6}\right) + 22$ find the times in the first turn when $h = 32$. **Student work:** $\cos \frac{\pi t}{6} = -\frac{1}{2}$, so $\frac{\pi t}{6} = \frac{2\pi}{3}$ and $t = 4$. **Where it went wrong:** cosine equals $-\frac{1}{2}$ at two angles on the unit circle, $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. The student found only one. **Correct:** $t = 4$ and $t = 8$.

Real-World Connection

The voltage in a household outlet follows a sinusoid such as $V(t) = 170 \sin(120\pi t)$ volts with t in seconds. The period is $\frac{2\pi}{120\pi} = \frac{1}{60}$ second, so the frequency is 60 cycles per second. The amplitude 170 is the peak voltage. Engineers use these same four parameters to compare signals and design circuits.

Challenge Corner

Problem: For the wheel in Example 1, for how many minutes during each turn is the rider higher than 37 m? **Solution:** Solve $-20 \cos \frac{\pi t}{6} + 22 > 37$: $\cos \frac{\pi t}{6} < -\frac{3}{4}$. Let $\alpha = \arccos\left(-\frac{3}{4}\right) \approx 2.419$. The cosine is less than $-\frac{3}{4}$ for angles between α and $2\pi - \alpha$, so $\frac{\pi t}{6}$ runs from 2.419 to 3.864 and t runs from about 4.62 to 7.38 minutes. Duration $\approx 7.38 - 4.62 = 2.76$ minutes.

Practice**PRACTICE Modeling with Trigonometric Functions**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- A wheel's height is $h(t) = -15 \cos\left(\frac{\pi t}{5}\right) + 17$ meters. What is the lowest height?
(A) 2 m (B) 17 m (C) 32 m (D) -15 m
- In $h(t) = 4 \cos\left(\frac{\pi}{6}(t - 2)\right) + 7$, with t in hours, how much time passes between consecutive high tides?
(A) 6 hours (B) 12 hours (C) 2π hours (D) 24 hours
- The monthly high temperature of a city varies between a low of 46° and a high of 82° . What is the midline of a sinusoidal model?
(A) 36° (B) 18° (C) 64° (D) 128°
- A temperature model is $T(d) = 20 \cos\left(\frac{2\pi}{365}(d - 200)\right) + 55$, where d is the day of the year. Which statement is true?
(A) The coldest day is day 200. (B) The hottest day is about day 18.
(C) The hottest day is day 365. (D) The coldest day is about day 18.
- A weight on a spring has $h(t) = 6 \cos(4\pi t) + 20$ with t in seconds. How many full oscillations occur each second?
(A) 4π (B) 2 (C) $\frac{1}{2}$ (D) 4
- A rider's height is $h(t) = -20 \cos\left(\frac{\pi t}{6}\right) + 22$. What is the first time $t > 0$ when $h = 32$?
(A) 2 (B) 3 (C) 8 (D) 4
- A tide depth is $d(t) = 2 \cos\left(\frac{\pi}{6}(t - 3)\right) + 3$ feet. A boat needs a depth of at least 4 ft. For how many hours of each 12-hour cycle is the depth at least 4 ft?
(A) 4 (B) 2 (C) 6 (D) 8
- A quantity has maximum 70 at $t = 0$, minimum 30, and a period of 8. Which equation models it?
(A) $y = 20 \cos\left(\frac{\pi t}{8}\right) + 50$ (B) $y = 20 \sin\left(\frac{\pi t}{4}\right) + 50$ (C) $y = 20 \cos\left(\frac{\pi t}{4}\right) + 50$
(D) $y = 40 \cos\left(\frac{\pi t}{4}\right) + 50$
- A regression gives $y = 3.2 \sin(0.52x + 1.1) + 9.8$. What is the predicted maximum value?
(A) 13.0 (B) 3.2 (C) 9.8 (D) 6.6

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- A Ferris wheel has diameter 30 m, its lowest point is 2 m above the ground, and one turn takes 8 minutes. A rider starts at the bottom. Write $h(t)$ and find the height at $t = 2$.
- The temperature in a room ranges from 12° at 4 a.m. to 28° at 4 p.m. Using t as hours after midnight, write a cosine model and find the temperature at $t = 10$.
- For $d(t) = 2 \cos\left(\frac{\pi}{6}(t - 3)\right) + 3$, find all t in $[0, 12)$ with $d(t) = 2$.
- A spring has height $h(t) = 5 \cos(2\pi t) + 12$ cm. Find the first time $t > 0$ when $h = 14.5$ cm.
- The data points $(0, 30)$, $(3, 50)$, $(6, 70)$, $(9, 50)$, $(12, 30)$ come from a sinusoid. Write a cosine model $y = A \cos(Bt) + D$.
- A regression gives $y = 2.5 \sin(0.9x - 0.45) + 6$. State the amplitude, midline, approximate period, phase shift, and maximum.

Answers and Explanations

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

25.1 Radians, Degrees, and the Unit Circle

PRACTICE · RADIANS, DEGREES, AND THE UNIT CIRCLE

- B. $225 \cdot \frac{\pi}{180} = \frac{5\pi}{4}$. Choice A is 135° , and C is 150° .
- A. The angle is in quadrant IV, so reference $= 2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$. Choice B subtracts from π , which is the quadrant II rule.
- C. Reference angle $\frac{\pi}{3}$ gives $\frac{1}{2}$ and quadrant II makes cosine negative, so $-\frac{1}{2}$. Choice D is the sine value, not the cosine value.
- D. $s = r\theta = 8 \cdot \frac{3\pi}{4} = 6\pi$. Choice B divides instead of multiplying.
- A. Reference angle $\frac{\pi}{4}$ gives $\tan = 1$. In quadrant IV sine is negative and cosine positive, so tangent is -1 . Choice C is a sine or cosine value.
- B. Add one full turn: $-\frac{\pi}{3} + 2\pi = \frac{5\pi}{3}$. Choices A and C reflect the angle across an axis, which is not the same terminal side.
- C. $\cos^2 t = 1 - \frac{9}{25} = \frac{16}{25}$, so $\cos t = \pm \frac{4}{5}$. Quadrant II has negative cosine. Choice A ignores the sign.
- A. $x^2 + y^2 = 1$ gives $y^2 = 1 - \frac{25}{169} = \frac{144}{169}$, and $y > 0$ gives $\frac{12}{13}$. Choice B violates $y > 0$.
- D. $A = \frac{1}{2}r^2\theta = \frac{1}{2} \cdot 36 \cdot \frac{2\pi}{3} = 12\pi$. Choice A forgets the $\frac{1}{2}$. Choice B is the arc length $r\theta$.
- 105° . Multiply by $\frac{180}{\pi}$: $\frac{7 \cdot 180}{12} = 7 \cdot 15 = 105$.
0. $-\frac{5\pi}{3}$ is coterminal with $\frac{\pi}{3}$, so the cosine is $\frac{1}{2}$. $\frac{7\pi}{6}$ is in quadrant III with reference $\frac{\pi}{6}$, so the sine is $-\frac{1}{2}$. Sum: $\frac{1}{2} - \frac{1}{2} = 0$.
- $t = \frac{3\pi}{4}$ and $t = \frac{5\pi}{4}$. Reference angle $\frac{\pi}{4}$; cosine is negative in quadrants II and III, giving $\pi - \frac{\pi}{4}$ and $\pi + \frac{\pi}{4}$.
- $-\frac{\sqrt{5}}{5}$. Use $1 + \tan^2 t = \sec^2 t$: $\sec^2 t = 5$, so $|\cos t| = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$. Cosine is negative in quadrant III. (Equivalent: legs 1 and 2 give hypotenuse $\sqrt{5}$.)
- 120 π meters per minute. Each revolution is 2π radians, so the angle rate is $150 \cdot 2\pi = 300\pi$ radians per minute. Speed $= r \cdot (\text{angle rate}) = 0.4 \cdot 300\pi = 120\pi$.
- $-\frac{\sqrt{7}}{3}$. $\cos^2 t = 1 - \frac{7}{16} = \frac{9}{16}$, and cosine is positive in quadrant IV, so $\cos t = \frac{3}{4}$. Then $\tan t = \frac{-\sqrt{7}/4}{3/4} = -\frac{\sqrt{7}}{3}$.

25.2 Graphs of Sine and Cosine

PRACTICE · GRAPHS OF SINE AND COSINE

- B. Amplitude is $|A| = 5$. Choice A keeps the negative sign, but a distance cannot be negative. Choice D adds the midline.
- A. Period $= \frac{2\pi}{\pi/4} = 8$. Choice D uses B as the period.
- C. Factor: $2x + \pi = 2\left(x + \frac{\pi}{2}\right)$, so the shift is $\frac{\pi}{2}$ left. Choice A forgets to divide by 2.
- A. Midline 4 and amplitude 2 give $4 - 2 = 2$ to $4 + 2 = 6$. The negative sign flips the wave but does not change the range.
- D. $D = 5$, $|A| = 4$, $B = \frac{2\pi}{12} = \frac{\pi}{6}$, cosine starts at its max. Choice A would have period 24, and C has period $\frac{\pi}{6}$.
- B. The maximum needs $2x = \frac{\pi}{2}$, so $x = \frac{\pi}{4}$. Choice A forgets that the period has been halved.
- C. Shifting sine left by $\frac{\pi}{2}$ moves its peak to $x = 0$, as cosine has. Choice A shifts the wrong way and gives $-\cos x$.
- A. The period is $\frac{2\pi}{3}$, and $\frac{4\pi}{2\pi/3} = 6$. Choice C uses B as the count without checking the interval length.
- D. $2 \cdot \frac{\pi}{2} - \pi = 0$, and $\cos 0 = 1$, so $y = 3 + 2 = 5$. Choice A uses $\cos \pi$.
- Amplitude $\frac{1}{2}$, period $\frac{1}{2}$. Amplitude is $|\frac{1}{2}|$, and the period is $\frac{2\pi}{4\pi} = \frac{1}{2}$.
- $y = -4 \cos\left(\frac{\pi x}{5}\right) + 2$. Half period is 5, so the period is 10 and $B = \frac{\pi}{5}$. Midline $\frac{6+(-2)}{2} = 2$, amplitude 4. Starting at a minimum means $A = -4$ with no shift. Check: $x = 5$ gives $-4(-1) + 2 = 6$.
- Maximum 4, minimum -2 , first maximum at $x = \frac{2\pi}{3}$. Max $= 1 + 3$, min $= 1 - 3$. The maximum needs $x - \frac{\pi}{6} = \frac{\pi}{2}$, so $x = \frac{2\pi}{3}$.
- Lowest temperature 50 degrees, at $t = 9$. Minimum $= 60 - 10 = 50$. The cosine is -1 when $\frac{\pi}{6}(t - 3) = \pi$, so $t - 3 = 6$ and $t = 9$ (3 p.m.).
- $x = \frac{\pi}{9}$ and $x = \frac{5\pi}{9}$. Then $\cos 3x = \frac{1}{2}$. Let $u = 3x$ in $[0, 2\pi]$: $u = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. Divide by 3.
- $y = 3 \sin\left(\frac{\pi}{3}\left(x - \frac{1}{2}\right)\right) - 1$. $B = \frac{2\pi}{6}$. Sine peaks a quarter period (1.5) after its upward midline crossing, so $C = 2 - 1.5 = \frac{1}{2}$. Check: $x = 2$ gives $3 \sin \frac{\pi}{2} - 1 = 2$.

25.3 Tangent and the Reciprocal Functions

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PRACTICE · TANGENT AND THE RECIPROCAL FUNCTIONS

1. C. Quadrant II with reference angle $\frac{\pi}{3}$: $\tan \frac{\pi}{3} = \sqrt{3}$ and the sign is negative. Choice A has the wrong sign, and B is the tangent of the wrong reference angle.
2. A. Period $= \frac{\pi}{B} = \frac{\pi}{3}$. Choice B uses the sine and cosine formula.
3. D. The inside $\frac{\pi}{2}$ must equal $\frac{\pi}{2}$, so $x = \pi$. Choice A applies the asymptote of $\tan x$ without adjusting.
4. B. $\sin \frac{\pi}{6} = \frac{1}{2}$, and $\csc = \frac{1}{1/2} = 2$. Choice C is $\sec \frac{\pi}{6}$, which uses cosine.
5. D. Since $|\cos x| \leq 1$, $|\sec x| \geq 1$, and the values ± 1 are reached when $\cos x = \pm 1$. Choice C leaves out the endpoints.
6. C. $\cos \frac{5\pi}{4} = \sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$, so the quotient is 1. Choice A forgets that two negatives give a positive.
7. B. $\cos t = -\frac{1}{2}$, $\sin t = -\frac{\sqrt{3}}{2}$ (negative in QIII), so $\tan t = \sqrt{3}$. Choice A has the wrong sign: tangent is positive in quadrant III.
8. A. Cotangent is $\frac{\cos x}{\sin x}$, undefined when $\sin x = 0$, that is at every multiple of π . Choice B is the domain of tangent.
9. C. Set $x - \frac{\pi}{4} = \frac{\pi}{2}$, so $x = \frac{3\pi}{4}$. Choice A is a zero of the function, not an asymptote.
10. 0. $\sin \frac{7\pi}{6} = -\frac{1}{2}$, so $\csc = -2$. $\cos \frac{\pi}{3} = \frac{1}{2}$, so $\sec = 2$. Sum $= 0$.
11. $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$. Set $2x = \frac{\pi}{2} + k\pi$, so $x = \frac{\pi}{4} + \frac{k\pi}{2}$; $k = 0, 1$ give values in $(0, \pi)$.
12. $-\frac{5}{4}$. $\tan t = \frac{4}{3}$, so the legs are 3 and 4 and the hypotenuse is 5. In quadrant III, $\sin t = -\frac{4}{5}$, so $\csc t = -\frac{5}{4}$.
13. $y = 2 \tan \left(\frac{\pi}{4}(x - 1) \right) - 1$. Period $= 3 - (-1) = 4$, so $B = \frac{\pi}{4}$; center at $x = 1$. Then $A + D = 1$ and $-A + D = -3$, giving $D = -1$ and $A = 2$.
14. $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$. $\csc x = -2$ means $\sin x = -\frac{1}{2}$. Reference angle $\frac{\pi}{6}$, sine negative in quadrants III and IV.
15. Domain: all real x with $x \neq \frac{\pi}{2} + k\pi$. Range: $y \leq -1$ or $y \geq 5$. Since $\sec x \geq 1$ or ≤ -1 , $3 \sec x \geq 3$ or ≤ -3 ; adding 2 gives $y \geq 5$ or $y \leq -1$.

25.4 Modeling with Trigonometric Functions**PRACTICE · MODELING WITH TRIGONOMETRIC FUNCTIONS**

1. A. The minimum is midline minus amplitude: $17 - 15 = 2$. Choice C is the maximum, and D forgets the midline.
2. B. Period $= \frac{2\pi}{\pi/6} = 12$. Choice A is only a half cycle, from high to low.
3. C. Midline $= \frac{82+46}{2} = 64$. Choice A is the full range $82 - 46$, and B is the amplitude.
4. D. The maximum is at $d = 200$ (the cosine is 1), and the minimum is half a period earlier or later: $200 - 182.5 \approx 18$. Choice A confuses max with min.
5. B. Period $= \frac{2\pi}{4\pi} = \frac{1}{2}$ s, so frequency is 2 per second. Choice C is the period, and D uses B without dividing by π .
6. D. $\cos \frac{\pi t}{6} = -\frac{1}{2}$, so $\frac{\pi t}{6} = \frac{2\pi}{3}$ and $t = 4$. Choice C is the second time the rider is at 32, not the first.
7. A. $\cos \left(\frac{\pi}{6}(t - 3) \right) \geq \frac{1}{2}$ means $|t - 3| \leq 2$, a window of 4 hours. Choice B is half the window, from the peak to one edge.
8. C. Amplitude 20, midline 50, $B = \frac{2\pi}{8} = \frac{\pi}{4}$, and cosine starts at its maximum. Choice A has period 16. Choice B starts on the midline. Choice D has amplitude 40.
9. A. Maximum $=$ midline $+$ amplitude $= 9.8 + 3.2 = 13.0$. Choice D is the minimum.
10. $h(t) = -15 \cos \left(\frac{\pi t}{12} \right) + 17$ and $h(2) = 17$ m. Radius 15, center $2 + 15 = 17$, $B = \frac{2\pi}{8}$. At $t = 2$, $\cos \frac{\pi}{2} = 0$, so the rider is at the center height.
11. $T(t) = 8 \cos \left(\frac{\pi}{12}(t - 16) \right) + 20$; $T(10) = 20^\circ$. Midline 20, amplitude 8, period 24, maximum at $t = 16$. At $t = 10$ the angle is $-\frac{\pi}{2}$, whose cosine is 0.
12. $t = 7$ and $t = 11$. Then $\cos \left(\frac{\pi}{6}(t - 3) \right) = -\frac{1}{2}$, so $\frac{\pi}{6}(t - 3) = \pm \frac{2\pi}{3}$ and $t - 3 = \pm 4$. That gives $t = 7$ and $t = -1$; adding the period 12 gives $t = 11$.
13. $t = \frac{1}{6}$ second. $5 \cos(2\pi t) = 2.5$ so $\cos(2\pi t) = \frac{1}{2}$, the smallest positive angle is $\frac{\pi}{3}$, and $2\pi t = \frac{\pi}{3}$ gives $t = \frac{1}{6}$.
14. $y = -20 \cos \left(\frac{\pi t}{6} \right) + 50$. Min 30, max 70: midline 50, amplitude 20. Period 12 so $B = \frac{\pi}{6}$. The data start at a minimum, so $A = -20$. Check $t = 3$: $-20 \cdot 0 + 50 = 50$.
15. Amplitude 2.5; midline $y = 6$; period $\frac{2\pi}{0.9} \approx 6.98$; phase shift 0.5 to the right (since $0.9x - 0.45 = 0.9(x - 0.5)$); maximum $6 + 2.5 = 8.5$.

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