



Counting, Probability, and Combinatorics

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

In this booster

29.1 Counting Principles, Permutations, and Combinations

29.2 Probability Rules

29.3 Expected Value and Binomial Probability

29.4 Probability Puzzles and Common Misconceptions

Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

29.1 Counting Principles, Permutations, and Combinations

Learn to count outcomes without listing them. Master the multiplication principle, factorials, permutations, combinations, and arrangements with repeated items, and learn how to decide which tool a problem needs.

Why This Matters

Counting sounds like the easiest topic in math, yet it is where many strong students stumble. The reason is that counting problems rarely come with a formula attached. You have to read a story, decide what is being chosen, decide whether the order of the choices matters, and only then pick a tool. A wrong decision at that point gives a clean, confident, wrong answer.

This skill is used everywhere. Computer scientists count passwords and possible network paths. Biologists count gene arrangements. Game designers count possible hands and boards. Every probability you will ever compute begins with counting how many outcomes are possible and how many are favorable. The most common struggles are mixing up permutations and combinations, forgetting to divide out repeated items, and adding when you should multiply. This section gives you a clear decision process so you stop guessing.

Core Ideas

Start with the simplest idea in counting. When a task happens in stages, you multiply the number of choices at each stage. Picture a tree: each branch splits again at the next stage, and the number of ends of the tree is the product of the splits.

The Multiplication Principle

If a task has k stages, and stage 1 can be done in n_1 ways, stage 2 in n_2 ways (no matter how stage 1 went), and so on, then the whole task can be done in

$$n_1 \cdot n_2 \cdots n_k$$

ways.

Use it when you see the word "and": choose a shirt **and** a pair of pants **and** shoes. If the problem says "or" with no overlap (a vowel **or** a consonant), you add the counts instead.

Many stages involve placing distinct things in a line, one after another. The number of ways to line up n different objects is $n \times (n-1) \times \cdots \times 2 \times 1$, because you have n choices for the first spot, then $n-1$ for the next, and so on. This product is so common that it has its own name and symbol.

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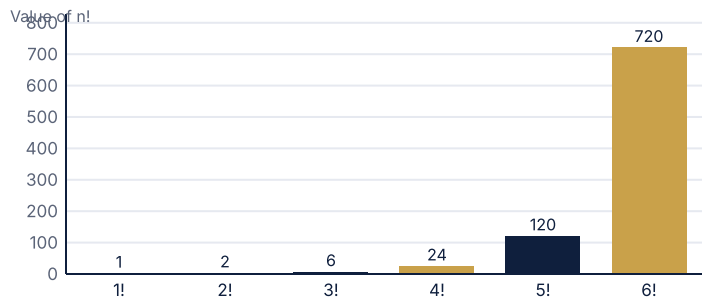
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Factorial

For a whole number $n \geq 1$,

$$n! = n(n - 1)(n - 2) \cdots 2 \cdot 1, \quad 0! = 1.$$

The number of different ways to arrange n distinct objects in a row is $n!$. Factorials grow very fast, as the chart shows.



Factorials explode. Six objects already have 720 orderings, and ten objects have more than three million. Often you do not line up all of the objects. You pick only r of n and place them in order, like choosing a first, second, and third place winner from a race. That is a **permutation**. If you pick r of n and do not care about order, like choosing a team, that is a **combination**. The only difference is whether swapping two chosen items gives a new outcome.

Permutations and Combinations

Order matters (an arrangement of r items chosen from n distinct items):

$$P(n, r) = \frac{n!}{(n - r)!}.$$

Order does not matter (a group of r items chosen from n distinct items):

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n - r)!}.$$

Connection: $P(n, r) = r! \cdot C(n, r)$. Every group of r items can be ordered in $r!$ ways, so permutations are combinations multiplied by $r!$.

Question to ask	Order matters (permutation)	Order does not matter (combination)
Typical words	arrange, line up, rank, first/second/third, code, password	choose, select, committee, team, hand of cards, subset
Swap two chosen items	Gives a different outcome	Gives the same outcome
Formula	$\frac{n!}{(n - r)!}$	$\frac{n!}{r!(n - r)!}$
Example, 5 choose 3	$P(5, 3) = 60$	$C(5, 3) = 10$

Notice that $C(5, 3) = 10$ and $P(5, 3) = 60$, and $60 = 3! \times 10$. The 10 groups each produce $3! = 6$ different orderings. Combinations also have a handy symmetry: $C(n, r) = C(n, n - r)$, because choosing which r items to take is the same as choosing which $n - r$ items to leave behind. So $C(10, 8) = C(10, 2) = 45$.

Arrangements with Repeated Items

If you arrange n objects in a row and n_1 of them are identical of one kind, n_2 identical of another kind, and so on, the number of distinct arrangements is

$$\frac{n!}{n_1! n_2! \cdots}.$$

You divide by $n_i!$ because swapping identical items does not create a new arrangement.

Two more habits will save you. First, when a problem has a restriction (the first digit cannot be zero, two people must sit together), handle the restricted choice **first**, then count the rest. Second, when a problem says "at least" or "not", count the opposite and subtract from the total. You will see both habits in the examples.

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Worked Examples

EXAMPLE 1: MULTIPLICATION PRINCIPLE

A school cafe offers 4 sandwiches, 3 soups, and 2 drinks. A lunch special is one of each. How many different specials are possible?

SOLUTION

Three independent stages: sandwich, then soup, then drink.

Multiply the choices: $4 \cdot 3 \cdot 2 = 24$.

There are **24** different specials.

EXAMPLE 2: PERMUTATION

Eight runners compete in a final. How many ways can the gold, silver, and bronze medals be awarded?

SOLUTION

Order matters because gold, silver, and bronze are different prizes.

$$P(8, 3) = 8 \cdot 7 \cdot 6 = 336.$$

Check by stages: 8 choices for gold, 7 left for silver, 6 left for bronze. There are **336** ways.

EXAMPLE 3: COMBINATION

A club has 9 members. How many different 3-person committees can be formed?

SOLUTION

A committee has no ranking, so order does not matter. Use a combination.

$$C(9, 3) = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = \frac{504}{6} = 84.$$

There are **84** committees.

EXAMPLE 4: REPEATED LETTERS

How many distinct arrangements are there of the letters in the word BALLOON?

SOLUTION

BALLOON has 7 letters. The L appears twice and the O appears twice; B, A, N appear once.

$$\text{Distinct arrangements} = \frac{7!}{2!2!} = \frac{5040}{4} = 1260.$$

There are **1260** distinct arrangements.

EXAMPLE 5: SEVERAL GROUPS

A class has 6 girls and 7 boys. A team needs 2 girls and 3 boys. How many teams are possible?

SOLUTION

Choosing girls **and** choosing boys are two stages, so multiply. Within each stage order does not matter.

$$\text{Girls: } C(6, 2) = 15. \text{ Boys: } C(7, 3) = 35.$$

$$\text{Teams: } 15 \cdot 35 = \mathbf{525}.$$

EXAMPLE 6: A RESTRICTION

How many 4-digit numbers have no repeated digit and are even? (The first digit cannot be 0.)

SOLUTION

Handle the restricted spots first. Split by the last digit.

Case 1, last digit 0: the other three spots use any of the remaining 9 digits in order: $9 \cdot 8 \cdot 7 = 504$.

Case 2, last digit is 2, 4, 6, or 8 (4 choices): the first digit cannot be 0 or the digit used, so 8 choices; the middle two spots then have 8 and 7 choices: $4 \cdot 8 \cdot 8 \cdot 7 = 1792$.

Add the cases, since they cannot happen together: $504 + 1792 = \mathbf{2296}$.

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n	P(n,2)	C(n,2)	Ratio
4	12	6	2
5	20	10	2
6	30	15	2
7	42	21	2

Coach's Playbook

- 1 Name the task.** Say in words what is being built: a line, a group, a code. Decide the number of stages.
- 2 Order test.** Imagine two outcomes that contain the same items in a different order. Are they the same result? If yes, use a combination. If no, use a permutation or the multiplication principle.
- 3 Repeated items?** If some objects are identical, divide by the factorial of each repeat count.
- 4 Restrictions first.** Fill the most restricted position before the free ones.
- 5 Check size.** A combination can never exceed the matching permutation, and a count of arrangements of n objects can never exceed $n!$.
 - Cancel before you multiply. $\frac{12!}{9!} = 12 \cdot 11 \cdot 10$, so never compute the full factorials.
 - Use the symmetry $C(n, r) = C(n, n - r)$ to keep numbers small.
 - For "at least one" problems, compute total minus the count with none.
 - Small test: if a formula is unclear, try it on 3 or 4 objects where you can list everything.

Common traps

1. Using a permutation when the group has no order (committees, hands of cards). 2. Forgetting to divide by $n_i!$ when letters or items repeat. 3. Adding the stages when you should multiply, or multiplying when the cases are separate and should be added. 4. Counting a restricted spot last, which lets a forbidden value (like 0 in the leading digit) sneak in.

Mistake Clinic

Spot the error 1

A student says: "Ten runners, and we award first, second, and third place. That is $C(10, 3) = 120$ ways." **Where is the error?** The student used a combination, but gold, silver, and bronze are different prizes, so order matters. The runner who wins gold is a different outcome from the runner who finishes third. Correct: $P(10, 3) = 10 \cdot 9 \cdot 8 = 720$. Notice $720 = 3! \times 120$, which is exactly the orderings the student ignored.

Spot the error 2

A student says: "BANANA has 6 letters, so there are $6! = 720$ arrangements." **Where is the error?** BANANA has three A's and two N's. Swapping two A's does not create a new word, but $6!$ counts it as new. Correct: $\frac{6!}{3!2!} = \frac{720}{12} = 60$ distinct arrangements.

Spot the error 3

A student says: "I must choose 3 of 7 electives and 2 of 5 clubs, so I add: $C(7, 3) + C(5, 2) = 35 + 10 = 45$." **Where is the error?** Choosing electives **and** clubs are two stages that both happen, so the counts multiply. Correct: $35 \cdot 10 = 350$ possible schedules.

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Real-World Connection

A phone passcode has 4 digits and digits may repeat. That is $10 \cdot 10 \cdot 10 \cdot 10 = 10,000$ codes. If a system forbids repeats, the count drops to $P(10, 4) = 5040$, about half. Forbidding repeats looks like extra security but it actually makes guessing easier, because there are fewer codes to try.

Challenge Corner

Four boys and four girls stand in a line for a photo. In how many ways can they line up so that no two boys stand next to each other? **Solution.** Place the four girls first in $4! = 24$ ways. They create 5 gaps (before, between, and after them). The four boys must go in four different gaps, and they are different people, so choose and order the gaps: $P(5, 4) = 120$. Total: $24 \cdot 120 = 2880$ lineups. (Checked by a computer count.)

Practice**PRACTICE Counting Principles, Permutations, and Combinations**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- A cafe offers 5 sandwiches, 4 soups, and 3 drinks. A meal is one sandwich, one soup, and one drink. How many meals are possible?
(A) 12 (B) 60 (C) 20 (D) 120
- Evaluate $\frac{9!}{7!2!}$.
(A) 72 (B) 18 (C) 36 (D) 144
- How many ways can a president, a vice president, and a secretary be chosen from 7 candidates (no one holds two jobs)?
(A) 210 (B) 35 (C) 343 (D) 21
- A coach picks 4 players from a club of 12 to travel to a meet. Order does not matter. How many different groups are possible?
(A) 11880 (B) 495 (C) 48 (D) 20736
- How many distinct arrangements are there of the letters in LEVEL?
(A) 30 (B) 120 (C) 60 (D) 10
- Six friends sit in a row of six seats. Two of them, Mia and Jo, insist on sitting next to each other. How many seatings are possible?
(A) 120 (B) 240 (C) 720 (D) 480
- A committee of 4 is chosen from 5 seniors and 6 juniors. How many committees have exactly 2 seniors?
(A) 150 (B) 330 (C) 25 (D) 300
- A plate has 2 different letters followed by 3 different digits. How many plates are possible?
(A) 676000 (B) 1370 (C) 468000 (D) 650000
- For what value of n is $C(n, 2) = 45$?
(A) 9 (B) 10 (C) 11 (D) 90
- In how many ways can 4 books be chosen from 8 different books and placed in order on a shelf?
- How many distinct arrangements are there of the letters in STATISTICS?
- A student must choose 3 of 7 electives and 2 of 5 clubs. How many different choices are possible?
- At a meeting, each of 15 people shakes hands once with every other person. How many handshakes occur?
- How many 5-digit numbers (the first digit cannot be 0, digits may repeat) are multiples of 5?
- A committee of 5 is chosen from 6 men and 4 women. How many committees include at least one woman?

29.2 Probability Rules

Learn the core probability rules: complements, the addition rule, conditional probability, the multiplication rule, and independence. Use two-way tables to handle Bayes-style problems with confidence.

Why This Matters

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Probability measures how likely something is, on a scale from 0 (impossible) to 1 (certain). Weather forecasts, medical tests, insurance prices, and sports statistics all rest on a handful of probability rules. Students usually get stuck for three reasons: they cannot tell when to add and when to multiply, they treat "and" and "given that" as the same thing, and they confuse independent events with mutually exclusive events.

The rules in this section are the toolbox for the rest of probability. If you can use a two-way table and the multiplication rule comfortably, you can solve nearly every textbook problem, and you can reason about everyday claims such as "the test is 95 percent accurate, so I probably have the disease." That claim is usually wrong, and you will see exactly why.

Core Ideas

When all outcomes are equally likely, the probability of an event A is the number of favorable outcomes divided by the number of possible outcomes. Counting tools from the last section often supply those numbers. Two basic facts follow: every probability lies between 0 and 1, and the probability that A does **not** happen is one minus the probability that it does.

Complement Rule

$$P(\text{not } A) = 1 - P(A).$$

If the chance of rain is 0.35, the chance of no rain is 0.65.

To find the chance that **at least one of two** events happens, add their probabilities, but remember that outcomes in both events were counted twice. Subtract the overlap once.

Addition Rule

For any two events A and B ,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

If A and B are **mutually exclusive** (they cannot both happen), then $P(A \text{ and } B) = 0$ and the rule becomes $P(A) + P(B)$.

Next comes the idea that information changes probabilities. The **conditional probability** $P(A | B)$ is the probability of A given that B is already known to have happened. In a table, it means you shrink your world to just the B group and ask what fraction of that group is also in A .

Conditional Probability and the Multiplication Rule

$$P(A | B) = \frac{P(A \text{ and } B)}{P(B)}, \quad P(A \text{ and } B) = P(B) \cdot P(A | B).$$

In words: the chance both happen is the chance the first happens times the chance the second happens **given** the first happened. This is the rule for drawing without replacement.

Independent Events

Events A and B are **independent** if knowing one happened does not change the probability of the other:

$$P(A | B) = P(A) \iff P(A \text{ and } B) = P(A) \cdot P(B).$$

Independent is not the same as mutually exclusive. If two events with positive probability are mutually exclusive, they are actually dependent, because if one happens the other cannot.

A two-way table makes all of these rules visible. Here are 200 students sorted by whether they play a sport and whether they play a musical instrument.

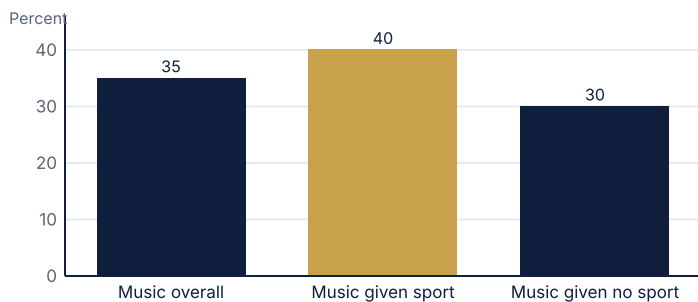
	Plays a sport	No sport	Total
Plays music	40	30	70
No music	60	70	130
Total	100	100	200

From the table: $P(\text{music}) = \frac{70}{200} = 0.35$, $P(\text{sport}) = \frac{100}{200} = 0.5$, and $P(\text{music and sport}) = \frac{40}{200} = 0.2$. Given that a student plays a sport, the chance they also play music is $\frac{40}{100} = 0.4$, which is not 0.35. The two events are not independent: knowing a student plays a sport raises the chance they play music. Check with the product rule: $0.5 \times 0.35 = 0.175 \neq 0.2$.

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Playing a sport changes the chance of playing music, so the events are dependent.

One more habit pays off in probability: write the event in plain words before you touch a formula. "At least one" means one or more. "Exactly one" means one and no more. "Neither" means not the first and not the second. A single word can change the answer, and the translation takes only a few seconds. Also decide whether the situation is a single trial or a sequence. A sequence usually calls for the multiplication rule, and you must ask whether each step changes what is left.

Finally, a word on tree diagrams. A tree lists the first stage as branches, then draws new branches from each end for the second stage. Multiply along a path to get the probability of that whole path, and add the paths that belong to your event. The Bayes-style calculation in Example 6 is exactly this: add two paths to find the total chance of a flag, then ask what share of that total comes from the defective path.

Worked Examples

EXAMPLE 1: ADDITION RULE WITH OVERLAP

A fair die is rolled. What is the probability that the result is even or greater than 4?

SOLUTION

Even: {2, 4, 6}. Greater than 4: {5, 6}. The outcome 6 is in both.

$$P = \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}.$$

Check by listing: {2, 4, 5, 6} has 4 outcomes, so $\frac{4}{6} = \frac{2}{3}$.

EXAMPLE 2: CARDS

One card is drawn from a standard 52-card deck. Find the probability that it is a heart or a face card (jack, queen, king).

SOLUTION

Hearts: 13 cards. Face cards: 12 cards. Overlap (heart face cards): 3.

$$P = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52} = \frac{11}{26}.$$

EXAMPLE 3: CONDITIONAL PROBABILITY FROM A TABLE

Use the table of 200 students. Given that a student plays a sport, what is the probability that the student also plays music?

SOLUTION

Restrict to the 100 students who play a sport.

Of those, 40 play music.

$$P(\text{music} \mid \text{sport}) = \frac{40}{100} = \frac{2}{5}.$$

EXAMPLE 4: MULTIPLICATION RULE, NO REPLACEMENT

A bag holds 5 red and 3 blue marbles. Two are drawn without replacement. Find the probability that both are red.

SOLUTION

First draw red: $\frac{5}{8}$. After removing one red, 4 red remain among 7 marbles.

$$P = \frac{5}{8} \cdot \frac{4}{7} = \frac{20}{56} = \frac{5}{14}.$$

If the first marble had been put back, the answer would have been $\frac{5}{8} \cdot \frac{5}{8}$. Without replacement the second probability changes.

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EXAMPLE 5: TESTING INDEPENDENCE

Are playing a sport and playing music independent for the 200 students?

SOLUTION

Independence means $P(\text{sport and music}) = P(\text{sport})P(\text{music})$.

Left side: $\frac{40}{200} = 0.20$. Right side: $0.5 \cdot 0.35 = 0.175$.

They are not equal, so the events are **not independent**.

EXAMPLE 6: A BAYES-STYLE PROBLEM

A factory makes devices. 3 percent are defective. A scanner flags 95 percent of defective devices and also wrongly flags 4 percent of good ones. A device is flagged. What is the probability it is actually defective?

SOLUTION

Imagine 10,000 devices. Defective: 300. Good: 9700.

Flagged defective: $0.95 \cdot 300 = 285$. Flagged good: $0.04 \cdot 9700 = 388$.

Total flagged: $285 + 388 = 673$.

$P(\text{defective} \mid \text{flagged}) = \frac{285}{673} \approx 0.424$.

Even with a 95 percent detection rate, a flagged device is defective less than half the time, because good devices are so much more common.

	Flagged	Not flagged	Total
Defective	285	15	300
Good	388	9312	9700
Total	673	9327	10000

Coach's Playbook

- 1 Draw it.** Build a two-way table or a tree. Use a fake population of 1000 or 10000 people to turn percents into counts.
 - 2 Find the denominator.** For "given that", the denominator is the group you are told about, not the whole population.
 - 3 And, or, given.** "And" usually multiplies (with a conditional if needed). "Or" adds and subtracts the overlap. "Given" divides by the condition.
 - 4 Without replacement?** If items are not returned, the second probability uses the smaller total.
 - 5 Sanity check.** Every probability must be between 0 and 1, and $P(A \text{ or } B)$ must be at least as big as each of $P(A)$ and $P(B)$.
- ◆ Say it aloud: "out of those who ...". This points you to the right denominator.
 - ◆ To test independence, compute both $P(A)P(B)$ and $P(A \text{ and } B)$ and compare.
 - ◆ When a problem gives rates (95 percent, 4 percent), convert to counts before reasoning.
 - ◆ Use $1 - P$ when the complement is easier to count.

Common traps

1. Adding probabilities of overlapping events without subtracting the overlap. 2. Confusing $P(A \mid B)$ with $P(B \mid A)$; they usually differ. 3. Thinking mutually exclusive events are independent. 4. Forgetting that the second draw uses a smaller total when there is no replacement.

Mistake Clinic

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Spot the error 1

A student says: "The probability of drawing a heart or a king is $\frac{13}{52} + \frac{4}{52} = \frac{17}{52}$." **Where is the error?** The king of hearts is in both groups and was counted twice. Correct: $\frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$.

Spot the error 2

A student says: "Drawing two cards without replacement, the chance both are aces is $\frac{4}{52} \cdot \frac{4}{52}$." **Where is the error?** After one ace is removed, only 3 aces remain among 51 cards. Correct: $\frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$.

Spot the error 3

A student says: "Events A and B cannot both happen, so they are independent, and $P(A \text{ and } B) = 0.3 \cdot 0.4 = 0.12$."

Where is the error? If they cannot both happen, then $P(A \text{ and } B) = 0$, not 0.12. Since $0 \neq 0.12$, they are actually dependent. Mutually exclusive and independent are different ideas.

Real-World Connection

A health screening has a low false-alarm rate, yet most positive results can still be false alarms when the condition is rare. Doctors use the same table method as Example 6 to explain results, and then order a second test to separate the true cases from the false ones.

Challenge Corner

A box holds 5 red and 3 green balls. Three are drawn without replacement. Find the probability that exactly 2 are red.

Solution. Counting method: $\frac{C(5,2)C(3,1)}{C(8,3)} = \frac{10 \cdot 3}{56} = \frac{15}{28}$. Sequence method: the order RRG, RGR, GRR each has probability $\frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} = \frac{5}{28}$, and $3 \cdot \frac{5}{28} = \frac{15}{28}$. Both methods agree.

Practice**PRACTICE Probability Rules**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- The probability of rain tomorrow is 0.35. What is the probability that it does not rain?
(A) 0.65 (B) 0.35 (C) 0.55 (D) 1.35
- Events A and B are mutually exclusive with $P(A) = 0.5$ and $P(B) = 0.3$. Find $P(A \text{ or } B)$.
(A) 0.15 (B) 0.8 (C) 0.65 (D) 0.2
- If $P(A) = 0.6$, $P(B) = 0.5$, and $P(A \text{ and } B) = 0.3$, find $P(A \text{ or } B)$.
(A) 1.1 (B) 0.3 (C) 0.8 (D) 0.2
- Two cards are drawn from a standard deck without replacement. What is the probability both are aces?
(A) $\frac{1}{169}$ (B) $\frac{1}{13}$ (C) $\frac{1}{26}$ (D) $\frac{1}{221}$
- If $P(A \text{ and } B) = 0.12$ and $P(B) = 0.4$, find $P(A | B)$.
(A) 0.3 (B) 0.048 (C) 0.28 (D) 0.52
- Events A and B are independent with $P(A) = 0.4$ and $P(B) = 0.25$. What is $P(A \text{ and } B)$?
(A) 0.65 (B) 0.1 (C) 0.15 (D) 0.4

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7. Sixty students are sorted by pet ownership and how they get to school (table below). Given that a student has a pet, what is the probability the student walks?

	Walks	Takes the bus
Has a pet	12	18
No pet	14	16

- (A) $\frac{1}{5}$ (B) $\frac{6}{13}$ (C) $\frac{2}{5}$ (D) $\frac{1}{2}$

8. For events A and B , $P(A) = 0.5$, $P(B) = 0.4$, and $P(A \text{ or } B) = 0.7$. Which statement is true?

- (A) A and B are mutually exclusive
 (B) A and B are independent but not mutually exclusive
 (C) A and B are both independent and mutually exclusive
 (D) A and B are neither independent nor mutually exclusive

9. Bag 1 has 3 red and 1 blue marble. Bag 2 has 1 red and 3 blue. A bag is chosen at random and one marble is drawn; it is red. What is the probability it came from Bag 1?

- (A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{3}{8}$ (D) $\frac{1}{4}$

10. Two fair dice are rolled. Find the probability that the sum is 9 or the dice show a double.

11. A bag has 4 green and 6 yellow marbles. Three are drawn without replacement. Find the probability that all three are yellow.

12. Using the table of 200 students (music and sport), find the probability that a student plays music given that the student does not play a sport.

	Plays a sport	No sport	Total
Plays music	40	30	70
No music	60	70	130
Total	100	100	200

13. A player makes each free throw with probability 0.8, independently. Find the probability that the player makes exactly one of two shots.
14. A condition affects 2 percent of players. A test is positive for 90 percent of players who have it and for 5 percent of players who do not. Given a positive test, what is the probability the player has the condition? Give a fraction and a decimal to three places.
15. Events A and B are independent, $P(A) = 0.4$, and $P(A \text{ or } B) = 0.76$. Find $P(B)$.

29.3 Expected Value and Binomial Probability

Compute expected value and judge whether a game is fair. Recognize binomial experiments, find exact and at-least probabilities, and expand powers of binomials using Pascal's triangle and the binomial theorem.

Why This Matters

Probability tells you how likely an event is. Expected value goes one step further and answers the question that matters in games, business, and science: **what happens on average if this situation is repeated many times?** Casinos, insurance companies, and game designers all price their products using expected value, which is why it is worth understanding before you hand over money to play a game.

Students usually get stuck in three places. They average the outcomes and forget to weight each by its probability. They use the binomial probability formula but leave out the counting factor. And they find the binomial theorem scary until they notice that its coefficients are just the counts you learned in the last two sections. This section connects all of them.

Core Ideas

A **random variable** X is a number that depends on chance, such as the winnings from one play of a game. Its **expected value** $E(X)$ is the long-run average: multiply each possible value by its probability and add.

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Expected Value

If X takes values $x_1, x_2, \dots, x_k$ with probabilities $p_1, p_2, \dots, p_k$ (which sum to 1), then

$$E(X) = x_1p_1 + x_2p_2 + \dots + x_kp_k.$$

A game is **fair** when the expected net gain is 0. If a game costs c to play, then fair means $c = E(\text{prize})$.

Expected value need not be a possible outcome. A fair die has $E = 3.5$, though no roll shows 3.5. It is the balance point of the distribution, and over thousands of rolls the average gets very close to it.

Many chance experiments repeat the same two-outcome trial. Flip a coin 10 times and count heads. Take a 12-question quiz by guessing and count correct answers. Watch 50 free throws and count made shots. These all share four features, the **binomial conditions**: a fixed number of trials n , each trial has two outcomes (success or failure), the trials are independent, and the success probability p is the same each time.

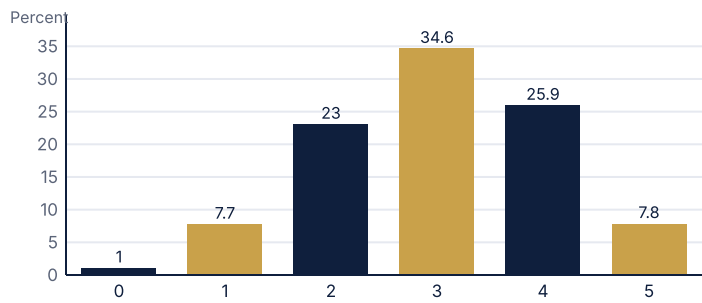
Binomial Probability

If X counts successes in n independent trials, each with success probability p , then

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n - k}, \quad E(X) = np.$$

The factor $\binom{n}{k}$ counts which k of the n trials are the successes. The rest of the formula is the probability of any one specific arrangement.

Here is the whole distribution for $n = 5$ trials with success probability $p = 0.6$. Notice that the probabilities add to 100 percent and peak near $np = 3$.



Number of successes in 5 trials with $p = 0.6$. The most likely result is 3, which equals np .

The numbers $\binom{n}{k}$ appear in a famous pattern. In **Pascal's triangle**, each entry is the sum of the two entries above it, and row n lists $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$.

Row n	Entries
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1

Binomial Theorem

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k.$$

The coefficients come from row n of Pascal's triangle. Powers of a go down from n to 0 while powers of b go up from 0 to n . The general term is $\binom{n}{k} a^{n-k} b^k$. The coefficients in row n add to 2^n .

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Why does it work? When you multiply n copies of $(a + b)$, each term of the result picks either a or b from every factor. The number of ways to pick b from exactly k of the n factors is $\binom{n}{k}$, and that is the coefficient. This is the same counting idea as binomial probability, with $a = 1 - p$ and $b = p$.

Expected value also behaves nicely when you transform a variable. If every prize is doubled, the expected prize doubles. If a fixed fee of c is subtracted from every outcome, the expected net gain drops by exactly c . And the expected value of a sum of two games is the sum of their expected values, even when the games are connected in some way. These facts let you break a complicated situation into small pieces, find each expected value, and add.

A quick way to feel binomial probabilities: the distribution peaks near np and is symmetric when $p = 0.5$. With $n = 10$ and $p = 0.5$, getting exactly 5 heads has probability about 24.6 percent, while 0 heads has probability under 0.1 percent. Results far from the mean are rare, and this is why a streak far from the average is a signal worth investigating.

Worked Examples

EXAMPLE 1: EXPECTED VALUE

A spinner game pays 10 points with probability $\frac{1}{10}$, 2 points with probability $\frac{3}{10}$, and 0 points otherwise. Find the expected number of points per spin.

SOLUTION

$$E = 10 \cdot \frac{1}{10} + 2 \cdot \frac{3}{10} + 0 \cdot \frac{6}{10}.$$

$$E = 1 + 0.6 + 0 = 1.6 \text{ points.}$$

Interpretation: over many spins, the average is about 1.6 points per spin.

EXAMPLE 2: A FAIR PRICE

A carnival game pays 24 dollars with probability $\frac{1}{6}$ and nothing otherwise. What entry fee makes the game fair?

SOLUTION

$$E(\text{prize}) = 24 \cdot \frac{1}{6} = 4 \text{ dollars.}$$

A fair game has expected net gain 0, so the fee equals the expected prize: **4 dollars**.

A fee above 4 dollars favors the host. A fee below favors the player.

EXAMPLE 3: EXACT BINOMIAL PROBABILITY

A basketball player makes 70 percent of free throws, independently. In 5 attempts, find the probability of exactly 3 makes.

SOLUTION

Conditions hold: $n = 5$, $p = 0.7$, $k = 3$.

$$P = \binom{5}{3}(0.7)^3(0.3)^2 = 10 \cdot 0.343 \cdot 0.09.$$

$$P = 0.3087, \text{ about 31 percent.}$$

EXAMPLE 4: AT LEAST

A weak signal reaches a sensor with probability 0.1 each of 5 independent tries. Find the probability of at least 2 successes.

SOLUTION

"At least 2" is the complement of 0 or 1 success.

$$P(0) = 0.9^5 = 0.59049 \text{ and } P(1) = 5(0.1)(0.9)^4 = 0.32805.$$

$$P(\geq 2) = 1 - 0.59049 - 0.32805 = 0.08146, \text{ about 8.1 percent.}$$

EXAMPLE 5: BINOMIAL THEOREM EXPANSION

Expand $(x + 2)^4$.

SOLUTION

Row 4 of Pascal's triangle: 1, 4, 6, 4, 1. Here $a = x$, $b = 2$.

$$x^4 + 4x^3(2) + 6x^2(4) + 4x(8) + 16.$$

$$(x + 2)^4 = x^4 + 8x^3 + 24x^2 + 32x + 16.$$

$$\text{Check at } x = 1: 3^4 = 81 \text{ and } 1 + 8 + 24 + 32 + 16 = 81.$$

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EXAMPLE 6: ONE COEFFICIENT

Find the coefficient of x^3 in $(2x - 1)^6$.

SOLUTION

General term: $\binom{6}{k}(2x)^{6-k}(-1)^k$. For x^3 we need $6 - k = 3$, so $k = 3$.

$$\binom{6}{3}(2x)^3(-1)^3 = 20 \cdot 8x^3 \cdot (-1).$$

The coefficient is **-160**. The negative sign comes from $(-1)^3$.

Outcome	Prize	Probability	Prize x Probability
Top	10	0.1	1.0
Mid	2	0.3	0.6
None	0	0.6	0.0
Total		1.0	1.6

Coach's Playbook

- 1 Make a table.** List every value of X with its probability. Check that the probabilities sum to 1.
 - 2 Weight, then add.** Multiply each value by its own probability. Never just average the values.
 - 3 Check binomial conditions.** Fixed n , two outcomes, independent trials, constant p . Drawing without replacement from a small group fails independence.
 - 4 Use the complement.** For "at least one" or "at least 2", compute $1 - P(\text{too few})$.
 - 5 Test your expansion.** Plug in $x = 1$ to compare with $(a + b)^n$ numerically.
- ◆ Row n of Pascal's triangle gives all the coefficients at once. Build it by adding neighbors.
 - ◆ In a binomial probability, the exponents k and $n - k$ always add to n .
 - ◆ Expected value of a binomial is np . You do not need to sum the distribution.
 - ◆ A negative or alternating sign inside $(a - b)^n$ gives alternating signs in the expansion.

Common traps

1. Averaging the outcomes without weighting by probability. 2. Dropping the $\binom{n}{k}$ factor in a binomial probability. 3. Forgetting to raise the second term (including its coefficient and sign) to the power in the binomial theorem, as in $(2x)^3 = 8x^3$, not $2x^3$. 4. Calling an experiment binomial when the trials are not independent.

Mistake Clinic**Spot the error 1**

A student says: "A game pays 1 dollar with probability 0.6, 5 dollars with probability 0.3, and 10 dollars with probability 0.1. The expected prize is $\frac{1+5+10}{3} \approx 5.33$." **Where is the error?** The student averaged the prizes as if they were equally likely. The cheap prize happens most of the time. Correct: $1(0.6) + 5(0.3) + 10(0.1) = 0.6 + 1.5 + 1 = 3.1$ dollars.

Spot the error 2

A student says: "The probability of exactly 2 successes in 5 trials with $p = 0.3$ is $(0.3)^2(0.7)^3 = 0.0309$." **Where is the error?** That is the probability of one specific order such as SSFFFF. There are $\binom{5}{2} = 10$ orders. Correct: $10(0.3)^2(0.7)^3 = 0.3087$.

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Spot the error 3

A student says: " $(x + 2)^3 = x^3 + 2^3 = x^3 + 8$." **Where is the error?** Powers do not distribute over addition. Use Pascal's row 1, 3, 3, 1: $(x + 2)^3 = x^3 + 3x^2(2) + 3x(4) + 8 = x^3 + 6x^2 + 12x + 8$. Check with $x = 1$: 27 versus $1 + 6 + 12 + 8 = 27$.

Real-World Connection

A phone repair plan costs 20 dollars and covers a repair that costs 150 dollars and happens with probability 0.1. The expected payout is $0.1 \cdot 150 = 15$ dollars, so the plan has expected net gain of $15 - 20 = -5$ dollars for the buyer. People still buy such plans because they want protection against a large, unexpected cost, which is a risk decision rather than an average-value decision.

Challenge Corner

A game flips a fair coin up to 3 times and stops at the first head. If the first head comes on flip k , you win 2^k dollars. If there is no head in three flips, you win nothing. What entry fee makes the game fair? **Solution.** $P(\text{first head on flip } k) = \left(\frac{1}{2}\right)^k$, so $E = 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + 0 \cdot \frac{1}{8} = 1 + 1 + 1 = 3$. The fair fee is **3 dollars**.

Practice**PRACTICE Expected Value and Binomial Probability**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- A variable X takes the values 0, 1, and 2 with probabilities 0.2, 0.5, and 0.3. What is $E(X)$?
(A) 1.0 (B) 1.1 (C) 1.5 (D) 3.0
- A game pays 10 dollars with probability 0.25 and nothing otherwise. What entry fee makes it fair?
(A) 2.50 dollars (B) 10 dollars (C) 7.50 dollars (D) 4 dollars
- In Pascal's triangle, row 6 is 1, 6, 15, 20, 15, 6, 1. What is the sum of the entries in row 6?
(A) 36 (B) 32 (C) 64 (D) 128
- Which of these is a binomial experiment?
(A) The number of heads in 10 flips of a fair coin
(B) The number of cards drawn until the first ace appears
(C) The heights of 10 randomly chosen students
(D) The number of red cards when 5 are dealt from a 10-card deck
- A fair coin is flipped 4 times. What is the probability of exactly 2 heads?
(A) $\frac{1}{16}$ (B) $\frac{1}{2}$ (C) $\frac{3}{8}$ (D) $\frac{1}{4}$
- A binomial experiment has $n = 50$ trials and $p = 0.12$. What is the expected number of successes?
(A) 0.12 (B) 6 (C) 44 (D) 5.28
- Expand $(x + 3)^3$.
(A) $x^3 + 9x^2 + 27x + 27$ (B) $x^3 + 27$ (C) $x^3 + 3x^2 + 9x + 27$
(D) $x^3 + 9x^2 + 9x + 27$
- What is the coefficient of x^2 in the expansion of $(x + 4)^5$?
(A) 160 (B) 80 (C) 640 (D) 1024
- A player makes each of 3 independent shots with probability 0.5. What is the probability of at least one make?
(A) $\frac{1}{8}$ (B) $\frac{3}{8}$ (C) $\frac{1}{2}$ (D) $\frac{7}{8}$
- A spinner gives 5 points with probability $\frac{1}{2}$, 10 points with probability $\frac{1}{4}$, and 20 points with probability $\frac{1}{4}$. Find the expected score per spin.
- A seed sprouts with probability 0.25, independently. Out of 8 seeds, find the probability that exactly 2 sprout. Give your answer to three decimal places.

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12. A raffle sells 500 tickets. One ticket wins 300 dollars and four tickets each win 50 dollars. A ticket costs 1.50 dollars. Find the expected net gain for one ticket.
13. A player makes each free throw with probability 0.6, independently. In 4 attempts, find the probability of making at least 2. Give three decimal places.
14. Find the coefficient of x^3 in the expansion of $(3x - 2)^5$.
15. Each trial succeeds with probability 0.2, independently. What is the smallest number of trials n so that the probability of at least one success is at least 0.9?

29.4 Probability Puzzles and Common Misconceptions

Use the at-least-one rule, understand independence and the gambler's fallacy, compare sampling with and without replacement, and see why the birthday problem surprises almost everyone.

Why This Matters

Probability is full of results that feel wrong. Most people think you need about 180 people in a room before two share a birthday, but 23 is enough. Many people believe a coin that has landed heads five times is "due" for tails, but the coin has no memory. And many people add probabilities when they should use the complement. Puzzles like these are the best way to test whether you really understand probability or are just following recipes.

The same mistakes cost real money and real safety. Gamblers chase losses because of the gambler's fallacy. Engineers who ignore the "at least one failure" calculation underestimate the risk in systems made of many parts. Doctors and jurors misjudge evidence when they confuse one conditional probability with another. This section trains you to slow down, set up the correct model, and trust the math over your gut.

Core Ideas

Three tools solve most puzzles in this family. The first is the **at-least-one rule**. Computing "at least one" directly means adding many cases. Computing "none" is a single product, and then you subtract from 1.

At-Least-One Rule

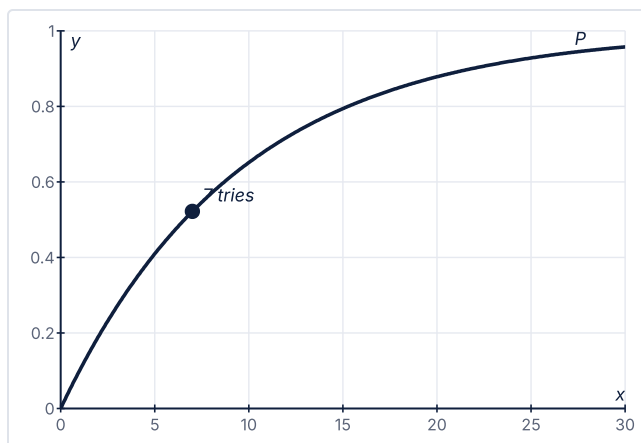
If the events are independent,

$$P(\text{at least one occurs}) = 1 - P(\text{none occur}).$$

For n independent trials with the same success probability p ,

$$P(\text{at least one success}) = 1 - (1 - p)^n.$$

The graph shows how fast this climbs. Even when each trial succeeds only 10 percent of the time, repeating it many times makes at least one success very likely. The curve rises quickly at first and then flattens, but it never quite reaches 1.



Chance of at least one success when $p = 0.1$ per try: $P = 1 - 0.9^n$. After 7 tries the chance is just over one half.

The second tool is **independence and memory**. If trials are independent, the past never changes the next probability. A fair coin has probability $\frac{1}{2}$ of heads on every flip, no matter what happened before. What does shrink is the probability of a long streak **before** it starts: five heads in a row has probability $(\frac{1}{2})^5 = \frac{1}{32}$, but once four are done, the chance that the fifth is heads is still $\frac{1}{2}$.

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The Gambler's Fallacy

For independent trials, $P(\text{next} \mid \text{past results}) = P(\text{next})$.
The **law of large numbers** says the **proportion** of heads gets close to $\frac{1}{2}$ in the long run. It does not say the counts must balance out, and it does not say that a tails is owed. Early streaks get diluted by later flips, not corrected by them.

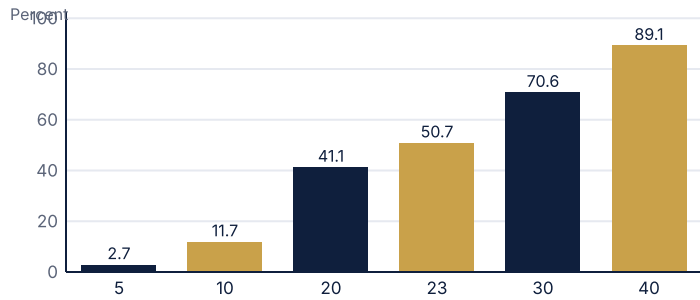
The third tool is the difference between sampling **with replacement** and **without replacement**. With replacement, you put the item back, so the probabilities stay the same and the draws are independent. Without replacement, each draw changes what is left, so you must use conditional probabilities.

With and Without Replacement

With replacement: multiply the same fractions, such as $\frac{3}{8} \cdot \frac{3}{8}$.
Without replacement: reduce the total, and the count of the type drawn, after each draw, such as $\frac{3}{8} \cdot \frac{2}{7}$.

Finally, the birthday problem. Suppose n people have birthdays spread evenly over 365 days. The probability that **nobody** shares is $\frac{365}{365} \cdot \frac{364}{365} \cdots \frac{365-n+1}{365}$, and the probability that **at least two** share is one minus that. The reason the numbers jump so fast is that the number of pairs grows quickly: n people create $\binom{n}{2}$ pairs, and 23 people already form 253 pairs.

People in the room	5	10	15	20	23	30	40
P(at least two share a birthday)	2.7%	11.7%	25.3%	41.1%	50.7%	70.6%	89.1%



Birthday matches become more likely than not at just 23 people.

Worked Examples

EXAMPLE 1: AT LEAST ONE

A die is rolled 4 times. What is the probability of at least one 6?

SOLUTION
Complement: no 6 on any roll. Each roll avoids a 6 with probability $\frac{5}{6}$.
 $P(\text{no } 6) = \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$.
 $P(\text{at least one } 6) = 1 - \frac{625}{1296} = \frac{671}{1296} \approx 0.518$.
So it is slightly better than even odds. A common wrong answer is $\frac{4}{6}$, which adds probabilities that overlap.

EXAMPLE 2: THE GAMBLER'S FALLACY

A fair coin lands heads 6 times in a row. (a) What is the probability that the 7th flip is heads? (b) Before any flips, what was the probability of 7 heads in a row?

SOLUTION
(a) Flips are independent, so the earlier flips do not matter: $\frac{1}{2}$.
(b) $\left(\frac{1}{2}\right)^7 = \frac{1}{128} \approx 0.0078$.
The streak was unlikely in advance, but it does not change the next flip. The coin is not "due" for tails.

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EXAMPLE 3: A SMALL BIRTHDAY PROBLEM

Five students are in a room. What is the probability that at least two share a birthday? Ignore leap years.

SOLUTION

No match: $\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \approx 0.9729$.

At least one match: $1 - 0.9729 \approx 0.0271$.

About 2.7 percent. That is small for 5 people, but the chance grows rapidly with more people.

EXAMPLE 4: WITH VERSUS WITHOUT REPLACEMENT

A bag holds 5 red and 3 blue marbles. Two marbles are drawn. Find the probability that both are blue (a) with replacement and (b) without replacement.

SOLUTION

(a) $\frac{3}{8} \cdot \frac{3}{8} = \frac{9}{64} \approx 0.141$.

(b) After one blue is removed: $\frac{3}{8} \cdot \frac{2}{7} = \frac{6}{56} = \frac{3}{28} \approx 0.107$.

Without replacement the second draw has fewer blue marbles, so the probability is lower.

EXAMPLE 5: AT LEAST ONE WITH DIFFERENT PROBABILITIES

Three independent machines fail on a given day with probabilities 0.1, 0.2, and 0.05. What is the probability that at least one fails?

SOLUTION

None fail: $0.9 \cdot 0.8 \cdot 0.95 = 0.684$.

At least one fails: $1 - 0.684 = 0.316$.

The formula $1 - (1 - p)^n$ does not apply because the probabilities differ, but the idea of complementing is the same.

EXAMPLE 6: HOW MANY TRIES?

How many times must a die be rolled so that the probability of at least one 6 exceeds 0.9?

SOLUTION

Need $1 - \left(\frac{5}{6}\right)^n > 0.9$, so $\left(\frac{5}{6}\right)^n < 0.1$.

Take logs: $n > \frac{\ln 0.1}{\ln(5/6)} \approx 12.63$.

Check: $n = 12$ gives $1 - (5/6)^{12} \approx 0.888$ and $n = 13$ gives ≈ 0.907 .

The smallest whole number is **13** rolls.

Coch's Playbook

- 1 Spot the phrase.** "At least one", "at least two", "none", "exactly". Translate before you compute.
 - 2 Complement first.** For "at least one" of anything, compute the probability of none and subtract from 1.
 - 3 Replacement check.** Ask "Is the item returned?" If not, shrink the total and the count after every draw.
 - 4 Memory check.** Ask "Do these trials affect each other?" If no, the past is irrelevant to the next trial.
 - 5 Sanity check.** Is your answer between 0 and 1? Is it larger when there are more chances, and smaller when you need more things to happen together?
- ◆ For "how many tries" questions, set up an inequality and use logarithms. Then round **up**.
 - ◆ Keep fractions as long as you can. Cancel before you multiply.
 - ◆ To explain a surprise, count the number of chances: pairs, not people, for the birthday problem.
 - ◆ If a problem seems to depend on a story ("due", "hot streak"), look for independence.

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Common traps

1. Adding probabilities for "at least one" (for example $\frac{1}{6} + \frac{1}{6} + \frac{1}{6}$ for three rolls). 2. Believing a streak makes the opposite result "due". 3. Using the same fraction for every draw when items are not replaced. 4. Rounding down in a "smallest number of tries" problem.

Mistake Clinic**Spot the error 1**

A student says: "In 3 rolls of a die, the probability of at least one 6 is $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$." **Where is the error?** Adding counts the overlaps (rolls with two or three 6s) more than once. Use the complement: $1 - \left(\frac{5}{6}\right)^3 = 1 - \frac{125}{216} = \frac{91}{216} \approx 0.421$.

Spot the error 2

A student says: "A fair coin has landed tails 4 times. Heads is due, so the chance of heads next is greater than $\frac{1}{2}$." **Where is the error?** The coin has no memory. The flips are independent, so the probability of heads is exactly $\frac{1}{2}$ on the next flip. Over a very long run the **proportion** of heads settles near one half, but that does not require extra heads right now.

Spot the error 3

A student says: "A bag has 4 green and 6 yellow marbles. Two are drawn without replacement. The probability both are green is $\frac{4}{10} \cdot \frac{4}{10} = 0.16$." **Where is the error?** The second draw cannot be $\frac{4}{10}$ because one green marble is gone. Correct: $\frac{4}{10} \cdot \frac{3}{9} = \frac{12}{90} = \frac{2}{15} \approx 0.133$.

Real-World Connection

A bridge has 3 independent safety systems, each of which fails with probability 0.02 on a given test. The chance that **all three** fail is $0.02^3 = 0.000008$, but the chance that **at least one** fails on a given test is $1 - 0.98^3 \approx 0.0588$. Engineers care about both numbers: the first measures total protection, the second measures how often a repair is needed.

Challenge Corner

Five people each have a birth month chosen independently and uniformly from 12 months. What is the probability that at least two share a birth month? **Solution.** No match: $\frac{12}{12} \cdot \frac{11}{12} \cdot \frac{10}{12} \cdot \frac{9}{12} \cdot \frac{8}{12} = \frac{95040}{248832} \approx 0.382$. At least one match: $1 - 0.382 \approx 0.618$. Just five people give better than a 60 percent chance, the same effect as the birthday problem with fewer days.

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Practice

PRACTICE Probability Puzzles and Common Misconceptions

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- A fair coin has landed heads 5 times in a row. What is the probability that the next flip is heads?
 (A) $\frac{1}{2}$
 (B) Less than $\frac{1}{2}$, because tails is due
 (C) More than $\frac{1}{2}$, because the coin is on a streak
 (D) $\frac{1}{64}$
- A fair coin is flipped 3 times. What is the probability of at least one head?
 (A) $\frac{1}{8}$ (B) $\frac{3}{8}$ (C) $\frac{7}{8}$ (D) $\frac{1}{2}$
- A die is rolled twice. What is the probability of at least one 6?
 (A) $\frac{11}{36}$ (B) $\frac{1}{3}$ (C) $\frac{25}{36}$ (D) $\frac{1}{36}$
- A bag holds 4 red and 6 blue marbles. Two are drawn without replacement. What is the probability that both are red?
 (A) $\frac{4}{25}$ (B) $\frac{2}{5}$ (C) $\frac{2}{15}$ (D) $\frac{1}{3}$
- Why are 23 people enough to give better than even odds of a shared birthday?
 (A) Each person has a 50 percent chance of matching someone
 (B) 23 is about half of the 365 days in a year
 (C) The calculation only considers the first 23 days of the year
 (D) There are 253 pairs of people, and every pair is a chance for a match
- Four independent events each have probability 0.25. What is the probability that at least one occurs?
 (A) 1.0 (B) 0.316 (C) 0.684 (D) 0.75
- A bag has 3 red and 2 blue marbles. Two marbles are drawn without replacement. What is the probability that they are different colors?
 (A) $\frac{3}{5}$ (B) $\frac{6}{25}$ (C) $\frac{2}{5}$ (D) $\frac{1}{2}$
- What is the smallest number of fair coin flips needed so that the probability of at least one head exceeds 0.99?
 (A) 7 (B) 5 (C) 6 (D) 8
- In a class of 30 students, which is closest to the probability that at least two share a birthday?
 (A) 0.08 (B) 0.5 (C) 0.71 (D) 0.95
- A student guesses on 10 questions, each with a 1 in 5 chance of being right. Find the probability of getting at least one right. Give three decimal places.
- Four people are in a room. What is the probability that at least two share a birthday? Ignore leap years and give three decimal places.
- A deck has 5 red and 5 black cards. Three cards are drawn without replacement. Find the probability all three are red.
- Each trial succeeds with probability 0.3, independently. What is the smallest number of trials so that the probability of at least one success is greater than 0.95?
- A drawer has 6 white and 4 black socks. Two are drawn without replacement. Find the probability that they match in color.
- A defect passes three independent checks with probabilities 0.4, 0.3, and 0.2 (the chance it slips through each check). What is the probability the defect is caught by at least one check?

Answers Answers and Explanations

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

29.1 Counting Principles, Permutations, and Combinations

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PRACTICE · COUNTING PRINCIPLES, PERMUTATIONS, AND COMBINATIONS

1. B. Three stages multiply: $5 \cdot 4 \cdot 3 = 60$. Choice A (12) adds the numbers, which counts only single items, not meals.
2. C. $\frac{9!}{7!} = 9 \cdot 8 = 72$, then divide by $2! = 2$ to get 36. Choice A (72) forgot to divide by $2!$.
3. A. The jobs are different, so order matters: $P(7, 3) = 7 \cdot 6 \cdot 5 = 210$. Choice B (35) is $C(7, 3)$, which ignores who gets which job.
4. B. A travel group has no order: $C(12, 4) = \frac{12 \cdot 11 \cdot 10 \cdot 9}{24} = 495$. Choice A (11880) is $P(12, 4)$, which counts each group 24 times.
5. A. LEVEL has 5 letters with L twice and E twice: $\frac{5!}{2!2!} = 30$. Choice B (120) is $5!$, which ignores the repeats.
6. C. Glue Mia and Jo into one block. Five units can be arranged in $5! = 120$ ways, and the pair can be ordered in 2 ways: 240. Choice B (120) forgot the 2 orders inside the block.
7. A. Exactly 2 seniors and 2 juniors: $C(5, 2) \cdot C(6, 2) = 10 \cdot 15 = 150$. Choice B (330) is $C(11, 4)$ and ignores the condition; choice C (25) adds 10 and 15.
8. C. Multiply stages without repeats: $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 = 468,000$. Choice A (676000) lets letters and digits repeat.
9. B. $C(n, 2) = \frac{n(n-1)}{2} = 45$ gives $n(n-1) = 90$, so $n = 10$. Choice D (90) is the product $n(n-1)$, not n .
10. 1680. Order matters: $P(8, 4) = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$.
11. 50400. Letters: S three times, T three times, I twice, A and C once. $\frac{10!}{3!3!2!} = \frac{3628800}{72} = 50400$.
12. 350. $C(7, 3) \cdot C(5, 2) = 35 \cdot 10 = 350$.
13. 105. A handshake is an unordered pair: $C(15, 2) = \frac{15 \cdot 14}{2} = 105$.
14. 18000. First digit 9 choices, next three digits 10 each, last digit 0 or 5 (2 choices): $9 \cdot 10 \cdot 10 \cdot 10 \cdot 2 = 18000$.
15. 246. Total $C(10, 5) = 252$. Committees with no women: $C(6, 5) = 6$. At least one woman: $252 - 6 = 246$.

29.2 Probability Rules**PRACTICE · PROBABILITY RULES**

1. A. Complement rule: $1 - 0.35 = 0.65$. Choice D (1.35) adds instead of subtracting.
2. B. Mutually exclusive means no overlap, so add: $0.5 + 0.3 = 0.8$. Choice A (0.15) multiplies, which would be for independent events and an and-question.
3. C. $0.6 + 0.5 - 0.3 = 0.8$. Choice A (1.1) forgets to subtract the overlap and exceeds 1, which is impossible.
4. D. $\frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$. Choice A is the with-replacement value $\frac{4}{52} \cdot \frac{4}{52}$.
5. A. $P(A | B) = \frac{0.12}{0.4} = 0.3$. Choice B (0.048) multiplies instead of dividing.
6. B. For independent events multiply: $0.4 \cdot 0.25 = 0.1$. Choice A (0.65) is the sum.
7. C. There are 30 pet owners and 12 of them walk: $\frac{12}{30} = \frac{2}{5}$. Choice A uses all 60 students as the denominator. Choice B is $P(\text{pet} | \text{walks})$, the reverse.
8. B. $P(A \text{ and } B) = 0.5 + 0.4 - 0.7 = 0.2 = 0.5 \cdot 0.4$, so they are independent. The overlap is not 0, so they are not mutually exclusive.
9. A. Red from Bag 1: $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$. Red from Bag 2: $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$. Given red: $\frac{3/8}{3/8 + 1/8} = \frac{3}{4}$. Choice C (3/8) is the joint probability and is not divided by the total chance of red.
10. $\frac{5}{18}$. Sum 9: 4 outcomes. Doubles: 6 outcomes. No overlap (a double has an even sum). $\frac{10}{36} = \frac{5}{18}$.
11. $\frac{1}{6} \cdot \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} = \frac{120}{720} = \frac{1}{6}$.
12. $\frac{3}{10}$. Students with no sport: 100. Of them, 30 play music. $\frac{30}{100} = \frac{3}{10}$.
13. 0.32. Either make-miss or miss-make: $2 \cdot 0.8 \cdot 0.2 = 0.32$.
14. $\frac{18}{67} \approx 0.269$. Out of 10,000: 200 have it, 180 test positive; 9800 do not, 490 test positive. $\frac{180}{180 + 490} = \frac{18}{67}$.
15. 0.6. Independence gives $P(A \text{ or } B) = 0.4 + b - 0.4b = 0.4 + 0.6b = 0.76$, so $b = 0.6$.

29.3 Expected Value and Binomial Probability**PRACTICE · EXPECTED VALUE AND BINOMIAL PROBABILITY**

1. B. $0(0.2) + 1(0.5) + 2(0.3) = 1.1$. Choice A (1.0) averages 0, 1, 2 ignoring the probabilities.
2. A. Fair fee equals expected prize: $10(0.25) = 2.50$. Choice B is the top prize, not the average.
3. C. The sum of row n is 2^n , so $2^6 = 64$. Adding gives $1 + 6 + 15 + 20 + 15 + 6 + 1 = 64$. Choice B (32) is 2^5 .
4. A. Ten independent flips, two outcomes each, constant $p = 0.5$, fixed n . Choice B has no fixed number of trials. Choice D fails independence because dealing without replacement from a small deck changes the odds.
5. C. $\binom{4}{2} \left(\frac{1}{2}\right)^4 = \frac{6}{16} = \frac{3}{8}$. Choice A ($\frac{1}{16}$) is the probability of one specific order such as HHTT and leaves out the count 6.
6. B. $E = np = 50(0.12) = 6$. Choice D (5.28) is $np(1 - p)$, the variance, a different quantity. Choice C is $n(1 - p)$, the expected failures.
7. A. Row 3 is 1, 3, 3, 1: $x^3 + 3x^2(3) + 3x(9) + 27$. Choice B ($x^3 + 27$) incorrectly distributes the power.
8. C. The term is $\binom{5}{2} x^2 \cdot 4^3 = 10 \cdot 64 = 640$. Choice A used 4^2 instead of 4^3 .
9. D. Complement: $1 - \left(\frac{1}{2}\right)^3 = \frac{7}{8}$. Choice A is the probability of making none.
10. 10 points. $5 \cdot \frac{1}{2} + 10 \cdot \frac{1}{4} + 20 \cdot \frac{1}{4} = 2.5 + 2.5 + 5 = 10$.
11. 0.311. $\binom{8}{2} (0.25)^2 (0.75)^6 = 28(0.0625)(0.17798) \approx 0.3115$.

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12. -0.50 dollars. Expected prize: $\frac{300+4(50)}{500} = 1.00$. Net: $1.00 - 1.50 = -0.50$.
13. 0.821. $1 - (0.4)^4 - 4(0.6)(0.4)^3 = 1 - 0.0256 - 0.1536 = 0.8208$.
14. 1080. The term is $\binom{5}{3}(3x)^3(-2)^2 = 10 \cdot 27x^3 \cdot 4 = 1080x^3$.
15. 11. Need $1 - 0.8^n \geq 0.9$, so $0.8^n \leq 0.1$ and $n \geq \frac{\ln 0.1}{\ln 0.8} \approx 10.3$. Check: $1 - 0.8^{10} \approx 0.893$ (too small), $1 - 0.8^{11} \approx 0.914$.

29.4 Probability Puzzles and Common Misconceptions

PRACTICE · PROBABILITY PUZZLES AND COMMON MISCONCEPTIONS

- A. Flips are independent, so the next flip is heads with probability $\frac{1}{2}$. Choice D is the probability of 6 heads in a row before any flips, not the conditional chance now.
- C. The complement is all tails: $(\frac{1}{2})^3 = \frac{1}{8}$, so $1 - \frac{1}{8} = \frac{7}{8}$. Choice A is the probability of no heads.
- A. $1 - (\frac{5}{6})^2 = \frac{11}{36}$. Choice B ($\frac{1}{3}$) adds $\frac{1}{6} + \frac{1}{6}$ and double counts the case of two 6s. Choice C is the probability of no 6.
- C. $\frac{4}{10} \cdot \frac{3}{9} = \frac{12}{90} = \frac{2}{15}$. Choice A ($\frac{4}{25}$) is the with-replacement value $\frac{4}{10} \cdot \frac{4}{10}$.
- D. The number of pairs is $\binom{23}{2} = 253$, which is many more chances than 23. Choice B is false because half of 365 is about 182.
- C. None occur: $0.75^4 \approx 0.316$, so at least one occurs with probability $1 - 0.316 = 0.684$. Choice A adds $4 \cdot 0.25$, which cannot be the probability of anything that can fail to happen.
- A. Red then blue: $\frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}$. Blue then red: $\frac{2}{5} \cdot \frac{3}{4} = \frac{3}{10}$. Total $\frac{3}{5}$. Choice B is the joint probability of one specific order with replacement, $\frac{3}{5} \cdot \frac{2}{5}$.
- A. Need $1 - (\frac{1}{2})^n > 0.99$. For $n = 6$ the value is $\frac{63}{64} \approx 0.984$ (too small). For $n = 7$ it is $\frac{127}{128} \approx 0.992$. Choice D (8) also works but is not the smallest.
- C. The probability of no match is $\frac{365}{365} \cdot \frac{364}{365} \cdots \frac{336}{365} \approx 0.294$, so a match has probability about 0.706. Choice A ($\frac{30}{365} \approx 0.08$) treats the problem as one person matching a fixed day.
- 0.893. None right: $0.8^{10} \approx 0.1074$. At least one: $1 - 0.1074 = 0.893$.
- 0.016. No match: $\frac{365 \cdot 364 \cdot 363 \cdot 362}{365^4} \approx 0.9836$. Complement: 0.0164.
- $\frac{1}{12} \cdot \frac{5}{10} \cdot \frac{4}{9} \cdot \frac{3}{8} = \frac{60}{720} = \frac{1}{12}$.
9. Need $0.7^n < 0.05$, so $n > \frac{\ln 0.05}{\ln 0.7} \approx 8.4$. Check: $1 - 0.7^8 \approx 0.942$ and $1 - 0.7^9 \approx 0.960$.
- $\frac{7}{15}$. White pair: $\frac{6}{10} \cdot \frac{5}{9} = \frac{30}{90}$. Black pair: $\frac{4}{10} \cdot \frac{3}{9} = \frac{12}{90}$. Total $\frac{42}{90} = \frac{7}{15}$.
- 0.976. It slips through all three with probability $0.4 \cdot 0.3 \cdot 0.2 = 0.024$, so it is caught by at least one with probability $1 - 0.024 = 0.976$.

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