



The Normal Distribution and z-Scores

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

In this booster

32.1 Density Curves and the Empirical Rule

32.2 z-Scores and Normal Calculations

32.3 Sampling Distributions

32.4 Random Variables and Combining Distributions

Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

32.1 Density Curves and the Empirical Rule

Learn what a density curve is, how area measures proportion, and how to use the 68-95-99.7 rule with symmetry, sketching, and shading in normal distributions.

Why This Matters

Take the heights of every student in a large school, the weights of cereal boxes coming off a line, or the lifetimes of light bulbs. Draw the histogram of any of them with enough data and you often see the same shape: a smooth, symmetric mound. That shape is the **normal distribution**, the most important model in statistics, and it is the foundation for inference.

Students get stuck in two places. They treat a curve like a histogram of counts instead of an area that measures a proportion. And they memorize 68-95-99.7 without learning to use symmetry, so questions such as "what percent lie above two standard deviations?" become guesses. This section builds both skills, with sketching and shading you can use on every free-response question.

Core Ideas

Density curve

A **density curve** is a curve that lies on or above the horizontal axis and has total area exactly 1. The area under the curve between a and b is the proportion of values that fall between a and b .

The **median** splits the area in half. The **mean** is the balance point. For a symmetric curve they are equal. For a right-skewed curve the mean is greater than the median.

A density curve is a model. It smooths a histogram so that area, not count, measures how likely values are. Any single exact value has area 0, so $P(X = a) = 0$ and including or excluding an endpoint changes nothing for a continuous variable.

The normal distribution

A normal distribution $N(\mu, \sigma)$ has a symmetric, single-peaked, bell-shaped density curve. μ is the mean (and median) and σ is the standard deviation. The curve changes from curving down to curving up at $\mu \pm \sigma$.

Changing μ slides the curve left or right. Changing σ makes it narrower and taller (smaller σ) or wider and flatter (larger σ).

The empirical rule (68-95-99.7)

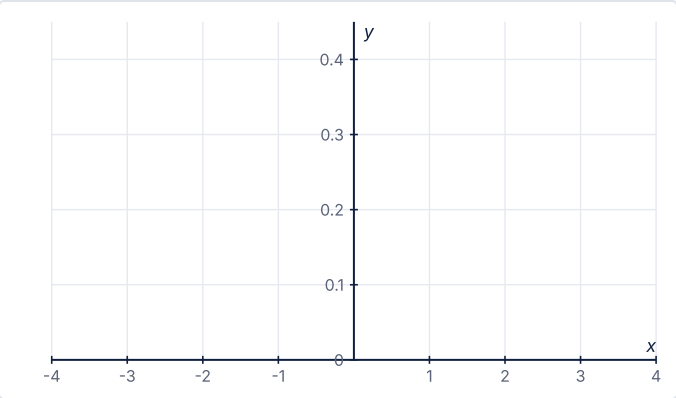
In any normal distribution, about **68%** of values fall within 1 standard deviation of the mean, **95%** within 2, and **99.7%** within 3.

In symbols: about 68% lie in $\mu \pm \sigma$, 95% in $\mu \pm 2\sigma$, and 99.7% in $\mu \pm 3\sigma$.

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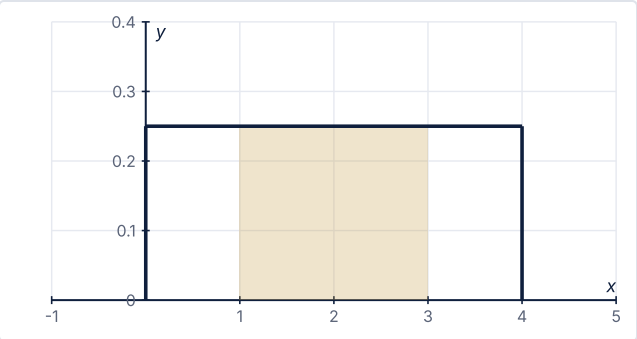


Standard normal curve (units are standard deviations from the mean). The shaded area within 1 standard deviation is about 68%.

Region (units of σ from μ)	Area
Within 1	about 68%
Within 2	about 95%
Within 3	about 99.7%
Between 1 and 2 above the mean	about 13.5%
Beyond 2 above the mean	about 2.5%
Beyond 3 above the mean	about 0.15%

How to use symmetry. The middle 68% leaves 32% in the two tails, so each tail has 16%. The middle 95% leaves 5%, so each tail has 2.5%. The middle 99.7% leaves 0.3%, so each tail has 0.15%. The slice between 1 and 2 standard deviations on one side is $\frac{95-68}{2} = 13.5\%$. **Always sketch and shade** before you compute.

A density curve can also be non-normal. The uniform curve below is flat on 0 to 4. Its height must be $\frac{1}{4}$ so the total area is $4 \cdot \frac{1}{4} = 1$. The shaded rectangle shows $P(1 < X < 3) = 2 \cdot \frac{1}{4} = 0.5$.



A uniform density curve on $[0, 4]$. Shaded area from 1 to 3 is 0.5.

Worked Examples

EXAMPLE 1: AREA UNDER A DENSITY CURVE

A density curve is uniform on the interval from 0 to 4. (a) Find its height. (b) Find $P(1 < X < 3)$. (c) Find the median.

SOLUTION

- (a) Total area = width $\times$ height = $4h = 1$, so $h = \frac{1}{4}$.
- (b) The region from 1 to 3 has width 2, so the area is $2 \cdot \frac{1}{4} = 0.5$.
- (c) The median has half of the area on each side, so it is at $x = 2$.

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EXAMPLE 2: DIRECT USE OF THE RULE

Heights of adult women in a large group are approximately normal with mean 64 inches and standard deviation 2.5 inches. Between what heights do the middle 68% and the middle 95% fall?

SOLUTION

68%: $\mu \pm \sigma = 64 \pm 2.5$, which is 61.5 to 66.5 inches.

95%: $\mu \pm 2\sigma = 64 \pm 5$, which is 59 to 69 inches.

EXAMPLE 3: TAIL AREAS

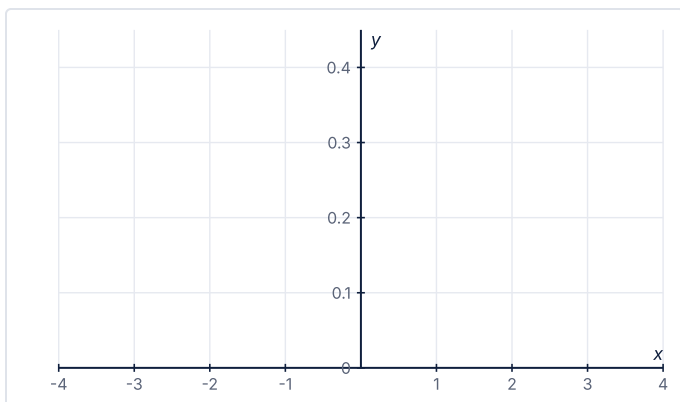
The lifetime of a type of bulb is approximately $N(1200, 100)$ hours. Find the percent of bulbs that last (a) more than 1300 hours, (b) less than 1000 hours, (c) between 1100 and 1400 hours.

SOLUTION

(a) 1300 is one standard deviation above the mean. The upper tail beyond $+1\sigma$ is $(100 - 68)/2 = 16\%$.

(b) 1000 is two standard deviations below the mean. The lower tail beyond -2σ is $(100 - 95)/2 = 2.5\%$.

(c) 1100 is -1σ and 1400 is $+2\sigma$. Area = $\frac{68}{2} + \frac{95}{2} = 34 + 47.5 = 81.5\%$.



Example 3(c): the area from 1 standard deviation below to 2 above the mean is $34\% + 47.5\% = 81.5\%$.

EXAMPLE 4: COUNTING VALUES

A batch of 500 bulbs has lifetimes $N(1200, 100)$. About how many last fewer than 1100 hours?

SOLUTION

1100 is one standard deviation below the mean, so about 16% are below it.

$0.16 \times 500 = 80$. About 80 bulbs fail before 1100 hours.

EXAMPLE 5: FINDING VALUES FROM PERCENTS

For the same bulbs, between what two lifetimes do the middle 95% fall? Below what lifetime are the lowest 0.15%?

SOLUTION

Middle 95%: $1200 \pm 2(100)$, so 1000 to 1400 hours.

The lowest 0.15% lie below $\mu - 3\sigma = 1200 - 300 = 900$ hours.

EXAMPLE 6: WORKING BACKWARD

The middle 95% of the lengths of a manufactured part runs from 47.2 mm to 52.8 mm. Assuming a normal distribution, find μ and σ . What percent of parts are longer than 51.4 mm?

SOLUTION

The mean is the midpoint: $\mu = (47.2 + 52.8)/2 = 50$.

The interval is $\mu \pm 2\sigma$, so $2\sigma = 2.8$ and $\sigma = 1.4$ mm.

$51.4 = 50 + 1.4$ is one standard deviation above the mean, so about 16% of parts are longer than 51.4 mm.

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Coach's Playbook

- 1 **Sketch a bell and label the mean.** Mark μ , then $\mu \pm \sigma$, $\mu \pm 2\sigma$, and $\mu \pm 3\sigma$ with the actual numbers.
- 2 **Locate the boundary values.** Convert each given number into "how many standard deviations from the mean."
- 3 **Shade the region you want.** Decide if you need an inside area, a tail, or a slice between two values.
- 4 **Add areas using symmetry.** Use 34, 13.5, and 2.35 for the slices on each side; they sum to 49.85 per side.
- 5 **Answer in context.** State the percent, the variable, and the units, for example "about 16% of the bulbs last more than 1300 hours."

Common traps

1) Using 68% when the interval is 2 standard deviations wide. 2) Forgetting to halve the leftover percent when you need one tail. 3) Applying the empirical rule to a distribution that is not approximately normal. 4) Drawing a density curve with the wrong height so the area is not 1.

Mistake Clinic

Spot the error 1

Problem: Scores are $N(40, 5)$. What percent are between 30 and 50? Wrong work: "30 and 50 are 10 from the mean, so 68%." What went wrong: The 68% figure is for 1 standard deviation (5 units). Here the endpoints are 2 standard deviations from the mean. Right solution: $40 \pm 2(5)$ is 30 to 50, so about 95% of scores lie there.

Spot the error 2

Problem: Scores are $N(40, 5)$. What percent are above 50? Wrong work: "95% are within 2 standard deviations, so 5% are above 50." What went wrong: The leftover 5% is split between both tails, and the question asks about only the upper tail. Right solution: Each tail has $5\%/2 = 2.5\%$, so about 2.5% of scores exceed 50.

Spot the error 3

Problem: A density curve is uniform on 0 to 5. What is its height? Wrong work: "A density curve has height 1, so the height is 1." What went wrong: The total area, not the height, must equal 1. Height 1 would give area 5. Right solution: $5h = 1$, so $h = 0.2$.

Real-World Connection

A cereal factory fills boxes with a mean of 510 grams and a standard deviation of 4 grams, and the fill weights are approximately normal. The label says 500 grams. The label weight is 2.5 standard deviations below the mean, so very few boxes (well under 1%) are underweight. Managers use the empirical rule to set the machine mean so that underfilled boxes stay rare without wasting product.

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Challenge Corner

Problem: A machine fills bags with weights that are $N(\mu, 0.3)$ ounces. The company wants only about 2.5% of bags below 16 ounces. What should μ be, and what percent of bags will then weigh more than 17.2 ounces? Solution: The lower 2.5% tail begins at $\mu - 2\sigma$, so $\mu - 2(0.3) = 16$ and $\mu = 16.6$ ounces. Then $17.2 = 16.6 + 2(0.3)$ is two standard deviations above the mean, and the upper tail beyond that is also 2.5%. In a batch of 1000 bags, about 25 would be underweight and about 25 would exceed 17.2 ounces.

Practice

PRACTICE Density Curves and the Empirical Rule

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is the total area under any density curve?
(A) 0.5 (B) 1 (C) 100
(D) It depends on the curve.
- Adult head circumference is approximately $N(50, 6)$ centimeters. Between which values do the middle 95% lie?
(A) 44 and 56 (B) 32 and 68 (C) 47 and 53 (D) 38 and 62
- Scores are approximately $N(70, 5)$. What percent of scores are between 65 and 80?
(A) 81.5% (B) 68% (C) 95% (D) 97.5%
- Heart rates are approximately $N(120, 15)$ beats per minute during a workout. About what percent are below 90?
(A) 0.15% (B) 5% (C) 2.5% (D) 16%
- A set of 800 normally distributed values is described by its mean and standard deviation. About how many fall within 1 standard deviation of the mean?
(A) 272 (B) 400 (C) 544 (D) 760
- Which statement about a density curve is true?
(A) The height at x is the proportion of values equal to x .
(B) The curve must be symmetric.
(C) The mean is always the point where half of the area is on each side.
(D) The area under the curve between a and b is the proportion of values between a and b .
- A density curve is uniform on the interval from 0 to 10. What is $P(X > 7)$?
(A) 0.3 (B) 0.07 (C) 0.7 (D) 0.1
- A density curve is skewed to the right. Which statement is true?
(A) Mean = median (B) Mean < median
(C) Mean > median (D) The mean cannot be found.
- A normal distribution has mean 100, and 84% of values are below 115. What is the standard deviation?
(A) 5 (B) 15 (C) 10 (D) 30
- Times for a trip are approximately $N(25, 3)$ minutes. What percent of trips take between 22 and 28 minutes? What percent take more than 31 minutes?
- The battery life of a tablet is approximately $N(10, 1.2)$ hours. Between what two values do the middle 95% of battery lives fall?
- A sample of 600 measurements is approximately normal with mean 72 and standard deviation 8. About how many of them fall between 64 and 88?
- A density curve is uniform on the interval from 2 to 10. Find its height, $P(3 < X < 7)$, and the median.
- The middle 95% of the weights of a type of fruit runs from 18 grams to 34 grams. Assuming a normal distribution, find the mean and standard deviation, and the percent of fruit heavier than 30 grams.
- Delivery times are approximately normal. About 16% of deliveries take less than 36 minutes and about 2.5% take more than 60 minutes. Find the mean and standard deviation.

32.2 z-Scores and Normal Calculations

Standardize values with z-scores, find proportions and percentiles for any normal distribution using a table or calculator, and solve inverse problems to find cutoffs and missing parameters.

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Why This Matters

The empirical rule works well when a value is a whole number of standard deviations from the mean. Real questions are messier: What percent of runners finish a mile faster than 7 minutes? How high must you score to be in the top 10%? To answer them you need a way to measure any distance from the mean in standard deviations, and a way to turn that distance into an area. That is what z-scores and normal calculations do.

Students get stuck in three places. They plug numbers into the z formula in the wrong order or with the wrong sign. They forget that tables and calculators report the area to the **left**, so right-tail questions need subtraction. And they confuse the forward problem (value to area) with the inverse problem (area to value). A picture of the bell with shading solves all three.

Core Ideas

The z-score

$$z = \frac{x - \mu}{\sigma}$$

A z-score says how many standard deviations a value is from the mean. Positive z means above the mean, negative means below. It has no units, so it lets you compare values from different distributions.

If X is $N(\mu, \sigma)$, then $Z = \frac{X - \mu}{\sigma}$ follows the **standard normal distribution** $N(0, 1)$. That is why one table, or one calculator command, handles every normal distribution: standardize first, then look up the area.

Normal calculations (forward)

To find a proportion: (1) state the distribution and sketch it; (2) shade the region; (3) standardize each boundary; (4) find the area. A table gives the area to the **left** of z .

Right tail: $P(Z > z) = 1 - P(Z < z)$. Between: $P(a < Z < b) = P(Z < b) - P(Z < a)$. A graphing calculator does this in one step with its normal-probability function (often called normalcdf, with inputs lower bound, upper bound, mean, standard deviation).

z	Area to the left
-2.00	0.0228
-1.25	0.1056
-0.50	0.3085
0.00	0.5000
0.50	0.6915
1.00	0.8413
1.25	0.8944
1.50	0.9332
1.64	0.9495
2.00	0.9772

Percentiles and the inverse problem

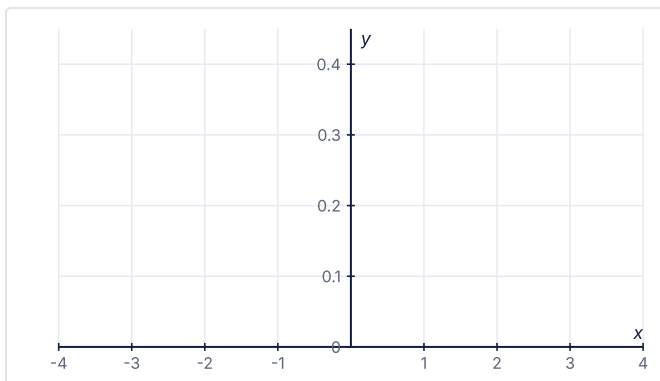
The p th **percentile** is the value with $p\%$ of the distribution at or below it. A z-score of 1.5 has area 0.9332 to its left, so it is at about the 93rd percentile.

For the inverse problem, start with the **area to the left**, find the matching z (inverse normal), then unstandardize: $x = \mu + z\sigma$.

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$P(Z < -1.25)$: the area to the left of $z = -1.25$ is about 0.1056.

Standardizing a whole data set. Replacing every value by its z-score gives a new data set with mean 0 and standard deviation 1. The shape does not change. This is the linear-transformation rule from earlier in action: subtract μ (shift) and divide by σ (scale).

Worked Examples

EXAMPLE 1: COMPARING WITH Z-SCORES

Mia scored 82 on a test with mean 75 and standard deviation 5. She scored 90 on a different test with mean 80 and standard deviation 8. On which test did she do better relative to her class?

SOLUTION

First test: $z = \frac{82 - 75}{5} = 1.4$.

Second test: $z = \frac{90 - 80}{8} = 1.25$.

Her score on the first test was 1.4 standard deviations above its mean, which is more than 1.25. She did relatively better on the first test even though her raw score was lower.

EXAMPLE 2: AREA TO THE LEFT

The time to run one mile for students is approximately $N(8.5, 1.2)$ minutes. What proportion run the mile in under 7 minutes?

SOLUTION

Standardize: $z = \frac{7 - 8.5}{1.2} = -1.25$.

From the table, the area to the left of -1.25 is 0.1056.

About 10.6% of students run the mile in under 7 minutes.

EXAMPLE 3: RIGHT TAIL AND BETWEEN TWO VALUES

The weight of apples is approximately $N(150, 12)$ grams. Find (a) the proportion heavier than 170 g and (b) the proportion between 140 g and 165 g.

SOLUTION

(a) $z = \frac{170 - 150}{12} \approx 1.667$ (keep the unrounded value). Area to the left is about 0.9522, so the right tail is $1 - 0.9522 = 0.0478$ (about 4.8%).

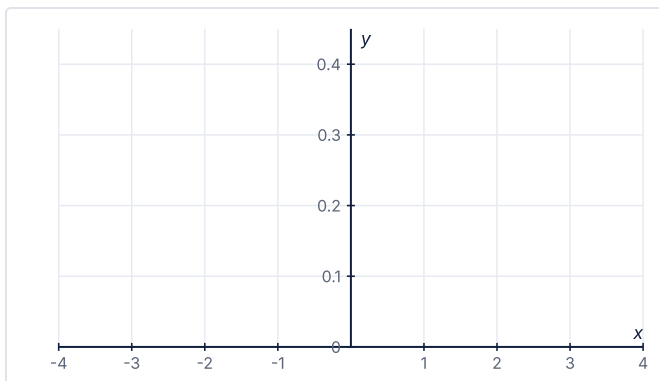
(b) $z_1 = \frac{140 - 150}{12} \approx -0.833$ and $z_2 = \frac{165 - 150}{12} = 1.25$.

Area = $0.8944 - 0.2023 \approx 0.692$, so about 69.2% of apples weigh between 140 and 165 grams.

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Example 3(b): the shaded area from $z = -0.83$ to $z = 1.25$ is about 0.692.

EXAMPLE 4: PERCENTILE OF A SCORE

Scores on a test are approximately $N(70, 8)$. What percentile is a score of 82?

SOLUTION

$$z = \frac{82 - 70}{8} = 1.5.$$

The area to the left of 1.5 is 0.9332.

A score of 82 is at about the 93rd percentile: roughly 93% of scores are at or below 82.

EXAMPLE 5: INVERSE NORMAL

For the same test, $N(70, 8)$, find the minimum score needed to be in the top 10%, and the score that marks the lowest 25%.

SOLUTION

Top 10% means 90% lies to the left. The z-score with area 0.90 to its left is $z \approx 1.2816$. Then $x = 70 + 1.2816(8) \approx 80.25$.

Lowest 25%: area 0.25 to the left gives $z \approx -0.6745$. Then $x = 70 - 0.6745(8) \approx 64.60$.

You need about 80.3 or higher for the top 10%, and scores below about 64.6 are in the lowest quarter.

EXAMPLE 6: FINDING A MISSING PARAMETER

The lengths of fish in a pond are approximately normal with mean 50 cm. The 20th percentile is 45 cm. Find the standard deviation.

SOLUTION

The 20th percentile has $z \approx -0.8416$ (area 0.20 to the left).

$$z = \frac{x - \mu}{\sigma} \text{ gives } -0.8416 = \frac{45 - 50}{\sigma}.$$

$$\sigma = \frac{-5}{-0.8416} \approx 5.94 \text{ cm.}$$

Coach's Playbook

- 1 **Always sketch and shade.** Draw the bell, mark the mean and the boundaries, and shade. The picture tells you if the answer should be small or large.
- 2 **Name the distribution.** Write $X \sim N(\mu, \sigma)$ with the numbers and units before computing, as AP graders expect.
- 3 **Standardize each boundary.** Use $z = (x - \mu)/\sigma$ and keep the sign.
- 4 **Convert to area, then check the direction.** Table values are left areas. A right-tail answer must be at most 0.5 when the boundary is above the mean.
- 5 **For inverse problems, work backward.** Area to the left, then z , then $x = \mu + z\sigma$.

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Common traps

1) Dividing by σ^2 or by μ instead of σ . 2) Reporting the left area when the question asks for the right tail. 3) Using the area to the right as the input to inverse normal for a "top" percent. 4) Using a normal model when the distribution is clearly skewed.

Mistake Clinic**Spot the error 1**

Problem: Scores are $N(70, 8)$. Find the z-score for a score of 86. Wrong work: " $z = (86 - 70)/64 = 0.25$." What went wrong: The student divided by the variance 64, not the standard deviation 8. Right solution: $z = (86 - 70)/8 = 2$.

Spot the error 2

Problem: Mile times are $N(8.5, 1.2)$. Find the proportion slower than 10 minutes. Wrong work: " $z = 1.25$, area = 0.8944, so 89.4%." What went wrong: 0.8944 is the area to the left (faster than 10 minutes). "Slower" is the right tail. Right solution: $1 - 0.8944 = 0.1056$, so about 10.6% are slower than 10 minutes.

Spot the error 3

Problem: Scores are $N(70, 8)$. What score marks the top 10%? Wrong work: "inverse normal of 0.10 gives $z = -1.28$, so $x = 70 - 1.28(8) = 59.8$." What went wrong: Inverse normal takes the area to the **left**. The top 10% has 90% to its left. Right solution: Use area 0.90: $z = 1.28$ and $x = 70 + 1.28(8) \approx 80.3$.

Real-World Connection

A factory makes bolts whose diameters are approximately $N(10.00, 0.05)$ mm. Bolts must be between 9.92 and 10.08 mm to be accepted. The limits are $z = \pm 1.6$, so about 0.8904 of bolts pass. Raising quality by reducing σ makes those z values larger and the pass rate higher, which is exactly how process improvement is measured.

Challenge Corner

Problem: Test scores are $N(500, 40)$. Find the two scores that bound the middle 90% of the distribution, and the proportion of scores between 450 and 560. Solution: The middle 90% leaves 5% in each tail, so use area 0.05 and 0.95: $z = \pm 1.6449$. Scores: $500 \pm 1.6449(40)$, which is about 434.2 and 565.8. For 450 to 560: $z_1 = -1.25$ and $z_2 = 1.5$. Area = $0.9332 - 0.1056 = 0.8276$, about 82.8%.

Practice**PRACTICE z-Scores and Normal Calculations**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- A score of 85 is on a test with mean 70 and standard deviation 12.5. What is the z-score?
(A) 1.2 (B) -1.2 (C) 0.84 (D) 15
- A value has z-score -0.5 in a distribution with mean 40 and standard deviation 6. What is the value?
(A) 34 (B) 37 (C) 43 (D) 43.5
- A student's z-score is 1.2, and the area to the left of 1.2 in the standard normal distribution is 0.8849. This student's score is at about which percentile?
(A) 12th (B) 50th (C) 99th (D) 88th

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4. For a standard normal variable, what is $P(-1 < Z < 2)$?
(A) 0.6827 (B) 0.8186 (C) 0.9545 (D) 0.9759
5. Test X has mean 75 and standard deviation 5. Test Y has mean 80 and standard deviation 8. A student scores 82 on X and 90 on Y. Which is a correct comparison?
(A) Y is better because 90 is greater than 82.
(B) The scores are equally good.
(C) X is better because its z-score is higher.
(D) Y is better because its standard deviation is higher.
6. For a standard normal variable, the area to the left of 1.64 is 0.9495. What is $P(Z > 1.64)$?
(A) 0.0505 (B) 0.9495 (C) 0.164 (D) 0.0050
7. A data set is converted to z-scores. What are the mean and standard deviation of the new data?
(A) Mean 0, standard deviation 1
(B) Mean 1, standard deviation 0
(C) Mean 0, standard deviation equal to the original
(D) The same as the original data
8. Scores are $N(70, 8)$. Which score is closest to the cutoff for the top 10%?
(A) 62 (B) 78 (C) 80 (D) 85
9. Which z-score is closest to the 25th percentile of a standard normal distribution?
(A) 0.67 (B) -1.28 (C) -0.25 (D) -0.67
10. Mile times are approximately $N(8.5, 1.2)$ minutes. Find the proportion of times between 7 and 10 minutes.
11. Daily commute times are approximately $N(34, 4)$ minutes. Find the proportion of commutes longer than 40 minutes and the proportion shorter than 30 minutes.
12. IQ-like scores are approximately $N(100, 15)$. Find the score at the 90th percentile and the two scores that bound the middle 90%. Round to the nearest tenth.
13. In a normal distribution with $\sigma = 6$, the 80th percentile is 55. Find the mean. Round to two decimal places.
14. Maya ran a 5-kilometer race in 24.5 minutes among runners with times $N(28, 4)$. Leo swam a lap in 11.2 seconds among swimmers with times $N(13, 1.5)$. Lower times are better. Who did better relative to their group?
15. Package weights are approximately normal. About 15% of packages weigh less than 48 ounces and 5% weigh more than 56 ounces. Find the mean and standard deviation. Use $z = -1.0364$ for the 15th percentile and $z = 1.6449$ for the 95th. Round to two decimal places.

32.3 Sampling Distributions

Understand how sample proportions and sample means vary from sample to sample, with their center, spread, and shape, the standard error, and the central limit theorem.

Why This Matters

Pollsters survey 1,000 people to learn about millions. A factory tests 25 bulbs to judge a whole shipment. In both cases a **sample** is used to learn about a **population**, and a different sample would give a different answer. Sampling distributions describe exactly how much those answers vary from sample to sample. Without them, confidence intervals and hypothesis tests would have no foundation.

This is the topic where students most often lose the thread. The ideas are abstract: the thing you graph is not your data but all the values a statistic could take. Students also mix up the standard deviation of the population with the standard deviation of the sample mean, and they forget to check conditions. Go slowly here and keep asking, "What is one dot in this distribution?" The answer is always one sample's statistic.

Core Ideas

A **parameter** is a number that describes a population (for example μ or p). A **statistic** is a number computed from a sample (for example $\bar{x}$ or $\hat{p}$). The **sampling distribution** of a statistic is the distribution of values it takes over all possible samples of the same size from the same population.

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Sampling distribution of a sample proportion $\hat{p}$

For samples of size n from a population with proportion p :

$$\mu_{\hat{p}} = p, \quad \sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}.$$

Conditions: the sample is random; the **10% condition** ($n \leq \frac{1}{10}$ of the population when sampling without replacement); and the **Large Counts condition** $np \geq 10$ and $n(1-p) \geq 10$, so the distribution of $\hat{p}$ is approximately normal.

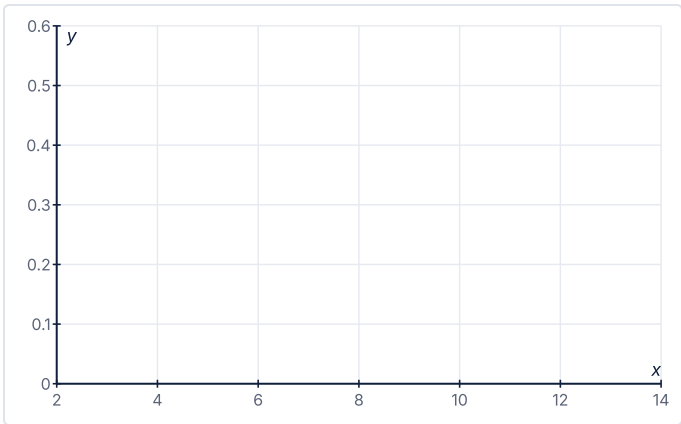
Sampling distribution of a sample mean $\bar{x}$

For random samples of size n from a population with mean μ and standard deviation σ :

$$\mu_{\bar{x}} = \mu, \quad \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}.$$

If the population is normal, $\bar{x}$ is normal for any n . Otherwise the **central limit theorem** says that for large n (a common guideline is $n \geq 30$) the distribution of $\bar{x}$ is approximately normal, whatever the population shape.

The **standard error** is an estimate of a standard deviation of a statistic computed from sample data. When the population value is known, we call it the standard deviation of the statistic. In every case the message is the same: **spread shrinks like $1/\sqrt{n}$** . To cut the spread in half you must quadruple the sample size.



Approximate sampling distributions of the sample mean for a population with mean 8 and standard deviation 6. Gray: $n = 4$ (standard deviation 3). Navy: $n = 16$ (1.5). Gold: $n = 64$ (0.75). Larger samples give narrower, taller curves.

Statistic	Center	Spread	Shape
$\hat{p}$	p	$\sqrt{p(1-p)/n}$	Approximately normal if $np \geq 10$ and $n(1-p) \geq 10$
$\bar{x}$	μ	$\sigma/\sqrt{n}$	Normal if the population is normal; approximately normal for large n (CLT)

Bias and variability. A statistic is **unbiased** if the center of its sampling distribution equals the parameter. Both $\hat{p}$ and $\bar{x}$ are unbiased when sampling is random. Larger samples reduce **variability** (the spread of the distribution) but cannot fix **bias** that comes from a poor sampling method, such as asking only volunteers.

Worked Examples

EXAMPLE 1: PARAMETER OR STATISTIC?

A school has 1,800 students. A random sample of 90 finds that 54 take the bus. (a) Identify the statistic. (b) If the principal's records show 62% of all students take the bus, identify the parameter.

SOLUTION

- (a) The statistic is $\hat{p} = 54/90 = 0.60$, computed from the sample.
(b) The parameter is $p = 0.62$, the proportion of the whole population.

The sample proportion differs from the parameter by sampling variability, and the sampling distribution describes how large such differences usually are.

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EXAMPLE 2: SAMPLING DISTRIBUTION OF P

In a large city, 30% of households recycle. A random sample of 200 households is selected. Describe the sampling distribution of $\hat{p}$ and find $P(\hat{p} > 0.35)$.

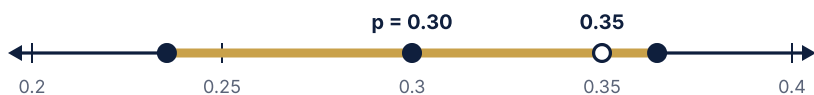
SOLUTION

Center: $\mu_{\hat{p}} = 0.30$. Spread: $\sigma_{\hat{p}} = \sqrt{\frac{0.30(0.70)}{200}} \approx 0.0324$.

Conditions: $np = 60 \geq 10$ and $n(1 - p) = 140 \geq 10$, and 200 is less than 10% of a large city. So $\hat{p}$ is approximately normal.

$$z = \frac{0.35 - 0.30}{0.0324} \approx 1.54, \text{ so } P(\hat{p} > 0.35) = 1 - 0.9386 \approx 0.061.$$

About 6.1% of samples of 200 households would show more than 35% recycling.



Sampling distribution of the proportion for $n = 200$: most sample proportions fall within 2 standard deviations of 0.30, between about 0.235 and 0.365.

EXAMPLE 3: EFFECT OF SAMPLE SIZE

A population proportion is $p = 0.60$. Compare the standard deviation of $\hat{p}$ for $n = 50$ and $n = 800$.

SOLUTION

$$n = 50: \sqrt{0.6(0.4)/50} \approx 0.0693.$$

$$n = 800: \sqrt{0.6(0.4)/800} \approx 0.0173.$$

The sample size was multiplied by 16 and the standard deviation was divided by $\sqrt{16} = 4$ ($0.0693/4 \approx 0.0173$). Bigger samples give more precise estimates.

EXAMPLE 4: MEAN OF A NORMAL POPULATION

The battery life of a tablet is normal with mean 12 hours and standard deviation 1.5 hours. A random sample of 9 tablets is tested. Find $P(\bar{x} > 12.8)$ and compare it with $P(X > 12.8)$ for one tablet.

SOLUTION

$$\sigma_{\bar{x}} = \frac{1.5}{\sqrt{9}} = 0.5. \text{ The population is normal, so } \bar{x} \text{ is normal with mean 12.}$$

$$z = \frac{12.8 - 12}{0.5} = 1.6, \text{ so } P(\bar{x} > 12.8) = 1 - 0.9452 \approx 0.055.$$

$$\text{For one tablet: } z = \frac{12.8 - 12}{1.5} \approx 0.533 \text{ and } P(X > 12.8) \approx 0.297.$$

An average of 9 tablets is much less likely to be extreme than one tablet. Averaging cancels out variation.

EXAMPLE 5: THE CENTRAL LIMIT THEOREM

Waiting times at a clinic are skewed right with mean 8 minutes and standard deviation 6 minutes. For a random sample of 40 patients, find the probability that the mean wait is less than 7 minutes.

SOLUTION

The population is skewed, but $n = 40 \geq 30$, so the CLT says $\bar{x}$ is approximately normal with mean 8 and $\sigma_{\bar{x}} = 6/\sqrt{40} \approx 0.949$.

$$z = \frac{7 - 8}{0.949} \approx -1.05, \text{ so the probability is about 0.146.}$$

State the conclusion in context: about a 14.6% chance that the mean wait for 40 patients is under 7 minutes. Note that this says nothing about a single patient's wait, which could easily be 2 minutes or 20.

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EXAMPLE 6: CHECKING CONDITIONS

Can we use a normal model for $\hat{p}$ in each case? (a) $p = 0.04$, $n = 150$. (b) $p = 0.20$, $n = 150$ from a town of 900 voters.

SOLUTION

(a) $np = 150(0.04) = 6 < 10$. The Large Counts condition fails, so the sampling distribution of $\hat{p}$ is likely skewed right and a normal model is not appropriate.

(b) $np = 30$ and $n(1 - p) = 120$, both at least 10. But $150 > \frac{1}{10}(900) = 90$, so the 10% condition fails and the standard deviation formula would overstate the spread. We cannot use it as written.

Condition	Why it matters
Random sample	Makes the statistic unbiased
10% of the population	Keeps observations nearly independent
Large Counts / large n or normal population	Makes the normal model reasonable

Coach's Playbook

- 1 Name the statistic.** Is the problem about a proportion $\hat{p}$ or a mean $\bar{x}$?
- 2 Write center, spread, shape.** Use the formulas in the key boxes and state the conditions that justify "approximately normal."
- 3 Remember one dot equals one sample.** The sampling distribution is not your data. It describes many hypothetical samples.
- 4 Standardize with the standard deviation of the statistic.** Use $\sigma/\sqrt{n}$ or $\sqrt{p(1-p)/n}$, never the population σ alone.
- 5 Answer in context.** Say "the probability that the mean wait for 40 patients is less than 7 minutes is about 0.146," not just a decimal.

Common traps

1) Using σ instead of $\sigma/\sqrt{n}$ when asked about a sample mean. 2) Forgetting to check Large Counts or the 10% condition before using a normal model. 3) Using the sample proportion $\hat{p}$ in place of p when p is given. 4) Thinking a bigger sample fixes a biased sampling method.

Mistake Clinic**Spot the error 1**

Problem: Tablet battery life is $N(12, 1.5)$. Find $P(\bar{x} > 12.8)$ for $n = 9$. Wrong work: " $z = (12.8 - 12)/1.5 = 0.53$, so the probability is 0.297." What went wrong: The student used the standard deviation of one tablet. A sample mean has a smaller standard deviation, $1.5/\sqrt{9} = 0.5$. Right solution: $z = 0.8/0.5 = 1.6$, so the probability is about 0.055.

Spot the error 2

Problem: A population has 40% successes. A sample of 100 is taken. Find the standard deviation of $\hat{p}$. Wrong work: " $\sqrt{0.4(0.6)}/100 = 0.0049$." What went wrong: The student divided after taking the square root. The n belongs inside the square root. Right solution: $\sqrt{0.4(0.6)/100} = \sqrt{0.0024} \approx 0.049$.

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Spot the error 3

Problem: A sample of 20 gives a skewed distribution of weights. A student says the sampling distribution of $\bar{x}$ is normal "because of the central limit theorem." What went wrong: The CLT needs a large sample. With $n = 20$ and a skewed population, the distribution of $\bar{x}$ may still be skewed. The student also confused the sample with the sampling distribution. Right solution: Say the shape of the sampling distribution is not necessarily normal unless the population is normal or n is large (about 30 or more).

Real-World Connection

Election polls report a margin of error. That margin is built from $\sqrt{p(1-p)/n}$. A poll of 1,000 voters has a standard deviation of about $\sqrt{0.25/1000} \approx 0.016$ when p is near 0.5, so results typically land within about 3 percentage points (two standard deviations). To halve that uncertainty, a pollster would need about 4,000 voters, and that expense is why polls rarely do it.

Challenge Corner

Problem: A population has $\sigma = 12$. How large a random sample is needed so that the standard deviation of $\bar{x}$ is at most 2? At most 1? Then find the probability that $\hat{p}$ is within 0.05 of $p = 0.70$ when $n = 150$. Solution: Need $12/\sqrt{n} \leq 2$, so $\sqrt{n} \geq 6$ and $n \geq 36$. For 1: $\sqrt{n} \geq 12$ and $n \geq 144$ (four times as large). For $\hat{p}$: $\sigma_{\hat{p}} = \sqrt{0.7(0.3)/150} \approx 0.0374$, so the limits are $z = \pm 0.05/0.0374 \approx \pm 1.336$ and the probability is $0.9093 - 0.0907 \approx 0.819$.

Practice

PRACTICE Sampling Distributions

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- A random sample of 120 students finds that 66 prefer online homework. The value 0.55 is a
(A) statistic (B) parameter (C) population proportion
(D) sampling distribution
- A population has proportion $p = 0.40$. What is the standard deviation of $\hat{p}$ for samples of size 100?
(A) 0.0024 (B) 0.24 (C) 0.49 (D) 0.049
- A sample size is made four times as large. What happens to the standard deviation of $\bar{x}$?
(A) It is multiplied by 4. (B) It is divided by 4. (C) It is divided by 2.
(D) It does not change.
- For which sample is a normal model for $\hat{p}$ appropriate? (Assume random samples from a very large population.)
(A) $n = 50$, $p = 0.05$ (B) $n = 200$, $p = 0.04$ (C) $n = 40$, $p = 0.90$
(D) $n = 150$, $p = 0.20$
- The central limit theorem says that, for a large random sample, the sampling distribution of $\bar{x}$ is approximately normal
(A) only if the population is normal (B) regardless of the shape of the population
(C) only if the population is skewed (D) only if σ is known
- A population has $\mu = 50$ and $\sigma = 12$. For a random sample of 36, what is the standard deviation of $\bar{x}$?
(A) 2 (B) 0.33 (C) 6 (D) 12
- A random sample is taken without replacement from a population of 500. Which sample size satisfies the 10% condition?
(A) 80 (B) 60 (C) 40 (D) 120
- A population has $p = 0.50$. For samples of size 100, what is $P(\hat{p} > 0.60)$?
(A) 0.0228 (B) 0.1587 (C) 0.0668 (D) 0.0013

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9. Taking a larger random sample primarily does which of the following?
 - (A) Removes bias caused by a poor sampling method
 - (B) Changes the mean of the population
 - (C) Reduces the variability of the statistic
 - (D) Makes the population normal
10. In a large school, 30% of students take the bus. A random sample of 200 is taken. Find the mean and standard deviation of $\hat{p}$ and the probability that $\hat{p} > 0.35$.
11. The weights of bags of rice are normal with mean 500 grams and standard deviation 40 grams. For a random sample of 16 bags, find the probability that the mean weight is less than 485 grams.
12. A skewed population has mean 20 and standard deviation 9. For a random sample of 36, find the probability that $\bar{x}$ is between 19 and 22.
13. A town has 900 voters and 6% support a measure. A random sample of 120 voters is taken. Check whether the Large Counts and 10% conditions are met, and say whether a normal model for $\hat{p}$ is appropriate.
14. A population has standard deviation 12. How large must a random sample be so that the standard deviation of $\bar{x}$ is at most 2? Then, how large for at most 1?
15. A population has $p = 0.70$. For a random sample of 150, find the probability that $\hat{p}$ is within 0.05 of 0.70 (between 0.65 and 0.75).

32.4 Random Variables and Combining Distributions

Describe discrete random variables with a probability distribution, mean, and standard deviation, then apply the rules for linear transformations and for combining independent random variables.

Why This Matters

Casinos, insurance companies, and engineers all ask the same question: if an uncertain quantity is measured many times, what is its long-run average and how much does it fluctuate? A **random variable** turns a random outcome into a number, and the mean and standard deviation summarize it. The rules for combining random variables are what let us say how the total of many parts behaves. Students get stuck on two things. First, the formulas look similar but behave differently: means add and subtract, but **variances** add even when you subtract the variables, and standard deviations never add directly. Second, students forget to weight by probabilities when finding the mean. This section gives you a table-based routine and clear rules for transformations and for combinations.

Core Ideas

A **random variable** X takes numerical values determined by chance. A **discrete** random variable has a countable list of possible values (the number of goals, the number of defective parts). Its **probability distribution** lists each value x_i with its probability p_i , where every p_i is between 0 and 1 and $\sum p_i = 1$.

Mean and standard deviation of a discrete random variable

$$\mu_X = E(X) = \sum x_i p_i, \quad \sigma_X = \sqrt{\sum (x_i - \mu_X)^2 p_i}.$$

The mean is a **long-run average** over many repetitions; it need not be a possible value of X . The standard deviation measures the typical distance of X from its mean.

Linear transformations

If $Y = a + bX$, then

$$\mu_Y = a + b\mu_X, \quad \sigma_Y = |b|\sigma_X.$$

Adding a shifts the center but does not change the spread. Multiplying by b scales both center and spread. A negative b flips the distribution but spread stays positive.

Combining independent random variables

If X and Y are **independent**, then

$$\mu_{X \pm Y} = \mu_X \pm \mu_Y, \quad \sigma_{X \pm Y}^2 = \sigma_X^2 + \sigma_Y^2, \text{ so } \sigma_{X \pm Y} = \sqrt{\sigma_X^2 + \sigma_Y^2}.$$

Means add or subtract. **Variances add** for both sums and differences. Standard deviations are never added directly. If X and Y are both normal, then $X \pm Y$ is also normal.

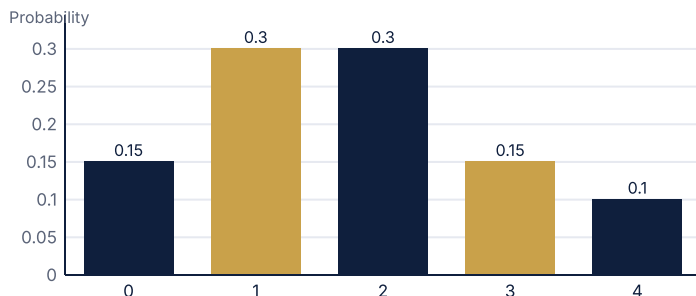
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Quantity	Mean	Standard deviation
$a + bX$	$a + b\mu_X$	$ b \sigma_X$
$X + Y$ (independent)	$\mu_X + \mu_Y$	$\sqrt{\sigma_X^2 + \sigma_Y^2}$
$X - Y$ (independent)	$\mu_X - \mu_Y$	$\sqrt{\sigma_X^2 + \sigma_Y^2}$

The bar chart below is the probability distribution of X , the number of goals a team scores in a game. The mean, 1.75, is the balance point of this distribution.



Probability distribution of the number of goals X . Probabilities add to 1.

Why variances add. The variability of $X - Y$ is not smaller than that of $X + Y$: subtracting an uncertain quantity adds more uncertainty, not less. That is why $\sigma_X^2 + \sigma_Y^2$ appears for both. The square root of a sum is not the sum of the square roots, so standard deviations cannot be added.

Worked Examples

EXAMPLE 1: A VALID DISTRIBUTION

The number of goals X has probabilities 0.15, 0.30, 0.30, 0.15 for $x = 0, 1, 2, 3$ and an unknown probability for $x = 4$. Find that probability, and $P(X \geq 2)$.

SOLUTION

The probabilities must sum to 1: $0.15 + 0.30 + 0.30 + 0.15 = 0.90$, so $P(X = 4) = 0.10$.

$P(X \geq 2) = 0.30 + 0.15 + 0.10 = 0.55$.

EXAMPLE 2: MEAN AND STANDARD DEVIATION

Find the mean and standard deviation of the number of goals X .

SOLUTION

$\mu_X = 0(0.15) + 1(0.30) + 2(0.30) + 3(0.15) + 4(0.10) = 0 + 0.30 + 0.60 + 0.45 + 0.40 = 1.75$.

$\sigma_X^2 = (0 - 1.75)^2(0.15) + (1 - 1.75)^2(0.30) + (2 - 1.75)^2(0.30) + (3 - 1.75)^2(0.15) + (4 - 1.75)^2(0.10) = 1.3875$.

$\sigma_X = \sqrt{1.3875} \approx 1.18$ goals.

Interpretation: in the long run the team averages 1.75 goals per game, and a typical game differs from that average by about 1.18 goals.

EXAMPLE 3: LINEAR TRANSFORMATION

A club pays a bonus of 5 dollars plus 3 dollars per goal. Let $W = 5 + 3X$. Find the mean and standard deviation of W .

SOLUTION

$\mu_W = 5 + 3(1.75) = 10.25$ dollars.

$\sigma_W = 3(1.18) \approx 3.53$ dollars. The 5 dollars shifts the center only, and the 3 scales both.

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EXAMPLE 4: SUM AND DIFFERENCE OF INDEPENDENT VARIABLES

Time to prepare part A has mean 12 minutes and standard deviation 3 minutes. Part B has mean 8 minutes and standard deviation 2 minutes. They are independent. Find the mean and standard deviation of (a) the total time $A + B$ and (b) the difference $A - B$.

SOLUTION

(a) $\mu = 12 + 8 = 20$ minutes. $\sigma = \sqrt{3^2 + 2^2} = \sqrt{13} \approx 3.61$ minutes.

(b) $\mu = 12 - 8 = 4$ minutes. $\sigma = \sqrt{3^2 + 2^2} = \sqrt{13} \approx 3.61$ minutes. The standard deviation is the same, because variances add.

EXAMPLE 5: NORMAL COMBINATIONS

In Example 4, assume both times are normal. Find the probability that the total time $A + B$ exceeds 25 minutes.

SOLUTION

$A + B$ is normal with mean 20 and standard deviation 3.61.

$$z = \frac{25 - 20}{3.61} \approx 1.39.$$

$P(A + B > 25) = 1 - 0.9172 \approx 0.083$. About 8.3% of the time the total exceeds 25 minutes.

EXAMPLE 6: EXPECTED VALUE OF A GAME

A game costs 2 dollars to play. A spinner pays 10 dollars with probability 0.10, 3 dollars with probability 0.30, and nothing with probability 0.60. Find the expected net gain and its standard deviation.

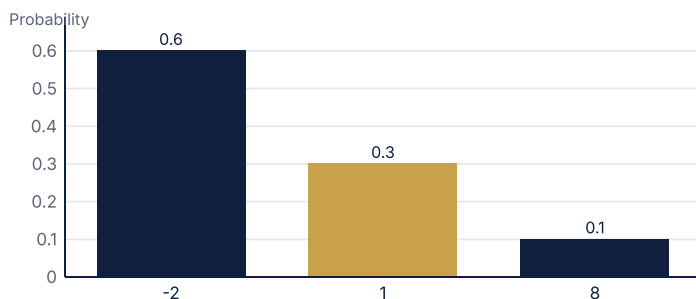
SOLUTION

Net gain $G = \text{payout} - 2$, so G takes values 8, 1, -2 with probabilities 0.10, 0.30, 0.60.

$$\mu_G = 8(0.10) + 1(0.30) + (-2)(0.60) = 0.80 + 0.30 - 1.20 = -0.10.$$

$$\sigma_G^2 = (8.1)^2(0.10) + (1.1)^2(0.30) + (-1.9)^2(0.60) = 6.561 + 0.363 + 2.166 = 9.09, \text{ so } \sigma_G \approx 3.01.$$

On average a player loses 10 cents per play, though single results swing by about 3 dollars.



Probability distribution of the net gain G in dollars.

Coach's Playbook

- 1 Set up a table.** Columns for x , p , $x \cdot p$, and $(x - \mu)^2 p$. Check that the probabilities add to 1 before you do anything else.
- 2 Find the mean first.** You need μ before you can compute any deviation.
- 3 Choose the right rule.** One variable changed by $a + bX$? Use the linear rules. Two variables? Check independence, then add means and add variances.
- 4 Convert at the end.** Add variances, then take the square root once to return to the standard deviation.
- 5 Interpret.** A mean is a long-run average. Say what it means in the context, with units.

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Common traps

1) Averaging the x values instead of weighting by p . 2) Adding standard deviations instead of variances. 3) Subtracting variances for $X - Y$. 4) Applying the variance rule when the variables are not independent.

Mistake Clinic**Spot the error 1**

Problem: A has $\sigma = 3$ and B has $\sigma = 2$ (independent). Find the standard deviation of $A + B$. Wrong work: " $3 + 2 = 5$." What went wrong: Standard deviations do not add. Variances do. Right solution: $\sqrt{3^2 + 2^2} = \sqrt{13} \approx 3.61$.

Spot the error 2

Problem: A has $\sigma = 3$ and B has $\sigma = 2$ (independent). Find the standard deviation of $A - B$. Wrong work: " $\sqrt{9 - 4} = \sqrt{5} \approx 2.24$." What went wrong: The student subtracted the variances. Subtracting a random variable still adds uncertainty. Right solution: $\sqrt{9 + 4} = \sqrt{13} \approx 3.61$.

Spot the error 3

Problem: $X = 0, 1, 2, 3, 4$ with probabilities 0.15, 0.30, 0.30, 0.15, 0.10. Find the mean. Wrong work: " $(0 + 1 + 2 + 3 + 4)/5 = 2$." What went wrong: The student ignored the probabilities. Values with a higher probability must count more. Right solution: $\mu = \sum xp = 1.75$.

Real-World Connection

An insurance company sells a policy for 300 dollars. The chance of a 10,000-dollar claim in a year is 0.02. The expected cost per policy is $10000(0.02) = 200$ dollars, so the expected profit is 100 dollars per policy. On one policy the result swings widely, but across many independent policies the average gets close to 100 dollars. That is why large companies can predict results that individuals cannot.

Challenge Corner

Problem: X has mean 40 and standard deviation 6. Compare $2X$ with $X_1 + X_2$, where X_1 and X_2 are two independent copies of X . Then find $P(X_1 + X_2 < 70)$ if X is normal. Solution: Both have mean 80. For $2X$, $\sigma = 2(6) = 12$. For $X_1 + X_2$, $\sigma = \sqrt{36 + 36} = \sqrt{72} \approx 8.49$. Doubling one measurement doubles its error, but two independent measurements partly cancel. For the probability, $z = \frac{70 - 80}{8.49} \approx -1.18$, so $P \approx 0.119$.

Practice**PRACTICE Random Variables and Combining Distributions**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Which table could be a probability distribution for $X = 0, 1, 2, 3$?
 (A) 0.2, 0.3, 0.4, 0.2 (B) 0.3, 0.3, 0.3, 0.3 (C) 0.1, 0.4, 0.3, 0.2
 (D) -0.1, 0.4, 0.4, 0.3
- X takes values 1, 2, 3 with probabilities 0.2, 0.5, 0.3. What is μ_X ?
 (A) 2 (B) 2.1 (C) 2.3 (D) 6
- X has standard deviation 2. What is the standard deviation of $3X - 4$?
 (A) 2 (B) 4 (C) 10 (D) 6

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4. X and Y are independent with $\sigma_X = 3$ and $\sigma_Y = 4$. What is σ_{X+Y} ?
 (A) 5 (B) 7 (C) 12 (D) 25
5. Variances of two random variables can be added to find the variance of their sum when the variables are
 (A) both normal (B) independent (C) discrete
 (D) identically distributed
6. The mean of a game's net gain is -0.10 dollars per play. Which interpretation is correct?
 (A) You lose 10 cents on every play.
 (B) You will lose after exactly 10 plays.
 (C) The game is fair.
 (D) In the long run, the average result is a loss of 10 cents per play.
7. X has mean 20 and standard deviation 4, and $Y = 50 - 2X$. What are the mean and standard deviation of Y ?
 (A) Mean 10, standard deviation 8 (B) Mean 10, standard deviation -8
 (C) Mean -40 , standard deviation 8 (D) Mean 10, standard deviation 4
8. $X \sim N(30, 4)$ and $Y \sim N(20, 3)$ are independent. Which describes $X + Y$?
 (A) $N(50, 7)$ (B) $N(50, 25)$ (C) $N(50, 5)$ (D) $N(10, 5)$
9. Which of these is a discrete random variable?
 (A) The number of heads in 10 coin flips (B) The exact time to run a mile
 (C) The weight of a backpack (D) The height of a plant
10. Find the mean and standard deviation of X where $x = 0, 1, 2, 3$ have probabilities 0.1, 0.3, 0.4, 0.2.
11. A random variable X takes values 1, 2, 3, 4, 5 with probabilities 0.10, 0.25, k , 0.30, 0.15. Find k and $P(X \geq 3)$.
12. A student earns 4 dollars plus 2.5 dollars for each item sold. The number of items X has mean 3.2 and standard deviation 1.1. Find the mean and standard deviation of the earnings.
13. Task A has mean 15 minutes, standard deviation 4. Task B has mean 11 minutes, standard deviation 3. They are independent and both normal. Find the mean and standard deviation of $A - B$, and $P(A > B)$.
14. A game costs 2 dollars. A spinner pays 10 dollars with probability 0.10, 3 dollars with probability 0.30, and nothing otherwise. Find the expected net gain and its standard deviation. Round to two decimal places.
15. Each of four independent bags of flour has weight $N(250, 6)$ grams. Find $P(\text{total of the four bags} < 980)$. Then find $P(4X < 980)$ for the weight X of one bag, and explain why the two answers differ.

Answers**Answers and Explanations**

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

32.1 Density Curves and the Empirical Rule**PRACTICE · DENSITY CURVES AND THE EMPIRICAL RULE**

1. B. By definition the total area under a density curve is exactly 1 (100% of the data).
2. D. $50 \pm 2(6) = 38$ to 62. Choice A is only 1 standard deviation (68%). Choice B is 3 standard deviations (99.7%).
3. A. 65 is -1σ and 80 is $+2\sigma$: $34 + 47.5 = 81.5\%$. Choice B (68%) ignores the extra slice from 1 to 2 standard deviations, and choice C (95%) treats both sides as 2.
4. C. 90 is 2 standard deviations below 120, and the lower tail has $(100 - 95)/2 = 2.5\%$. Choice B forgets to halve.
5. C. $0.68 \times 800 = 544$. Choice D uses 95%.
6. D. Area measures proportion; height alone does not. Choice A confuses density with probability, choice B is false because density curves can be skewed, and choice C describes the median, not the mean.
7. A. The width of 7 to 10 is 3 and the height is 0.1, so the area is 0.3. Choice C is the area below 7.
8. C. A long right tail pulls the mean above the median.
9. B. 84% below means the point is $\mu + 1\sigma$ ($50\% + 34\%$). So $115 - 100 = 15$. Choice D (30) would mean 115 is half a standard deviation above the mean.
10. 22 and 28 are 1 standard deviation from 25, so 68%. 31 is $25 + 2(3)$, so the upper tail is 2.5%.
11. 7.6 to 12.4 hours. Compute $10 \pm 2(1.2) = 10 \pm 2.4$.
12. About 489. 64 is -1σ and 88 is $+2\sigma$, giving 81.5% of 600, which is $0.815 \times 600 = 489$.
13. Height $= 1/8 = 0.125$ (width 8). $P(3 < X < 7) = 4/8 = 0.5$. Median $= 6$.
14. Mean $= (18 + 34)/2 = 26$ grams. $2\sigma = 8$, so $\sigma = 4$ grams. 30 is $+1\sigma$, so about 16% of fruit are heavier than 30 grams.
15. Mean 44 minutes, standard deviation 8 minutes. 36 is $\mu - \sigma$ and 60 is $\mu + 2\sigma$, so $3\sigma = 24$, $\sigma = 8$, and $\mu = 36 + 8 = 44$. Check: $44 + 16 = 60$.

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32.2 z-Scores and Normal Calculations

PRACTICE · Z-SCORES AND NORMAL CALCULATIONS

1. $A. z = (85 - 70)/12.5 = 1.2$. Choice B has the wrong sign, and choice D is the raw distance.
2. $B. x = \mu + z\sigma = 40 + (-0.5)(6) = 37$. Choice C adds instead of subtracting.
3. D. Area to the left of 1.2 is 0.8849, so the score is at about the 88th percentile.
4. $B. P(Z < 2) - P(Z < -1) = 0.9772 - 0.1587 = 0.8186$. Choice A is the area within 1 standard deviation, and choice C is within 2.
5. $C. z_X = (82 - 75)/5 = 1.4$ and $z_Y = (90 - 80)/8 = 1.25$. Higher z means better relative performance. Choice A compares raw scores, which have different scales.
6. $A. 1 - 0.9495 = 0.0505$. Choice B is the left area.
7. A. Subtracting the mean and dividing by the standard deviation gives mean 0 and standard deviation 1. Choice D is true only for the shape.
8. C. The top 10% starts at $z = 1.2816$: $70 + 1.2816(8) \approx 80.3$. Choice D is about 1.9 standard deviations above the mean, and choice B is only 1 standard deviation above.
9. D. The 25th percentile has area 0.25 to the left, so $z \approx -0.6745$. Choice A has the wrong sign.
10. 0.789. $z = -1.25$ and $z = 1.25$, so area $= 0.8944 - 0.1056 = 0.7887$ (about 78.9%).
11. Longer than 40: $z = 1.5$, so $1 - 0.9332 = 0.0668$ (6.7%). Shorter than 30: $z = -1$, so 0.1587 (15.9%).
12. 90th percentile: $100 + 1.2816(15) \approx 119.2$. Middle 90%: $100 \pm 1.6449(15)$, which is about 75.3 and 124.7.
13. 49.95. The 80th percentile has $z = 0.8416$, so $55 = \mu + 0.8416(6)$ and $\mu = 55 - 5.05 = 49.95$.
14. Leo. Maya: $z = (24.5 - 28)/4 = -0.875$. Leo: $z = (11.2 - 13)/1.5 = -1.2$. Leo is 1.2 standard deviations faster than average and Maya is 0.875 faster, so Leo did better relative to his group.
15. Mean ≈ 51.09 and standard deviation ≈ 2.98 ounces. Set $\mu - 1.0364\sigma = 48$ and $\mu + 1.6449\sigma = 56$. Subtract: $2.6813\sigma = 8$, so $\sigma = 2.98$. Then $\mu = 48 + 1.0364(2.98) = 51.09$.

32.3 Sampling Distributions

PRACTICE · SAMPLING DISTRIBUTIONS

1. A. 0.55 is computed from a sample (66/120), so it is a statistic. Choices B and C describe the population value.
2. D. $\sqrt{0.4(0.6)/100} = \sqrt{0.0024} \approx 0.049$. Choice A forgets the square root, and choice B forgets to divide by n .
3. C. $\sigma/\sqrt{4n} = \frac{1}{2} \cdot \sigma/\sqrt{n}$, so the standard deviation is divided by 2, not 4.
4. D. Large Counts: $np = 30$ and $n(1 - p) = 120$, both at least 10. The others have $np < 10$ or $n(1 - p) < 10$ (for example $200(0.04) = 8$).
5. B. For large n the shape of the population does not matter. Choice A is the condition for small samples.
6. A. $12/\sqrt{36} = 12/6 = 2$. Choice D is the population standard deviation.
7. C. 10% of 500 is 50, so the sample must be at most 50. Only 40 qualifies.
8. A. $\sigma_{\hat{p}} = \sqrt{0.25/100} = 0.05$. $z = (0.60 - 0.50)/0.05 = 2$, and $P(Z > 2) = 0.0228$. Choice B is the tail for $z = 1$.
9. C. Larger samples reduce variability. They cannot repair bias from a bad sampling method (choice A).
10. Mean 0.30; standard deviation $\sqrt{0.3(0.7)/200} \approx 0.0324$. $z \approx 1.54$, so $P(\hat{p} > 0.35) \approx 0.061$.
11. 0.0668. $\sigma_{\bar{x}} = 40/\sqrt{16} = 10$. $z = (485 - 500)/10 = -1.5$, and the area to the left is 0.0668.
12. 0.656. $\sigma_{\bar{x}} = 9/\sqrt{36} = 1.5$. By the CLT ($n = 36 \geq 30$), $z = (19 - 20)/1.5 = -0.667$ and $z = (22 - 20)/1.5 = 1.333$. Area $= 0.9088 - 0.2525 \approx 0.656$.
13. Large Counts fails: $np = 120(0.06) = 7.2 < 10$. The 10% condition also fails: 120 is more than 10% of 900 (which is 90). A normal model is not appropriate.
14. At most 2: $12/\sqrt{n} \leq 2$ gives $n \geq 36$. At most 1: $12/\sqrt{n} \leq 1$ gives $n \geq 144$. Halving the standard deviation requires four times the sample size.
15. 0.819. $\sigma_{\hat{p}} = \sqrt{0.7(0.3)/150} \approx 0.0374$. $z = \pm 0.05/0.0374 = \pm 1.336$. Area $= 0.9093 - 0.0907 \approx 0.819$.

32.4 Random Variables and Combining Distributions

PRACTICE · RANDOM VARIABLES AND COMBINING DISTRIBUTIONS

1. C. A valid distribution has probabilities between 0 and 1 that add to 1. Choice A sums to 1.1, choice B to 1.2, and choice D has a negative value.
2. B. $1(0.2) + 2(0.5) + 3(0.3) = 0.2 + 1.0 + 0.9 = 2.1$. Choice A averages the values without weights.
3. D. Adding a constant does not change spread, and multiplying by 3 triples it: $3(2) = 6$. Choice C adds 4.
4. A. $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$. Choice B adds the standard deviations.
5. B. The rule $\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2$ requires independence. Normality helps with probabilities but is not required.
6. D. The mean is a long-run average, not what happens on each play. Choice A treats it as a certain result of each play.
7. A. $\mu_Y = 50 - 2(20) = 10$ and $\sigma_Y = |-2|(4) = 8$. Standard deviation cannot be negative, so choice B is impossible.
8. C. The mean is $30 + 20 = 50$ and the standard deviation is $\sqrt{4^2 + 3^2} = 5$. Choice A adds standard deviations.
9. A. The number of heads takes the counted values 0, 1, ..., 10. The others are measurements that can take any value in an interval.
10. Mean $0(0.1) + 1(0.3) + 2(0.4) + 3(0.2) = 1.7$. Variance $= 0.1(2.89) + 0.3(0.49) + 0.4(0.09) + 0.2(1.69) = 0.81$, so the standard deviation is 0.9.
11. $k = 1 - (0.10 + 0.25 + 0.30 + 0.15) = 0.20$. $P(X \geq 3) = 0.20 + 0.30 + 0.15 = 0.65$.
12. Mean $= 4 + 2.5(3.2) = 12$ dollars. Standard deviation $= 2.5(1.1) = 2.75$ dollars.

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13. Mean 4 minutes and standard deviation $\sqrt{16+9} = 5$ minutes. $A - B \sim N(4, 5)$, so $P(A - B > 0) = P(z > -0.8) = 0.788$.
14. Expected net gain: $8(0.10) + 1(0.30) + (-2)(0.60) = -0.10$ dollars. Variance = 9.09, so the standard deviation is about 3.01 dollars.
15. First: the total is $N(1000, \sqrt{4 \cdot 36} = 12)$, so $z = (980 - 1000)/12 = -1.67$ and $P \approx 0.048$. Second: $4X \sim N(1000, 24)$, $z = -0.83$, and $P \approx 0.202$. Four different bags partly cancel each other's errors, but multiplying one bag by 4 multiplies its error by 4.

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