



Conic Sections

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

In this booster

27.1 Parabolas

27.2 Circles and Ellipses

27.3 Hyperbolas

27.4 Parametric Equations and Polar Coordinates

Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

27.1 Parabolas

Learn the distance definition of a parabola, read and write its standard forms, find the vertex, focus, and directrix, and use the focus property in real designs.

Why This Matters

Parabolas are the first curves most students meet that are not lines, and they show up everywhere: the path of a thrown ball, the shape of a bridge cable under load, the curve of a satellite dish. Most students know $y = x^2$ well. The stumbling block is that a conic-section parabola is defined by a **geometric rule** (distance to a point equals distance to a line), and its equation can open up, down, left, or right.

The most common struggles are mixing up which variable is squared, forgetting that the number in front is $4p$ and not p , losing the sign of p when the parabola opens down or left, and making errors while completing the square. This section gives you a clear system so that every parabola problem becomes the same short checklist. Parabolas are also the gateway to the other conics, so the habits you build here will pay off in the next three sections.

Core Ideas

A **parabola** is the set of all points in a plane that are the same distance from a fixed point, the **focus**, and a fixed line, the **directrix**. The focus is never on the directrix. The line through the focus that is perpendicular to the directrix is the **axis of symmetry**. The point of the parabola halfway between the focus and the directrix is the **vertex**. The distance from the vertex to the focus is called p .

Definition by distance

A point P is on the parabola exactly when

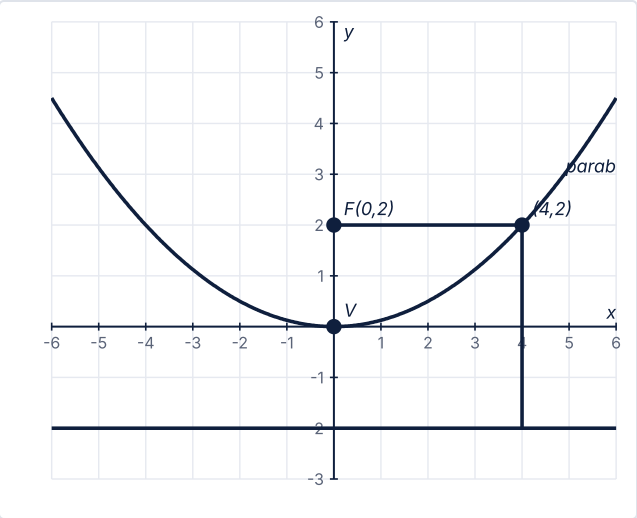
$$\text{distance from } P \text{ to the focus} = \text{distance from } P \text{ to the directrix.}$$

The vertex is the midpoint between the focus and the directrix, so the focus and the directrix are each $|p|$ units from the vertex, on opposite sides.

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The parabola $x^2 = 8y$ has focus $(0,2)$ and directrix $y = -2$. The point $(4,2)$ is 4 units from the focus and 4 units from the directrix.

Now we turn that rule into equations. Place the vertex at the origin. If the parabola opens upward, the focus is $(0,p)$ and the directrix is $y = -p$. A point (x,y) on the curve satisfies $\sqrt{x^2 + (y - p)^2} = |y + p|$. Squaring both sides and simplifying gives $x^2 = 4py$. The same work with the focus on the x -axis gives $y^2 = 4px$.

Standard forms with vertex (h,k)

Opens up or down (vertical axis):

$$(x - h)^2 = 4p(y - k).$$

Focus $(h, k + p)$, directrix $y = k - p$, axis $x = h$. Opens up if $p > 0$, down if $p < 0$.

Opens right or left (horizontal axis):

$$(y - k)^2 = 4p(x - h).$$

Focus $(h + p, k)$, directrix $x = h - p$, axis $y = k$. Opens right if $p > 0$, left if $p < 0$.

The squared variable tells you the direction: squared x means the axis is vertical, squared y means the axis is horizontal.

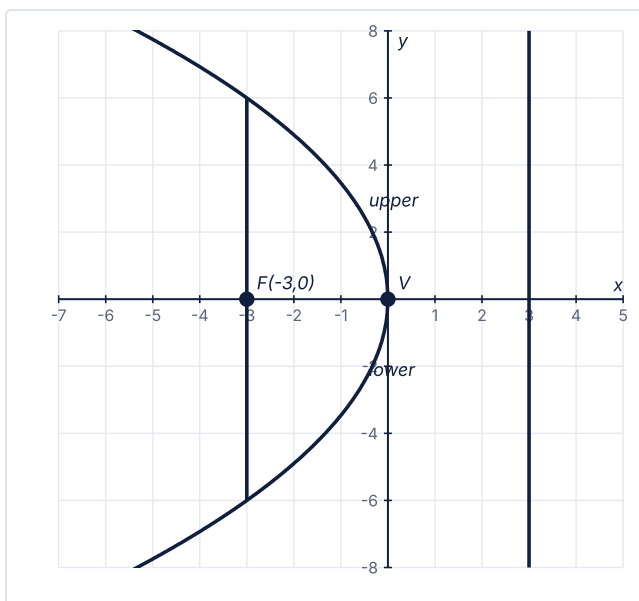
Equation (vertex at origin)	Opens	Focus	Directrix
$x^2 = 4py, p > 0$	up	$(0, p)$	$y = -p$
$x^2 = 4py, p < 0$	down	$(0, p)$	$y = -p$
$y^2 = 4px, p > 0$	right	$(p, 0)$	$x = -p$
$y^2 = 4px, p < 0$	left	$(p, 0)$	$x = -p$

Two more facts are useful. First, the chord through the focus that is perpendicular to the axis is the **latus rectum**, and its length is always $|4p|$. It is a quick way to find two more points for a sketch: they sit $|2p|$ on each side of the axis, at the height (or width) of the focus. Second, when an equation is not in standard form, you complete the square on the variable that is squared, then factor the other side to expose the vertex.

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The parabola $y^2 = -12x$ has $4p = -12$, so $p = -3$. It opens left, with focus $(-3,0)$ and directrix $x = 3$. The vertical segment through the focus is the latus rectum, length 12.

Worked Examples

EXAMPLE 1: READING A PARABOLA AT THE ORIGIN

Find the vertex, focus, directrix, and opening direction of $x^2 = 12y$.

SOLUTION

The x is squared, so the axis is vertical and the form is $x^2 = 4py$.

Match $4p = 12$, so $p = 3$.

The vertex is $(0,0)$. Since $p > 0$, the parabola opens **up**. The focus is $(0,3)$ and the directrix is $y = -3$.

Check: the latus rectum has length 12, so the points $(\pm 6, 3)$ are on the curve. Indeed $6^2 = 36 = 12 \cdot 3$.

EXAMPLE 2: A PARABOLA THAT OPENS LEFT

Find the focus and directrix of $y^2 = -8x$ and tell which way it opens.

SOLUTION

The y is squared, so the axis is horizontal: $y^2 = 4px$.

Match $4p = -8$, so $p = -2$. Keep the negative sign.

Opens **left**. Focus $(p, 0) = (-2, 0)$. Directrix $x = -p = 2$.

The focus is inside the curve and the directrix is on the opposite side of the vertex.

EXAMPLE 3: EQUATION FROM FOCUS AND DIRECTRIX

Write the equation of the parabola with focus $(2, 5)$ and directrix $y = 1$.

SOLUTION

The directrix is horizontal, so the axis is vertical: $(x - h)^2 = 4p(y - k)$.

The vertex is halfway between the focus and the directrix on the line $x = 2$: $k = \frac{5+1}{2} = 3$. So the vertex is $(2, 3)$.

The focus is above the vertex by $p = 5 - 3 = 2$, so $p = 2$ and the parabola opens up.

$(x - 2)^2 = 4(2)(y - 3)$, that is

$$(x - 2)^2 = 8(y - 3).$$

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EXAMPLE 4: COMPLETING THE SQUARE (VERTICAL AXIS)

Find the vertex, focus, and directrix of $x^2 - 6x - 8y + 25 = 0$.

SOLUTION

Get the squared variable's terms alone: $x^2 - 6x = 8y - 25$.

Add $\left(\frac{-6}{2}\right)^2 = 9$ to both sides: $x^2 - 6x + 9 = 8y - 16$.

Factor: $(x - 3)^2 = 8(y - 2)$.

So $h = 3$, $k = 2$, $4p = 8$, $p = 2$. Vertex $(3, 2)$, opens up, focus $(3, 4)$, directrix $y = 0$.

EXAMPLE 5: COMPLETING THE SQUARE (HORIZONTAL AXIS)

Find the vertex, focus, and directrix of $y^2 + 4y - 12x + 28 = 0$.

SOLUTION

Isolate: $y^2 + 4y = 12x - 28$.

Add $\left(\frac{4}{2}\right)^2 = 4$ to both sides: $(y + 2)^2 = 12x - 24 = 12(x - 2)$.

So $k = -2$, $h = 2$, $4p = 12$, $p = 3$. Vertex $(2, -2)$, opens right.

Focus $(h + p, k) = (5, -2)$. Directrix $x = h - p = -1$.

EXAMPLE 6: A REFLECTOR

A reflector for a lamp has a cross section that is a parabola with its vertex at the bottom. The opening is 2 meters across and 0.5 meters deep. How far above the vertex should the bulb (the focus) be placed?

SOLUTION

Put the vertex at the origin with the parabola opening up: $x^2 = 4py$.

The rim is at $x = \pm 1$ (half of 2 meters) and $y = 0.5$. Substitute: $1^2 = 4p(0.5)$.

So $1 = 2p$ and $p = 0.5$.

The bulb should be **0.5 meters** above the vertex, which is exactly at the level of the rim.

Coach's Playbook

- 1 Find the squared variable.** It sets the direction of the axis: x^2 means vertical, y^2 means horizontal.
 - 2 Get standard form.** Complete the square on the squared variable only. Factor the other side so it reads $4p(\text{variable} - \text{shift})$.
 - 3 Read h , k , and $4p$.** Divide the number in front by 4 to get p . Keep its sign.
 - 4 Use the sign of p .** Positive means up or right. Negative means down or left.
 - 5 Place focus and directrix.** The focus is $|p|$ from the vertex inside the curve. The directrix is $|p|$ from the vertex on the other side.
 - 6 Check with one point.** Plug in a point such as an endpoint of the latus rectum, or test equal distances.
- ◆ Sketch fast: plot the vertex, the focus, and the two latus rectum endpoints. Those four points give a very good curve.
 - ◆ Memory trick: the parabola always "hugs" the focus. It bends around the focus and away from the directrix.
 - ◆ If a problem gives a point on the parabola and the vertex, substitute the point into the standard form and solve for p .
 - ◆ The distance definition is your backup: if you forget a formula, rebuild it from "distance to focus equals distance to directrix."

Common traps

1) Treating the number in front as p . In $x^2 = 8y$ the number is $4p$, so $p = 2$. 2) Dropping the sign of p for parabolas that open down or left. 3) Adding the completing-the-square number to only one side of the equation. 4) Using $(x + h)$ when the vertex is (h, k) : the form is $(x - h)^2$, so $(x + 3)^2$ means $h = -3$.

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Mistake Clinic

Spot the error 1

A student is asked for the focus of $x^2 = 8y$ and writes: "Compare with $x^2 = 4py$. Then $p = 8$, so the focus is $(0, 8)$." Where is the error? The student matched p with the whole number 8, but the form is $4p = 8$. The correct work is $p = \frac{8}{4} = 2$, so the focus is $(0, 2)$ and the directrix is $y = -2$. This is the same parabola drawn in the first graph above.

Spot the error 2

A student completes the square for $x^2 + 6x - 4y + 17 = 0$ and writes: " $x^2 + 6x = 4y - 17$. Add $6^2 = 36$: $(x + 6)^2 = 4y + 19$, so the vertex has $h = -6$." The error is the amount added. You add the **square of half** the coefficient: $(\frac{6}{2})^2 = 9$. The correct work is $x^2 + 6x + 9 = 4y - 17 + 9$, so $(x + 3)^2 = 4y - 8 = 4(y - 2)$. The vertex is $(-3, 2)$ and $p = 1$.

Spot the error 3

A student is given $x^2 = -12y$ and writes: " $4p = -12$, so $p = -3$, but a distance cannot be negative, so $p = 3$. The focus is $(0, 3)$ and the directrix is $y = -3$." The error is throwing away the sign. In this form, p is a **signed** number that tells the direction. With $p = -3$ the parabola opens **down**, the focus is $(0, -3)$, and the directrix is $y = 3$. The student's answer would put the focus outside the curve.

Real-World Connection

A solar cooker uses a parabolic mirror so that all sunlight arriving parallel to the axis bounces to one point, the focus. If the mirror is modeled by $x^2 = 60y$ (units in centimeters), then $p = 15$, so a pot should sit 15 centimeters above the vertex to catch all of the reflected light. This same focusing property is why dishes, car headlights, and telescopes use parabolas.

Challenge Corner

A parabola has a vertical axis, vertex $(1, -1)$, and passes through $(5, 3)$. Find its equation, focus, directrix, and the endpoints of its latus rectum. Solution: the form is $(x - 1)^2 = 4p(y + 1)$. Substitute $(5, 3)$: $16 = 4p(4)$, so $p = 1$. The equation is $(x - 1)^2 = 4(y + 1)$. The focus is $(1, 0)$ and the directrix is $y = -2$. For the latus rectum, set $y = 0$: $(x - 1)^2 = 4$, so $x = 3$ or $x = -1$. The endpoints are $(-1, 0)$ and $(3, 0)$, and the length is $4 = |4p|$.

Practice

PRACTICE Parabolas

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is the focus of the parabola $x^2 = 16y$?
 (A) $(0, 16)$ (B) $(0, 4)$ (C) $(4, 0)$ (D) $(0, -4)$
- What is the directrix of $y^2 = 20x$?
 (A) $y = -5$ (B) $x = 5$ (C) $x = -5$ (D) $y = 5$
- Which way does the parabola $x^2 = -6y$ open?
 (A) Up (B) Down (C) Left (D) Right
- A parabola has its vertex at the origin and its focus at $(0, -2)$. What is its equation?
 (A) $x^2 = 8y$ (B) $x^2 = -2y$ (C) $y^2 = -8x$ (D) $x^2 = -8y$
- What is the vertex of $(x + 1)^2 = 8(y - 3)$?
 (A) $(1, 3)$ (B) $(-1, -3)$ (C) $(-1, 3)$ (D) $(3, -1)$

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6. The equation $x^2 + 8x - 4y + 8 = 0$ describes a parabola. What is its focus?
 (A) $(-4, -1)$ (B) $(-4, -3)$ (C) $(-3, -2)$ (D) $(-5, -2)$
7. A parabola has focus $(3, 1)$ and directrix $x = -1$. What is its equation?
 (A) $(y - 1)^2 = 4(x - 1)$ (B) $(x - 1)^2 = 8(y - 1)$ (C) $(y - 1)^2 = 8(x + 1)$
 (D) $(y - 1)^2 = 8(x - 1)$
8. A dish has a parabolic cross section with its vertex at the origin, opening up. It is 12 units across the rim and 4.5 units deep. How far above the vertex is the focus?
 (A) 8 (B) 4 (C) 2 (D) 3
9. Which point lies on the parabola $(y + 1)^2 = 12(x - 2)$?
 (A) $(5, 5)$ (B) $(5, 3)$ (C) $(3, 5)$ (D) $(6, 2)$
10. Find the focus and directrix of $x^2 = -24y$.
11. Write the equation of the parabola with vertex $(-3, 2)$ and focus $(-3, 5)$.
12. Find the vertex, focus, and directrix of $y^2 - 6y - 8x + 25 = 0$.
13. The parabola with focus $(0, 3)$ and directrix $y = -3$ meets the line $y = 3$ at two points. Find them and the distance between them.
14. A flashlight reflector is a paraboloid whose cross section has vertex at the bottom and the bulb 1.5 cm above the vertex at the focus. The reflector is 6 cm deep. How wide is the opening, in centimeters?
15. A parabola with a vertical axis passes through $(0, 0)$ and $(4, 0)$ and has its vertex at $(2, -3)$. Find its focus.

27.2 Circles and Ellipses

Write and read the equations of circles and ellipses, find foci, axes, and eccentricity, and complete the square to put any conic of this type in standard form.

Why This Matters

Circles feel easy, so students often rush them. Ellipses then feel like a different world. They are in fact one family: a circle is an ellipse whose two foci have merged into one center. Planets move in ellipses, a bicycle gear can be elliptical, a stadium or track is an elliptical shape, and a whispering gallery uses the focus property so that a whisper at one spot can be heard clearly at another.

The usual struggles are mixing up a and b , forgetting that a goes with the **larger** denominator, using the wrong formula for c (the ellipse and the hyperbola have different ones), and slipping on completing the square when the coefficients are not 1. Learn a clear process here and you can handle every circle and ellipse problem.

Core Ideas

A **circle** is the set of all points at a fixed distance r (the radius) from a fixed point, the center. The distance formula turns that rule straight into an equation.

Circle

Center (h, k) and radius r :

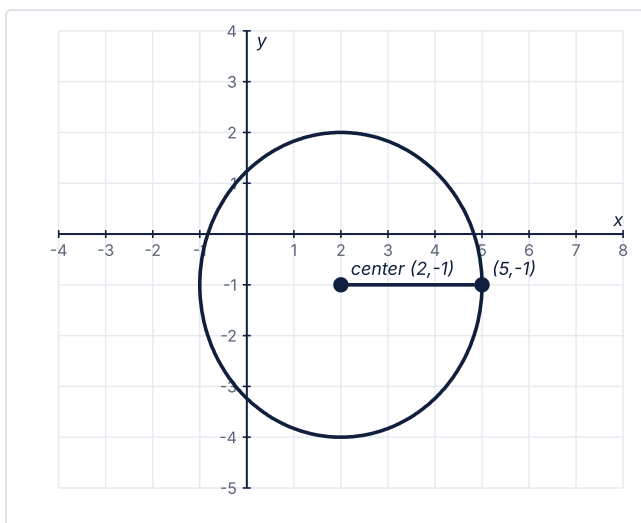
$$(x - h)^2 + (y - k)^2 = r^2.$$

General form: $x^2 + y^2 + Dx + Ey + F = 0$. Complete the square in x and in y to find the center and radius. The right side must come out positive, or there is no circle.

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The circle $(x-2)^2 + (y+1)^2 = 9$ has center $(2, -1)$ and radius 3. Every point is 3 units from the center.

An **ellipse** is the set of all points where the **sum** of the distances to two fixed points, the **foci**, is a constant. Think of pinning the ends of a loose string at two points and tracing a pencil around, keeping the string tight. The longest diameter is the **major axis**, and its length is $2a$. The shortest diameter is the **minor axis**, with length $2b$. The endpoints of the major axis are the **vertices**, and the endpoints of the minor axis are the **co-vertices**.

Ellipse (center (h, k) , with $a > b > 0$)

Horizontal major axis:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Vertices $(h \pm a, k)$, co-vertices $(h, k \pm b)$, foci $(h \pm c, k)$.

Vertical major axis:

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.$$

Vertices $(h, k \pm a)$, co-vertices $(h \pm b, k)$, foci $(h, k \pm c)$.

In both cases a^2 is the **larger** denominator, and

$$c^2 = a^2 - b^2.$$

Eccentricity

The eccentricity of an ellipse is

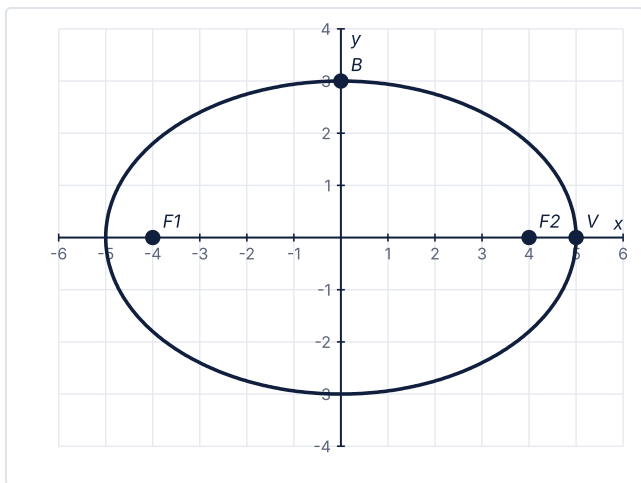
$$e = \frac{c}{a}, \quad 0 \leq e < 1.$$

An eccentricity near 0 means the ellipse is almost a circle. An eccentricity near 1 means it is long and thin. A circle has $c = 0$, so $e = 0$.

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The ellipse $x^2/25 + y^2/9 = 1$: $a = 5$, $b = 3$, $c = 4$. The foci $(4,0)$ and $(-4,0)$ lie on the major axis. Note that a , b , and c form a right triangle with hypotenuse a .

Feature	Horizontal major	Vertical major
Larger denominator under	x term	y term
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Foci	$(h \pm c, k)$	$(h, k \pm c)$
Major axis length	$2a$	$2a$
Minor axis length	$2b$	$2b$

A useful way to remember the ellipse: the foci always sit on the major axis, and the bigger the gap between the foci, the more stretched the ellipse. When the two foci coincide, $c = 0$ and $a = b = r$, so the ellipse becomes a circle of radius r . This is why the circle equation is just the ellipse equation with equal denominators: $\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$.

When you are given an equation that is not yet in standard form, ask two questions. Are the coefficients of x^2 and y^2 equal and positive? Then it is a circle. Are they different but both positive? Then it is an ellipse. If one is negative, it is not an ellipse at all: that is a hyperbola, which is the subject of the next section.

Worked Examples

EXAMPLE 1: A CIRCLE FROM CENTER AND RADIUS

Write the equation of the circle with center $(-2, 3)$ and radius 5.

SOLUTION

Use $(x - h)^2 + (y - k)^2 = r^2$ with $h = -2$, $k = 3$, $r = 5$.

$$(x - (-2))^2 + (y - 3)^2 = 25, \text{ so}$$

$$(x + 2)^2 + (y - 3)^2 = 25.$$

Notice the sign flip: a center coordinate of -2 appears as $x + 2$.

EXAMPLE 2: A CIRCLE IN GENERAL FORM

Find the center and radius of $x^2 + y^2 - 6x + 4y - 12 = 0$.

SOLUTION

$$\text{Group: } (x^2 - 6x) + (y^2 + 4y) = 12.$$

Complete each square. Half of -6 is -3 , and $(-3)^2 = 9$. Half of 4 is 2 , and $2^2 = 4$. Add both to **both sides**: $(x^2 - 6x + 9) + (y^2 + 4y + 4) = 12 + 9 + 4$.

$$(x - 3)^2 + (y + 2)^2 = 25.$$

The center is $(3, -2)$ and the radius is $\sqrt{25} = 5$.

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EXAMPLE 3: AN ELLIPSE WITH A HORIZONTAL MAJOR AXIS

For $\frac{x^2}{25} + \frac{y^2}{9} = 1$, find the vertices, co-vertices, foci, and eccentricity.

SOLUTION

The larger denominator, 25, is under x , so the major axis is horizontal. $a^2 = 25$, so $a = 5$, and $b^2 = 9$, so $b = 3$.

$$c^2 = a^2 - b^2 = 25 - 9 = 16, \text{ so } c = 4.$$

Vertices $(\pm 5, 0)$. Co-vertices $(0, \pm 3)$. Foci $(\pm 4, 0)$.

$$e = \frac{c}{a} = \frac{4}{5}. \text{ See the graph above.}$$

EXAMPLE 4: AN ELLIPSE WITH A VERTICAL MAJOR AXIS

For $\frac{x^2}{16} + \frac{y^2}{36} = 1$, find the vertices, foci, and eccentricity.

SOLUTION

The larger denominator, 36, is under y , so the major axis is vertical: $a^2 = 36$, $a = 6$, and $b^2 = 16$, $b = 4$.

$$c^2 = 36 - 16 = 20, \text{ so } c = \sqrt{20} = 2\sqrt{5}.$$

Vertices $(0, \pm 6)$. Foci $(0, \pm 2\sqrt{5}) \approx (0, \pm 4.47)$.

$$e = \frac{2\sqrt{5}}{6} = \frac{\sqrt{5}}{3} \approx 0.745.$$

EXAMPLE 5: WRITE THE EQUATION FROM THE VERTICES AND FOCI

An ellipse has vertices $(1, 4)$ and $(1, -8)$ and foci $(1, 2)$ and $(1, -6)$. Find its equation.

SOLUTION

All four points share $x = 1$, so the major axis is vertical. The center is the midpoint of the vertices: $\left(1, \frac{4+(-8)}{2}\right) = (1, -2)$.

The distance from the center to a vertex is $a = 6$. The distance from the center to a focus is $c = 4$.

$$b^2 = a^2 - c^2 = 36 - 16 = 20.$$

$$\frac{(x-1)^2}{20} + \frac{(y+2)^2}{36} = 1.$$

EXAMPLE 6: AN ELLIPSE IN GENERAL FORM

Find the center, foci, and eccentricity of $4x^2 + 9y^2 - 16x + 18y - 11 = 0$.

SOLUTION

Group and factor out the leading coefficients: $4(x^2 - 4x) + 9(y^2 + 2y) = 11$.

Complete the squares. Inside the first parentheses add 4, which adds $4 \cdot 4 = 16$ to the left side. Inside the second add 1, which adds $9 \cdot 1 = 9$. Add the same amounts to the right: $4(x-2)^2 + 9(y+1)^2 = 11 + 16 + 9 = 36$.

Divide by 36:

$$\frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} = 1.$$

Center $(2, -1)$. $a^2 = 9$, $b^2 = 4$, horizontal major axis. $c = \sqrt{9-4} = \sqrt{5}$.

Foci $(2 \pm \sqrt{5}, -1)$. Eccentricity $e = \frac{\sqrt{5}}{3}$.

Coach's Playbook

- 1 Divide to get a 1.** Every ellipse equation must have a 1 on the right side before you read a and b .
- 2 Compare denominators.** The bigger one is a^2 . Its variable tells you the direction of the major axis.
- 3 Find c with a minus sign.** For an ellipse, $c^2 = a^2 - b^2$. Since $c < a$, the foci are always inside the vertices.
- 4 Find the center first.** Everything else is measured from it, and it is the midpoint of the vertices and of the foci.

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5 Sketch. Plot the center, the four axis endpoints, and the foci. Then draw a smooth oval through the endpoints.

- ◆ When completing the square with a leading coefficient, factor it out first. Whatever you add inside the parentheses is multiplied by that coefficient on the other side.
- ◆ Sanity checks: $a > b$, $c < a$, and $0 \leq e < 1$. If you get $e > 1$ you have used the wrong formula for c .
- ◆ Right-triangle picture: the segment from a co-vertex to a focus has length a , so $b^2 + c^2 = a^2$.
- ◆ For a circle, divide the whole equation by the common coefficient so that x^2 and y^2 have coefficient 1.

Common traps

1) Calling the number under x always a^2 . It is a^2 only when it is the larger one. 2) Using $c^2 = a^2 + b^2$, which belongs to hyperbolas. 3) Writing the radius as r^2 , for example saying the radius is 25 instead of 5. 4) Forgetting to multiply the added number by the leading coefficient when completing the square on an ellipse.

Mistake Clinic

Spot the error 1

A student looks at $\frac{x^2}{9} + \frac{y^2}{25} = 1$ and writes: " $a^2 = 9$, $b^2 = 25$, so the vertices are $(\pm 3, 0)$ and the foci are on the x -axis." The error is assuming the first denominator is a^2 . The larger denominator is 25, under y , so $a = 5$, $b = 3$, and the major axis is **vertical**. Then $c = \sqrt{25 - 9} = 4$, the vertices are $(0, \pm 5)$, and the foci are $(0, \pm 4)$.

Spot the error 2

For $\frac{x^2}{25} + \frac{y^2}{9} = 1$ a student writes: " $c^2 = a^2 + b^2 = 25 + 9 = 34$, so the foci are $(\pm\sqrt{34}, 0)$." That is the hyperbola formula. For an ellipse the foci must be inside the vertices, but $\sqrt{34} \approx 5.83$ is farther out than the vertex at 5. The correct formula is $c^2 = a^2 - b^2 = 25 - 9 = 16$, so $c = 4$ and the foci are $(\pm 4, 0)$.

Spot the error 3

A student completes the square for $x^2 + y^2 - 6x + 4y - 12 = 0$ and writes: " $(x - 3)^2 + (y + 2)^2 = 12$, so $r = \sqrt{12}$." The student forgot to add 9 and 4 to the right side after adding them on the left. The correct line is $(x - 3)^2 + (y + 2)^2 = 12 + 9 + 4 = 25$, so $r = 5$. Whatever you add to one side you must add to the other.

Real-World Connection

In a whispering gallery the ceiling is shaped like part of an ellipse. If one person stands at one focus and whispers, the sound waves reflect off the curved ceiling and all meet at the other focus, so a second person standing there hears the whisper clearly even across a large room. Planets also orbit the sun on ellipses, with the sun at one focus, and the eccentricity measures how stretched the orbit is.

Challenge Corner

An ellipse is centered at the origin with a horizontal major axis, and it passes through $(3, \frac{12}{5})$ and has a focus at $(4, 0)$. Find its equation. Solution: $c = 4$, so $a^2 - b^2 = 16$. The equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. A quick route uses the distance definition: the point is $\sqrt{(3-4)^2 + \frac{144}{25}} = \frac{13}{5}$ from $(4, 0)$ and $\sqrt{49 + \frac{144}{25}} = \frac{37}{5}$ from $(-4, 0)$. The sum is $2a = 10$, so $a = 5$, $b^2 = 25 - 16 = 9$. The equation is $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

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Practice

PRACTICE Circles and Ellipses

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What are the center and radius of $(x - 4)^2 + (y + 7)^2 = 49$?
 (A) Center $(4, -7)$, radius 7 (B) Center $(4, -7)$, radius 49 (C) Center $(4, 7)$, radius 7
 (D) Center $(-4, 7)$, radius 7
- Which equation describes the circle with center $(0, -3)$ and radius 2?
 (A) $x^2 + (y - 3)^2 = 4$ (B) $x^2 + (y + 3)^2 = 2$ (C) $x^2 + (y + 3)^2 = 4$
 (D) $x^2 + y^2 = 4$
- What are the foci of $\frac{x^2}{25} + \frac{y^2}{9} = 1$?
 (A) $(\pm 5, 0)$ (B) $(0, \pm 4)$ (C) $(\pm \sqrt{34}, 0)$ (D) $(\pm 4, 0)$
- What is the length of the major axis of $\frac{x^2}{16} + \frac{y^2}{49} = 1$?
 (A) 8 (B) 14 (C) 7 (D) 4
- What is the eccentricity of $\frac{x^2}{169} + \frac{y^2}{144} = 1$?
 (A) $\frac{5}{13}$ (B) $\frac{12}{13}$ (C) $\frac{5}{12}$ (D) $\frac{1}{13}$
- What is the radius of the circle $x^2 + y^2 + 10x - 8y + 5 = 0$?
 (A) $\sqrt{46}$ (B) 36 (C) 6 (D) 4
- An ellipse has vertices $(\pm 6, 0)$ and foci $(\pm 4, 0)$. What is its equation?
 (A) $\frac{x^2}{36} + \frac{y^2}{16} = 1$ (B) $\frac{x^2}{36} + \frac{y^2}{20} = 1$ (C) $\frac{x^2}{20} + \frac{y^2}{36} = 1$
 (D) $\frac{x^2}{36} + \frac{y^2}{52} = 1$
- A running track is shaped like an ellipse that is 100 meters long and 60 meters wide. How far is each focus from the center?
 (A) 20 m (B) 50 m (C) 80 m (D) 40 m
- Which point is a focus of $4x^2 + 9y^2 - 16x + 18y - 11 = 0$?
 (A) $(2, -1 + \sqrt{5})$ (B) $(2 + \sqrt{13}, -1)$ (C) $(2 + \sqrt{5}, -1)$ (D) $(5, -1)$
- A circle has a diameter with endpoints $(-1, 2)$ and $(7, 6)$. Write its equation.
- Find the foci of $9x^2 + 4y^2 = 36$.
- An ellipse is centered at the origin with vertices $(0, \pm 10)$ and eccentricity $\frac{3}{5}$. Write its equation.
- The ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ has foci $(\pm 4, 0)$. Find the sum of the distances from the point $(3, \frac{12}{5})$ to the two foci.
- Find the points where the circle $x^2 + y^2 = 25$ meets the line $y = x + 1$.
- Find the center, foci, and eccentricity of $x^2 + 4y^2 + 6x - 8y + 9 = 0$.

27.3 Hyperbolas

Learn the standard forms of a hyperbola, find vertices, foci, and asymptotes, and decide whether any given conic equation is a circle, parabola, ellipse, or hyperbola.

Why This Matters

The hyperbola is the conic students find hardest to picture, because it has two separate branches that open away from each other and two invisible guide lines called asymptotes. Yet hyperbolas are everywhere: the shape of a cooling tower, the paths of some comets that pass a star and never return, the pattern formed by two overlapping ripples, and navigation methods that locate a ship from the time difference between two signals.

Typical struggles are confusing the hyperbola formula with the ellipse formula, believing that a goes with the larger denominator (it does not), flipping the slopes of the asymptotes, and not knowing which of the four conics an equation represents. By the end of this section you will have a short checklist for each of these.

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Core Ideas

A **hyperbola** is the set of all points where the **difference** of the distances to two fixed points, the **foci**, is a constant. Compare that with an ellipse, where the **sum** is constant. The line through the foci is the **transverse axis**. The hyperbola crosses it at the two **vertices**, and the midpoint of the vertices is the **center**. The distance from the center to a vertex is a , and the distance from the center to a focus is c , where $c > a$.

Hyperbola (center (h, k))

Horizontal transverse axis:

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1.$$

Vertices $(h \pm a, k)$, foci $(h \pm c, k)$, asymptotes $y - k = \pm \frac{b}{a}(x - h)$.

Vertical transverse axis:

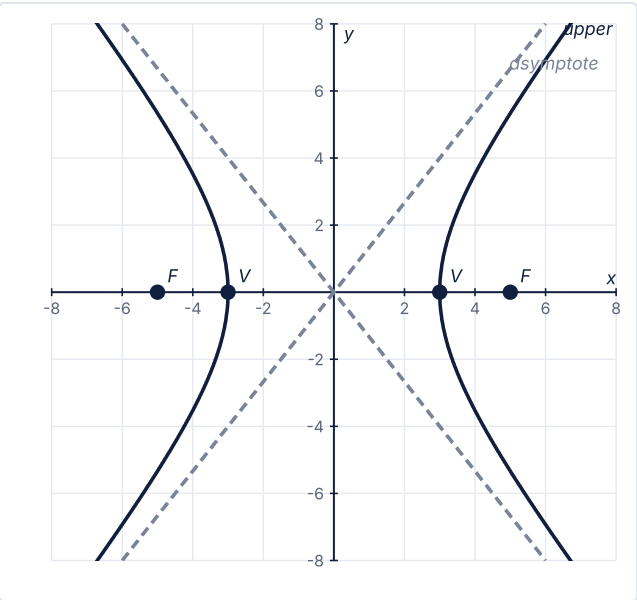
$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1.$$

Vertices $(h, k \pm a)$, foci $(h, k \pm c)$, asymptotes $y - k = \pm \frac{a}{b}(x - h)$.

In both cases a^2 is the denominator of the **positive** term (not necessarily the larger one), and

$$c^2 = a^2 + b^2.$$

The eccentricity is $e = \frac{c}{a} > 1$.



The hyperbola $x^2/9 - y^2/16 = 1$: $a = 3$, $b = 4$, $c = 5$. The dashed lines $y = 4x/3$ and $y = -4x/3$ are the asymptotes. The branches get closer and closer to them.

The asymptotes come from a simple trick called the **central rectangle**. Draw a rectangle centered at the center with half-width a in the direction of the transverse axis and half-height b in the other direction (for the picture above, 3 across and 4 up). The diagonals of that rectangle, extended, are the asymptotes. The vertices touch the rectangle, and the branches curve away from it and hug the diagonals.

Feature	Ellipse	Hyperbola
Defining rule	sum of distances is $2a$	difference of distances is $2a$
Equation signs	plus	minus
Relation of a, b, c	$c^2 = a^2 - b^2$	$c^2 = a^2 + b^2$
Eccentricity	$0 \leq e < 1$	$e > 1$
Which denominator is a^2 ?	the larger one	the one with the positive term

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Why do asymptotes exist? Far from the center, the "1" in the equation becomes tiny compared with the huge x^2 and y^2 terms, so the curve behaves almost exactly like $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, which is the pair of lines $y = \pm \frac{b}{a}x$. The branches approach these lines but never touch them. The segment of length $2b$ perpendicular to the transverse axis through the center is called the **conjugate axis**, and it is the second side of the central rectangle.

Identifying a conic from $Ax^2 + Cy^2 + Dx + Ey + F = 0$

If there is no xy term, look at A and C (the coefficients of x^2 and y^2):

$A = C$: circle. $A \neq C$ with the same sign: ellipse. A and C with opposite signs: hyperbola. Exactly one of A, C equal to 0: parabola.

Only one variable is squared in a parabola. Complete the square to confirm the exact shape, and watch for degenerate cases (a single point or a pair of lines).

Worked Examples

EXAMPLE 1: HORIZONTAL HYPERBOLA

For $\frac{x^2}{9} - \frac{y^2}{16} = 1$, find the vertices, foci, asymptotes, and eccentricity.

SOLUTION

The x term is positive, so the transverse axis is horizontal. $a^2 = 9$, $a = 3$. $b^2 = 16$, $b = 4$.

$c^2 = a^2 + b^2 = 9 + 16 = 25$, so $c = 5$.

Vertices $(\pm 3, 0)$. Foci $(\pm 5, 0)$.

Asymptotes $y = \pm \frac{b}{a}x = \pm \frac{4}{3}x$. Eccentricity $e = \frac{5}{3}$.

These match the graph above.

EXAMPLE 2: VERTICAL HYPERBOLA

For $\frac{y^2}{25} - \frac{x^2}{144} = 1$, find the vertices, foci, and asymptotes.

SOLUTION

The y term is positive, so the transverse axis is vertical. $a^2 = 25$, $a = 5$, and $b^2 = 144$, $b = 12$.

$c^2 = 25 + 144 = 169$, so $c = 13$.

Vertices $(0, \pm 5)$. Foci $(0, \pm 13)$.

For a vertical hyperbola the asymptote slope is $\pm \frac{a}{b}$:

$$y = \pm \frac{5}{12}x.$$

EXAMPLE 3: EQUATION FROM VERTICES AND FOCI

Find the equation of the hyperbola with vertices $(\pm 6, 0)$ and foci $(\pm 10, 0)$, and give its asymptotes.

SOLUTION

The vertices and foci are on the x -axis, so the form is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with center $(0, 0)$.

$a = 6$, $c = 10$. From $c^2 = a^2 + b^2$: $b^2 = 100 - 36 = 64$, so $b = 8$.

$$\frac{x^2}{36} - \frac{y^2}{64} = 1.$$

Asymptotes: $y = \pm \frac{8}{6}x = \pm \frac{4}{3}x$.

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EXAMPLE 4: A SHIFTED HYPERBOLA

Find the center, vertices, foci, and asymptotes of $\frac{(x-1)^2}{4} - \frac{(y+3)^2}{9} = 1$.

SOLUTION

Center $(h, k) = (1, -3)$. Horizontal axis, $a = 2$, $b = 3$, and $c = \sqrt{4+9} = \sqrt{13}$.

Vertices $(1 \pm 2, -3)$: $(3, -3)$ and $(-1, -3)$.

Foci $(1 \pm \sqrt{13}, -3)$.

Asymptotes: $y + 3 = \pm \frac{3}{2}(x - 1)$.

EXAMPLE 5: FROM GENERAL FORM TO STANDARD FORM

Put $9x^2 - 16y^2 - 18x - 64y - 199 = 0$ in standard form and find the foci.

SOLUTION

Group: $9(x^2 - 2x) - 16(y^2 + 4y) = 199$.

Complete the squares. Inside the first parentheses add 1, which adds $9 \cdot 1 = 9$. Inside the second add 4, which adds $-16 \cdot 4 = -64$. So $9(x-1)^2 - 16(y+2)^2 = 199 + 9 - 64 = 144$.

Divide by 144:

$$\frac{(x-1)^2}{16} - \frac{(y+2)^2}{9} = 1.$$

Center $(1, -2)$, $a = 4$, $b = 3$, $c = 5$. Foci $(1 \pm 5, -2)$: $(6, -2)$ and $(-4, -2)$.

EXAMPLE 6: WHICH CONIC IS IT?

Identify each equation as a circle, parabola, ellipse, or hyperbola: (a) $x^2 + y^2 - 10x + 4y + 4 = 0$; (b) $3x^2 + y^2 - 6x + 4y - 2 = 0$; (c) $4x^2 - y^2 + 8x + 6y - 9 = 0$; (d) $y^2 - 8x + 2y + 9 = 0$.

SOLUTION

(a) The coefficients of x^2 and y^2 are equal, 1 and 1: a **circle**. Completing squares gives $(x-5)^2 + (y+2)^2 = 25$.

(b) Coefficients 3 and 1 are different with the same sign: an **ellipse**. It becomes $3(x-1)^2 + (y+2)^2 = 9$.

(c) Coefficients 4 and -1 have opposite signs: a **hyperbola**. It becomes $4(x+1)^2 - (y-3)^2 = 4$, or $(x+1)^2 - \frac{(y-3)^2}{4} = 1$.

(d) Only y is squared: a **parabola**. It becomes $(y+1)^2 = 8(x-1)$.

Coach's Playbook

- 1 Find the positive term.** The squared variable with the plus sign gives the direction of the transverse axis and the value of a^2 .
 - 2 Get b^2 from the other denominator.** Then use the plus rule $c^2 = a^2 + b^2$.
 - 3 Draw the central rectangle.** Use half-width a and half-height b (or the reverse for vertical). Draw its diagonals as the asymptotes.
 - 4 Write asymptote slopes from the rectangle.** Slope is rise over run in the rectangle: $\pm \frac{b}{a}$ for horizontal and $\pm \frac{a}{b}$ for vertical.
 - 5 Sketch the branches.** Start at the vertices, curve outward, and bend toward the asymptotes without crossing them.
- ◆ A quick test for asymptotes: replace the 1 on the right side with 0 and solve. For $\frac{x^2}{9} - \frac{y^2}{16} = 0$ you get $y = \pm \frac{4}{3}x$.
 - ◆ The foci are always **outside** the vertices on the same axis, since $c > a$. If your c comes out smaller than a , you used the wrong sign.
 - ◆ Check a vertex in the equation. For a horizontal hyperbola with center $(0, 0)$ the point $(a, 0)$ must make the equation true.
 - ◆ For identification problems, do not rely on memory of pictures. Use the signs of the x^2 and y^2 coefficients.

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Common traps

1) Assuming a^2 is the larger denominator. For hyperbolas it is the positive term's denominator. 2) Using $c^2 = a^2 - b^2$ (the ellipse rule), which gives a negative number when $b > a$. 3) Inverting the asymptote slope for a vertical hyperbola. 4) Forgetting the shift: asymptotes pass through the center (h, k) , not the origin.

Mistake Clinic**Spot the error 1**

A student looks at $\frac{y^2}{16} - \frac{x^2}{9} = 1$ and says: "The x^2 term is the one being subtracted, so the hyperbola opens left and right, and the vertices are $(\pm 3, 0)$." The error is using the subtracted term. The **positive** term decides the direction. Here y^2 is positive, so the transverse axis is vertical, $a = 4$, and the vertices are $(0, \pm 4)$. Since $c^2 = 16 + 9 = 25$, the foci are $(0, \pm 5)$.

Spot the error 2

For $\frac{y^2}{25} - \frac{x^2}{144} = 1$ a student writes the asymptotes as $y = \pm \frac{12}{5}x$. The error is using $\frac{b}{a}$ for a vertical hyperbola. Use the central rectangle: it goes 12 across and 5 up from the center, so the diagonal has slope $\frac{5}{12}$. The correct asymptotes are $y = \pm \frac{5}{12}x$. A good check is that the vertex $(0, 5)$ is on the curve, and a line of slope $\frac{12}{5}$ would pass far above it, which does not look right.

Spot the error 3

For $\frac{x^2}{9} - \frac{y^2}{16} = 1$ a student writes: " $c^2 = a^2 - b^2 = 9 - 16 = -7$, so there are no foci." The student used the ellipse relationship. For a hyperbola the foci are farther from the center than the vertices, so $c^2 = a^2 + b^2 = 9 + 16 = 25$ and $c = 5$. The foci are $(\pm 5, 0)$.

Real-World Connection

Two radio stations 26 km apart send a signal at the same instant. A ship measures the difference in arrival times and finds that it is 10 km farther from one station than from the other. The ship must lie on a hyperbola with the stations as foci: $2a = 10$, $c = 13$, so $b = 12$ and the path is $\frac{x^2}{25} - \frac{y^2}{144} = 1$. A second pair of stations gives a second hyperbola, and the ship is where the two curves cross.

Challenge Corner

A hyperbola is centered at the origin with vertices $(\pm 4, 0)$ and asymptotes $y = \pm \frac{3}{2}x$. Find its equation, its foci, and the points on it with $x = 8$. Solution: $a = 4$ and $\frac{b}{a} = \frac{3}{2}$, so $b = 6$. The equation is $\frac{x^2}{16} - \frac{y^2}{36} = 1$. Then $c^2 = 16 + 36 = 52$, so the foci are $(\pm 2\sqrt{13}, 0)$. For $x = 8$: $\frac{64}{16} - \frac{y^2}{36} = 1$, so $\frac{y^2}{36} = 3$ and $y^2 = 108$, so $y = \pm 6\sqrt{3}$. The points are $(8, 6\sqrt{3})$ and $(8, -6\sqrt{3})$.

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Practice

PRACTICE Hyperbolas

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What are the vertices of $\frac{x^2}{49} - \frac{y^2}{25} = 1$?
 (A) $(\pm 5, 0)$ (B) $(0, \pm 7)$ (C) $(\pm 7, 0)$ (D) $(\pm \sqrt{74}, 0)$
- What are the foci of $\frac{x^2}{20} - \frac{y^2}{16} = 1$?
 (A) $(\pm 2, 0)$ (B) $(\pm 6, 0)$ (C) $(0, \pm 6)$ (D) $(\pm \sqrt{20}, 0)$
- What are the asymptotes of $\frac{x^2}{4} - \frac{y^2}{25} = 1$?
 (A) $y = \pm \frac{5}{2}x$ (B) $y = \pm \frac{25}{4}x$ (C) $y = \pm \frac{2}{5}x$ (D) $y = \pm \frac{4}{25}x$
- Where are the vertices of $9y^2 - 4x^2 = 36$?
 (A) $(\pm 2, 0)$ (B) $(0, \pm 2)$ (C) $(\pm 3, 0)$ (D) $(0, \pm 3)$
- What is the eccentricity of $\frac{x^2}{16} - \frac{y^2}{9} = 1$?
 (A) $\frac{4}{5}$ (B) $\frac{3}{4}$ (C) $\frac{5}{4}$ (D) $\frac{5}{3}$
- Which equation has vertices $(0, \pm 3)$ and foci $(0, \pm 5)$?
 (A) $\frac{y^2}{9} - \frac{x^2}{16} = 1$ (B) $\frac{x^2}{9} - \frac{y^2}{16} = 1$ (C) $\frac{y^2}{9} - \frac{x^2}{25} = 1$
 (D) $\frac{y^2}{25} - \frac{x^2}{9} = 1$
- Which conic is represented by $3x^2 - 2y^2 + 12x + 4y - 20 = 0$?
 (A) Circle (B) Hyperbola (C) Ellipse (D) Parabola
- Which point is a focus of $\frac{(x+2)^2}{9} - \frac{(y-5)^2}{4} = 1$?
 (A) $(-2 + \sqrt{5}, 5)$ (B) $(2 + \sqrt{13}, -5)$ (C) $(-2 + \sqrt{13}, 5)$ (D) $(-2, 5 + \sqrt{13})$
- A hyperbola has center $(1, -2)$, a vertex at $(1, 3)$, and a focus at $(1, 4)$. What is its equation?
 (A) $\frac{(y+2)^2}{25} - \frac{(x-1)^2}{61} = 1$ (B) $\frac{(y+2)^2}{25} - \frac{(x-1)^2}{25} = 1$ (C) $\frac{(x-1)^2}{25} - \frac{(y+2)^2}{11} = 1$
 (D) $\frac{(y+2)^2}{25} - \frac{(x-1)^2}{11} = 1$
- Find the foci and the asymptotes of $\frac{x^2}{64} - \frac{y^2}{36} = 1$.
- Write the equation of the hyperbola centered at the origin with vertices $(\pm 3, 0)$ and asymptotes $y = \pm 2x$.
- Find the center and foci of $16x^2 - 9y^2 + 64x + 18y - 89 = 0$.
- Find the points where $\frac{x^2}{9} - \frac{y^2}{16} = 1$ meets the line $y = 4$.
- Two stations are at $(-13, 0)$ and $(13, 0)$. A point moves so that the difference of its distances to the stations is always 10. Write the equation of its path.
- For $4y^2 - x^2 + 8y + 2x - 13 = 0$, write the standard form and find the foci and the asymptotes.

27.4 Parametric Equations and Polar Coordinates

Describe curves with a parameter, eliminate the parameter, convert between polar and rectangular coordinates, and recognize the common polar graphs.

Why This Matters

So far you have described a curve by an equation relating x and y . Two powerful ideas break that habit. In **parametric equations**, a third variable t (often time) tells you where a moving object is: x and y are each a function of t . In **polar coordinates**, you locate a point by its distance from the origin and its angle, which makes circles, spirals, and flower shapes simple to write down.

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Students struggle with three things: eliminating the parameter cleanly (especially with sine and cosine), finding the correct angle when converting a point to polar form (the calculator gives the wrong quadrant half the time), and remembering what a polar graph looks like. Physics, animation, robotics, radar, and later calculus all use these ideas, so it pays to master them now.

Core Ideas

A **parametric curve** is given by $x = f(t)$ and $y = g(t)$ for t in some interval. As t increases, the point (x, y) traces the curve, which gives it a direction called its **orientation**. To see the shape, make a table of values and plot points, or **eliminate the parameter** to get an equation in x and y alone.

Eliminating the parameter

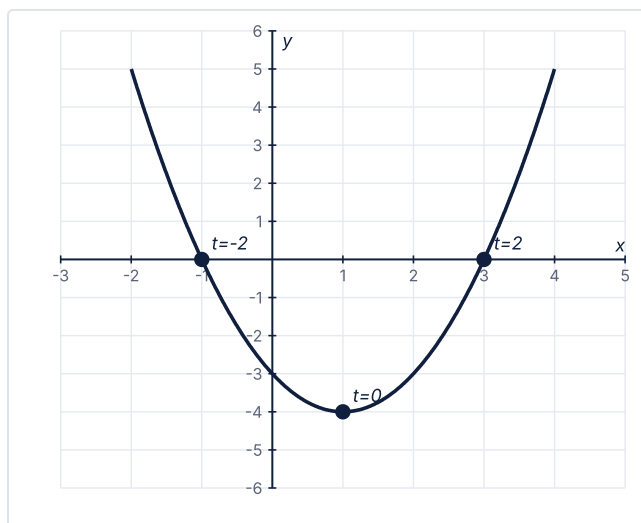
Algebraic way: solve one equation for t and substitute into the other.

Trig way: if $x = h + a \cos t$ and $y = k + b \sin t$, use $\cos^2 t + \sin^2 t = 1$ to get

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Always check whether the interval for t limits the curve to part of the graph, and note the direction of travel.

t	-2	-1	0	1	2
$x = t + 1$	-1	0	1	2	3
$y = t^2 - 4$	0	-3	-4	-3	0



The curve $x = t + 1$, $y = t^2 - 4$ for $-3 \leq t \leq 3$. Eliminating t gives $y = (x-1)^2 - 4$, a parabola. The point moves left to right as t increases.

Polar coordinates describe a point as (r, θ) : go out a signed distance r from the origin along the ray at angle θ measured counterclockwise from the positive x -axis. Unlike rectangular coordinates, a point has many polar names: (r, θ) , $(r, \theta + 2\pi)$, and $(-r, \theta + \pi)$ all name the same point.

Converting between systems

Polar to rectangular:

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Rectangular to polar:

$$r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x} \quad (x \neq 0).$$

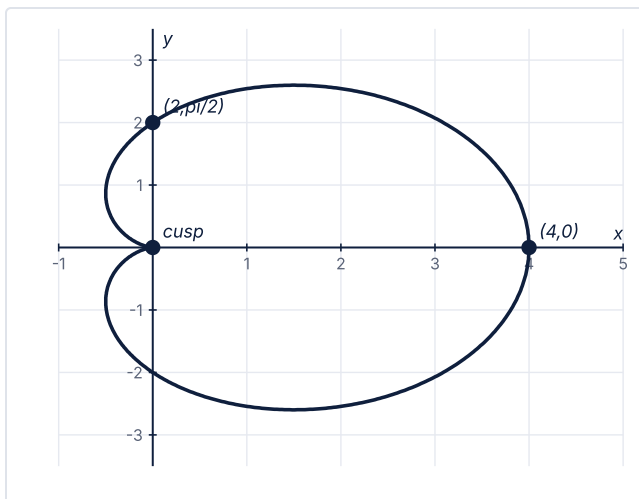
For θ , first sketch the point to see its quadrant. The calculator's inverse tangent only returns angles in Quadrants I and IV, so add π when the point has $x < 0$.

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Polar equation	Graph	Key facts
$r = a$	circle centered at origin	radius $ a $
$\theta = c$	line through origin	angle c
$r = 2a \cos \theta$	circle through origin	center $(a, 0)$, radius a
$r = a(1 + \cos \theta)$	cardioid	cusp at origin, reaches $r = 2a$
$r = a \cos n\theta$	rose	n petals if n odd, $2n$ if n even



The cardioid $r = 2 + 2\cos(\theta)$. It is symmetric about the x -axis, reaches its farthest point $r = 4$ at $\theta = 0$, and returns to the origin at $\theta = \pi$.

Worked Examples

EXAMPLE 1: ELIMINATE THE PARAMETER (ALGEBRA)

Eliminate t from $x = t + 1$, $y = t^2 - 4$ and describe the curve.

SOLUTION

Solve the simple equation for t : $t = x - 1$.

Substitute into y : $y = (x - 1)^2 - 4 = x^2 - 2x - 3$.

This is a parabola with vertex $(1, -4)$ opening up. Since t can be any real number, the whole parabola is traced, from left to right as t increases.

If the problem restricted $t \geq 0$, only the part with $x \geq 1$ (the right half) would be drawn.

EXAMPLE 2: ELIMINATE THE PARAMETER (TRIG)

Eliminate t from $x = 3 \cos t + 1$, $y = 2 \sin t - 2$ for $0 \leq t \leq 2\pi$.

SOLUTION

Isolate the trig functions: $\cos t = \frac{x-1}{3}$, $\sin t = \frac{y+2}{2}$.

Use $\cos^2 t + \sin^2 t = 1$:

$$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1.$$

This is an ellipse centered at $(1, -2)$ with $a = 3$ and $b = 2$.

At $t = 0$ the point is $(4, -2)$. At $t = \frac{\pi}{2}$ it is $(1, 0)$. So the ellipse is traced **counterclockwise**, once.

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EXAMPLE 3: A THROWN BALL

A ball's position is $x = 10t$, $y = -5t^2 + 20t$ (meters, t in seconds). Find its path, its maximum height, and how far it travels horizontally before landing.

SOLUTION

Eliminate t : $t = \frac{x}{10}$, so $y = -5 \cdot \frac{x^2}{100} + 20 \cdot \frac{x}{10} = -\frac{x^2}{20} + 2x$. The path is a parabola opening down.

The maximum height is at the vertex of $y = -5t^2 + 20t$: $t = \frac{-20}{2(-5)} = 2$ s, and $y(2) = -20 + 40 = 20$ m.

The ball lands when $y = 0$: $-5t(t - 4) = 0$, so $t = 4$ s. Then $x = 10 \cdot 4 = 40$ m.

The ball reaches a height of 20 m and lands 40 m away.

EXAMPLE 4: CONVERTING POINTS

(a) Convert the polar point $(4, \frac{2\pi}{3})$ to rectangular coordinates. (b) Convert $(-3, 3)$ to polar form with $r > 0$ and $0 \leq \theta < 2\pi$.

SOLUTION

(a) $x = 4 \cos \frac{2\pi}{3} = 4(-\frac{1}{2}) = -2$ and $y = 4 \sin \frac{2\pi}{3} = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$. The point is $(-2, 2\sqrt{3})$, in Quadrant II as expected.

(b) $r = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$.

The point is in Quadrant II. The reference angle satisfies $\tan = \frac{3}{3} = 1$, so it is $\frac{\pi}{4}$, and $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

The polar form is $(3\sqrt{2}, \frac{3\pi}{4})$.

EXAMPLE 5: CONVERTING EQUATIONS

Convert to rectangular form: (a) $r = 6 \cos \theta$; (b) $r \sin \theta = 3$.

SOLUTION

(a) Multiply both sides by r : $r^2 = 6r \cos \theta$. Replace r^2 by $x^2 + y^2$ and $r \cos \theta$ by x : $x^2 + y^2 = 6x$.

Complete the square: $(x - 3)^2 + y^2 = 9$. This is a circle with center $(3, 0)$ and radius 3.

(b) Replace $r \sin \theta$ by y : $y = 3$. It is a horizontal line.

EXAMPLE 6: SKETCHING A CARDIOID

Make a table and sketch $r = 2 + 2 \cos \theta$ for $0 \leq \theta \leq \pi$, then use symmetry.

SOLUTION

Compute r : at $\theta = 0$, $r = 4$. At $\frac{\pi}{6}$, $r = 2 + \sqrt{3} \approx 3.73$. At $\frac{\pi}{3}$, $r = 3$. At $\frac{\pi}{2}$, $r = 2$. At $\frac{2\pi}{3}$, $r = 1$. At π , $r = 0$.

Plot these points by going out r along each angle. The curve sweeps from $(4, 0)$ up over the top, through $(0, 2)$, and into the origin.

Cosine is even, so $r(-\theta) = r(\theta)$ and the graph is symmetric about the x -axis. Reflect the top half to get the bottom half.

The result is the heart-shaped curve in the graph above.

Coach's Playbook

- 1 Parametric: decide the goal.** Table of values for orientation, elimination for the shape.
 - 2 Eliminate smartly.** Use substitution if one variable is simple. Use the Pythagorean identity if you see sine and cosine.
 - 3 Check the range.** The values of t may show only part of the curve. Check the endpoints.
 - 4 Polar: sketch the point first.** Know the quadrant before you compute the angle.
 - 5 Convert with the four formulas.** To convert an equation, aim for products like $r \cos \theta$, $r \sin \theta$, and r^2 , and multiply both sides by r if needed.
- ◆ Quick orientation test: plug in two or three values of t and watch the direction the point moves.
 - ◆ For rose curves, count petals with the rule on the table, and find the petal tips by looking for $|r|$ at its maximum.
 - ◆ When $r < 0$, plot the point in the opposite direction. This is a common source of mistakes with roses and limacons.
 - ◆ Always verify an elimination by plugging a table point into your final equation.

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Common traps

1) Taking $\arctan \frac{y}{x}$ on a calculator and not adjusting for Quadrant II or III. 2) Solving for t and forgetting to substitute into **both** places it appears. 3) Replacing $\cos^2 t + \sin^2 t = 1$ with $x^2 + y^2 = 1$ when the coefficients are not 1, so you must divide by a and b first. 4) Forgetting that a negative r points the opposite way.

Mistake Clinic**Spot the error 1**

A student converts $(-3, 3)$ to polar form and writes: " $r = 3\sqrt{2}$, $\theta = \arctan \frac{3}{-3} = -\frac{\pi}{4}$, so the point is $(3\sqrt{2}, -\frac{\pi}{4})$." The angle $-\frac{\pi}{4}$ points into Quadrant IV, where $x > 0$ and $y < 0$, but the original point has $x < 0$ and $y > 0$. The inverse tangent cannot tell Quadrant II from Quadrant IV. Add π : $\theta = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$. The correct answer is $(3\sqrt{2}, \frac{3\pi}{4})$.

Spot the error 2

A student eliminates t from $x = 2t$, $y = t^2$ and writes: " $y = t^2 = x^2$." The error is replacing t by x . From $x = 2t$ we get $t = \frac{x}{2}$, so $y = (\frac{x}{2})^2 = \frac{x^2}{4}$. Check with $t = 2$: the point is $(4, 4)$. The student's equation gives $y = 16$ at $x = 4$, which is wrong. The correct equation $y = \frac{x^2}{4}$ gives 4.

Spot the error 3

A student eliminates t from $x = 3 \cos t$, $y = 2 \sin t$ and writes: " $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the curve is the unit circle." The student forgot the coefficients. We have $\cos t = \frac{x}{3}$ and $\sin t = \frac{y}{2}$, so $\frac{x^2}{9} + \frac{y^2}{4} = 1$, an ellipse. The point at $t = 0$ is $(3, 0)$, which is not on the unit circle.

Real-World Connection

Video game designers and animators use parametric equations to move characters and cameras: the position is given by $x(t)$ and $y(t)$, and the program changes t every frame. A drone that flies in a circle of radius 50 meters around a tower can be modeled by $x = 50 \cos t$, $y = 50 \sin t$. Radar screens, on the other hand, naturally report objects in polar form: a distance and a bearing angle.

Challenge Corner

Find all points where the cardioid $r = 2 + 2 \cos \theta$ meets the circle $r = 2$, and give them in rectangular coordinates.

Solution: set $2 + 2 \cos \theta = 2$, so $\cos \theta = 0$ and $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. At these angles $r = 2$. The points are $(2, \frac{\pi}{2})$ and $(2, \frac{3\pi}{2})$, which are $(0, 2)$ and $(0, -2)$ in rectangular form. The origin is on the cardioid (at $\theta = \pi$) but not on the circle $r = 2$, so it is not an intersection point. There are exactly two intersection points.

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Practice

PRACTICE Parametric Equations and Polar Coordinates

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Eliminate the parameter from $x = t - 2$, $y = 3t + 1$.
 (A) $y = 3x + 7$ (B) $y = 3x + 1$ (C) $y = 3x - 5$ (D) $y = \frac{x}{3} + 7$
- What is the graph of $x = \cos t$, $y = \sin t$ for $0 \leq t \leq \pi$?
 (A) The full unit circle (B) The right half of the unit circle
 (C) The upper half of the unit circle (D) A line segment
- Eliminate t from $x = 2t$, $y = t^2 + 1$.
 (A) $y = x^2 + 1$ (B) $y = \frac{x^2}{2} + 1$ (C) $y = 2x^2 + 1$ (D) $y = \frac{x^2}{4} + 1$
- What are the rectangular coordinates of the polar point $(6, \frac{\pi}{3})$?
 (A) $(3\sqrt{3}, 3)$ (B) $(3, 3\sqrt{3})$ (C) $(-3, 3\sqrt{3})$ (D) $(3, 3)$
- Which polar point names $(-1, -1)$, with $r > 0$ and $0 \leq \theta < 2\pi$?
 (A) $(\sqrt{2}, \frac{\pi}{4})$ (B) $(\sqrt{2}, \frac{3\pi}{4})$ (C) $(\sqrt{2}, \frac{5\pi}{4})$ (D) $(2, \frac{5\pi}{4})$
- Which rectangular equation matches $r = 4 \sin \theta$?
 (A) $x^2 + (y - 2)^2 = 4$ (B) $(x - 2)^2 + y^2 = 4$ (C) $x^2 + y^2 = 4$
 (D) $x^2 + y^2 = 16$
- How many petals does the rose $r = 5 \cos 3\theta$ have?
 (A) 6 (B) 3 (C) 5 (D) 10
- A ball's position is $x = 10t$, $y = -5t^2 + 20t$. What is its maximum height?
 (A) 10 (B) 40 (C) 20 (D) 25
- Which equation describes $x = 3 \cos t + 1$, $y = 3 \sin t - 2$?
 (A) $(x + 1)^2 + (y - 2)^2 = 9$ (B) $(x - 1)^2 + (y + 2)^2 = 9$ (C) $x^2 + y^2 = 9$
 (D) $(x - 1)^2 + (y + 2)^2 = 9$
- Eliminate t from $x = t^2 + 1$, $y = 2t$ and say which way the curve opens.
- For $x = 1 + 4t$, $y = 4 - 8t$, find y in terms of x and the point where $t = \frac{1}{2}$.
- Convert $(2\sqrt{3}, 2)$ to polar form with $r > 0$ and $0 \leq \theta < 2\pi$.
- Convert $r = 4 \cos \theta + 2 \sin \theta$ to rectangular form and identify the curve.
- Write $2x - 3y = 6$ in the form $r = f(\theta)$.
- Find the distance between the points with polar coordinates $(4, \frac{\pi}{6})$ and $(6, \frac{\pi}{2})$.

Answers Answers and Explanations

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

27.1 Parabolas**PRACTICE · PARABOLAS**

- B. $4p = 16$ gives $p = 4$, and the focus of $x^2 = 4py$ is $(0, p) = (0, 4)$. Choice A uses the whole number 16 as p .
- C. $4p = 20$ gives $p = 5$. The axis is horizontal, so the directrix is the vertical line $x = -p = -5$. Choice B has the wrong sign, and A and D are horizontal lines.
- B. The x is squared, so the axis is vertical. Since $4p = -6 < 0$, the parabola opens down. Up would need a positive coefficient.
- D. The focus is on the y -axis, so the form is $x^2 = 4py$ with $p = -2$. Then $4p = -8$ and the equation is $x^2 = -8y$. Choice B forgets the factor 4.
- C. Compare with $(x - h)^2 = 4p(y - k)$: $x + 1 = x - (-1)$ gives $h = -1$, and $y - 3$ gives $k = 3$. The vertex is $(-1, 3)$. Choice A flips the sign of h .
- A. $x^2 + 8x = 4y - 8$. Add 16: $(x + 4)^2 = 4y + 8 = 4(y + 2)$. Vertex $(-4, -2)$, $p = 1$, opens up, so the focus is $(-4, -1)$. Choice B is below the vertex, which is where the directrix side is.
- D. The vertex is halfway between $x = -1$ and $x = 3$, so $h = 1$, and $k = 1$. Then $p = 3 - 1 = 2$ and $4p = 8$, giving $(y - 1)^2 = 8(x - 1)$. Choice A uses $p = 1$, and B swaps the variables.

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8. C. Use $x^2 = 4py$ at the rim point $(6, 4.5)$: $36 = 4p(4.5) = 18p$, so $p = 2$. Choice A is $4p$, not p .
9. A. For $(5, 5)$: $(5 + 1)^2 = 36$ and $12(5 - 2) = 36$. For $(5, 3)$, $16 \neq 36$. For $(3, 5)$, $36 \neq 12$. For $(6, 2)$, $9 \neq 48$.
10. Focus $(0, -6)$, directrix $y = 6$. Here $4p = -24$, so $p = -6$. The focus is $(0, p)$ and the directrix is $y = -p$.
11. $(x + 3)^2 = 12(y - 2)$. The focus is 3 units above the vertex, so $p = 3$ and $4p = 12$. The axis is vertical.
12. Vertex $(2, 3)$, focus $(4, 3)$, directrix $x = 0$. Work: $y^2 - 6y + 9 = 8x - 16$, so $(y - 3)^2 = 8(x - 2)$ with $p = 2$.
13. Points $(-6, 3)$ and $(6, 3)$, distance 12. The equation is $x^2 = 12y$. Set $y = 3$: $x^2 = 36$, so $x = \pm 6$. The distance is $12 = |4p|$.
14. 12 cm. Here $p = 1.5$, so $x^2 = 6y$. At depth $y = 6$: $x^2 = 36$, $x = \pm 6$. The opening is 12 cm wide.
15. Focus $(2, -\frac{8}{3})$. The form is $(x - 2)^2 = 4p(y + 3)$. Using $(0, 0)$: $4 = 12p$, so $p = \frac{1}{3}$. The focus is $(2, -3 + \frac{1}{3}) = (2, -\frac{8}{3})$.

27.2 Circles and Ellipses

PRACTICE · CIRCLES AND ELLIPSES

1. A. The form $(x - h)^2 + (y - k)^2 = r^2$ gives $h = 4$, $k = -7$, and $r = \sqrt{49} = 7$. Choice B reports r^2 as the radius.
2. C. With $h = 0$, $k = -3$ we get $x^2 + (y + 3)^2 = 2^2 = 4$. Choice A has the wrong sign on k , and B uses r instead of r^2 .
3. D. $a^2 = 25$ is under x , so the major axis is horizontal. $c^2 = 25 - 9 = 16$, $c = 4$. Choice C uses $a^2 + b^2$, and A gives the vertices.
4. B. The larger denominator is 49, so $a = 7$ and the major axis has length $2a = 14$. Choice C is a and choice A is the minor axis length.
5. A. $a = 13$, $b = 12$, $c = \sqrt{169 - 144} = 5$, so $e = \frac{c}{a} = \frac{5}{13}$. Choice C divides by b and B divides b by a .
6. C. $(x + 5)^2 + (y - 4)^2 = -5 + 25 + 16 = 36$, so $r = 6$. Choice B stops at r^2 . Choice A moves the constant with the wrong sign $(25 + 16 + 5)$.
7. B. $a = 6$, $c = 4$, so $b^2 = 36 - 16 = 20$. Choice A uses $b = c$. Choice B adds $a^2 + c^2$. Choice C has the major axis vertical.
8. D. $a = 50$ and $b = 30$, so $c = \sqrt{50^2 - 30^2} = \sqrt{1600} = 40$ m. Choice B is a , and C is the distance between the two foci.
9. C. Completing squares gives $\frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} = 1$. The major axis is horizontal and $c = \sqrt{9 - 4} = \sqrt{5}$. The focus is $(2 + \sqrt{5}, -1)$. Choice C is a vertex.
10. $(x - 3)^2 + (y - 4)^2 = 20$. The center is the midpoint $(3, 4)$. The radius squared is $\left(\frac{\sqrt{8^2 + 4^2}}{2}\right)^2 = \frac{80}{4} = 20$.
11. $(0, \pm\sqrt{5})$. Divide by 36: $\frac{x^2}{4} + \frac{y^2}{9} = 1$. The larger denominator is under y , so $a = 3$, $b = 2$, $c = \sqrt{9 - 4} = \sqrt{5}$, and the major axis is vertical.
12. $\frac{x^2}{64} + \frac{y^2}{100} = 1$. Here $a = 10$ and $c = ea = 6$, so $b^2 = 100 - 36 = 64$. The major axis is vertical.
13. 10. The point is on the ellipse because $\frac{9}{25} + \frac{144/25}{9} = \frac{9}{25} + \frac{16}{25} = 1$. Distances: $\sqrt{1 + \frac{144}{25}} = \frac{13}{5}$ and $\sqrt{49 + \frac{144}{25}} = \frac{37}{5}$. Sum = $\frac{50}{5} = 10 = 2a$.
14. $(3, 4)$ and $(-4, -3)$. Substitute: $x^2 + (x + 1)^2 = 25$, so $2x^2 + 2x - 24 = 0$, $x^2 + x - 12 = 0$, $(x + 4)(x - 3) = 0$. Then $y = x + 1$ gives -3 and 4 .
15. Center $(-3, 1)$, foci $(-3 \pm \sqrt{3}, 1)$, $e = \frac{\sqrt{3}}{2}$. Work: $(x + 3)^2 + 4(y - 1)^2 = -9 + 9 + 4 = 4$, so $\frac{(x+3)^2}{4} + \frac{(y-1)^2}{1} = 1$, $a = 2$, $b = 1$, $c = \sqrt{3}$.

27.3 Hyperbolas

PRACTICE · HYPERBOLAS

1. C. The x term is positive, so the axis is horizontal and $a^2 = 49$, $a = 7$. The vertices are $(\pm 7, 0)$. Choice D is the focus distance $\sqrt{49 + 25}$.
2. B. $c^2 = a^2 + b^2 = 20 + 16 = 36$, so $c = 6$ and the foci are $(\pm 6, 0)$. Choice A uses $20 - 16 = 4$.
3. A. Horizontal hyperbola: slope $\pm \frac{b}{a} = \pm \frac{5}{2}$. Choice C inverts the slope.
4. B. Divide by 36: $\frac{y^2}{4} - \frac{x^2}{9} = 1$. The y term is positive, so the axis is vertical and $a = 2$. The vertices are $(0, \pm 2)$. Choice D uses b .
5. C. $c = \sqrt{16 + 9} = 5$ and $a = 4$, so $e = \frac{c}{a} = \frac{5}{4}$. A hyperbola has $e > 1$, which rules out A and B.
6. A. The axis is vertical, so y^2 is positive, $a = 3$, $c = 5$, $b^2 = 25 - 9 = 16$. Choice B has a horizontal axis. Choice C uses $b = c$.
7. B. The coefficients of x^2 and y^2 are 3 and -2 , which have opposite signs, so it is a hyperbola. Completing the squares gives $3(x + 2)^2 - 2(y - 1)^2 = 30$.
8. C. The center is $(-2, 5)$, the axis is horizontal, and $c = \sqrt{9 + 4} = \sqrt{13}$. The foci are $(-2 \pm \sqrt{13}, 5)$. Choice A uses $9 - 4$.
9. D. Vertical axis. $a = 3 - (-2) = 5$ and $c = 4 - (-2) = 6$, so $b^2 = c^2 - a^2 = 36 - 25 = 11$. Choice A adds $a^2 + c^2$.
10. Foci $(\pm 10, 0)$; asymptotes $y = \pm \frac{3}{4}x$. $c^2 = 64 + 36 = 100$, and the slope is $\frac{b}{a} = \frac{6}{8}$.
11. $\frac{x^2}{9} - \frac{y^2}{36} = 1$. $a = 3$ and $\frac{b}{a} = 2$, so $b = 6$.
12. Center $(-2, 1)$, foci $(3, 1)$ and $(-7, 1)$. $16(x + 2)^2 - 9(y - 1)^2 = 144$, so $\frac{(x+2)^2}{9} - \frac{(y-1)^2}{16} = 1$, $c = 5$.
13. $(3\sqrt{2}, 4)$ and $(-3\sqrt{2}, 4)$. Substitute: $\frac{x^2}{9} - 1 = 1$, so $x^2 = 18$ and $x = \pm 3\sqrt{2}$.
14. $\frac{x^2}{25} - \frac{y^2}{144} = 1$. $2a = 10$ gives $a = 5$, $c = 13$, and $b^2 = 169 - 25 = 144$.
15. Standard form $\frac{(y+1)^2}{4} - \frac{(x-1)^2}{16} = 1$. Foci $(1, -1 \pm 2\sqrt{5})$. Asymptotes $y + 1 = \pm \frac{1}{2}(x - 1)$. Work: $4(y + 1)^2 - (x - 1)^2 = 16$, $a = 2$, $b = 4$, $c = \sqrt{20}$, slope $\frac{a}{b} = \frac{1}{2}$.

27.4 Parametric Equations and Polar Coordinates

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PRACTICE · PARAMETRIC EQUATIONS AND POLAR COORDINATES

1. A. From $x = t - 2$, $t = x + 2$. Then $y = 3(x + 2) + 1 = 3x + 7$. Choice C uses $t = x - 2$, the wrong sign.
2. C. Since $x^2 + y^2 = 1$ the points are on the unit circle. For $0 \leq t \leq \pi$, $\sin t \geq 0$, so $y \geq 0$: the upper half. The point starts at $(1, 0)$ and ends at $(-1, 0)$.
3. D. $t = \frac{\pi}{2}$, so $y = \frac{\pi^2}{4} + 1$. Choice A forgets to divide by 2 before squaring.
4. B. $x = 6 \cos \frac{\pi}{3} = 3$ and $y = 6 \sin \frac{\pi}{3} = 3\sqrt{3}$. Choice A swaps sine and cosine.
5. C. $r = \sqrt{1+1} = \sqrt{2}$. The point is in Quadrant III, so $\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$. Choice A is what the calculator gives from $\arctan 1$.
6. A. Multiply by r : $r^2 = 4r \sin \theta$, so $x^2 + y^2 = 4y$, which becomes $x^2 + (y - 2)^2 = 4$. Choice B would come from $r = 4 \cos \theta$.
7. B. For $r = a \cos n\theta$ with n odd there are n petals, so 3. The number 5 is the petal length. Choice A applies the even- n doubling rule to an odd n .
8. C. The vertex is at $t = 2$, and $y(2) = -20 + 40 = 20$. Choice B is the horizontal distance at landing.
9. D. $\cos t = \frac{x-1}{3}$ and $\sin t = \frac{y+2}{3}$, so $(x - 1)^2 + (y + 2)^2 = 9$. Choice A has both signs flipped, and B uses a instead of a^2 .
10. $x = \frac{y^2}{4} + 1$, a parabola opening to the right with vertex $(1, 0)$. Work: $t = \frac{y}{2}$, so $x = \frac{y^2}{4} + 1$.
11. $y = -2x + 6$; the point is $(3, 0)$. Work: $t = \frac{x-1}{4}$, so $y = 4 - 2(x - 1) = 6 - 2x$. At $t = \frac{1}{2}$: $x = 3$, $y = 0$.
12. $(4, \frac{\pi}{6})$. $r = \sqrt{12+4} = 4$. $\tan \theta = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$, and the point is in Quadrant I, so $\theta = \frac{\pi}{6}$.
13. $(x - 2)^2 + (y - 1)^2 = 5$, a circle with center $(2, 1)$ and radius $\sqrt{5}$. Multiply by r : $x^2 + y^2 = 4x + 2y$, then complete the squares.
14. $r = \frac{6}{2 \cos \theta - 3 \sin \theta}$. Substitute $x = r \cos \theta$ and $y = r \sin \theta$: $r(2 \cos \theta - 3 \sin \theta) = 6$.
15. $2\sqrt{7}$. In rectangular coordinates the points are $(2\sqrt{3}, 2)$ and $(0, 6)$, so $d^2 = 12 + 16 = 28$ and $d = 2\sqrt{7}$. (The law of cosines gives the same: $16 + 36 - 48 \cos \frac{\pi}{3} = 28$.)

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