



Trigonometric Identities and Equations

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

In this booster

26.1 Fundamental and Pythagorean Identities

26.2 Sum, Difference, Double-Angle, and Half-Angle Formulas

26.3 Proving Trigonometric Identities

26.4 Solving Trigonometric Equations

Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

26.1 Fundamental and Pythagorean Identities

Learn the reciprocal, quotient, Pythagorean, even-odd, and cofunction identities as a few connected families. Use them to simplify expressions, find all six trig values from one, and avoid the classic cancellation errors.

Why This Matters

An identity is an equation that is true for every value of the variable where both sides are defined. The identities of trigonometry are the grammar of the subject: they let you rewrite an expression in a simpler or more useful form. In calculus you will constantly replace $\sec^2 x - 1$ by $\tan^2 x$ or swap $\sin^2 x$ for $1 - \cos^2 x$ to make an integral possible. Physicists use them to show that energy is conserved in oscillations, and engineers use them to simplify signal formulas.

Students usually struggle here for four reasons. They try to **memorize** every identity as an isolated fact instead of seeing that there are only a few families. They **cancel terms** that are added, as if $\frac{\sin x + \cos x}{\cos x}$ were $\sin x$. They are unsure **when to convert** everything to sine and cosine. And they apply a Pythagorean identity with the **wrong sign or the wrong function**, such as writing $1 + \tan^2 x = \csc^2 x$. This section organizes the identities into three small families and gives you a dependable strategy.

Core Ideas

The first family comes straight from the definitions. Because $\tan$, $\cot$, $\sec$, and $\csc$ are all built from $\sin$ and $\cos$, you can always rewrite any trigonometric expression using only those two.

Reciprocal and Quotient Identities

$$\csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{1}{\tan x}, \quad \tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}.$$

These hold wherever the denominators are not zero.

The second family is the heart of the subject. Every point on the unit circle is $(\cos x, \sin x)$ and satisfies $x^2 + y^2 = 1$. That gives the first Pythagorean identity, and dividing it by $\cos^2 x$ or $\sin^2 x$ gives the other two.

Pythagorean Identities

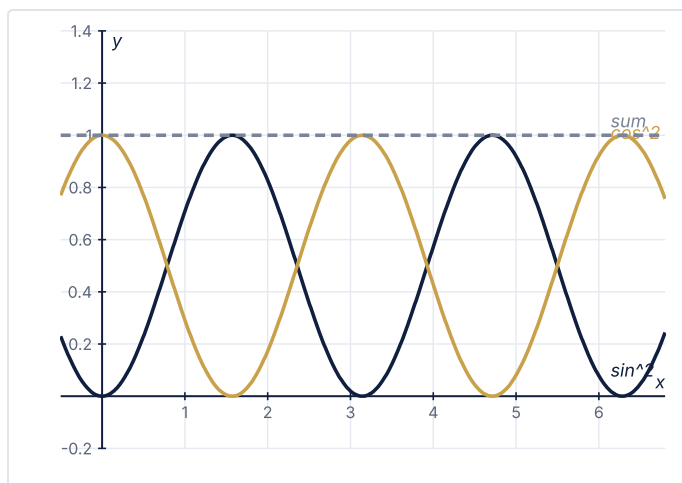
$$\sin^2 x + \cos^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x.$$

Useful rearrangements: $\sin^2 x = 1 - \cos^2 x$, $\cos^2 x = 1 - \sin^2 x$, $\tan^2 x = \sec^2 x - 1$, $\cot^2 x = \csc^2 x - 1$. Notation: $\sin^2 x$ means $(\sin x)^2$.

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The graphs of $\sin^2 x$ (navy) and $\cos^2 x$ (gold) always add to the constant 1 (dashed line).

The third family describes symmetry. Cosine is **even** and sine and tangent are **odd**, because reflecting an angle across the x -axis flips the sign of the y -coordinate but not the x -coordinate. Finally, the **cofunction** identities say that the sine of an angle is the cosine of its complement.

Even-Odd and Cofunction Identities

$$\begin{aligned} \cos(-x) &= \cos x, & \sin(-x) &= -\sin x, & \tan(-x) &= -\tan x. \\ \sin\left(\frac{\pi}{2} - x\right) &= \cos x, & \cos\left(\frac{\pi}{2} - x\right) &= \sin x, & \tan\left(\frac{\pi}{2} - x\right) &= \cot x. \end{aligned}$$

Family	What it does	When to use it
Reciprocal and quotient	Turns any function into sin and cos	Mixed expressions, getting started
Pythagorean	Swaps squares: $\sin^2 \leftrightarrow \cos^2$, $\tan^2 \leftrightarrow \sec^2$	Squared terms, $1 \pm$ expressions
Even-odd	Removes negative angles	Arguments like $-x$ or $-\frac{\pi}{3}$
Cofunction	Swaps $\sin \leftrightarrow \cos$ at complements	Angles like $\frac{\pi}{2} - x$

How to simplify. There is no single recipe, but this order solves most problems: (1) look for a Pythagorean pattern like $1 - \cos^2 x$; (2) otherwise convert everything to sine and cosine; (3) combine fractions over a common denominator; (4) factor, including difference of squares and sums or differences of cubes; (5) cancel only **factors**, never terms. The goal is an expression with fewer terms and fewer different functions.

Worked Examples

EXAMPLE 1: CONVERT TO SINE AND COSINE

Simplify $\sin x \cdot \cot x \cdot \sec x$.

SOLUTION

Rewrite each factor: $\sin x \cdot \frac{\cos x}{\sin x} \cdot \frac{1}{\cos x}$.

Cancel $\sin x$ with $\sin x$ and $\cos x$ with $\cos x$. Everything cancels, leaving 1.

So $\sin x \cot x \sec x = 1$ (for every x where all three functions are defined).

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EXAMPLE 2: FINDING ALL SIX VALUES

If $\sin x = \frac{5}{13}$ and x is in quadrant II, find the other five trigonometric functions of x .

SOLUTION

From $\cos^2 x = 1 - \sin^2 x = 1 - \frac{25}{169} = \frac{144}{169}$, $\cos x = \pm \frac{12}{13}$. Quadrant II has negative cosine, so $\cos x = -\frac{12}{13}$.

$$\tan x = \frac{5/13}{-12/13} = -\frac{5}{12}.$$

Reciprocals: $\csc x = \frac{13}{5}$, $\sec x = -\frac{13}{12}$, $\cot x = -\frac{12}{5}$.

Sign check for quadrant II: only $\sin$ and $\csc$ are positive. Correct.

EXAMPLE 3: SPOTTING A PYTHAGOREAN PATTERN

Simplify $\frac{1 - \cos^2 x}{\sin x \cos x}$.

SOLUTION

The numerator is $\sin^2 x$ by the first Pythagorean identity.

$$\frac{\sin^2 x}{\sin x \cos x} = \frac{\sin x}{\cos x} = \tan x \text{ after cancelling the common factor } \sin x.$$

EXAMPLE 4: DIFFERENCE OF SQUARES

Simplify $\cos^4 x - \sin^4 x$ to an expression in $\cos x$ only.

SOLUTION

Treat it as a difference of squares: $\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$.

The second factor is 1, so the expression is $\cos^2 x - \sin^2 x$.

Replace $\sin^2 x$ by $1 - \cos^2 x$: $\cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1$.

Check at $x = 0$: $\cos^4 0 - \sin^4 0 = 1$ and $2(1) - 1 = 1$.

EXAMPLE 5: ADDING FRACTIONS

Write $\frac{1}{1 - \sin x} + \frac{1}{1 + \sin x}$ as a single simple expression.

SOLUTION

Common denominator $(1 - \sin x)(1 + \sin x) = 1 - \sin^2 x$.

Numerator: $(1 + \sin x) + (1 - \sin x) = 2$. So the sum is $\frac{2}{1 - \sin^2 x}$.

Since $1 - \sin^2 x = \cos^2 x$, the result is $\frac{2}{\cos^2 x} = 2\sec^2 x$.

Check at $x = \frac{\pi}{6}$: $\frac{1}{1/2} + \frac{1}{3/2} = 2 + \frac{2}{3} = \frac{8}{3}$, and $2\sec^2 \frac{\pi}{6} = 2 \cdot \frac{4}{3} = \frac{8}{3}$.

EXAMPLE 6: A MULTI-STEP SIMPLIFICATION

Simplify $\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x}$.

SOLUTION

Common denominator $\sin x(1 + \cos x)$. Numerator: $\sin^2 x + (1 + \cos x)^2 = \sin^2 x + 1 + 2\cos x + \cos^2 x$.

Use $\sin^2 x + \cos^2 x = 1$: numerator $= 1 + 1 + 2\cos x = 2 + 2\cos x = 2(1 + \cos x)$.

So the sum is $\frac{2(1 + \cos x)}{\sin x(1 + \cos x)}$. Cancel the factor $1 + \cos x$ to get $\frac{2}{\sin x} = 2\csc x$.

Check at $x = \frac{\pi}{2}$: $\frac{1}{1} + \frac{1}{1} = 2$ and $2\csc \frac{\pi}{2} = 2$.

Coach's Playbook

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- 1 **Scan for patterns.** Look for $1 - \cos^2 x$, $\sec^2 x - 1$, $\sin^2 x + \cos^2 x$, and similar. Replace them first.
 - 2 **Convert to sine and cosine.** If the expression mixes several functions, rewrite all of them in $\sin$ and $\cos$.
 - 3 **Combine and factor.** Add fractions over a common denominator, then factor numerator and denominator.
 - 4 **Cancel factors, not terms.** Only a common factor of the whole numerator and the whole denominator may be cancelled.
 - 5 **Test with a number.** Plug in an easy angle like $\frac{\pi}{6}$ or $\frac{\pi}{4}$ into both the original and your answer. They must match.
- ◆ **Conjugate trick.** When you see $1 \pm \sin x$ or $1 \pm \cos x$ in a denominator, multiply top and bottom by the conjugate to produce $1 - \sin^2 x$ or $1 - \cos^2 x$.
 - ◆ **Which Pythagorean form?** Squared $\tan$ goes with $\sec$; squared $\cot$ goes with $\csc$. The 'co' pairs with the 'co'.
 - ◆ **Sign from the quadrant.** When taking a square root, $\sqrt{\cos^2 x} = |\cos x|$. Choose the sign from the quadrant.

Common traps

- (1) Cancelling terms, as in $\frac{\sin x + \cos x}{\cos x} = \sin x$. (2) Forgetting the middle term when squaring a binomial: $(\sin x + \cos x)^2 \neq 1$.
 (3) Writing $1 + \tan^2 x = \csc^2 x$ instead of $\sec^2 x$. (4) Taking $\sqrt{\sin^2 x} = \sin x$ without checking the sign.

Mistake Clinic

Spot the error

Problem: Simplify $(\sin x + \cos x)^2$. **Student work:** $(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x = 1$. **Where it went wrong:** squaring a sum gives three terms, not two. The student dropped the middle term $2 \sin x \cos x$. **Correct:** $\sin^2 x + 2 \sin x \cos x + \cos^2 x = 1 + 2 \sin x \cos x$. Check at $x = \frac{\pi}{4}$: the left side is $(\sqrt{2})^2 = 2$ and the right side is $1 + 2 \cdot \frac{1}{2} = 2$.

Spot the error

Problem: Simplify $\frac{\sin x + \cos x}{\cos x}$. **Student work:** Cancel $\cos x$ to get $\sin x$. **Where it went wrong:** $\cos x$ is a term in the numerator, not a factor of the whole numerator. Cancellation is only allowed for factors. **Correct:** split the fraction:
 $\frac{\sin x}{\cos x} + \frac{\cos x}{\cos x} = \tan x + 1$.

Spot the error

Problem: Simplify $\sqrt{1 - \cos^2 x}$ and evaluate at $x = \frac{7\pi}{6}$. **Student work:** $\sqrt{1 - \cos^2 x} = \sqrt{\sin^2 x} = \sin x$, so the value is $\sin \frac{7\pi}{6} = -\frac{1}{2}$. **Where it went wrong:** a square root is never negative, and $\sqrt{\sin^2 x} = |\sin x|$. **Correct:** $\sqrt{1 - \cos^2 x} = |\sin x|$, which equals $|\frac{1}{2}| = \frac{1}{2}$ at $x = \frac{7\pi}{6}$.

Real-World Connection

A mass on a spring has kinetic energy $K = E \sin^2(\omega t)$ and potential energy $U = E \cos^2(\omega t)$, where E is the total energy and ω is a constant. By the first Pythagorean identity, $K + U = E(\sin^2 + \cos^2) = E$ at every instant. The identity is exactly the statement that energy is conserved: whatever kinetic energy the spring loses, it gains back as potential energy.

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Challenge Corner

Problem: Given $\sec x + \tan x = 2$ with $0 < x < \frac{\pi}{2}$, find $\tan x$. **Solution:** Since $\sec^2 x - \tan^2 x = 1$, factor: $(\sec x - \tan x)(\sec x + \tan x) = 1$, so $\sec x - \tan x = \frac{1}{2}$. Add the two equations: $2\sec x = \frac{5}{2}$, so $\sec x = \frac{5}{4}$. Subtract: $2\tan x = \frac{3}{2}$, so $\tan x = \frac{3}{4}$. Check: $\frac{25}{16} - \frac{9}{16} = 1$ and $\frac{5}{4} + \frac{3}{4} = 2$. Bonus use in calculus: substituting $x = 4 \tan \theta$ turns $\sqrt{16 + x^2}$ into $\sqrt{16 + 16 \tan^2 \theta} = 4 \sec \theta$.

Practice

PRACTICE Fundamental and Pythagorean Identities

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Which expression is equal to $\tan x \cos x$?
(A) $\sec x$ (B) $\cos^2 x$ (C) $\sin x$ (D) 1
- Which expression is equal to $1 - \sin^2 x$?
(A) $\cos^2 x$ (B) $\cos x$ (C) $-\cos^2 x$ (D) $1 + \cos^2 x$
- Which expression is equal to $\sec^2 x - 1$?
(A) $\cot^2 x$ (B) $\sin^2 x$ (C) $\csc^2 x$ (D) $\tan^2 x$
- If $\cos x = \frac{3}{5}$ and x is in quadrant IV, what is $\sin x$?
(A) $\frac{4}{5}$ (B) $-\frac{4}{5}$ (C) $-\frac{3}{4}$ (D) $\frac{2}{5}$
- Which expression is equal to $\frac{\csc x}{\sec x}$?
(A) $\cot x$ (B) $\tan x$ (C) 1 (D) $\sin x \cos x$
- Simplify $(\sin x + \cos x)^2 - 2 \sin x \cos x$.
(A) 0 (B) 2 (C) $1 + 2 \sin x \cos x$ (D) 1
- Which expression is equal to $\tan x + \cot x$?
(A) 1 (B) $\sec x \csc x$ (C) $\sec^2 x$ (D) $\sin x \cos x$
- Simplify $(1 - \cos^2 x)(1 + \cot^2 x)$.
(A) $\sin^2 x$ (B) $\csc^2 x$ (C) 1 (D) 0
- Which expression is equal to $\cos(-x) + \sin(-x)$?
(A) $\cos x - \sin x$ (B) $\cos x + \sin x$ (C) $-\cos x - \sin x$ (D) $-\cos x + \sin x$
- Given $\tan x = \frac{3}{4}$ and x in quadrant III, find $\sin x$ and $\cos x$.
- Simplify $\frac{\sec x - \cos x}{\sin x}$.
- Simplify $\frac{\cos x}{1 + \sin x} + \tan x$.
- Simplify $\cot^2 x (1 - \cos^2 x)$.
- For $0 < \theta < \frac{\pi}{2}$, write $\sqrt{16 + x^2}$ in terms of θ if $x = 4 \tan \theta$.
- If $\sec x + \tan x = 2$ with $0 < x < \frac{\pi}{2}$, find $\sec x$.

26.2 Sum, Difference, Double-Angle, and Half-Angle Formulas

Derive the sum and difference formulas from the unit circle, then use them with the double-angle, power-reducing, and half-angle formulas to find exact values and simplify expressions.

Why This Matters

The angle formulas answer a question that every student eventually asks: what is $\sin 75^\circ$ exactly, when 75 is not one of the angles on the unit circle? The answer is to split 75° into $45^\circ + 30^\circ$ and combine known values. These formulas are also the machinery behind alternating-current circuits, the beats you hear when two guitar strings are slightly out of tune, the range of a thrown ball, and nearly every integral involving $\sin^2 x$ in calculus.

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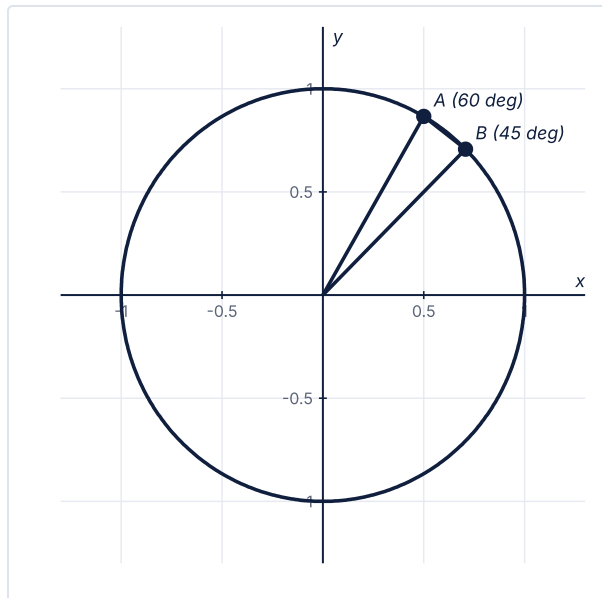
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Students typically stumble in four spots. The first is the false belief that **trigonometric functions distribute**, so $\sin(a + b) = \sin a + \sin b$. The second is mixing up the **signs** in the cosine formulas, which flip. The third is choosing among the **three forms of the cosine double-angle formula**. The fourth is the **sign of the square root** in the half-angle formulas. This section derives the main formulas once so you trust them, then teaches you to pick the right tool.

Core Ideas

Everything starts with the difference formula for cosine. Place two points on the unit circle, $A = (\cos a, \sin a)$ and $B = (\cos b, \sin b)$. The distance between them depends only on the angle $a - b$ between the radii. Computing the squared chord length two ways, once with the distance formula and once with the law of cosines, gives $2 - 2\cos(a - b) = 2 - 2(\cos a \cos b + \sin a \sin b)$.



Points A and B on the unit circle. The chord AB depends only on the angle between the radii, here 15 degrees.

Sum and Difference Formulas

$$\begin{aligned}\cos(a - b) &= \cos a \cos b + \sin a \sin b, & \cos(a + b) &= \cos a \cos b - \sin a \sin b \\ \sin(a + b) &= \sin a \cos b + \cos a \sin b, & \sin(a - b) &= \sin a \cos b - \cos a \sin b \\ \tan(a \pm b) &= \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}\end{aligned}$$

Sine formulas keep the sign of the operation. Cosine formulas **flip** it.

Setting $b = a$ in the sum formulas gives the double-angle formulas at once. For cosine there are three forms, linked by $\sin^2 a + \cos^2 a = 1$.

Double-Angle Formulas

$$\begin{aligned}\sin 2a &= 2 \sin a \cos a, & \tan 2a &= \frac{2 \tan a}{1 - \tan^2 a} \\ \cos 2a &= \cos^2 a - \sin^2 a = 2 \cos^2 a - 1 = 1 - 2 \sin^2 a\end{aligned}$$

Solving the last two forms for $\cos^2 a$ and $\sin^2 a$ gives the **power-reducing** formulas, and replacing a by $\frac{a}{2}$ gives the half-angle formulas.

Half-Angle and Power-Reducing Formulas

$$\begin{aligned}\sin^2 a &= \frac{1 - \cos 2a}{2}, & \cos^2 a &= \frac{1 + \cos 2a}{2} \\ \sin \frac{a}{2} &= \pm \sqrt{\frac{1 - \cos a}{2}}, & \cos \frac{a}{2} &= \pm \sqrt{\frac{1 + \cos a}{2}}, & \tan \frac{a}{2} &= \frac{1 - \cos a}{\sin a} = \frac{\sin a}{1 + \cos a}\end{aligned}$$

Choose + or – from the **quadrant of $\frac{a}{2}$** , not the quadrant of a .

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Angle	Split	sin	cos
15°	45° - 30°	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$
75°	45° + 30°	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$
105°	60° + 45°	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{2}-\sqrt{6}}{4}$

The table shows the three most common non-standard angles. You do not need to memorize it: you can rebuild any row in thirty seconds from the first-quadrant values, and it is a good check on your work that $\sin 75^\circ = \cos 15^\circ$.

Worked Examples

EXAMPLE 1: EXACT VALUE FROM A SUM

Find the exact value of $\sin 75^\circ$.

SOLUTION

Split $75^\circ = 45^\circ + 30^\circ$ and use $\sin(a+b) = \sin a \cos b + \cos a \sin b$.

$$\sin 75^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

Decimal check: $\frac{2.449+1.414}{4} \approx 0.966$, and $\sin 75^\circ \approx 0.966$.

EXAMPLE 2: COSINE IN RADIANs

Find the exact value of $\cos \frac{7\pi}{12}$.

SOLUTION

$$\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4} \text{ (since } \frac{4\pi+3\pi}{12} = \frac{7\pi}{12} \text{)}.$$

$$\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{3} \sin \frac{\pi}{4} = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}.$$

$$= \frac{\sqrt{2}-\sqrt{6}}{4}, \text{ which is negative, as it must be because } \frac{7\pi}{12} \text{ is in quadrant II.}$$

EXAMPLE 3: SUM FORMULA WITH GIVEN INFORMATION

Angle a is in quadrant I with $\sin a = \frac{3}{5}$. Angle b is in quadrant II with $\cos b = -\frac{5}{13}$. Find $\sin(a+b)$, $\cos(a+b)$, and the quadrant of $a+b$.

SOLUTION

Find the missing values. $\cos a = \frac{4}{5}$ (quadrant I). For b : $\sin^2 b = 1 - \frac{25}{169} = \frac{144}{169}$, and sine is positive in quadrant II, so $\sin b = \frac{12}{13}$.

$$\sin(a+b) = \frac{3}{5} \left(-\frac{5}{13}\right) + \frac{4}{5} \cdot \frac{12}{13} = \frac{-15+48}{65} = \frac{33}{65}.$$

$$\cos(a+b) = \frac{4}{5} \left(-\frac{5}{13}\right) - \frac{3}{5} \cdot \frac{12}{13} = \frac{-20-36}{65} = -\frac{56}{65}.$$

Sine is positive and cosine is negative, so $a+b$ is in quadrant II. Check: $33^2 + 56^2 = 4225 = 65^2$.

EXAMPLE 4: DOUBLE-ANGLE VALUES

If $\sin \theta = \frac{4}{5}$ and θ is in quadrant II, find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

SOLUTION

In quadrant II, $\cos \theta = -\frac{3}{5}$.

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \cdot \frac{4}{5} \cdot \left(-\frac{3}{5}\right) = -\frac{24}{25}.$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}.$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-24/25}{-7/25} = \frac{24}{7}. \text{ Both are negative, so } 2\theta \text{ is in quadrant III.}$$

EXAMPLE 5: HALF-ANGLE FORMULAS

Find the exact values of $\sin \frac{\pi}{8}$ and $\tan 15^\circ$.

SOLUTION

$$\frac{\pi}{8} = \frac{1}{2} \cdot \frac{\pi}{4}, \text{ in quadrant I, so the root is positive: } \sin \frac{\pi}{8} = \sqrt{\frac{1-\cos(\pi/4)}{2}} = \sqrt{\frac{1-\sqrt{2}/2}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

$$15^\circ = \frac{1}{2} \cdot 30^\circ: \tan 15^\circ = \frac{1-\cos 30^\circ}{\sin 30^\circ} = \frac{1-\sqrt{3}/2}{1/2} = 2 - \sqrt{3}.$$

Decimal check: $\frac{\sqrt{0.5858}}{2} \approx 0.3827$ and $2 - 1.732 = 0.268$. Both match the calculator.

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EXAMPLE 6: RECOGNIZING A FORMULA IN REVERSE

Simplify $\cos(x + y) \cos y + \sin(x + y) \sin y$, then evaluate $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ$.

SOLUTION

The pattern is $\cos A \cos B + \sin A \sin B = \cos(A - B)$ with $A = x + y$ and $B = y$.

So the expression equals $\cos((x + y) - y) = \cos x$.

For the number: $A = 80^\circ$, $B = 20^\circ$, so the value is $\cos 60^\circ = \frac{1}{2}$.

Coach's Playbook

- 1 Decide which formula the problem wants.** Angle like 75° or $\frac{7\pi}{12}$: sum or difference. Angle double a known one: double-angle. Angle half a known one: half-angle.
 - 2 Find all four pieces.** Before substituting, write down $\sin a$, $\cos a$, $\sin b$, $\cos b$, using Pythagoras and the quadrant when only one is given.
 - 3 Pick the best cosine form.** Given $\cos \theta$, use $2 \cos^2 \theta - 1$. Given $\sin \theta$, use $1 - 2 \sin^2 \theta$. Given both, use $\cos^2 - \sin^2$.
 - 4 Set the sign last.** For half-angles, find the quadrant of $\frac{a}{2}$, then attach $+$ or $-$.
 - 5 Check with a decimal.** Use a calculator in the right mode to confirm.
- ◆ **Useful splits.** $15^\circ = 45^\circ - 30^\circ$, $75^\circ = 45^\circ + 30^\circ$, $105^\circ = 60^\circ + 45^\circ$, $165^\circ = 120^\circ + 45^\circ$.
 - ◆ **Tangent shortcut.** If you know $\tan a$ and $\tan b$ already, use the tangent formula directly.
 - ◆ **Reverse reading.** If an expression looks like $\sin A \cos B + \cos A \sin B$, collapse it to $\sin(A + B)$.

Common traps

(1) Writing $\sin(a + b) = \sin a + \sin b$ (test with $30^\circ + 60^\circ$). (2) Not flipping the sign in the cosine formulas. (3) Writing $\sin 2a = 2 \sin a$ (test with $a = 30^\circ$). (4) Choosing the half-angle root sign by the quadrant of a instead of $\frac{a}{2}$.

Mistake Clinic**Spot the error**

Problem: Find $\sin 90^\circ$ using $90^\circ = 30^\circ + 60^\circ$. **Student work:** $\sin(30^\circ + 60^\circ) = \sin 30^\circ + \sin 60^\circ = \frac{1}{2} + \frac{\sqrt{3}}{2} \approx 1.366$. **Where it went wrong:** sine does not distribute, and the result exceeds 1, which is impossible. **Correct:** $\sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ = \frac{1}{4} + \frac{3}{4} = 1$.

Spot the error

Problem: If $\sin \theta = \frac{1}{2}$ with $\theta = 30^\circ$, find $\sin 2\theta$. **Student work:** $\sin 2\theta = 2 \sin \theta = 1$. **Where it went wrong:** the double-angle formula is $2 \sin \theta \cos \theta$, not $2 \sin \theta$. **Correct:** $2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$, which equals $\sin 60^\circ$.

Spot the error

Problem: Find $\cos 22.5^\circ$ with a half-angle formula. **Student work:** $\cos 22.5^\circ = -\sqrt{\frac{1 + \cos 45^\circ}{2}} \approx -0.924$, using the negative root. **Where it went wrong:** 22.5° is in quadrant I, where cosine is positive. The sign depends on the quadrant of the half angle. **Correct:** $\cos 22.5^\circ = +\sqrt{\frac{1 + \cos 45^\circ}{2}} = \frac{\sqrt{2 + \sqrt{2}}}{2} \approx 0.924$.

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Real-World Connection

A ball launched at speed v and angle θ lands at range $R = \frac{v^2 \sin 2\theta}{g} = \frac{2v^2 \sin \theta \cos \theta}{g}$. With $v = 20$ m/s and $g = 9.8$ m/s²: $\theta = 30^\circ$ gives $R = \frac{400 \sin 60^\circ}{9.8} \approx 35.3$ m, and $\theta = 60^\circ$ gives the same range because $\sin 120^\circ = \sin 60^\circ$. The largest range, $\frac{400}{9.8} \approx 40.8$ m, occurs at 45° , where $\sin 90^\circ = 1$.

Challenge Corner

Problem: If $\tan a = \frac{1}{2}$ and $\tan b = \frac{1}{3}$ with a and b acute, find $a + b$. **Solution:** $\tan(a + b) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} = \frac{5/6}{5/6} = 1$. Both angles are acute with positive tangent less than 1, so each is less than 45° and $0 < a + b < 90^\circ$. The only angle in that range with tangent 1 is $\frac{\pi}{4}$. Answer: $a + b = \frac{\pi}{4}$.

Practice

PRACTICE Sum, Difference, Double-Angle, and Half-Angle Formulas

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

1. What is the exact value of $\cos 75^\circ$?

(A) $\frac{\sqrt{6} + \sqrt{2}}{4}$

(B) $\frac{\sqrt{2} - \sqrt{6}}{4}$

(C) $\frac{\sqrt{6} - \sqrt{2}}{4}$

(D) $\frac{1}{2}$

2. What is $\sin \frac{\pi}{3} \cos \frac{\pi}{6} + \cos \frac{\pi}{3} \sin \frac{\pi}{6}$?

(A) 1

(B) $\frac{1}{2}$

(C) $\frac{\sqrt{3}}{2}$

(D) 0

3. If $\sin \theta = \frac{1}{3}$, what is $\cos 2\theta$?

(A) $\frac{2}{3}$

(B) $\frac{8}{9}$

(C) $\frac{1}{9}$

(D) $\frac{7}{9}$

4. If $\tan \theta = \frac{3}{4}$, what is $\tan 2\theta$?

(A) $\frac{3}{2}$

(B) $\frac{24}{7}$

(C) $\frac{24}{25}$

(D) $\frac{7}{24}$

5. What is the exact value of $\sin 15^\circ \cos 15^\circ$?

(A) $\frac{1}{2}$

(B) $\frac{1}{8}$

(C) $\frac{1}{4}$

(D) $\frac{\sqrt{3}}{4}$

6. What is the exact value of $\cos \frac{\pi}{8}$?

(A) $\frac{\sqrt{2 + \sqrt{2}}}{2}$

(B) $\frac{\sqrt{2 + \sqrt{2}}}{2}$

(C) $\frac{\sqrt{2 - \sqrt{2}}}{2}$

(D) $\frac{\sqrt{2 + \sqrt{2}}}{4}$

7. Which expression equals $\cos^2 x - \sin^2 x$?

(A) $\sin 2x$

(B) 1

(C) $2 \cos x$

(D) $\cos 2x$

8. If $\tan a = 2$ and $\tan b = 3$, what is $\tan(a + b)$?

(A) 1

(B) -1

(C) 5

(D) -5

9. What is the exact value of $\cos 165^\circ$?

(A) $\frac{\sqrt{2 + \sqrt{3}}}{2}$

(B) $\frac{\sqrt{2 - \sqrt{3}}}{2}$

(C) $-\frac{\sqrt{2 + \sqrt{3}}}{2}$

(D) $-\frac{\sqrt{2 - \sqrt{3}}}{2}$

10. Find the exact value of $\tan 75^\circ$.

11. If $\cos \theta = \frac{12}{13}$ and θ is in quadrant IV, find $\sin 2\theta$.

12. Find the exact value of $\sin \frac{\pi}{12}$.

13. If $\cos \theta = -\frac{7}{25}$ and θ is in quadrant II, find $\sin \frac{\theta}{2}$.

14. Simplify $\cos(x + y) \cos y + \sin(x + y) \sin y$.

15. Use power-reducing formulas to rewrite $\sin^4 x$ with no powers of trigonometric functions.

26.3 Proving Trigonometric Identities

Learn a step-by-step strategy for proving identities: work one side, convert to sine and cosine, combine, factor, and use conjugates. Avoid the classic pitfalls of moving terms across the equal sign or relying on number checks.

Why This Matters

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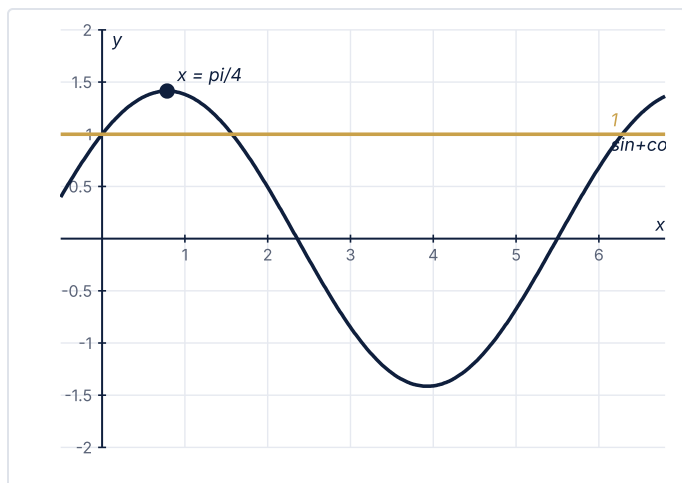
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Proving an identity is the first time many students write a real mathematical argument: you are given a claim and must show, with valid steps, that it is always true. The skill is more than a classroom exercise. It is the same reasoning programmers use to check that two formulas for the same quantity agree, that engineers use to reduce complicated signal expressions, and that mathematicians use in every proof. In calculus, a clever identity can turn an impossible integral into a routine one.

Students struggle in four ways. They do not know **where to start**. They **treat the identity as an equation to solve**, moving terms across the equal sign, which assumes what they are trying to prove. They **test one value** and think that proves it. And they make algebra slips, such as cancelling terms or multiplying only part of a fraction. This section gives you a clear strategy and a clean proof format so that you always have a next step.

Core Ideas

An **identity** is true for all values for which both sides are defined. A **conditional equation** is true only for some values. To prove an identity you must transform one side, using valid steps, into the other. Checking numbers or graphs can **disprove** a claim (one counterexample is enough) or build your confidence, but it never proves an identity.



The graphs of $y = \sin x + \cos x$ and $y = 1$ meet at some points but not everywhere, so $\sin x + \cos x = 1$ is a conditional equation. One counterexample, $x = \pi/4$, disproves it.

Rules of a Valid Proof

1. Start with **one side** only (usually the more complicated one) and transform it step by step until it matches the other side.
2. **Never** add, subtract, multiply, or cross-multiply across the equal sign. Doing so assumes the identity is true.
3. Each step must use a true fact: an algebra rule or a known identity. Write the reason beside each step.
4. A numerical check or graph can disprove, never prove.

Strategy Toolkit

(a) Convert every function to $\sin$ and $\cos$. **(b) Combine** fractions over a common denominator. **(c) Factor**, including difference of squares and $a^2 - b^2$ patterns. **(d) Conjugate**: multiply top and bottom by $1 \pm \sin x$ or $1 \pm \cos x$ to use $1 - \sin^2 x = \cos^2 x$. **(e) Substitute** a Pythagorean, double-angle, or sum identity when you see one. **(f) Work both sides** toward a common middle expression when stuck, but write the proof as one chain from one side.

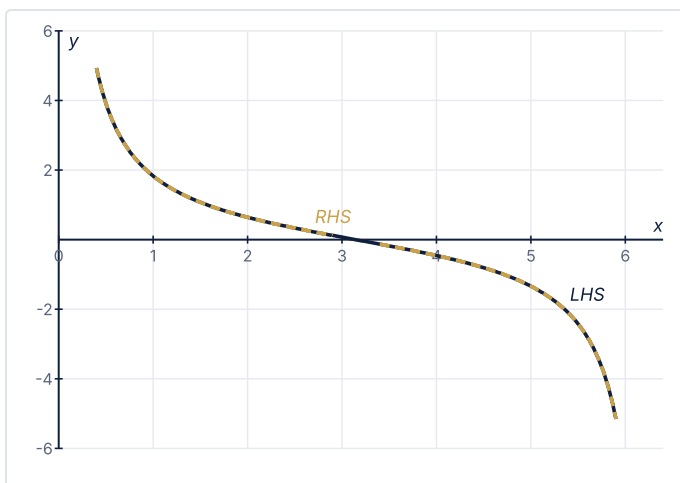
Many proofs fall into a few shapes. Recognizing the shape tells you the first step.

Shape you see	First move	Example
Sum of fractions	Common denominator, then Pythagorean identity	$\tan x + \cot x$
Fraction with $1 \pm \sin x$ or $1 \pm \cos x$	Multiply by the conjugate	$\frac{\sin x}{1 - \cos x}$
Squares or fourth powers	Factor as a difference of squares	$\sin^4 x - \cos^4 x$
$\sin 2x$, $\cos 2x$, $\sin(a + b)$	Expand with the formulas	$\frac{\sin 2x}{1 + \cos 2x}$
Mixed functions	Rewrite in $\sin$ and $\cos$	$\sec x - \cos x$

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The curves of $\sin x / (1 - \cos x)$ (solid) and $\csc x + \cot x$ (dashed) lie on top of each other. That supports the identity; Example 2 proves it.

Writing the proof. Put the claim at the top, then a chain of equal expressions starting from one side. Finish with the target expression. Use complete reasons such as 'common denominator', 'Pythagorean identity', or 'factor out $\sin x$ '. Examiners and teachers care as much about the logic as about the final line.

Worked Examples

EXAMPLE 1: COMBINE FRACTIONS

Prove that $\tan x + \cot x = \sec x \csc x$.

SOLUTION

Start with the left side and convert: $\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$.

Common denominator $\sin x \cos x$: $= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$.

Pythagorean identity: $= \frac{1}{\sin x \cos x} = \frac{1}{\cos x} \cdot \frac{1}{\sin x} = \sec x \csc x$. ■

EXAMPLE 2: MULTIPLY BY A CONJUGATE

Prove that $\frac{\sin x}{1 - \cos x} = \csc x + \cot x$.

SOLUTION

Start with the left side. Multiply top and bottom by the conjugate $1 + \cos x$: $\frac{\sin x(1 + \cos x)}{(1 - \cos x)(1 + \cos x)}$.

The denominator is a difference of squares: $1 - \cos^2 x = \sin^2 x$. So we have $\frac{\sin x(1 + \cos x)}{\sin^2 x} = \frac{1 + \cos x}{\sin x}$ after cancelling the factor $\sin x$.

Split the fraction: $\frac{1}{\sin x} + \frac{\cos x}{\sin x} = \csc x + \cot x$. ■

EXAMPLE 3: TWO SIDES THAT LOOK DIFFERENT

Prove that $\frac{1 + \sin x}{\cos x} = \frac{\cos x}{1 - \sin x}$.

SOLUTION

Work on the right side, which has the conjugate-ready denominator. Multiply top and bottom by $1 + \sin x$: $\frac{\cos x(1 + \sin x)}{(1 - \sin x)(1 + \sin x)}$.

The denominator becomes $1 - \sin^2 x = \cos^2 x$: $\frac{\cos x(1 + \sin x)}{\cos^2 x} = \frac{1 + \sin x}{\cos x}$, which is the left side. ■

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EXAMPLE 4: FRACTIONS WITH $1 - \tan x$

Prove that $\frac{\cos x}{1 - \tan x} + \frac{\sin x}{1 - \cot x} = \sin x + \cos x$.

SOLUTION

Write $\tan x = \frac{\sin x}{\cos x}$. First term: $\frac{\cos x}{1 - \frac{\sin x}{\cos x}} = \frac{\cos x \cdot \cos x}{\cos x - \sin x} = \frac{\cos^2 x}{\cos x - \sin x}$.

Write $\cot x = \frac{\cos x}{\sin x}$. Second term: $\frac{\sin x}{1 - \frac{\cos x}{\sin x}} = \frac{\sin^2 x}{\sin x - \cos x} = -\frac{\sin^2 x}{\cos x - \sin x}$.

Add: $\frac{\cos^2 x - \sin^2 x}{\cos x - \sin x} = \frac{(\cos x - \sin x)(\cos x + \sin x)}{\cos x - \sin x} = \cos x + \sin x$. ■

EXAMPLE 5: USING A DOUBLE-ANGLE FORMULA

Prove that $\frac{\sin 2x}{1 + \cos 2x} = \tan x$.

SOLUTION

Replace the numerator with $2 \sin x \cos x$ and the denominator with $1 + \cos 2x = 1 + (2 \cos^2 x - 1) = 2 \cos^2 x$.

The fraction becomes $\frac{2 \sin x \cos x}{2 \cos^2 x} = \frac{\sin x}{\cos x} = \tan x$. ■

The choice of $2 \cos^2 x - 1$ out of the three cosine forms is what makes the 1 cancel. That is a good habit: pick the form that cancels.

EXAMPLE 6: USING SUM AND DIFFERENCE FORMULAS

Prove that $\sin(a + b) \sin(a - b) = \sin^2 a - \sin^2 b$.

SOLUTION

Expand each factor: $(\sin a \cos b + \cos a \sin b)(\sin a \cos b - \cos a \sin b)$.

This is a difference of squares: $\sin^2 a \cos^2 b - \cos^2 a \sin^2 b$.

Replace $\cos^2 b = 1 - \sin^2 b$ and $\cos^2 a = 1 - \sin^2 a$: $\sin^2 a(1 - \sin^2 b) - (1 - \sin^2 a) \sin^2 b = \sin^2 a - \sin^2 a \sin^2 b - \sin^2 b + \sin^2 a \sin^2 b = \sin^2 a - \sin^2 b$. ■

Coach's Playbook

- 1 **Pick the busier side.** The side with more terms, more functions, or more fractions gives you more to simplify. Work on that side.
 - 2 **Ask what the target looks like.** If the other side has only $\sin$ and $\cos$, convert. If it has a single fraction, combine.
 - 3 **Try the standard moves in order.** Convert, combine, factor, conjugate, then substitute.
 - 4 **Keep a running comparison.** After each step, look at the target to see if you are closer. If you are not, back up.
 - 5 **Check with a number, then finish.** Test your last expression at $x = \frac{\pi}{6}$ to catch algebra slips, then write the final line.
- ◆ **Stuck?** Write the target in terms of $\sin$ and $\cos$ too. The two sides often meet in the middle, and you can reverse the second half.
 - ◆ **Squares hint.** Whenever you see $\sin^2 x$ or $\cos^2 x$ by itself, there is probably a Pythagorean swap waiting.
 - ◆ **Domain.** Mention restrictions if you multiply by an expression like $1 - \cos x$: it must not be 0 for the step to be valid.

Common traps

(1) Treating the identity as an equation and moving terms across. (2) Testing one value and calling it a proof. (3) Multiplying only the numerator or only the denominator by a conjugate. (4) Cancelling a term instead of a factor.

Mistake Clinic

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Spot the error

Problem: Prove $\sec x - \cos x = \sin x$. **Student work:** At $x = 0$: $\sec 0 - \cos 0 = 0$ and $\sin 0 = 0$. At $x = \frac{\pi}{4}$: both sides equal $\frac{\sqrt{2}}{2}$. Proved. **Where it went wrong:** checking values never proves an identity, and here the claim is actually false. At $x = \frac{\pi}{3}$ the left side is $2 - \frac{1}{2} = \frac{3}{2}$ but $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$. **Correct statement:** $\sec x - \cos x = \sin x \tan x$, proved by $\frac{1}{\cos x} - \cos x = \frac{1 - \cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x} = \sin x \cdot \frac{\sin x}{\cos x}$.

Spot the error

Problem: Prove $\frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$. **Student work:** Cross-multiply: $\sin^2 x = (1 - \cos x)(1 + \cos x) = 1 - \cos^2 x$. This is true, so the identity is proved. **Where it went wrong:** cross-multiplying begins with the claim and manipulates both sides, assuming what is to be shown. **Correct proof:** start with the left side, multiply top and bottom by $1 - \cos x$:

$$\frac{\sin x(1 - \cos x)}{1 - \cos^2 x} = \frac{\sin x(1 - \cos x)}{\sin^2 x} = \frac{1 - \cos x}{\sin x}.$$

Spot the error

Problem: Simplify $\frac{\sin x}{1 - \cos x}$ using a conjugate. **Student work:** Multiply the numerator by $1 + \cos x$ to get

$$\frac{\sin x(1 + \cos x)}{1 - \cos x} = \frac{\sin x(1 + \cos x)}{\sin^2 x}.$$
 Where it went wrong: the student multiplied only the numerator. That changes the value of the fraction by the factor $1 + \cos x$. **Correct:** multiply numerator and denominator, giving $\frac{\sin x(1 + \cos x)}{1 - \cos^2 x} = \frac{1 + \cos x}{\sin x}.$

Real-World Connection

Graphics software rotates a point by angle a and then by angle b . The combined rotation must equal one rotation by $a + b$. Writing both rotations as matrices and multiplying them produces entries like $\cos a \cos b - \sin a \sin b$, and the identities $\cos(a + b) = \cos a \cos b - \sin a \sin b$ and $\sin(a + b) = \sin a \cos b + \cos a \sin b$ are exactly what prove the two ways agree.

Challenge Corner

Problem: Prove that $\frac{\sin 3x}{\sin x} = 3 - 4 \sin^2 x$, then use it to find $\sin 3x$ when $\sin x = \frac{1}{2}$. **Solution:** Write $\sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x = 2 \sin x \cos^2 x + (1 - 2 \sin^2 x) \sin x$. Replace $\cos^2 x = 1 - \sin^2 x$: $2 \sin x(1 - \sin^2 x) + \sin x - 2 \sin^3 x = 3 \sin x - 4 \sin^3 x$. Divide by $\sin x$ (nonzero): $3 - 4 \sin^2 x$. With $\sin x = \frac{1}{2}$: $\sin 3x = \frac{1}{2} (3 - 4 \cdot \frac{1}{4}) = \frac{1}{2} \cdot 2 = 1$. Check: $x = 30^\circ$ gives $\sin 90^\circ = 1$.

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Practice

PRACTICE Proving Trigonometric Identities

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Which equation is an identity?
 (A) $\sin x + \cos x = 1$ (B) $\tan^2 x + 1 = \sec^2 x$ (C) $\sin 2x = 2 \sin x$
 (D) $\cos(x + y) = \cos x + \cos y$
- To prove $\frac{1 + \cos x}{\sin x} = \frac{\sin x}{1 - \cos x}$, which is a valid first step?
 (A) Multiply the top and bottom of the right side by $1 + \cos x$
 (B) Add 1 to both sides
 (C) Square both sides
 (D) Replace $\sin x$ by $1 - \cos x$
- Simplify $(\sec x - 1)(\sec x + 1)$.
 (A) 1 (B) $\sin^2 x$ (C) $\tan^2 x$ (D) $\sec^2 x + 1$
- Simplify $\frac{(1 - \sin x)(1 + \sin x)}{\cos x}$.
 (A) $\sin x$ (B) $\sec x$ (C) $\cos^2 x$ (D) $\cos x$
- A student claims $\sin x + \cos x = 1$ for all x . Which value of x is a counterexample?
 (A) 0 (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{4}$ (D) 2π
- Simplify $\frac{\sin 2x}{2 \sin x}$.
 (A) 1 (B) $\cos x$ (C) $\sin x$ (D) $\cos 2x$
- Simplify $\frac{\cos 2x}{\cos x - \sin x}$.
 (A) $\cos x - \sin x$ (B) 1 (C) $2 \cos x$ (D) $\cos x + \sin x$
- Which is NOT a valid way to begin proving an identity?
 (A) Adding the same expression to both sides of the claimed identity
 (B) Rewriting one side in terms of sine and cosine
 (C) Multiplying the numerator and denominator of one side by a conjugate
 (D) Factoring one side
- Simplify $\frac{\cos x}{1 + \sin x} + \frac{1 + \sin x}{\cos x}$.
 (A) $2 \csc x$ (B) $2 \sec x$ (C) $2 \cos x$ (D) 1
- Simplify $(1 + \tan^2 x) \cos^2 x$.
- Prove that $\sec x - \cos x = \sin x \tan x$.
- Prove that $\frac{\cos x}{1 - \sin x} = \sec x + \tan x$.
- Show that $\sin 2x = 2 \sin x$ is not an identity by giving a value of x for which it fails.
- Prove that $\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$.
- Prove the triple-angle identity $\cos 3x = 4 \cos^3 x - 3 \cos x$.

26.4 Solving Trigonometric Equations

Solve trigonometric equations on an interval and in general. Use the unit circle, factoring, identities, multiple-angle ranges, and inverse trigonometric functions, and learn to avoid lost and extraneous solutions.

Why This Matters

An identity is true for every angle. An **equation** is true only for certain angles, and finding those angles is the point. When will a Ferris wheel rider be 30 meters up? On which days does a city have exactly 14 hours of daylight? At what times does the voltage in a circuit hit a given value? Each question becomes a trigonometric equation. Because trigonometric functions repeat, the answer is usually not one number but a whole list, or a formula that describes infinitely many angles.

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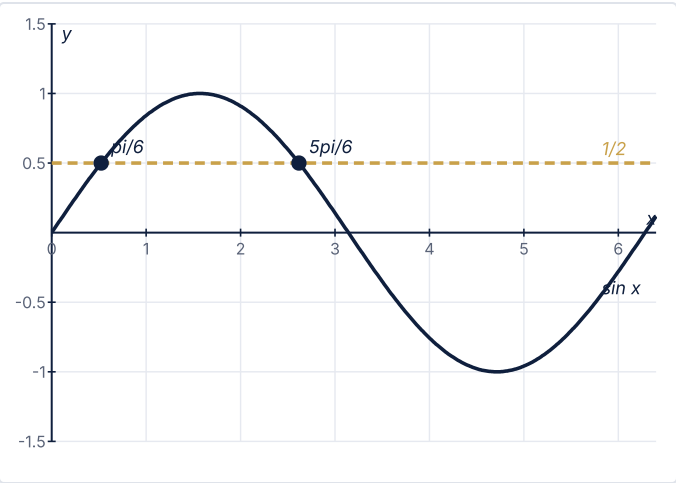
Students typically struggle for four reasons. They **find only one solution** when the unit circle gives two per lap. They **divide by a trigonometric factor** and silently lose solutions. They forget to **account for a multiple angle**, so they solve $\sin 2x = \frac{1}{2}$ but find only half the answers. And they misuse **inverse trigonometric functions**, thinking that $\arcsin(-\frac{1}{2})$ is the whole answer. This section builds a reliable method: isolate, use the unit circle, list every lap, and check.

Core Ideas

The simplest equations have the form $\sin x = k$, $\cos x = k$, or $\tan x = k$. You solve them with the unit circle and a reference angle, exactly as in the first booster section. On $[0, 2\pi)$, sine and cosine equations (with $|k| < 1$) have **two** solutions, and tangent equations have two as well, spaced π apart. To describe **all** real solutions, add whole numbers of periods.

General Solutions

Suppose β is any one angle that solves the equation. Then, for every integer n :
 $\sin x = \sin \beta$ gives $x = \beta + 2\pi n$ or $x = \pi - \beta + 2\pi n$. $\cos x = \cos \beta$ gives $x = \pm \beta + 2\pi n$. $\tan x = \tan \beta$ gives $x = \beta + \pi n$.



Solving $\sin x = 1/2$ on $[0, 2\pi)$: the line $y = 1/2$ crosses the sine curve at $x = \pi/6$ and $x = 5\pi/6$.

Inverse Trigonometric Functions

$y = \arcsin x$ means $\sin y = x$ with $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. $y = \arccos x$ means $\cos y = x$ with $0 \leq y \leq \pi$. $y = \arctan x$ means $\tan y = x$ with $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
A calculator returns only this **one** principal value. The other solutions come from symmetry on the unit circle, so you must find them yourself.

For more complicated equations the plan is to **reduce to a basic equation**. If there is a quadratic in a trigonometric function, treat $\sin x$ (or $\cos x$) as a single unknown u and factor. If two different functions appear, use an identity to get one function. If the angle is $2x$ or $3x$ or $x + \frac{\pi}{4}$, let u be the whole angle, solve for u over the right range, and then solve for x .

The Range Rule for Multiple Angles

If $0 \leq x < 2\pi$ and the equation involves nx , then $0 \leq nx < 2n\pi$. Solve for the angle $u = nx$ over **all n laps**, then divide by n .
An equation like $\cos 3x = \frac{\sqrt{3}}{2}$ has $2 \cdot 3 = 6$ solutions on $[0, 2\pi)$.

Equation type	Method	Watch out for
Linear in $\sin x$ or $\cos x$	Isolate, use the unit circle	Two solutions per lap
Quadratic in a trig function	Factor or use the quadratic formula in u	Reject $ u > 1$
Two different functions	Use an identity to make one function	Pick the identity that avoids square roots
Common factor like $\sin x \cos x = \sin x$	Subtract and factor	Never divide by $\sin x$
Multiple angle nx	Solve for nx over n laps	Count solutions
Squared both sides	Check every answer	Extraneous solutions

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Worked Examples

EXAMPLE 1: A BASIC EQUATION

Solve $2 \sin x - 1 = 0$ on $[0, 2\pi)$, then give all real solutions.

SOLUTION

Isolate: $\sin x = \frac{1}{2}$. The reference angle is $\frac{\pi}{6}$, and sine is positive in quadrants I and II.

On $[0, 2\pi)$: $x = \frac{\pi}{6}$ and $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

All real solutions: $x = \frac{\pi}{6} + 2\pi n$ or $x = \frac{5\pi}{6} + 2\pi n$, where n is any integer.

EXAMPLE 2: GENERAL SOLUTIONS FOR COSINE AND TANGENT

Find all real solutions of (a) $\cos x = -\frac{\sqrt{3}}{2}$ and (b) $\tan x = -1$.

SOLUTION

(a) Reference angle $\frac{\pi}{6}$. Cosine is negative in quadrants II and III: $x = \frac{5\pi}{6}$ and $x = \frac{7\pi}{6}$. General: $x = \frac{5\pi}{6} + 2\pi n$ or $x = \frac{7\pi}{6} + 2\pi n$ (equivalently $x = \pm \frac{5\pi}{6} + 2\pi n$).

(b) Reference angle $\frac{\pi}{4}$. Tangent is negative in quadrants II and IV, and the solutions are only π apart: $x = \frac{3\pi}{4} + \pi n$. This one formula covers both $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

EXAMPLE 3: A MULTIPLE ANGLE

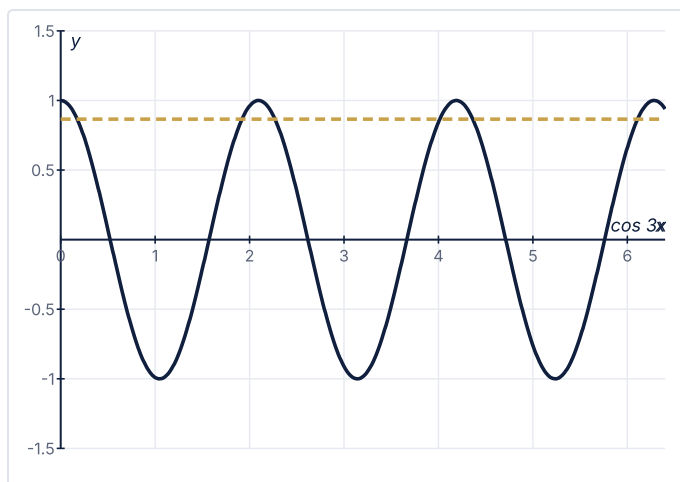
Solve $\cos 3x = \frac{\sqrt{3}}{2}$ on $[0, 2\pi)$.

SOLUTION

Let $u = 3x$. Since $0 \leq x < 2\pi$, we need $0 \leq u < 6\pi$, three full laps.

$\cos u = \frac{\sqrt{3}}{2}$ has solutions $u = \frac{\pi}{6}$ and $u = \frac{11\pi}{6}$ on each lap: $\frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}, \frac{35\pi}{6}$.

Divide by 3: $x = \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \frac{35\pi}{18}$. That is 6 solutions, as the range rule predicted.



Graph for Example 3. The curve $y = \cos 3x$ meets the line $y = \sqrt{3}/2$ six times on $[0, 2\pi)$.

EXAMPLE 4: A QUADRATIC IN SINE

Solve $2 \sin^2 x - \sin x - 1 = 0$ on $[0, 2\pi)$.

SOLUTION

Let $u = \sin x$. Then $2u^2 - u - 1 = 0$ factors as $(2u + 1)(u - 1) = 0$, so $u = -\frac{1}{2}$ or $u = 1$.

$\sin x = 1$: $x = \frac{\pi}{2}$. $\sin x = -\frac{1}{2}$: reference angle $\frac{\pi}{6}$, sine negative in quadrants III and IV, so $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$.

Solutions: $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

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The three solutions of Example 4 on $[0, 2\pi)$.

EXAMPLE 5: USING A DOUBLE-ANGLE IDENTITY

Solve $\sin 2x = \sin x$ on $[0, 2\pi)$.

SOLUTION

Replace $\sin 2x$ with $2 \sin x \cos x$ and move everything to one side: $2 \sin x \cos x - \sin x = 0$.

Factor (do not divide by $\sin x$): $\sin x(2 \cos x - 1) = 0$.

$\sin x = 0$ gives $x = 0$ and $x = \pi$. $\cos x = \frac{1}{2}$ gives $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$.

Solutions: $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

EXAMPLE 6: INVERSE FUNCTIONS AND A CALCULATOR

Solve $3 \sin x = -2$ on $[0, 2\pi)$. Round to three decimals (radian mode).

SOLUTION

$\sin x = -\frac{2}{3}$. The calculator gives $\arcsin(-\frac{2}{3}) \approx -0.730$, which is not in $[0, 2\pi)$. Use it only to find the reference angle $\alpha = 0.730$.

Sine is negative in quadrants III and IV. Quadrant III: $x = \pi + 0.730 \approx 3.871$. Quadrant IV: $x = 2\pi - 0.730 \approx 5.553$.

Check: $3 \sin 3.871 \approx 3(-0.667) = -2.0$. Solutions: $x \approx 3.871$ and $x \approx 5.553$.

Coach's Playbook

- 1 Isolate a trig function.** Get it alone on one side, or move everything to one side and factor.
 - 2 Reduce to one function.** Use $\sin^2 x + \cos^2 x = 1$, double-angle, or other identities so that only one function remains.
 - 3 Solve the basic equation on the right range.** For nx , solve for nx over n laps. Use the unit circle for both solutions in each lap.
 - 4 Convert back and list.** Divide by n , or subtract shifts, then list solutions in order.
 - 5 Check.** Substitute each answer back, especially if you squared both sides or used a calculator.
- ◆ **Counting check.** For $\sin(nx) = k$ with $|k| < 1$ on $[0, 2\pi)$, expect $2n$ solutions.
 - ◆ **Reject impossible values.** If factoring gives $\sin x = 3$ or $\cos x = -2$, those give no solutions.
 - ◆ **Squaring warning.** Squaring both sides can add false solutions, so always test.

Common traps

(1) Giving one solution when there are two per lap. (2) Dividing both sides by $\sin x$ or $\cos x$, which loses solutions. (3) Forgetting extra laps for multiple angles. (4) Giving $\arcsin(-\frac{1}{2}) = -\frac{\pi}{6}$ as the only answer when the problem asks for $[0, 2\pi)$.

Mistake Clinic

Spot the error

Problem: Solve $\sin x \cos x = \sin x$ on $[0, 2\pi)$. **Student work:** Divide both sides by $\sin x$ to get $\cos x = 1$, so $x = 0$. **Where it went wrong:** dividing by $\sin x$ discards every solution where $\sin x = 0$, and it is not even allowed there. **Correct:** $\sin x(\cos x - 1) = 0$ gives $\sin x = 0$ ($x = 0, \pi$) or $\cos x = 1$ ($x = 0$). Solutions: $x = 0$ and $x = \pi$.

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Spot the error

Problem: Solve $\sin x = -\frac{1}{2}$ on $[0, 2\pi)$. **Student work:** $x = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$. **Where it went wrong:** $-\frac{\pi}{6}$ is outside the interval, and the calculator gives only the principal value. There are two solutions in $[0, 2\pi)$. **Correct:** sine is negative in quadrants III and IV, so $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$.

Spot the error

Problem: Solve $\sin 2x = \frac{1}{2}$ on $[0, 2\pi)$. **Student work:** $2x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, so $x = \frac{\pi}{12}$ or $\frac{5\pi}{12}$. **Where it went wrong:** for $0 \leq x < 2\pi$ the angle $2x$ ranges over $[0, 4\pi)$, two laps. The student used only the first lap. **Correct:** also $2x = \frac{13\pi}{6}, \frac{17\pi}{6}$, so $x = \frac{13\pi}{12}, \frac{17\pi}{12}$. Four solutions in total.

Real-World Connection

The hours of daylight in a city are modeled by $D(t) = 12 + 3 \sin\left(\frac{2\pi}{365}(t - 80)\right)$, with t the day of the year. To find when $D = 14$, solve $\sin \theta = \frac{2}{3}$ where $\theta = \frac{2\pi}{365}(t - 80)$. The two angles are $\theta \approx 0.730$ and $\pi - 0.730 \approx 2.412$, giving $t \approx 80 + 42.4 \approx 122$ and $t \approx 80 + 140.1 \approx 220$. The city has 14 hours of daylight on about day 122 (early May) and day 220 (early August).

Challenge Corner

Problem: Solve $\sin x + \sqrt{3} \cos x = 1$ on $[0, 2\pi)$. **Solution:** Notice $2 \sin\left(x + \frac{\pi}{3}\right) = 2\left(\sin x \cdot \frac{1}{2} + \cos x \cdot \frac{\sqrt{3}}{2}\right) = \sin x + \sqrt{3} \cos x$. So the equation is $\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$. Then $x + \frac{\pi}{3} = \frac{\pi}{6} + 2\pi n$ or $\frac{5\pi}{6} + 2\pi n$. That gives $x = -\frac{\pi}{6} + 2\pi n$ or $x = \frac{\pi}{2} + 2\pi n$. In $[0, 2\pi)$: $x = \frac{\pi}{2}$ and $x = \frac{11\pi}{6}$. Check $\frac{11\pi}{6}$: $-\frac{1}{2} + \sqrt{3} \cdot \frac{\sqrt{3}}{2} = 1$.

Practice**PRACTICE Solving Trigonometric Equations**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Solve $2 \cos x = -\sqrt{2}$ on $[0, 2\pi)$.
 (A) $\frac{\pi}{4}, \frac{7\pi}{4}$ (B) $\frac{5\pi}{6}, \frac{7\pi}{6}$ (C) $\frac{3\pi}{4}, \frac{5\pi}{4}$ (D) $\frac{3\pi}{4}, \frac{7\pi}{4}$
- Which gives all real solutions of $\tan x = \sqrt{3}$?
 (A) $x = \frac{\pi}{3} + \pi n$ (B) $x = \frac{\pi}{3} + 2\pi n$ (C) $x = \frac{2\pi}{3} + \pi n$ (D) $x = \pm \frac{\pi}{3} + 2\pi n$
- How many solutions does $\sin x = 0.3$ have on $[0, 2\pi)$?
 (A) 1 (B) 2 (C) 3 (D) 4
- Solve $\sin 2x = 1$ on $[0, 2\pi)$.
 (A) $\frac{\pi}{4}$ only (B) $\frac{\pi}{4}, \frac{3\pi}{4}$ (C) $\frac{\pi}{2}, \frac{5\pi}{2}$ (D) $\frac{\pi}{4}, \frac{5\pi}{4}$
- The equation $2 \sin^2 x - \sin x - 1 = 0$ is solved by which values of $\sin x$?
 (A) -1 and $\frac{1}{2}$ (B) 1 and $\frac{1}{2}$ (C) 1 and $-\frac{1}{2}$ (D) 2 and -1
- What is $\arcsin\left(-\frac{1}{2}\right)$?
 (A) $\frac{11\pi}{6}$ (B) $\frac{7\pi}{6}$ (C) $\frac{\pi}{6}$ (D) $-\frac{\pi}{6}$
- Solve $\sin x \cos x = \sin x$ on $[0, 2\pi)$.
 (A) 0 only (B) $0, \pi$ (C) $\frac{\pi}{2}, \frac{3\pi}{2}$ (D) $0, \frac{\pi}{2}, \pi$
- How many solutions does $\cos 2x = \frac{1}{2}$ have on $[0, 2\pi)$?
 (A) 4 (B) 2 (C) 3 (D) 6

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9. Which gives all real solutions of $\cos x = \frac{1}{2}$?

(A) $x = \frac{\pi}{3} + \pi n$

(B) $x = \frac{\pi}{3} + 2\pi n$

(C) $x = \pm \frac{\pi}{3} + 2\pi n$

(D) $x = \frac{\pi}{3} + 2\pi n$ or $\frac{2\pi}{3} + 2\pi n$

10. Solve $2\sin x + \sqrt{3} = 0$ on $[0, 2\pi)$.

11. Solve $2\cos^2 x + 3\sin x - 3 = 0$ on $[0, 2\pi)$.

12. Solve $\tan^2 x - 3 = 0$ on $[0, \pi)$.

13. Solve $\cos 2x = \cos x$ on $[0, 2\pi)$.

14. Solve $4\cos x = 1$ on $[0, 2\pi)$. Round to three decimals (radian mode).

15. Solve $\sin x + \cos x = 1$ on $[0, 2\pi)$. Check for extraneous solutions.

Answers

Answers and Explanations

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

26.1 Fundamental and Pythagorean Identities

PRACTICE · FUNDAMENTAL AND PYTHAGOREAN IDENTITIES

- C. Write $\tan x = \frac{\sin x}{\cos x}$. Then $\frac{\sin x}{\cos x} \cdot \cos x = \sin x$. The cosines cancel as factors.
- A. Rearranging $\sin^2 x + \cos^2 x = 1$ gives $\cos^2 x$. The tempting answer $\cos x$ forgets the square.
- D. Dividing $\sin^2 x + \cos^2 x = 1$ by $\cos^2 x$ gives $\tan^2 x + 1 = \sec^2 x$. Then $\sec^2 x - 1 = \tan^2 x$. The cotangent version uses $\csc$, not $\sec$.
- B. $\sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$, so $\sin x = \pm \frac{4}{5}$. Sine is negative in quadrant IV. The tempting answer $\frac{4}{5}$ ignores the quadrant.
- A. $\frac{1/\sin x}{1/\cos x} = \frac{\cos x}{\sin x} = \cot x$. The tempting answer $\tan x$ inverts the quotient.
- D. Expand: $\sin^2 x + 2\sin x \cos x + \cos^2 x - 2\sin x \cos x = \sin^2 x + \cos^2 x = 1$. The middle terms cancel.
- B. $\frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x} = \sec x \csc x$. Adding fractions requires a common denominator; the sum is not 1.
- C. $1 - \cos^2 x = \sin^2 x$ and $1 + \cot^2 x = \csc^2 x$. The product is $\sin^2 x \cdot \frac{1}{\sin^2 x} = 1$.
- A. Cosine is even, so $\cos(-x) = \cos x$. Sine is odd, so $\sin(-x) = -\sin x$. The sum is $\cos x - \sin x$.
- $\sin x = -\frac{3}{5}$ and $\cos x = -\frac{4}{5}$. $\sec^2 x = 1 + \frac{9}{16} = \frac{25}{16}$, so $|\cos x| = \frac{4}{5}$. In quadrant III both are negative. Then $\sin x = \tan x \cos x = \frac{3}{4} \left(-\frac{4}{5}\right) = -\frac{3}{5}$.
- $\tan x \cdot \sec x - \cos x = \frac{1 - \cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x}$. Divide by $\sin x$: $\frac{\sin x}{\cos x} = \tan x$.
- $\sec x$. Common denominator $\cos x(1 + \sin x)$: numerator $\cos^2 x + \sin x(1 + \sin x) = \cos^2 x + \sin x + \sin^2 x = 1 + \sin x$. Cancel $1 + \sin x$ to get $\frac{1}{\cos x} = \sec x$.
- $\cos^2 x \cdot 1 - \cos^2 x = \sin^2 x$, and $\cot^2 x \sin^2 x = \frac{\cos^2 x}{\sin^2 x} \cdot \sin^2 x = \cos^2 x$.
- $4\sec \theta$. $16 + 16\tan^2 \theta = 16(1 + \tan^2 \theta) = 16\sec^2 \theta$, and the square root is $4\sec \theta$ (positive since $\sec \theta > 0$ on this interval).
- $\frac{5}{4}$. Since $\sec^2 x - \tan^2 x = 1$, we get $\sec x - \tan x = \frac{1}{2}$. Adding the two equations gives $2\sec x = \frac{5}{2}$, so $\sec x = \frac{5}{4}$. Check: $\tan x = \frac{3}{4}$ and $\frac{25}{16} - \frac{9}{16} = 1$.

26.2 Sum, Difference, Double-Angle, and Half-Angle Formulas

PRACTICE · SUM, DIFFERENCE, DOUBLE-ANGLE, AND HALF-ANGLE FORMULAS

- C. $\cos(45^\circ + 30^\circ) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$. The plus version is $\sin 75^\circ$, and the reversed subtraction is negative, which is impossible in quadrant I.
- A. This is $\sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin \frac{\pi}{2} = 1$. Check by direct arithmetic: $\frac{3}{4} + \frac{1}{4} = 1$.
- D. $\cos 2\theta = 1 - 2\sin^2 \theta = 1 - 2\left(\frac{2}{9}\right) = \frac{5}{9}$. Using $1 - \sin^2 \theta = \cos^2 \theta$, $\frac{5}{9}$ is the value of $\cos^2 \theta$, not $\cos 2\theta$.
- B. $\tan 2\theta = \frac{2 \cdot \frac{4}{7}}{1 - \frac{16}{49}} = \frac{8/7}{7/49} = \frac{24}{7}$. The tempting answer $\frac{3}{2}$ just doubles the tangent.
- C. $\sin 15^\circ \cos 15^\circ = \frac{1}{2} \sin 30^\circ = \frac{1}{4}$, from $\sin 2a = 2\sin a \cos a$.
- A. $\cos \frac{\pi}{8} = \sqrt{\frac{1+\cos(\pi/4)}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2}$, positive because $\frac{\pi}{8}$ is in quadrant I. The minus-sign version ignores the quadrant.
- D. This is the double-angle formula for cosine. It is not 1, which would be $\cos^2 x + \sin^2 x$.
- B. $\tan(a+b) = \frac{2+3}{1-6} = -\frac{5}{5} = -1$. Adding the tangents but forgetting the denominator gives 5.
- C. $165^\circ = \frac{330^\circ}{2}$ and $\cos 330^\circ = \frac{\sqrt{3}}{2}$, so the magnitude is $\sqrt{\frac{1+\sqrt{3}/2}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}$. The angle 165° is in quadrant II, so cosine is negative.
- $2 + \sqrt{3}$. $\tan(45^\circ + 30^\circ) = \frac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}} = \frac{3+\sqrt{3}}{3-\sqrt{3}}$. Multiply by the conjugate: $\frac{(3+\sqrt{3})^2}{9-3} = \frac{12+6\sqrt{3}}{6} = 2 + \sqrt{3}$.
- $-\frac{120}{169}$. $\sin \theta = -\frac{5}{13}$ in quadrant IV. $\sin 2\theta = 2\left(-\frac{5}{13}\right)\left(\frac{12}{13}\right) = -\frac{120}{169}$. (Also $\cos 2\theta = \frac{144-25}{169} = \frac{119}{169}$.)
- $\frac{\sqrt{6}-\sqrt{2}}{4}$. $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$, so $\sin \frac{\pi}{12} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$.
- $\frac{4}{5}$. Since θ is in quadrant II, $\frac{\theta}{2}$ is in quadrant I, so the root is positive. $\sin \frac{\theta}{2} = \sqrt{\frac{1+\cos \theta}{2}} = \sqrt{\frac{1+\frac{4}{5}}{2}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}} = \frac{4}{5}$.
- $\cos x$. This matches $\cos A \cos B + \sin A \sin B = \cos(A-B)$ with $A = x+y$ and $B = y$, so $A-B = x$.
- $\frac{3-4\cos 2x+\cos 4x}{8}$. $\sin^4 x = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1-2\cos 2x+\cos^2 2x}{4}$, and $\cos^2 2x = \frac{1+\cos 4x}{2}$. So $\sin^4 x = \frac{1-2\cos 2x+\frac{1+\cos 4x}{2}}{4} = \frac{3-4\cos 2x+\cos 4x}{8}$.

26.3 Proving Trigonometric Identities

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PRACTICE · PROVING TRIGONOMETRIC IDENTITIES

1. B. Dividing $\sin^2 x + \cos^2 x = 1$ by $\cos^2 x$ gives $\tan^2 x + 1 = \sec^2 x$. The others fail at simple values: $x = \frac{\pi}{4}$ for the first, $x = \frac{\pi}{6}$ for the second.
2. A. The conjugate turns $1 - \cos x$ into $1 - \cos^2 x = \sin^2 x$, which simplifies the right side to match the left. Operating on both sides assumes the claim.
3. C. Difference of squares gives $\sec^2 x - 1$, which equals $\tan^2 x$ by the Pythagorean identity.
4. D. The numerator is $1 - \sin^2 x = \cos^2 x$. Then $\frac{\cos^2 x}{\cos x} = \cos x$. Forgetting to divide leaves $\cos^2 x$.
5. C. At $\frac{\pi}{4}$ the sum is $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \neq 1$. At $0, \frac{\pi}{2}$, and 2π the sum happens to equal 1, which is why testing a few values proves nothing.
6. B. $\sin 2x = 2 \sin x \cos x$. Dividing by $2 \sin x$ leaves $\cos x$.
7. D. $\cos 2x = \cos^2 x - \sin^2 x = (\cos x - \sin x)(\cos x + \sin x)$. Cancel the common factor.
8. A. Adding to both sides assumes the claim is true. The other methods change only one side into an equivalent form.
9. B. Common denominator: $\frac{\cos^2 x + (1 + \sin x)^2}{\cos x(1 + \sin x)} = \frac{2 + 2 \sin x}{\cos x(1 + \sin x)} = \frac{2}{\cos x}$.
10. 1. $1 + \tan^2 x = \sec^2 x$, and $\sec^2 x \cos^2 x = 1$.
11. Proved. Left side: $\frac{1}{\cos x} - \cos x = \frac{1 - \cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x} = \sin x \cdot \frac{\sin x}{\cos x} = \sin x \tan x$.
12. Proved. Multiply top and bottom of the left side by $1 + \sin x$: $\frac{\cos x(1 + \sin x)}{1 - \sin^2 x} = \frac{\cos x(1 + \sin x)}{\cos^2 x} = \frac{1 + \sin x}{\cos x} = \sec x + \tan x$.
13. For example, $x = \frac{\pi}{6}$: $\sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} \approx 0.866$, but $2 \sin \frac{\pi}{6} = 1$. The sides differ, so it is not an identity.
14. Proved. Right side: $\frac{1 - \frac{\sin^2 x}{\cos^2 x}}{\cos^2 x}$, since $1 + \tan^2 x = \sec^2 x$. Multiply top and bottom by $\cos^2 x$: $\cos^2 x - \sin^2 x = \cos 2x$.
15. Proved. $\cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x = (2 \cos^2 x - 1) \cos x - 2 \sin^2 x \cos x$. Replace $\sin^2 x = 1 - \cos^2 x$: $2 \cos^3 x - \cos x - 2 \cos x + 2 \cos^3 x = 4 \cos^3 x - 3 \cos x$.

26.4 Solving Trigonometric Equations**PRACTICE · SOLVING TRIGONOMETRIC EQUATIONS**

1. C. $\cos x = -\frac{\sqrt{2}}{2}$, reference angle $\frac{\pi}{4}$, cosine negative in quadrants II and III. The pair with positive cosine uses the wrong quadrants.
2. A. Tangent has period π , and $\tan \frac{\pi}{3} = \sqrt{3}$. Using $2\pi n$ misses the solutions in quadrant III.
3. B. Sine is positive in quadrants I and II, giving one solution in each.
4. D. $2x = \frac{\pi}{2}$ or $\frac{\pi}{2} + 2\pi = \frac{5\pi}{2}$, so $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. Using one lap only misses the second solution.
5. C. Factor: $(2 \sin x + 1)(\sin x - 1) = 0$. The sign flips in the other pairs come from sign slips in factoring.
6. D. The range of arcsine is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so the only angle there with sine $-\frac{1}{2}$ is $-\frac{\pi}{6}$. The others have the right sine but lie outside the range.
7. B. Factor $\sin x(\cos x - 1) = 0$. Then $\sin x = 0$ gives $0, \pi$ and $\cos x = 1$ gives 0 . Dividing by $\sin x$ would give only 0 .
8. A. $2x$ runs over $[0, 4\pi)$, two laps, with two solutions per lap.
9. C. Cosine is even, so the solutions are symmetric about 0 : $\pm \frac{\pi}{3}$, then add full laps. The last choice is the sine pattern.
10. $x = \frac{4\pi}{3}, \frac{5\pi}{3}$. $\sin x = -\frac{\sqrt{3}}{2}$, reference angle $\frac{\pi}{3}$, sine negative in quadrants III and IV.
11. $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$. Replace $\cos^2 x = 1 - \sin^2 x$: $-2 \sin^2 x + 3 \sin x - 1 = 0$, so $(2 \sin x - 1)(\sin x - 1) = 0$. Then $\sin x = \frac{1}{2}$ or 1 .
12. $x = \frac{\pi}{3}, \frac{2\pi}{3}$. $\tan x = \pm \sqrt{3}$. On $[0, \pi)$, $\tan x = \sqrt{3}$ gives $\frac{\pi}{3}$ and $\tan x = -\sqrt{3}$ gives $\frac{2\pi}{3}$.
13. $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$. Use $\cos 2x = 2 \cos^2 x - 1$: $2 \cos^2 x - \cos x - 1 = 0$, so $(2 \cos x + 1)(\cos x - 1) = 0$. Then $\cos x = -\frac{1}{2}$ gives $\frac{2\pi}{3}, \frac{4\pi}{3}$ and $\cos x = 1$ gives 0 .
14. $x \approx 1.318$ and $x \approx 4.965$. $\cos x = 0.25$, $\arccos 0.25 \approx 1.318$ (quadrant I). Cosine is also positive in quadrant IV: $2\pi - 1.318 \approx 4.965$.
15. $x = 0$ and $x = \frac{\pi}{2}$. Squaring: $1 + 2 \sin x \cos x = 1$, so $\sin 2x = 0$ and $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Test in the original: 0 gives 1 (works), $\frac{\pi}{2}$ gives 1 (works), π gives -1 (fails), $\frac{3\pi}{2}$ gives -1 (fails).

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