



Vectors, Matrices, and Linear Systems

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

In this booster

28.1 Vectors in the Plane

28.2 Matrix Operations

28.3 Determinants and Inverses

28.4 Matrices as Transformations and Data

Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

28.1 Vectors in the Plane

Work with vectors in component form, find magnitude and direction, add and scale vectors, use unit vectors and the dot product, and solve force and velocity problems.

Why This Matters

Some quantities need more than a number. If a boat travels at 5 meters per second, you still need to know in which direction. A **vector** carries both a size and a direction, and it is the language of physics, engineering, computer graphics, and navigation. Every time a game character moves, a drone holds its position against the wind, or a bridge is designed to carry loads, vectors are at work.

Students most often struggle with finding the direction angle in the right quadrant, with the difference between a vector and a point, with remembering that the dot product gives a **number** (not a vector), and with setting up force and velocity word problems. This section gives you reliable steps for each.

Core Ideas

A **vector** in the plane is written in component form as $\langle a, b \rangle$. It describes a movement of a units horizontally and b units vertically. The vector from a point $P(x_1, y_1)$ to a point $Q(x_2, y_2)$ is $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$: always **end minus start**. Two vectors are equal when they have the same components, wherever they are drawn.

Magnitude and direction

The **magnitude** (length) of $\mathbf{v} = \langle a, b \rangle$ is

$$|\mathbf{v}| = \sqrt{a^2 + b^2}.$$

The **direction angle** θ (measured counterclockwise from the positive x -axis) satisfies $\tan \theta = \frac{b}{a}$, with the quadrant chosen from the signs of a and b .

From magnitude and direction back to components:

$$\mathbf{v} = \langle |\mathbf{v}| \cos \theta, |\mathbf{v}| \sin \theta \rangle.$$

Operations

For $\mathbf{u} = \langle u_1, u_2 \rangle$, $\mathbf{v} = \langle v_1, v_2 \rangle$ and a number c :

$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$, and $c\mathbf{u} = \langle cu_1, cu_2 \rangle$. Multiplying by c stretches the length by $|c|$ and reverses direction if $c < 0$.

A **unit vector** has length 1. The unit vector in the direction of $\mathbf{v}$ is

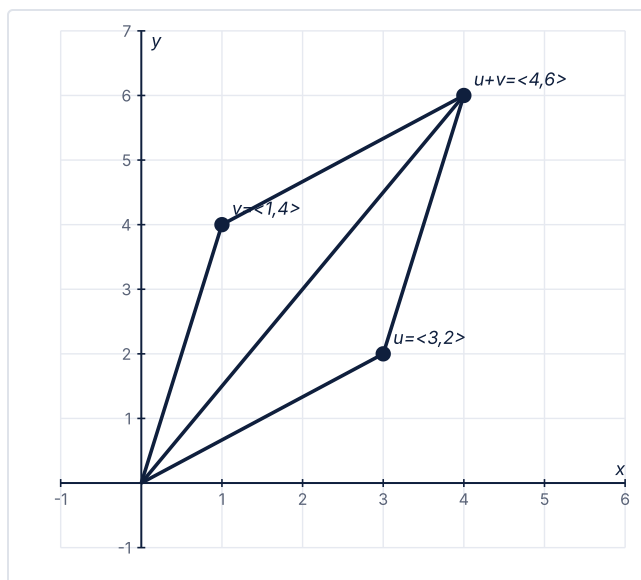
$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

The vectors $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ are the standard unit vectors, so $\langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$.

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →



Adding vectors by the parallelogram rule. Place the tail of v at the head of u (or the reverse): the diagonal from the origin is $u + v = \langle 4, 6 \rangle$.

Finding the direction angle takes one extra step. The inverse tangent on a calculator only returns angles between -90° and 90° , which describes vectors pointing right ($a > 0$). If the vector points left ($a < 0$), add 180° to the calculator value. A sketch of the vector settles the question every time: Quadrant I and IV need no change, Quadrant II and III need $+180^\circ$.

Two nonzero vectors are **parallel** when one is a scalar multiple of the other, for example $\langle 2, -3 \rangle$ and $\langle -6, 9 \rangle = -3\langle 2, -3 \rangle$. Vectors also have a **projection**: the part of u that points along v has signed length $\frac{u \cdot v}{|v|}$. This is exactly what is needed for work, since work equals force along the direction of motion times distance.

Dot product

$$u \cdot v = u_1v_1 + u_2v_2 = |u||v| \cos \theta,$$

where θ is the angle between the vectors ($0 \leq \theta \leq \pi$). The result is a **number**.

So $\cos \theta = \frac{u \cdot v}{|u||v|}$. Two nonzero vectors are **perpendicular (orthogonal)** exactly when $u \cdot v = 0$. A positive dot product means an acute angle, and a negative one means an obtuse angle.

Task	Formula	Example with $\langle 3, 4 \rangle$
Length	$\sqrt{a^2 + b^2}$	5
Unit vector	$v/ v $	$\langle \frac{3}{5}, \frac{4}{5} \rangle$
Direction angle	$\tan^{-1} \frac{b}{a}$	$\approx 53.1^\circ$
Perpendicular vector	swap, negate one	$\langle -4, 3 \rangle$

Worked Examples

EXAMPLE 1: VECTOR BETWEEN TWO POINTS

Find the component form, magnitude, and unit vector of $\overrightarrow{PQ}$ for $P(1, 2)$ and $Q(5, 5)$.

SOLUTION

End minus start: $\overrightarrow{PQ} = \langle 5 - 1, 5 - 2 \rangle = \langle 4, 3 \rangle$.

$$|\overrightarrow{PQ}| = \sqrt{16 + 9} = 5.$$

$$\text{Unit vector: } \frac{1}{5}\langle 4, 3 \rangle = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle.$$

$$\text{Check: } \frac{16}{25} + \frac{9}{25} = 1.$$

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

EXAMPLE 2: FROM MAGNITUDE AND DIRECTION

A vector has magnitude 12 and direction angle 60° . Find its components.

SOLUTION

$$x = 12 \cos 60^\circ = 12 \cdot \frac{1}{2} = 6.$$

$$y = 12 \sin 60^\circ = 12 \cdot \frac{\sqrt{3}}{2} = 6\sqrt{3}.$$

$$\mathbf{v} = \langle 6, 6\sqrt{3} \rangle \approx \langle 6, 10.39 \rangle. \text{ Check the length: } \sqrt{36 + 108} = 12.$$

EXAMPLE 3: COMBINING VECTORS

Let $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -1, 5 \rangle$. Find $2\mathbf{u} - 3\mathbf{v}$ and $|\mathbf{u} + \mathbf{v}|$.

SOLUTION

$$2\mathbf{u} = \langle 6, -4 \rangle \text{ and } 3\mathbf{v} = \langle -3, 15 \rangle.$$

$$2\mathbf{u} - 3\mathbf{v} = \langle 6 - (-3), -4 - 15 \rangle = \langle 9, -19 \rangle.$$

$$\mathbf{u} + \mathbf{v} = \langle 2, 3 \rangle, \text{ so } |\mathbf{u} + \mathbf{v}| = \sqrt{4 + 9} = \sqrt{13}.$$

EXAMPLE 4: DOT PRODUCT, ANGLE, AND PERPENDICULAR VECTORS

(a) Find the angle between $\mathbf{u} = \langle 2, 3 \rangle$ and $\mathbf{v} = \langle 4, -1 \rangle$. (b) For what k is $\langle k, 6 \rangle$ perpendicular to $\langle 3, -2 \rangle$?

SOLUTION

$$(a) \mathbf{u} \cdot \mathbf{v} = 8 - 3 = 5. |\mathbf{u}| = \sqrt{13} \text{ and } |\mathbf{v}| = \sqrt{17}.$$

$$\cos \theta = \frac{5}{\sqrt{221}} \approx 0.3363, \text{ so } \theta \approx 70.3^\circ.$$

$$(b) \text{ Set the dot product equal to 0: } 3k + 6(-2) = 0, \text{ so } 3k = 12 \text{ and } k = 4.$$

EXAMPLE 5: VELOCITY WITH WIND

A plane heads due east with an airspeed of 200 mph. A wind blows due south at 40 mph. Find the plane's ground speed and direction.

SOLUTION

Plane: $\langle 200, 0 \rangle$. Wind: $\langle 0, -40 \rangle$ (south is negative y).

Ground velocity is the sum: $\langle 200, -40 \rangle$.

$$\text{Speed: } \sqrt{200^2 + 40^2} = \sqrt{41600} = 40\sqrt{26} \approx 204.0 \text{ mph.}$$

$$\text{Direction: } \tan^{-1} \frac{-40}{200} \approx -11.3^\circ, \text{ that is } 11.3^\circ \text{ south of east.}$$

EXAMPLE 6: RESULTANT FORCE

Two people pull on a ring. One pulls with 50 N at 30° and the other with 80 N at 120° (angles from the positive x -axis). Find the resultant force.

SOLUTION

$$\mathbf{F}_1 = \langle 50 \cos 30^\circ, 50 \sin 30^\circ \rangle = \langle 25\sqrt{3}, 25 \rangle \approx \langle 43.30, 25 \rangle.$$

$$\mathbf{F}_2 = \langle 80 \cos 120^\circ, 80 \sin 120^\circ \rangle = \langle -40, 40\sqrt{3} \rangle \approx \langle -40, 69.28 \rangle.$$

$$\text{Sum: } \langle 3.30, 94.28 \rangle. \text{ Magnitude } \approx 94.3 \text{ N, direction } \approx 88.0^\circ.$$

$$\text{Check: the forces are } 90^\circ \text{ apart, so the magnitude is } \sqrt{50^2 + 80^2} = \sqrt{8900} \approx 94.3 \text{ N.}$$

Notice how the same idea works in every example: break each vector into an x -part and a y -part, do the arithmetic on each part, and rebuild. That component habit removes almost all of the difficulty in vector word problems, because every question (a resultant, a ground speed, an angle) becomes a short computation with numbers.

Coch's Playbook

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

- 1 **Draw it.** A quick sketch tells you the quadrant and whether an answer is reasonable.
 - 2 **Break into components.** For word problems, write every vector as $\langle x, y \rangle$ first, using $\cos$ for x and $\sin$ for y .
 - 3 **Add components, then convert back.** Combine x with x and y with y , and only then find the magnitude and direction.
 - 4 **Choose the right product.** Scalar multiplication gives a vector. The dot product gives a number.
 - 5 **Check with length.** A unit vector must have length 1. A resultant of perpendicular forces follows the Pythagorean theorem.
- ◆ To make a vector perpendicular to $\langle a, b \rangle$, swap the components and negate one: $\langle -b, a \rangle$ or $\langle b, -a \rangle$.
 - ◆ The sign of the dot product tells you the angle type: positive is acute, zero is right, negative is obtuse.
 - ◆ For a direction like "35° west of north", draw the compass first and find the angle from the positive x -axis.
 - ◆ Keep exact values ($\sqrt{3}$, fractions) until the last step, then round once.

Common traps

1) Computing $\overrightarrow{PQ}$ as start minus end. 2) Reading the direction angle straight from $\tan^{-1}$ when the vector points left (add 180°). 3) Treating the dot product as a vector or adding the components wrongly. 4) Forgetting the minus sign for south and west in wind and current problems.

Mistake Clinic

Spot the error 1

A student finds the magnitude of $\langle -3, 4 \rangle$ and writes: " $\sqrt{-3^2 + 4^2} = \sqrt{-9 + 16} = \sqrt{7}$." The error is squaring only the 3 and keeping the sign. A negative number squared is positive: $(-3)^2 = 9$. The correct work is $\sqrt{9 + 16} = \sqrt{25} = 5$. A length can never be smaller than the size of its individual components, and $\sqrt{7} \approx 2.65$ is smaller than 4, which is a red flag.

Spot the error 2

A student finds the unit vector for $\langle 6, 8 \rangle$ and writes: "Add the components, $6 + 8 = 14$, so the unit vector is $\langle \frac{6}{14}, \frac{8}{14} \rangle$." The correct divisor is the magnitude $\sqrt{6^2 + 8^2} = 10$, not the sum. The unit vector is $\langle \frac{6}{10}, \frac{8}{10} \rangle = \langle \frac{3}{5}, \frac{4}{5} \rangle$. Check: $\frac{9}{25} + \frac{16}{25} = 1$, while the student's vector has length $\frac{10}{14} \neq 1$.

Spot the error 3

A student computes $\langle 2, 3 \rangle \cdot \langle 4, -1 \rangle$ and writes: " $\langle 2 \cdot 4, 3 \cdot (-1) \rangle = \langle 8, -3 \rangle$." That multiplies the components but forgets to add them. The dot product of two vectors is a single number: $2 \cdot 4 + 3 \cdot (-1) = 8 - 3 = 5$.

Real-World Connection

A pilot cannot just point the plane at the destination: the wind adds its own vector. To fly due north across a crosswind, the pilot aims slightly into the wind so that the air velocity plus the wind velocity gives a ground velocity that points north. The same vector addition explains how a swimmer crossing a river is carried downstream, and why a drone needs to tilt into a gust.

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

Challenge Corner

Vectors $\mathbf{a}$ and $\mathbf{b}$ satisfy $|\mathbf{a}| = 5$, $|\mathbf{b}| = 8$, and $\mathbf{a} \cdot \mathbf{b} = 20$. Find the angle between them, $|\mathbf{a} + \mathbf{b}|$, and $|\mathbf{a} - \mathbf{b}|$. Solution: $\cos \theta = \frac{20}{5 \cdot 8} = \frac{1}{2}$, so $\theta = 60^\circ$. Use $|\mathbf{a} \pm \mathbf{b}|^2 = |\mathbf{a}|^2 \pm 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2$. Then $|\mathbf{a} + \mathbf{b}|^2 = 25 + 40 + 64 = 129$, so $|\mathbf{a} + \mathbf{b}| = \sqrt{129} \approx 11.36$. And $|\mathbf{a} - \mathbf{b}|^2 = 25 - 40 + 64 = 49$, so $|\mathbf{a} - \mathbf{b}| = 7$.

Practice

PRACTICE Vectors in the Plane

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is the vector from $A(-2, 3)$ to $B(4, -1)$?
 (A) $\langle -6, 4 \rangle$ (B) $\langle 2, 2 \rangle$ (C) $\langle 6, -4 \rangle$ (D) $\langle 6, 4 \rangle$
- What is the magnitude of $\langle 5, -12 \rangle$?
 (A) 7 (B) 13 (C) 17 (D) $\sqrt{119}$
- What is the unit vector in the direction of $\langle 8, -6 \rangle$?
 (A) $\langle \frac{4}{5}, -\frac{3}{5} \rangle$ (B) $\langle \frac{1}{8}, -\frac{1}{6} \rangle$ (C) $\langle \frac{4}{5}, \frac{3}{5} \rangle$ (D) $\langle \frac{8}{14}, -\frac{6}{14} \rangle$
- If $\mathbf{u} = \langle 2, -1 \rangle$ and $\mathbf{v} = \langle -4, 5 \rangle$, what is $3\mathbf{u} - \mathbf{v}$?
 (A) $\langle 2, 2 \rangle$ (B) $\langle 10, -2 \rangle$ (C) $\langle 2, -8 \rangle$ (D) $\langle 10, -8 \rangle$
- What is $\langle 3, -4 \rangle \cdot \langle 2, 5 \rangle$?
 (A) 14 (B) $\langle 6, -20 \rangle$ (C) 26 (D) -14
- For which value of k are $\langle k, 3 \rangle$ and $\langle 2, -6 \rangle$ perpendicular?
 (A) -9 (B) 1 (C) 9 (D) -1
- What is the angle between $\langle 2, 2 \rangle$ and $\langle 0, 3 \rangle$?
 (A) 30° (B) 45° (C) 60° (D) 90°
- A boat heads due east at 3 m/s relative to the water, and the current flows due north at 4 m/s. What is the boat's speed relative to the ground?
 (A) 5 m/s (B) 1 m/s (C) 12 m/s (D) 7 m/s
- Which vector has magnitude 10 and direction angle 150° ?
 (A) $\langle 5\sqrt{3}, 5 \rangle$ (B) $\langle -5\sqrt{3}, 5 \rangle$ (C) $\langle -5, 5\sqrt{3} \rangle$ (D) $\langle 5, -5\sqrt{3} \rangle$
- Find the magnitude and the direction angle ($0^\circ \leq \theta < 360^\circ$) of $\langle -4, 4\sqrt{3} \rangle$.
- Let $\mathbf{u} = \langle -2, 5 \rangle$ and $\mathbf{v} = \langle 3, 1 \rangle$. Find $|2\mathbf{u} + \mathbf{v}|$.
- Find the angle between $\langle 1, 2 \rangle$ and $\langle 3, -1 \rangle$ to the nearest tenth of a degree.
- A plane heads north with an airspeed of 300 mph. A wind blows toward the east at 50 mph. Find the ground speed to the nearest tenth and the angle of the path east of north to the nearest tenth of a degree.
- A force of 200 N is applied at 40° above the horizontal to pull a cart 15 m along level ground. Use the dot product to find the work done, to the nearest joule.
- Find both vectors of length 10 that are perpendicular to $\langle 4, -3 \rangle$.

28.2 Matrix Operations

Learn matrix dimensions, add and scale matrices, multiply matrices row by column, understand why order matters, and use the identity matrix and matrix products on real data.

Why This Matters

A **matrix** is a rectangular table of numbers, and it is one of the most useful organizers in mathematics. A spreadsheet of sales, the pixels of a photo, a network of connections, and the rules of a computer game are all matrices. Operating on whole tables at once is how computers solve huge problems quickly, from graphics to search engines to machine learning.

Students find matrices strange for two reasons. First, matrix multiplication is **not** what you expect: you do not multiply entry by entry. Second, the order matters, since AB and BA are usually different. The most common errors are multiplying the wrong way, ignoring whether the sizes match, and assuming ordinary algebra rules (like commutativity) still hold. This section builds the rules carefully.

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

Core Ideas

A matrix with m rows and n columns has **dimensions** $m \times n$ (rows first). The entry in row i , column j is written a_{ij} . A **square** matrix has $m = n$. The **zero matrix** O has every entry 0, and the **identity matrix** I is a square matrix with 1 on the main diagonal and 0 elsewhere: for 2×2 , $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Addition and scalar multiplication

Addition and subtraction work entry by entry and require the **same dimensions**: $(A \pm B)_{ij} = a_{ij} \pm b_{ij}$.

Scalar multiplication multiplies every entry by the number: $(cA)_{ij} = c a_{ij}$.

These operations behave like ordinary algebra: $A + B = B + A$, $c(A + B) = cA + cB$.

Matrix multiplication

The product AB is defined only when the number of **columns of** A equals the number of **rows of** B . If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$.

Each entry is a row-times-column sum:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

In general $AB \neq BA$. For any square matrix A of the right size, $AI = IA = A$.

A is	B is	AB	BA
2×3	3×4	2×4	not defined
3×2	2×3	3×3	2×2
2×2	2×2	2×2	2×2 (but usually different)
1×3	3×1	1×1	3×3

The easiest way to remember the row-times-column rule: to get the entry in row i and column j of AB , run your left finger across row i of A , run your right finger down column j of B , multiply the pairs as you go, and add. For example, with $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}$, the entry in row 1, column 1 of AB is $1 \cdot 0 + 2 \cdot 5 = 10$.

Several familiar rules carry over. Multiplication is **associative**, $(AB)C = A(BC)$, and it **distributes** over addition, $A(B + C) = AB + AC$. Multiplying by the identity changes nothing, and multiplying by the zero matrix gives the zero matrix. What fails is commutativity. You can see why from the sizes alone: if A is 2×3 and B is 3×2 , then AB is 2×2 but BA is 3×3 , so they cannot even be compared. Even for square matrices of the same size the entries usually come out different, as Example 5 shows.

Matrices also show data clearly. Suppose a cafe sells sandwiches, drinks, and cookies on two days, and the quantities form a 2×3 matrix. A 3×1 matrix of prices turns that into a 2×1 matrix of daily revenue in a single multiplication. Example 6 shows this.

When you work with powers, remember that A^2 means $A \cdot A$, and it exists only for square matrices. For a diagonal matrix, such as one with 2 and 3 on the diagonal and zeros elsewhere, powers are easy because you just raise each diagonal entry to the power. For general matrices, you need to multiply row by column each time.

Worked Examples

EXAMPLE 1: DIMENSIONS AND DEFINED PRODUCTS

Matrix A is 2×3 , B is 3×4 , C is 4×2 . Which of AB , BC , AC , CA are defined, and what are their sizes?

SOLUTION

AB : columns of A (3) equal rows of B (3). Defined, size 2×4 .

BC : columns of B (4) equal rows of C (4). Defined, size 3×2 .

AC : columns of A (3) and rows of C (4) are different. **Not defined**.

CA : columns of C (2) equal rows of A (2). Defined, size 4×3 .

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

EXAMPLE 2: ADDITION AND SCALAR MULTIPLES

Let $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 \\ -2 & 1 \end{bmatrix}$. Find $3A - 2B$.

SOLUTION

$$3A = \begin{bmatrix} 6 & -3 \\ 0 & 9 \end{bmatrix} \text{ and } 2B = \begin{bmatrix} 8 & 10 \\ -4 & 2 \end{bmatrix}.$$

$$\text{Subtract entry by entry: } 3A - 2B = \begin{bmatrix} -2 & -13 \\ 4 & 7 \end{bmatrix}.$$

$$\text{Check one entry: } 3(-1) - 2(5) = -3 - 10 = -13.$$

EXAMPLE 3: MULTIPLYING TWO 2×2 MATRICES

Find AB for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}$.

SOLUTION

Row 1, column 1: $1 \cdot 0 + 2 \cdot 5 = 10$. Row 1, column 2: $1 \cdot (-1) + 2 \cdot 2 = 3$.

Row 2, column 1: $3 \cdot 0 + 4 \cdot 5 = 20$. Row 2, column 2: $3 \cdot (-1) + 4 \cdot 2 = 5$.

$$AB = \begin{bmatrix} 10 & 3 \\ 20 & 5 \end{bmatrix}.$$

EXAMPLE 4: A 2×3 TIMES A 3×2

Find AB and the size of BA for $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 4 & 5 \end{bmatrix}$.

SOLUTION

A is 2×3 and B is 3×2 , so AB is 2×2 .

Row 1 of A with the columns of B : $1 \cdot 2 + 0 \cdot 0 + 2 \cdot 4 = 10$ and $1 \cdot 1 + 0 \cdot (-1) + 2 \cdot 5 = 11$.

Row 2 of A : $(-1) \cdot 2 + 3 \cdot 0 + 1 \cdot 4 = 2$ and $(-1) \cdot 1 + 3 \cdot (-1) + 1 \cdot 5 = 1$.

$$AB = \begin{bmatrix} 10 & 11 \\ 2 & 1 \end{bmatrix}. \text{ The product } BA \text{ is } 3 \times 3, \text{ a different size, so } AB \neq BA.$$

EXAMPLE 5: ORDER MATTERS

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$. Find AB and BA .

SOLUTION

AB : row 1 is $[1 \cdot 1 + 2 \cdot 3, 1 \cdot 0 + 2 \cdot 1] = [7, 2]$. Row 2 is $[0 \cdot 1 + 1 \cdot 3, 0 \cdot 0 + 1 \cdot 1] = [3, 1]$. So $AB = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$.

BA : row 1 is $[1, 2]$. Row 2 is $[3 \cdot 1 + 1 \cdot 0, 3 \cdot 2 + 1 \cdot 1] = [3, 7]$. So $BA = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$.

$AB \neq BA$, even though both matrices are 2×2 . Matrix multiplication is not commutative.

EXAMPLE 6: USING A PRODUCT ON DATA

A cafe sold the following on Monday and Tuesday. Quantities (sandwiches, drinks, cookies): Monday 20, 15, 10; Tuesday 25, 18, 12. Prices in dollars: sandwich 5, drink 2, cookie 1. Find the revenue each day.

SOLUTION

Quantity matrix $Q = \begin{bmatrix} 20 & 15 & 10 \\ 25 & 18 & 12 \end{bmatrix}$ (2×3) and price matrix $P = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$ (3×1).

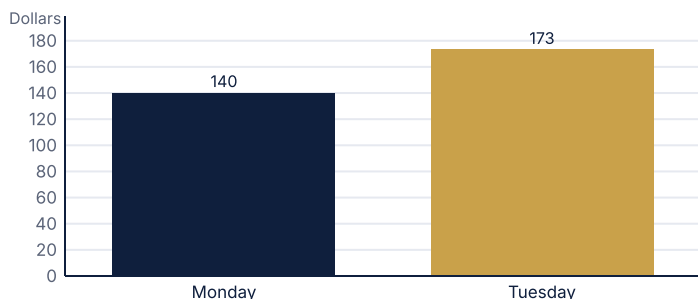
QP is 2×1 . Monday: $20 \cdot 5 + 15 \cdot 2 + 10 \cdot 1 = 100 + 30 + 10 = 140$. Tuesday: $25 \cdot 5 + 18 \cdot 2 + 12 \cdot 1 = 125 + 36 + 12 = 173$.

Revenue: $\begin{bmatrix} 140 \\ 173 \end{bmatrix}$, that is, 140 dollars on Monday and 173 dollars on Tuesday.

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →



Daily revenue computed from the matrix product QP in Example 6.

Coach's Playbook

- 1 Check dimensions first.** Write the sizes side by side, like $(2 \times 3)(3 \times 4)$. The inner numbers must match, and the outer numbers give the size of the answer.
 - 2 Set up the grid.** Draw an empty box of the answer's size, so you know how many entries to compute.
 - 3 Row times column.** For each entry, multiply pairs and add. Write the arithmetic out before you simplify.
 - 4 Check one entry a second way.** Recompute one entry, especially the last one, and verify signs.
 - 5 Avoid cancelling by habit.** Do not move matrices across the equal sign the way you move numbers. Only use properties that hold for matrices.
- ◆ Because $AB \neq BA$ in general, $(A + B)^2$ is $A^2 + AB + BA + B^2$, not $A^2 + 2AB + B^2$.
 - ◆ Use I the way you use 1: $AI = A$. The zero matrix works like 0 for addition.
 - ◆ To solve a matrix equation like $2X + A = B$, use the same steps as with numbers: $X = \frac{1}{2}(B - A)$.
 - ◆ A product of two nonzero matrices can be the zero matrix, so you cannot conclude that $AB = O$ means $A = O$ or $B = O$.

Common traps

1) Multiplying entry by entry (that is not matrix multiplication). 2) Not checking that the inner dimensions match. 3) Reversing the order of the factors. 4) Treating A^2 as squaring each entry; it means $A \cdot A$.

Mistake Clinic

Spot the error 1

A student finds AB for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}$ and writes $\begin{bmatrix} 0 & -2 \\ 15 & 8 \end{bmatrix}$, multiplying matching entries. That operation is not the matrix product. The correct entries come from rows times columns: the top-left is $1 \cdot 0 + 2 \cdot 5 = 10$, not 0. The right answer is $\begin{bmatrix} 10 & 3 \\ 20 & 5 \end{bmatrix}$.

Spot the error 2

A student reasons: " $(A + B)^2 = A^2 + 2AB + B^2$." For $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ this gives $\begin{bmatrix} 16 & 8 \\ 12 & 4 \end{bmatrix}$. The true value is $(A + B)^2 = \begin{bmatrix} 10 & 8 \\ 12 & 10 \end{bmatrix}$. The error is combining AB and BA into $2AB$, which requires $AB = BA$. Since $AB = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$, they are not equal. The correct expansion is $A^2 + AB + BA + B^2$.

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

Spot the error 3

A student says: "A and B are both 2×3 , so I can multiply them to get a 2×3 matrix." The error is using the outer sizes. For AB the inner numbers must match, and here they are 3 and 2. The product AB is **not defined**. (Adding $A + B$ is defined and gives a 2×3 matrix.)

Real-World Connection

A phone camera stores a photo as a grid of brightness numbers, which is a matrix. Brightening the photo is scalar multiplication, combining two photos is matrix addition, and many photo filters, such as blur and sharpen, are built from matrix products. Video games use matrix products to turn the objects in a scene into the picture that appears on your screen, millions of times per second.

Challenge Corner

Find all 2×2 matrices $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that commute with $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, that is, $JM = MJ$. Solution: $JM = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$ and $MJ = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$. Setting entries equal: $a+c = a$ gives $c = 0$, and $b+d = a+b$ gives $d = a$. The other two entries already agree. So $M = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ for any numbers a and b .

Practice**PRACTICE Matrix Operations**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- If A is 3×2 and B is 2×5 , what is the size of AB ?
 (A) 2×2 (B) 3×2 (C) 5×3 (D) 3×5
- What is $\begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix}$?
 (A) $\begin{bmatrix} 5 & -1 \\ -2 & 2 \end{bmatrix}$ (B) $\begin{bmatrix} 5 & -2 \\ -2 & 0 \end{bmatrix}$
 (C) $\begin{bmatrix} -3 & -3 \\ 8 & -2 \end{bmatrix}$ (D) $\begin{bmatrix} 4 & -2 \\ -15 & 0 \end{bmatrix}$
- Let $A = \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$. What is $2A - B$?
 (A) $\begin{bmatrix} 7 & 6 \\ 2 & -5 \end{bmatrix}$ (B) $\begin{bmatrix} 5 & -2 \\ -2 & -3 \end{bmatrix}$
 (C) $\begin{bmatrix} 5 & -2 \\ 2 & -5 \end{bmatrix}$ (D) $\begin{bmatrix} 5 & -2 \\ -2 & -5 \end{bmatrix}$
- A is 2×3 , B is 3×4 , C is 2×4 , and D is 4×2 . Which product is **not** defined?
 (A) AB (B) CD (C) BC (D) DA
- What is the entry in row 2, column 1 of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$?
 (A) 19 (B) 43 (C) 15 (D) 28
- What is $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$?
 (A) $\begin{bmatrix} 2 & 8 \\ 7 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 6 & 12 \\ 3 & -3 \end{bmatrix}$ (C) $\begin{bmatrix} 2 & 0 \\ 2 & -3 \end{bmatrix}$
 (D) $\begin{bmatrix} 2 & 8 \\ 5 & 1 \end{bmatrix}$

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

7. Solve $2X + \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 6 \end{bmatrix}$ for X .

(A) $\begin{bmatrix} 4 & 4 \\ 0 & 8 \end{bmatrix}$

(B) $\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} 8 & 8 \\ 0 & 16 \end{bmatrix}$

(D) $\begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix}$

8. A school store sold 12 notebooks and 8 pens in one day, recorded as the row $[12, 8]$. Notebooks cost 4 dollars and pens cost 6 dollars, recorded as the column $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$. What is the product, in dollars?

(A) 48

(B) 20

(C) 96

(D) 24

9. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, what is A^2 ?

(A) $\begin{bmatrix} 1 & 4 \\ 9 & 16 \end{bmatrix}$

(B) $\begin{bmatrix} 7 & 10 \\ 15 & 20 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

(D) $\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$

10. Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 4 \end{bmatrix}$. Find AB .

11. Find x and y if $\begin{bmatrix} x & 2 \\ 1 & y \end{bmatrix} + \begin{bmatrix} 3 & y \\ x & 4 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 3 & 9 \end{bmatrix}$.

12. Let $K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Find K^2 and K^4 .

13. Weekly fruit sales (apples, bananas, pears) are Week 1: 10, 6, 4 and Week 2: 8, 9, 5. Prices in dollars are apples 0.50, bananas 0.40, pears 0.80. Use a matrix product to find the revenue each week.

14. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix}$. Find $AB - BA$.

15. Find x and y if $\begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 9 & 6 \end{bmatrix}$.

28.3 Determinants and Inverses

Compute determinants of 2 by 2 and 3 by 3 matrices, find inverse matrices, connect determinants to area, and use inverses to solve systems of linear equations.

Why This Matters

In algebra you solve $3x = 12$ by dividing by 3, which is the same as multiplying by the reciprocal $\frac{1}{3}$. For matrices there is a similar idea, the **inverse matrix**, and a single number called the **determinant** that tells you whether the inverse exists at all. Together they let you solve whole systems of equations in one move, and they appear in computer graphics, economics, engineering, and statistics.

The usual stumbling blocks are mixing up the order in $ad - bc$, forgetting which entries swap and which change sign in the inverse, multiplying in the wrong order when solving $AX = B$, and getting lost in the arithmetic of a 3×3 determinant. This section gives you a short, reliable routine for each.

Core Ideas

Determinant of a 2×2 matrix

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\det A = ad - bc.$$

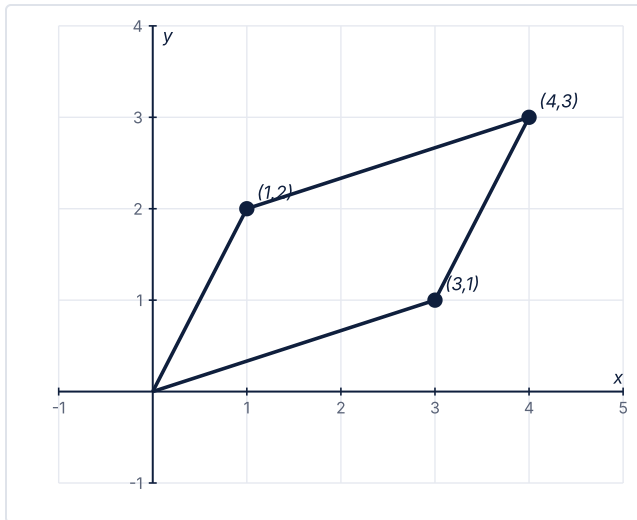
Think "main diagonal minus the other diagonal." The determinant is a single number. If $\det A = 0$ the matrix is **singular** and has **no inverse**.

The determinant has a geometric meaning. If the columns of A are the vectors $\langle a, c \rangle$ and $\langle b, d \rangle$, then $|\det A|$ is the **area of the parallelogram** they form. For the vectors $\langle 3, 1 \rangle$ and $\langle 1, 2 \rangle$ the determinant is $3 \cdot 2 - 1 \cdot 1 = 5$, so the parallelogram has area 5. A zero determinant means the vectors lie on one line and the parallelogram is flat, with no area.

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →



The parallelogram formed by the vectors $\langle 3,1 \rangle$ and $\langle 1,2 \rangle$ has area $|3 \cdot 2 - 1 \cdot 1| = 5$, which is the absolute value of the determinant.

Inverse of a 2×2 matrix

If $\det A = ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Swap a and d , change the signs of b and c , and divide by the determinant. Check: $AA^{-1} = A^{-1}A = I$.

Solving a system with matrices

A system can be written $AX = B$, where A holds the coefficients, X the unknowns, and B the constants. If A has an inverse, then

$$X = A^{-1}B.$$

Multiply on the **left** by A^{-1} . Order matters, because matrix multiplication is not commutative.

For a 3×3 matrix, the determinant can be found by **cofactor expansion** along any row or column. Using the first row:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg).$$

The signs alternate $+, -, +$ along the row. The pattern of signs for the whole matrix is a checkerboard.

Sign pattern	Col 1	Col 2	Col 3
Row 1	+	−	+
Row 2	−	+	−
Row 3	+	−	+

Cramer's Rule (2 by 2)

For the system $ax + by = e$, $cx + dy = f$ with $D = ad - bc \neq 0$:

$$x = \frac{ed - bf}{D}, \quad y = \frac{af - ec}{D}.$$

The numerator of x is the determinant you get by replacing the x -column of coefficients with the constants. It is a fast alternative to the inverse for a single unknown.

What does a zero determinant mean for a system? The two equations then describe lines with the same slope. If the lines are the same line, there are infinitely many solutions. If they are distinct parallel lines, there is no solution. Either way, there is not exactly one solution, and the inverse method cannot be used. A nonzero determinant guarantees exactly one solution.

A few rules save time. Swapping two rows changes the sign of the determinant. Multiplying one row by a number k multiplies the determinant by k . A matrix with a row of zeros, or with two equal rows, has determinant 0. For 3×3 inverses and larger, people use a graphing tool or row operations, but the idea is the same: $\det A \neq 0$ means the inverse exists.

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

Worked Examples

EXAMPLE 1: DETERMINANTS AND SINGULAR MATRICES

Find $\det \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ and $\det \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$.

SOLUTION

First: $3 \cdot 4 - 5 \cdot 2 = 12 - 10 = 2$.

Second: $2 \cdot 3 - 6 \cdot 1 = 6 - 6 = 0$.

The second matrix is singular. Its rows are proportional, $[2, 6] = 2[1, 3]$, so it has no inverse.

EXAMPLE 2: INVERSE OF A 2×2

Find the inverse of $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$.

SOLUTION

$\det A = 12 - 10 = 2 \neq 0$, so the inverse exists.

Swap 3 and 4, negate 5 and 2: $\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$.

Divide by 2: $A^{-1} = \begin{bmatrix} 2 & -\frac{5}{2} \\ -1 & \frac{3}{2} \end{bmatrix}$.

Check the top-left of AA^{-1} : $3 \cdot 2 + 5 \cdot (-1) = 1$. Good.

EXAMPLE 3: SOLVING A SYSTEM WITH AN INVERSE

Solve $2x + y = 7$ and $x - y = 2$ using matrices.

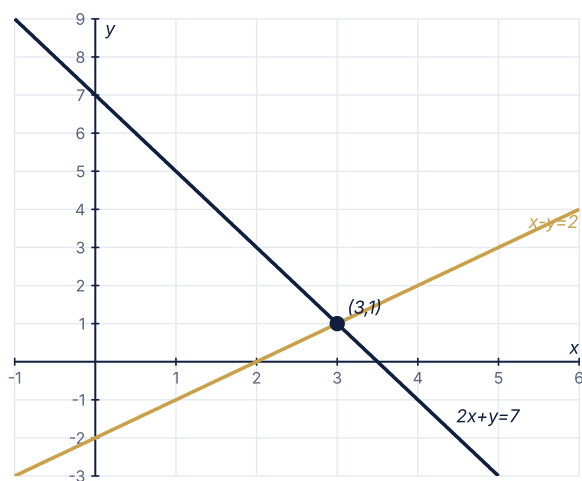
SOLUTION

Write $AX = B$ with $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

$\det A = -2 - 1 = -3$. So $A^{-1} = \frac{1}{-3} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$.

$X = A^{-1}B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, so $x = 3$ and $y = 1$.

Check: $2(3) + 1 = 7$ and $3 - 1 = 2$. The graph of the two lines confirms it.



The two equations of Example 3 are lines that cross at $(3,1)$, the solution found with the inverse matrix.

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

EXAMPLE 4: A 3×3 DETERMINANT

Find $\det \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ 1 & 2 & 0 \end{bmatrix}$.

SOLUTION

Expand along the first row: $2((-1)(0) - 4 \cdot 2) - 1(0 \cdot 0 - 4 \cdot 1) + 3(0 \cdot 2 - (-1)(1))$.
 $= 2(-8) - 1(-4) + 3(1)$.
 $= -16 + 4 + 3 = -9$.

EXAMPLE 5: AREA FROM A DETERMINANT

Find the area of the triangle with vertices $(0,0)$, $(4,1)$, and $(2,5)$.

SOLUTION

The sides from the origin are the vectors $\langle 4,1 \rangle$ and $\langle 2,5 \rangle$. Their determinant is $4 \cdot 5 - 1 \cdot 2 = 18$.
 The parallelogram has area $|18| = 18$, and the triangle is half of it.
 Area = 9 square units.

EXAMPLE 6: WHEN IS A MATRIX SINGULAR?

For what values of k does $\begin{bmatrix} k & 4 \\ 3 & k+1 \end{bmatrix}$ have no inverse?

SOLUTION

The matrix has no inverse exactly when its determinant is zero: $k(k+1) - 4 \cdot 3 = 0$.

$k^2 + k - 12 = 0$, which factors as $(k+4)(k-3) = 0$.

So $k = -4$ or $k = 3$.

For every other value of k the matrix is invertible. Check $k = 3$: $\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}$ has two equal rows, so the determinant is 0.

Coach's Playbook

- 1 Compute the determinant first.** It tells you immediately if the inverse exists, and you need it for the inverse formula anyway.
 - 2 Use the pattern for the inverse.** Swap the diagonal entries, negate the other two, and divide by the determinant.
 - 3 Check with a multiplication.** Multiply A by your inverse. If you do not get I , find the slip before moving on.
 - 4 Solve $AX = B$ as $X = A^{-1}B$.** Put A^{-1} on the left.
 - 5 Verify in the original equations.** Substituting the answer is the best final check.
- ◆ For 3×3 determinants choose the row or column with the most zeros, since each zero removes a term.
 - ◆ If a 2×2 determinant is 1, the inverse has no fractions, which makes a good test case.
 - ◆ The inverse of a product reverses the order: $(AB)^{-1} = B^{-1}A^{-1}$.
 - ◆ If the determinant is not zero but the system seems to have no solution, recheck the arithmetic: an invertible coefficient matrix always gives exactly one solution.

Common traps

1) Computing $ad + bc$ or $ac - bd$ instead of $ad - bc$. 2) Swapping the wrong entries in the inverse (only a and d swap; b and c stay in place but change sign). 3) Forgetting the factor $\frac{1}{\det A}$. 4) Writing $X = BA^{-1}$ instead of $A^{-1}B$.

Mistake Clinic

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

Spot the error 1

A student finds $\det \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ and writes $3 \cdot 2 - 5 \cdot 4 = -14$. The student multiplied the wrong pairs. The determinant is the product of the main diagonal ($3 \cdot 4$) minus the product of the other diagonal ($5 \cdot 2$): $12 - 10 = 2$.

Spot the error 2

A student inverts $\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ and writes $\frac{1}{2} \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$. The signs of 5 and 2 were changed correctly, but 3 and 4 were not swapped. The correct inverse is $\frac{1}{2} \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$. A quick check shows the error: the top-left entry of the product with A would be $\frac{1}{2}(3 \cdot 3 + 5 \cdot (-2)) = -\frac{1}{2}$, not 1.

Spot the error 3

A student solves $AX = B$ with $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ and writes $X = BA^{-1}$. This product is not even defined: B is 2×1 and A^{-1} is 2×2 , so the inner dimensions 1 and 2 do not match. Multiplying both sides of $AX = B$ on the left by A^{-1} gives $A^{-1}AX = A^{-1}B$, so $X = A^{-1}B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Real-World Connection

A school fundraiser sells two kinds of snack boxes. Box 1 holds 2 cookies and 1 fruit cup, and box 2 holds 1 cookie and 1 fruit cup. The contents form the matrix with columns $\langle 2, 1 \rangle$ and $\langle 1, 1 \rangle$. A volunteer who needs exactly 7 cookies and 5 fruit cups can multiply the inverse $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ by the order $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$ to get $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$: prepare 2 of box 1 and 3 of box 2. The same inverse works for any new order with a single multiplication, which is why inverse matrices are built into spreadsheet and design software.

Challenge Corner

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Show that $(AB)^{-1} = B^{-1}A^{-1}$ and that $A^{-1}B^{-1}$ gives a different matrix. Solution: $\det A = 1$, so $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. $\det B = 1$, so $B^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. $AB = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ has determinant 1, so $(AB)^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$. Now $B^{-1}A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$, which matches. But $A^{-1}B^{-1} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$, which does not. The order reverses.

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

Practice

PRACTICE Determinants and Inverses

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is $\det \begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix}$?
 (A) 5 (B) 11 (C) -11 (D) 8
- Which statement is true about $\begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}$?
 (A) It has an inverse because all entries are nonzero.
 (B) Its determinant is 36, so it has an inverse.
 (C) Its determinant is 0, so it has no inverse.
 (D) Its determinant is 0, so its inverse is the zero matrix.
- What is the inverse of $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$?
 (A) $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ (C) $\begin{bmatrix} -3 & 1 \\ 5 & -2 \end{bmatrix}$
 (D) $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$
- What is $\det \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$?
 (A) 13 (B) 1 (C) -13 (D) 7
- Use an inverse matrix to solve $x + 2y = 4$ and $3x + 5y = 11$.
 (A) (2, 1) (B) (1, 2) (C) (-2, -1) (D) (4, 11)
- For which values of k does $\begin{bmatrix} k & 2 \\ 8 & k \end{bmatrix}$ have no inverse?
 (A) $k = 4$ only (B) $k = \pm \sqrt{8}$ (C) $k = \pm 4$ (D) $k = 16$
- What is the area of the triangle with vertices (1, 1), (5, 2), and (3, 6)?
 (A) 18 (B) 11 (C) 9 (D) 7
- If $\det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = -2$, what is $\det \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$?
 (A) -2 (B) -4 (C) 0 (D) 2
- A is a 2×2 invertible matrix and B is a 2×1 matrix. Which expression solves $AX = B$?
 (A) $X = BA^{-1}$ (B) $X = A^{-1}B$ (C) $X = B - A$ (D) $X = \frac{B}{A}$
- Find the inverse of $\begin{bmatrix} 3 & 1 \\ 7 & 3 \end{bmatrix}$.
- Use an inverse matrix to solve $4x + 3y = 10$ and $2x + 2y = 6$.
- Find $\det \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & 2 \\ 0 & 4 & -1 \end{bmatrix}$.
- Find all values of k for which $\begin{bmatrix} k & 6 \\ 1 & k-5 \end{bmatrix}$ has no inverse.
- Find the area of the triangle with vertices (2, 1), (6, 3), and (3, 8).
- For what value of m does the system $x + 2y = 3$, $3x + my = 5$ fail to have exactly one solution? For that value, does the system have no solution or infinitely many?

28.4 Matrices as Transformations and Data

Use matrices to rotate, reflect, and dilate figures, compose transformations in the right order, relate the determinant to area, and model change with transition matrices.

Why This Matters

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

When you drag, rotate, or flip a picture on a screen, a computer multiplies the coordinates of every point by a small matrix. This is the most visual use of matrices: a 2×2 matrix can **transform** the whole plane. Matrices also model change over time, such as how people move between cities or subscribers move between services, and the same multiplication predicts the future.

Students struggle with three ideas here: which way a rotation turns and what its matrix looks like, the **order** in which transformations are composed (the first one is written last), and setting up a transition matrix with the columns, not the rows, adding to one. This section gives a clear process for each and shows how the determinant measures how much a transformation stretches area.

Core Ideas

Write a point (x, y) as a column vector $\begin{bmatrix} x \\ y \end{bmatrix}$. Multiplying by a 2×2 matrix M gives a new point $M \begin{bmatrix} x \\ y \end{bmatrix}$. This is called a **linear transformation**. The key fact is that the **columns of M are the images of the unit vectors**: the first column is where $(1, 0)$ goes, and the second column is where $(0, 1)$ goes.

Common transformations of the plane

Rotation by θ counterclockwise about the origin:

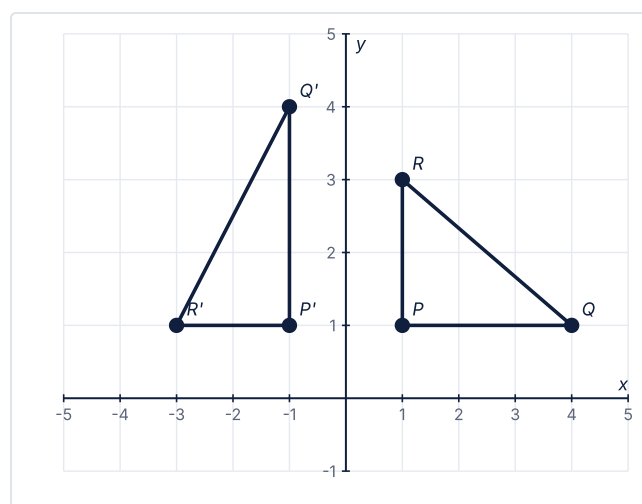
$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

For 90° this is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which sends (x, y) to $(-y, x)$.

Reflections: over the x -axis $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; over the y -axis $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$; over the line $y = x$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Dilation by factor k : $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, which sends (x, y) to (kx, ky) .

Transformation	Matrix	Image of $(2, 3)$
Rotate 90°	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$(-3, 2)$
Rotate 180°	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$(-2, -3)$
Reflect over x -axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$(2, -3)$
Reflect over $y = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$(3, 2)$
Dilate by 2	$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$(4, 6)$



Triangle PQR with vertices $(1,1)$, $(4,1)$, $(1,3)$ and its image after a 90 degree counterclockwise rotation: $P'(-1,1)$, $Q'(-1,4)$, $R'(-3,1)$. The shape and size do not change.

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

Compositions and area

If you apply transformation M_1 first and then M_2 , the combined matrix is

$$M_2 M_1.$$

The matrix written closest to the vector acts first.

A transformation with matrix M multiplies every area by $|\det M|$. If $\det M < 0$ the figure is also flipped (orientation reverses).

A second use of matrix multiplication is to model change over time. A **transition matrix** T has nonnegative entries and each **column** adds to 1: entry t_{ij} is the fraction of group j that moves to group i in one step. If $\mathbf{s}$ is the current state (a column of group sizes), then the next state is $T\mathbf{s}$, and the state after n steps is $T^n\mathbf{s}$. Over many steps, the state often settles into a **steady state** $\mathbf{s}$ with $T\mathbf{s} = \mathbf{s}$.

Worked Examples

EXAMPLE 1: ROTATION

Rotate the point $(3, 2)$ by 90° counterclockwise about the origin using a matrix.

SOLUTION

The matrix is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

The image is $(-2, 3)$. Check: the point $(3, 2)$ is in Quadrant I and a quarter turn counterclockwise puts it in Quadrant II.

EXAMPLE 2: REFLECTIONS

Find the image of $(4, -3)$ when it is reflected (a) across the x -axis and (b) across the line $y = x$.

SOLUTION

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. The y -coordinate changes sign: $(4, 3)$.

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. The coordinates swap: $(-3, 4)$.

EXAMPLE 3: DILATION AND AREA

Triangle PQR has vertices $(1, 1)$, $(4, 1)$, $(1, 3)$. A dilation by factor 2 is applied. By what factor does the area change, and what is the new area?

SOLUTION

The matrix is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and its determinant is 4.

The original triangle has base 3 and height 2, so area $\frac{1}{2} \cdot 3 \cdot 2 = 3$.

The new area is $4 \cdot 3 = 12$.

Check: the new vertices are $(2, 2)$, $(8, 2)$, $(2, 6)$, with base 6 and height 4, so the area is $\frac{1}{2} \cdot 6 \cdot 4 = 12$.

EXAMPLE 4: COMPOSITION

Find the single matrix for "reflect across the x -axis, then rotate 90° counterclockwise," and apply it to $(2, 5)$. What happens if the order is reversed?

SOLUTION

The first step is $F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and the second is $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The combined matrix is RF (the first step is written on the right).

$RF = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Notice that this is the reflection across $y = x$.

Apply to $(2, 5)$: $(5, 2)$. Check by steps: reflect to $(2, -5)$, then rotate: $(x, y) \rightarrow (-y, x)$ gives $(5, 2)$.

In the reverse order the matrix is $FR = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, and $(2, 5)$ goes to $(-5, -2)$. The order matters.

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

EXAMPLE 5: FINDING A MATRIX FROM IMAGES

A transformation sends $(1, 0)$ to $(2, 1)$ and $(0, 1)$ to $(-1, 3)$. Find its matrix, the image of $(3, 2)$, and the area scale factor.

SOLUTION

The columns of the matrix are the images of the unit vectors: $M = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$.

Image of $(3, 2)$: $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$, which is $(4, 9)$.

$\det M = 2 \cdot 3 - (-1)(1) = 7$. Areas are multiplied by 7, and the orientation is preserved since $7 > 0$.

EXAMPLE 6: A TRANSITION MATRIX

Each year, 80% of the customers of Service A stay with A and 20% switch to Service B. Of the customers of B, 30% switch to A and 70% stay. Start with 500 customers each. Find the numbers after one and two years, and the long-run split.

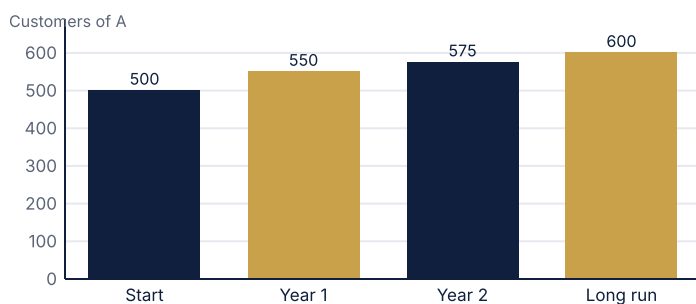
SOLUTION

Write $T = \begin{bmatrix} \frac{4}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{7}{10} \end{bmatrix}$, with columns = (from A, from B). Each column adds to 1. The state is $\begin{bmatrix} 500 \\ 500 \end{bmatrix}$.

Year 1: $Ts = \begin{bmatrix} 550 \\ 450 \end{bmatrix}$ (because $0.8 \cdot 500 + 0.3 \cdot 500 = 550$).

Year 2: $Ts = \begin{bmatrix} 575 \\ 425 \end{bmatrix}$ (because $0.8 \cdot 550 + 0.3 \cdot 450 = 440 + 135 = 575$).

Steady state: the A and B flows balance when $0.2a = 0.3b$, so $a : b = 3 : 2$. With 1000 customers, that is 600 for A and 400 for B. Check: $0.8 \cdot 600 + 0.3 \cdot 400 = 600$.



Customers of Service A in Example 6, approaching the steady state of 600.

Coach's Playbook

- 1 Use columns for images.** To build a transformation matrix, find where $(1, 0)$ and $(0, 1)$ land and write the results as the columns.
 - 2 Compose right to left.** The first transformation is the matrix closest to the vector.
 - 3 Use the determinant for area.** $|\det M|$ is the area factor. A negative value means a flip.
 - 4 Set up transition matrices by columns.** Each column is a "from" group and must add to 1. The rows are the "to" groups.
 - 5 Test with one point.** Apply your matrix to a simple point like $(1, 0)$ or $(2, 5)$ and check the result on a sketch.
- ◆ Quick reflexes: a 90° turn sends (x, y) to $(-y, x)$, a 180° turn sends it to $(-x, -y)$, and a 270° turn sends it to $(y, -x)$.
 - ◆ A rotation or a reflection has determinant 1 or -1 , so it keeps lengths and areas. A dilation by k has determinant k^2 .
 - ◆ Check any transition matrix by adding each column: all the sums must be 1.
 - ◆ To find the steady state, solve $Ts = s$ along with the total, or just multiply many times and watch the numbers settle.

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

Common traps

1) Using the wrong signs in the rotation matrix, which turns the point clockwise. 2) Composing in the wrong order. 3) Making the rows of a transition matrix sum to 1 when the columns should. 4) Saying a dilation by k multiplies area by k (it multiplies area by k^2).

Mistake Clinic**Spot the error 1**

A student rotates $(3, 2)$ by 90° counterclockwise using $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and gets $(2, -3)$. The matrix has its signs in the wrong places: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is the **clockwise** rotation, and $(2, -3)$ is in Quadrant IV, which is a quarter turn clockwise from Quadrant I. The counterclockwise matrix is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and the image is $(-2, 3)$.

Spot the error 2

A student writes the matrix for "reflect across the x -axis, then rotate 90° counterclockwise" as $FR = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ and applies it to $(2, 5)$ to get $(-5, -2)$. Checking by steps shows the problem: reflecting $(2, 5)$ gives $(2, -5)$, and rotating gives $(5, 2)$. The first transformation must be on the **right**, so the matrix is $RF = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which gives $(5, 2)$.

Spot the error 3

A student builds the transition matrix for Example 6 as $\begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$ because each row adds to 1. With the state $\begin{bmatrix} 500 \\ 500 \end{bmatrix}$ this gives $\begin{bmatrix} 500 \\ 500 \end{bmatrix}$, so nothing seems to change, which is wrong. Each **column** must add to 1 because each column describes where one group goes. The correct matrix is $\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$, which gives $\begin{bmatrix} 550 \\ 450 \end{bmatrix}$.

Real-World Connection

Every time a game character turns, a camera pans, or an image is flipped, the graphics card multiplies thousands of coordinate vectors by transformation matrices, and chains several of them together with matrix products. In the same way, a streaming service can use a transition matrix to estimate what share of viewers will move from one show to another over the next month.

Challenge Corner

Show with matrices that a rotation by 30° followed by a rotation by 60° is a rotation by 90° . Solution: with $c = \cos$ and $s = \sin$, the product is $R_{60}R_{30}$, where R_θ is the rotation matrix. The top-left entry is $\cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = 0$. The bottom-left entry is $\sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ = \frac{3}{4} + \frac{1}{4} = 1$. The full product is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, the 90° rotation. In general, multiplying rotation matrices adds the angles.

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

Practice

PRACTICE Matrices as Transformations and Data

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is the image of $(2, 3)$ after a 90° counterclockwise rotation about the origin?
 (A) $(3, -2)$ (B) $(-3, 2)$ (C) $(-2, -3)$ (D) $(3, 2)$
- Which matrix reflects points across the y -axis?
 (A) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (C) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
 (D) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
- A shape with area 5 is transformed by $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. What is the area of the image?
 (A) 15 (B) 14 (C) 45 (D) 8
- A transformation has matrix $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$. Which statement is true?
 (A) Areas are multiplied by 5 and orientation is reversed.
 (B) Areas are multiplied by -5 .
 (C) Areas are multiplied by 5 and orientation is kept.
 (D) Areas stay the same.
- A transformation sends $(1, 0)$ to $(3, 1)$ and $(0, 1)$ to $(-2, 4)$. What is its matrix?
 (A) $\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$ (B) $\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}$
 (D) $\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$
- What is the image of $(5, -2)$ when reflected across the line $y = x$?
 (A) $(5, 2)$ (B) $(-5, 2)$ (C) $(2, -5)$ (D) $(-2, 5)$
- The point $(3, 4)$ is rotated 180° about the origin and then reflected across the x -axis. What is the final image?
 (A) $(-3, 4)$ (B) $(-3, -4)$ (C) $(3, 4)$ (D) $(3, -4)$
- Group 1 has 300 people and group 2 has 200. Each step, 90% of group 1 stays and 10% moves to group 2, while 20% of group 2 moves to group 1 and 80% stays. How many are in group 1 after one step?
 (A) 290 (B) 270 (C) 310 (D) 330
- Which matrix could be a transition matrix for a two-group system?
 (A) $\begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.7 \end{bmatrix}$ (B) $\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$
 (C) $\begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$ (D) $\begin{bmatrix} 0.6 & 0.3 \\ 0.5 & 0.7 \end{bmatrix}$
- Rotate $(-2, 5)$ by 180° about the origin. Give the matrix used and the image.
- The triangle with vertices $(0, 0)$, $(4, 0)$, $(0, 3)$ is transformed by $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$. Find the image vertices and the area of the image.
- Write one matrix for "reflect across the x -axis, then dilate by 3" and use it to find the image of $(1, 2)$.
- Commuters use either the bus or the train. Each week, 70% of bus riders stay and 30% switch to the train, and 40% of train riders switch to the bus while 60% stay. Starting with 200 bus riders and 300 train riders, how many are in each group after two weeks?
- A transition matrix is $\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$ (columns are "from A" and "from B"). What fraction of the population is in group A in the long run?
- A transformation M sends $(1, 1)$ to $(3, 5)$ and $(1, -1)$ to $(1, -1)$. Find M .

Answers Answers and Explanations

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

28.1 Vectors in the Plane

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

PRACTICE · VECTORS IN THE PLANE

1. C. End minus start: $\langle 4 - (-2), -1 - 3 \rangle = \langle 6, -4 \rangle$. Choice A reverses the order, and B adds the coordinates.
2. B. $\sqrt{25 + 144} = \sqrt{169} = 13$. Choice A adds the components, and D subtracts the squares.
3. A. $|\langle 8, -6 \rangle| = 10$, so divide each component by 10: $\langle \frac{4}{5}, -\frac{3}{5} \rangle$. Choice C loses a sign.
4. D. $3\mathbf{u} = \langle 6, -3 \rangle$. Subtract: $\langle 6 - (-4), -3 - 5 \rangle = \langle 10, -8 \rangle$. Choice C forgets that subtracting -4 adds 4.
5. D. $3 \cdot 2 + (-4) \cdot 5 = 6 - 20 = -14$. Choice B is not a number. The dot product must be a scalar.
6. C. Set the dot product equal to 0: $2k - 18 = 0$, so $k = 9$. Choice A has the wrong sign.
7. B. The dot product is 6. The lengths are $2\sqrt{2}$ and 3, so $\cos \theta = \frac{6}{6\sqrt{2}} = \frac{1}{\sqrt{2}}$ and $\theta = 45^\circ$. Choice D would need a zero dot product.
8. A. The velocity is $\langle 3, 4 \rangle$, with magnitude $\sqrt{9 + 16} = 5$ m/s. Choice D adds the speeds, which only works if both point the same way.
9. B. $x = 10 \cos 150^\circ = -5\sqrt{3}$ and $y = 10 \sin 150^\circ = 5$. The angle is in Quadrant II, so $x < 0$ and $y > 0$. Choice C swaps sine and cosine.
10. Magnitude 8, direction 120° . $\sqrt{16 + 48} = 8$. $\tan \theta = -\sqrt{3}$ and the vector is in Quadrant II, so $\theta = 180^\circ - 60^\circ = 120^\circ$.
11. $\sqrt{122} \approx 11.05$. $2\mathbf{u} + \mathbf{v} = \langle -4 + 3, 10 + 1 \rangle = \langle -1, 11 \rangle$, and $\sqrt{1 + 121} = \sqrt{122}$.
12. 81.9° . The dot product is $3 - 2 = 1$. $|\mathbf{u}| = \sqrt{5}$, $|\mathbf{v}| = \sqrt{10}$, so $\cos \theta = \frac{1}{\sqrt{50}} \approx 0.1414$ and $\theta \approx 81.9^\circ$.
13. 304.1 mph, 9.5° east of north. The ground velocity is $\langle 50, 300 \rangle$ (east, north). Speed $\sqrt{2500 + 90000} = \sqrt{92500} \approx 304.1$. Angle $\tan^{-1} \frac{50}{300} \approx 9.5^\circ$.
14. 2298 J. $\mathbf{F} = \langle 200 \cos 40^\circ, 200 \sin 40^\circ \rangle$ and $\mathbf{d} = \langle 15, 0 \rangle$. Work $= \mathbf{F} \cdot \mathbf{d} = 200(15) \cos 40^\circ = 3000(0.7660) \approx 2298$ J.
15. $\langle 6, 8 \rangle$ and $\langle -6, -8 \rangle$. A perpendicular vector is $\langle 3, 4 \rangle$ (dot product $12 - 12 = 0$), which has length 5. Scale it by 2 to get length 10, and include the opposite direction.

28.2 Matrix Operations

PRACTICE · MATRIX OPERATIONS

1. D. The inner numbers 2 and 2 match, and the outer numbers give the size: 3×5 . Choice C reverses the order.
2. A. Add entry by entry: $1 + 4 = 5$, $-2 + 1 = -1$, $3 - 5 = -2$, $0 + 2 = 2$. Choice C subtracts, and D multiplies.
3. B. $2A = \begin{bmatrix} 6 & 2 \\ 0 & -4 \end{bmatrix}$. Subtract $B: \begin{bmatrix} 5 & -2 \\ -2 & -3 \end{bmatrix}$. Choice A adds B instead of subtracting.
4. C. BC would be $(3 \times 4)(2 \times 4)$ and the inner numbers 4 and 2 differ. The others are $(2 \times 3)(3 \times 4)$, $(2 \times 4)(4 \times 2)$, and $(4 \times 2)(2 \times 3)$, all defined.
5. B. Row 2 of the first times column 1 of the second: $3 \cdot 5 + 4 \cdot 7 = 15 + 28 = 43$. Choice A is the row 1, column 1 entry.
6. A. Row 1: $[2 \cdot 1 + 0 \cdot 2, 2 \cdot 4 + 0 \cdot (-1)] = [2, 8]$. Row 2: $[1 \cdot 1 + 3 \cdot 2, 1 \cdot 4 + 3 \cdot (-1)] = [7, 1]$. Choice C multiplies entries, and B is BA .
7. D. $2X = \begin{bmatrix} 4 & 4 \\ 0 & 8 \end{bmatrix}$, so $X = \begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix}$. Choice A forgets to divide by 2.
8. C. $[12 \ 8] \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 12 \cdot 4 + 8 \cdot 6 = 48 + 48 = 96$. Choice B adds the quantities and ignores the prices.
9. D. $A^2 = A \cdot A$. Row 1: $[1 + 6, 2 + 8] = [7, 10]$. Row 2: $[3 + 12, 6 + 16] = [15, 22]$. Choice A squares each entry, and C is $2A$.
10. $\begin{bmatrix} 1 & 8 \\ -2 & 13 \end{bmatrix}$. Row 1: $2 \cdot 1 + 0 \cdot 0 + 1 \cdot (-1) = 1$ and $2 \cdot 2 + 0 \cdot 1 + 1 \cdot 4 = 8$. Row 2: $1 \cdot 1 + (-1) \cdot 0 + 3 \cdot (-1) = -2$ and $1 \cdot 2 + (-1) \cdot 1 + 3 \cdot 4 = 13$.
11. $x = 2$, $y = 5$. Top left: $x + 3 = 5$. Top right: $2 + y = 7$. Check the bottom row: $1 + x = 3$ and $y + 4 = 9$.
12. $K^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$ and $K^4 = I$. K^2 : row 1 is $[0 \cdot 0 + 1 \cdot (-1), 0 \cdot 1 + 1 \cdot 0] = [-1, 0]$ and row 2 is $[0, -1]$. Then $K^4 = (K^2)^2 = (-I)^2 = I$.
13. Week 1: 10.60 dollars; Week 2: 11.60 dollars. Week 1: $10(0.5) + 6(0.4) + 4(0.8) = 5 + 2.4 + 3.2 = 10.6$. Week 2: $8(0.5) + 9(0.4) + 5(0.8) = 4 + 3.6 + 4 = 11.6$.
14. $\begin{bmatrix} 4 & 2 \\ 4 & -4 \end{bmatrix}$. $AB = \begin{bmatrix} 6 & 0 \\ 12 & 6 \end{bmatrix}$ and $BA = \begin{bmatrix} 2 & -2 \\ 8 & 10 \end{bmatrix}$. Subtract entry by entry.
15. $x = 2$, $y = 3$. The product is $\begin{bmatrix} y+x & 2x \\ 2y+3 & 6 \end{bmatrix}$. Then $2x = 4$ gives $x = 2$, and $y + x = 5$ gives $y = 3$. Check: $2y + 3 = 9$.

28.3 Determinants and Inverses

PRACTICE · DETERMINANTS AND INVERSES

1. B. $4 \cdot 2 - (-1)(3) = 8 + 3 = 11$. Choice A forgets that subtracting -3 adds 3.
2. C. $6 \cdot 3 - 9 \cdot 2 = 18 - 18 = 0$. A zero determinant means the matrix is singular. No matrix, zero or not, is its inverse.
3. D. $\det = 6 - 5 = 1$. Swap 2 and 3, negate 1 and 5: $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$. Choice B negates correctly but does not swap the diagonal.
4. A. Expand along row 1: $1(1 \cdot 1 - 0 \cdot 2) - 0 + 2(3 \cdot 2 - 1 \cdot 0) = 1 + 12 = 13$. Choice B forgets the third term.
5. A. $\det \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = -1$, so the inverse is $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$. Then $X = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Check: $2 + 2 = 4$ and $6 + 5 = 11$.
6. C. Set $k^2 - 16 = 0$, so $k = \pm 4$. Choice A misses the negative root, and D forgets the square root.
7. C. Vectors from $(1, 1)$: $\langle 4, 1 \rangle$ and $\langle 2, 5 \rangle$. $\det = 20 - 2 = 18$. The triangle is half: 9. Choice A forgets the half.
8. D. Swapping two rows changes the sign of the determinant, so the value is 2. Direct check: $4 \cdot 3 - 5 \cdot 2 = 2$.
9. B. Multiply both sides on the left by A^{-1} : $X = A^{-1}B$. Choice A is not even defined because the dimensions do not match.

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →

10. $\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{7}{2} & \frac{3}{2} \end{bmatrix}$. $\det = 9 - 7 = 2$. Swap the diagonal, negate the other entries, divide by 2.
11. $x = 1, y = 2$. $\det = 8 - 6 = 2$ and $A^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$. Then $X = \frac{1}{2} \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Check: $4 + 6 = 10, 2 + 4 = 6$.
12. -23. Row 1: $2(3 \cdot (-1) - 2 \cdot 4) - (-1)(1 \cdot (-1) - 2 \cdot 0) + 0 = 2(-11) + 1(-1) = -22 - 1 = -23$.
13. $k = 6$ or $k = -1$. Set $k(k - 5) - 6 = 0$: $k^2 - 5k - 6 = 0, (k - 6)(k + 1) = 0$.
14. 13 square units. Vectors from $(2, 1)$: $\langle 4, 2 \rangle$ and $\langle 1, 7 \rangle$. $\det = 28 - 2 = 26$. Half of 26 is 13.
15. $m = 6$; no solution. The determinant is $m - 6$, which is 0 when $m = 6$. Then the second equation is $3x + 6y = 5$, while three times the first is $3x + 6y = 9$. Those cannot both hold, so the lines are parallel and distinct.

28.4 Matrices as Transformations and Data

PRACTICE · MATRICES AS TRANSFORMATIONS AND DATA

1. B. A 90° counterclockwise turn sends (x, y) to $(-y, x)$, so $(2, 3) \rightarrow (-3, 2)$. Choice A is the clockwise image.
2. D. Reflecting across the y -axis sends (x, y) to $(-x, y)$, which is multiplication by $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Choice A reflects across the x -axis.
3. C. $\det = 9$, so the area is multiplied by 9: $5 \cdot 9 = 45$. Choice A multiplies by the scale factor 3 instead of by 3^2 .
4. A. $\det = 1 \cdot 1 - 2 \cdot 3 = -5$. The area factor is $|-5| = 5$, and a negative determinant means the figure is flipped. Choice B: an area cannot be negative.
5. B. The images of the unit vectors are the **columns**: first column $(3, 1)$, second column $(-2, 4)$. Choice A writes them as rows.
6. D. Reflection across $y = x$ swaps the coordinates: $(5, -2) \rightarrow (-2, 5)$. Choice C is the reflection across $y = -x$.
7. A. Rotating 180° gives $(-3, -4)$. Reflecting across the x -axis changes the sign of y : $(-3, 4)$. Choice B stops after the first step.
8. C. Group 1 keeps $0.9 \cdot 300 = 270$ and gains $0.2 \cdot 200 = 40$: 310. Choice A uses the wrong fractions (it uses 0.1 for group 2 moving into group 1).
9. B. Each column must add to 1. In B the columns add to 1.0 and 1.0. In C only the rows add to 1, and in A and D a column does not.
10. Matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, image $(2, -5)$. A half turn changes the sign of both coordinates.
11. Vertices $(0, 0)$, $(8, 0)$, $(3, 9)$. Area 36. The area of the original is 6 and $\det = 6$, so the new area is 36. Check: $\frac{1}{2} \cdot 8 \cdot 9 = 36$.
12. Matrix $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$; image $(3, -6)$. The dilation matrix $3I$ times the reflection matrix gives $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$. Check: reflect $(1, 2)$ to $(1, -2)$, then dilate to $(3, -6)$.
13. Bus 278, train 222. Week 1: bus $0.7(200) + 0.4(300) = 260$, train $0.3(200) + 0.6(300) = 240$. Week 2: bus $0.7(260) + 0.4(240) = 278$, train $0.3(260) + 0.6(240) = 222$.
14. $\frac{3}{5}$, or 60%. In the steady state the flows balance: $0.2a = 0.3b$. With $a + b = 1$, $0.2a = 0.3(1 - a)$, so $0.5a = 0.3$ and $a = 0.6$.
15. $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$. Adding the two equations: $M(2, 0) = (4, 4)$, so $M(1, 0) = (2, 2)$. Subtracting: $M(0, 2) = (2, 6)$, so $M(0, 1) = (1, 3)$. The columns are $(2, 2)$ and $(1, 3)$.

Want a plan built around **your** student?

Book a free consultation. The short intake takes about a minute and opens right away.

BOOK NOW →