



Limits and Continuity

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

In this booster

36.1 The Idea of a Limit

36.2 Evaluating Limits Algebraically

36.3 Continuity and Discontinuities

36.4 The Intermediate Value Theorem and Asymptotes

Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

36.1 The Idea of a Limit

Learn what a limit means, how to estimate limits from tables and graphs, how one-sided limits work, and the three ways a limit can fail to exist.

Why This Matters

Calculus is built on one idea: the **limit**. Every derivative is a limit of slopes of secant lines, and every integral is a limit of sums. A limit answers the question: what value is a function heading toward as the input gets close to some number, whether or not the function ever reaches it? Students get stuck because they confuse the limit with the value $f(a)$, they stop thinking when they see $\frac{0}{0}$, and they forget that a limit requires agreement from **both** sides.

Limits describe real situations: the speed shown on a speedometer is the limit of average speeds over shorter and shorter time intervals, and a population model near its carrying capacity is described by what it approaches. Learning to read limits from tables and graphs gives you the intuition that makes the algebraic methods of the next section feel natural.

Core Ideas

Definition (informal) of a limit

We write

$$\lim_{x \rightarrow a} f(x) = L$$

if $f(x)$ gets arbitrarily close to the single number L as x gets close to a from **both** sides, with $x \neq a$.

The value $f(a)$ is irrelevant to the limit. The function does not even have to be defined at a .

Read the notation aloud: ' $\lim_{x \rightarrow a} f(x) = L$ ' says 'the limit of $f(x)$ as x approaches a is L .' The arrow means 'gets closer and closer to', never 'equals'. The small plus and minus signs in a^+ and a^- tell you which side of a the input comes from: plus means from the right (larger values), minus means from the left (smaller values). Keep the three questions separate: what does f do near a (the limit), what happens on each side (one-sided limits), and what is f at a itself (the value).

One-sided limits and existence

$\lim_{x \rightarrow a^-} f(x)$ is the value approached as x increases toward a from the left. $\lim_{x \rightarrow a^+} f(x)$ is the value approached from the right.

$$\lim_{x \rightarrow a} f(x) = L \quad \text{exactly when} \quad \lim_{x \rightarrow a^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a^+} f(x) = L.$$

A table is the first tool. To estimate $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$, pick inputs on both sides of 2 and watch the outputs:

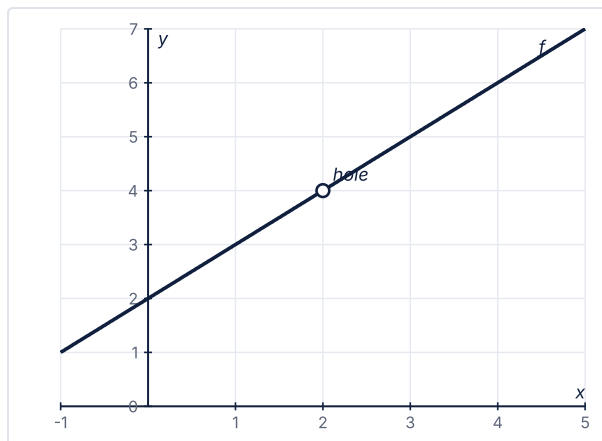
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x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	undefined	4.001	4.01	4.1

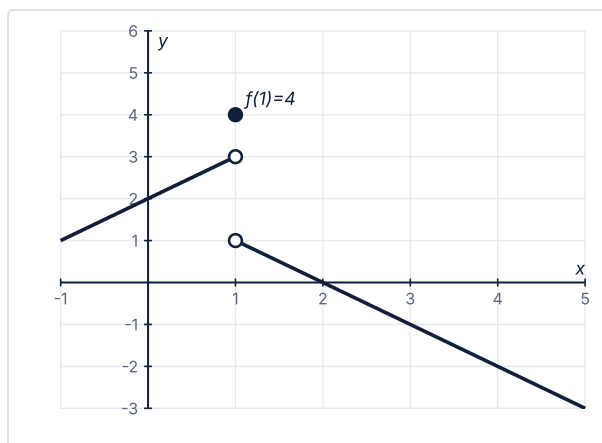
The outputs approach 4 from both sides, so the limit is 4, even though $f(2)$ does not exist. The graph shows why: $f(x) = x + 2$ with a hole at $(2, 4)$.



Graph of $f(x) = (x^2 - 4)/(x - 2)$. The limit at $x = 2$ is 4 even though $f(2)$ is undefined.

A limit can fail to exist (**DNE**) in three common ways. Each is worth recognizing at a glance:

- ◆ **Jump**: the left and right limits are different finite numbers, as in the graph below.
- ◆ **Unbounded behavior**: the outputs grow without bound, as with $\frac{1}{x^2}$ near 0. We may write $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$ to describe the behavior, but this limit does not exist as a number. If the left and right sides head to opposite infinities, as with $\frac{1}{x}$, we only say the limit does not exist.
- ◆ **Oscillation**: the outputs keep swinging between values, as with $\sin \frac{1}{x}$ near 0. For $x = \frac{1}{n\pi}$ the output is 0, while for $x = \frac{2}{(4n+1)\pi}$ it is 1, and both kinds of inputs get close to 0.



A jump at $x = 1$: the left limit is 3, the right limit is 1, and $f(1) = 4$.

Limits from graphs

Trace the curve toward $x = a$ with a finger from the left, then from the right. Where your finger **heads** is the one-sided limit. An open circle means the point is not part of the graph; a filled circle shows the actual value $f(a)$.

Worked Examples

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EXAMPLE 1: ESTIMATE A LIMIT FROM A TABLE

Estimate $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$ using $x = 2.9, 2.99, 3.01, 3.1$.

SOLUTION

At 2.9: $\frac{8.41 - 9}{-0.1} = 5.9$. At 2.99: 5.99. At 3.01: 6.01. At 3.1: 6.1.

The values approach 6 from both sides, so the limit is 6. Check: $\frac{x^2 - 9}{x - 3} = x + 3$ for $x \neq 3$.

EXAMPLE 2: READ A GRAPH WITH A JUMP

Use the jump graph above (left piece $y = x + 2$, right piece $y = 2 - x$, filled point $(1, 4)$). Find $\lim_{x \rightarrow 1^-} f(x)$, $\lim_{x \rightarrow 1^+} f(x)$, $\lim_{x \rightarrow 1} f(x)$ and $f(1)$.

SOLUTION

Left: the piece $y = x + 2$ heads to $1 + 2 = 3$.

Right: the piece $y = 2 - x$ heads to $2 - 1 = 1$.

The one-sided limits differ, so $\lim_{x \rightarrow 1} f(x)$ **does not exist**. The filled point gives $f(1) = 4$, which has no effect on the limit.

EXAMPLE 3: A PIECEWISE LIMIT THAT EXISTS

Let $f(x) = \begin{cases} x^2 + 1, & x < 2 \\ 3x - 1, & x \geq 2 \end{cases}$. Find $\lim_{x \rightarrow 2} f(x)$.

SOLUTION

Left: $\lim_{x \rightarrow 2^-} (x^2 + 1) = 5$. Right: $\lim_{x \rightarrow 2^+} (3x - 1) = 5$.

The one-sided limits agree, so $\lim_{x \rightarrow 2} f(x) = 5$. Also $f(2) = 5$, so the graph has no break there.

EXAMPLE 4: UNBOUNDED BEHAVIOR

Describe $\lim_{x \rightarrow 0} \frac{1}{x^2}$ and $\lim_{x \rightarrow 0} \frac{1}{x}$.

SOLUTION

$\frac{1}{x^2}$ is positive for every $x \neq 0$ and grows without bound on both sides: $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$ (no finite limit).

For $\frac{1}{x}$: $x = 0.01$ gives 100 and $x = -0.01$ gives -100 . Left limit $-\infty$, right limit $+\infty$; the limit **does not exist**.

EXAMPLE 5: AN ABSOLUTE VALUE QUOTIENT

Find the one-sided limits and the limit of $\frac{|x|}{x}$ as $x \rightarrow 0$.

SOLUTION

For $x > 0$: $\frac{|x|}{x} = \frac{x}{x} = 1$, so the right limit is 1.

For $x < 0$: $\frac{|x|}{x} = \frac{-x}{x} = -1$, so the left limit is -1 .

They disagree, so the limit does not exist.

EXAMPLE 6: MAKE A LIMIT EXIST

Find k so that $\lim_{x \rightarrow 3} f(x)$ exists, where $f(x) = \begin{cases} kx - 2, & x < 3 \\ x^2 - 2, & x \geq 3 \end{cases}$.

SOLUTION

Left: $\lim_{x \rightarrow 3^-} (kx - 2) = 3k - 2$. Right: $\lim_{x \rightarrow 3^+} (x^2 - 2) = 7$.

Set them equal: $3k - 2 = 7$, so $k = 3$.

Check: with $k = 3$ both sides approach 7 and $f(3) = 7$.

Coach's Playbook

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- 1 **Decide the question.** Is it $\lim_{x \rightarrow a}$, a one-sided limit, or the value $f(a)$? These are three different questions.
 - 2 **Approach from both sides.** Test inputs on each side of a , such as $a \pm 0.1$, $a \pm 0.01$, $a \pm 0.001$.
 - 3 **Compare the sides.** If the sides disagree, the limit does not exist. If they agree, that shared number is the limit.
 - 4 **Name the failure.** When a limit does not exist, say why: jump, unbounded, or oscillation.
 - 5 **Confirm with the graph.** Does the picture match your table? Open circles and filled dots tell you where the actual value sits.
- ◆ Tables suggest, they do not prove. Use values closer and closer to a (three or four steps).
 - ◆ Never stop at $\frac{0}{0}$. It means 'unknown yet', so simplify or use a table.
 - ◆ Write ∞ only to describe unbounded behavior. Strictly, the limit still does not exist as a real number.

Common traps

- 1) Believing $\lim_{x \rightarrow a} f(x) = f(a)$ always.
- 2) Using only one side of a in a table.
- 3) Saying a limit is 0 when it is $\frac{0}{0}$ or $\frac{\text{number}}{0}$.
- 4) Reading an open circle as the actual value.

Mistake Clinic

Spot the error 1

A student evaluates $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$ by substituting $x = 2$, gets $\frac{0}{0}$ and writes 'DNE'. The form $\frac{0}{0}$ is **indeterminate**: it tells you nothing yet. Factor: $\frac{(x-2)(x+2)}{x-2} = x + 2$ for $x \neq 2$, so the limit is $2 + 2 = 4$. The table in Core Ideas confirms it.

Spot the error 2

For the jump graph, a student answers: ' $\lim_{x \rightarrow 1} f(x) = 4$ because the dot at $x = 1$ is at height 4.' The dot is $f(1)$, not the limit. Limits come from the heights the curve approaches from each side: 3 from the left and 1 from the right. They disagree, so the limit does not exist.

Spot the error 3

A student writes $\lim_{x \rightarrow 0} \frac{1}{x} = \infty$ because 'the bottom gets small'. Check both sides: for $x = 0.001$, $\frac{1}{x} = 1000$, but for $x = -0.001$, $\frac{1}{x} = -1000$. The two sides head to opposite infinities, so the limit does not exist. Only $\frac{1}{x^2}$, which is positive on both sides, has the unbounded limit ∞ .

Real-World Connection

A car's position is $s(t) = t^2$ meters. The average speed from $t = 3$ to $t = 3 + h$ is $\frac{(3+h)^2 - 9}{h} = 6 + h$. As h shrinks to 0 these averages approach 6, so the instantaneous speed at $t = 3$ is 6 m/s. The speedometer reading is a limit, and the expression is $\frac{0}{0}$ at $h = 0$ even though its limit is perfectly good.

Challenge Corner

Find constants a and b so that $f(x) = \begin{cases} ax + b, & x < 1 \\ 7, & x = 1 \\ x^2 + 2a, & x > 1 \end{cases}$ has $\lim_{x \rightarrow 1} f(x) = f(1)$. Solution: left limit $a + b$, right limit $1 + 2a$,

and the value is 7. Set both equal to 7: $1 + 2a = 7$ gives $a = 3$, then $3 + b = 7$ gives $b = 4$. Check: left $3 + 4 = 7$, right $1 + 6 = 7$, value 7.

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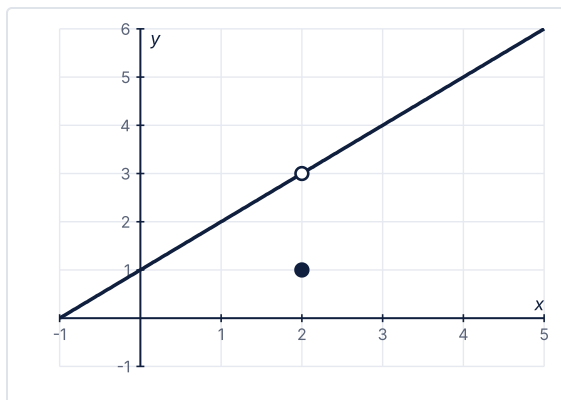
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Practice

PRACTICE The Idea of a Limit

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

1. A table gives $f(3.9) = 7.8$, $f(3.99) = 7.98$, $f(3.999) = 7.998$, $f(4.001) = 8.002$, $f(4.01) = 8.02$. What is the best estimate of $\lim_{x \rightarrow 4} f(x)$?
 (A) 7.8 (B) 8 (C) 4 (D) 0
2. What is $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$?
 (A) 0 (B) 2
 (C) The limit does not exist. (D) 4
3. The graph of f is the line $y = x + 1$ for $x \neq 2$, with an open circle at $(2, 3)$ and a filled point at $(2, 1)$. What is $\lim_{x \rightarrow 2} f(x)$?



- (A) 3 (B) 1 (C) 2
(D) It does not exist.
4. What is $\lim_{x \rightarrow 0^+} \frac{|x|}{x}$?
 (A) -1 (B) 0 (C) 1
 (D) It does not exist.
5. Which statement describes $\lim_{x \rightarrow 0} \frac{1}{x^2}$?
 (A) The limit is 0.
 (B) The values grow without bound, so there is no finite limit.
 (C) The limit is 1.
 (D) The limit is $-\infty$.
6. If $\lim_{x \rightarrow 3^-} f(x) = 5$, $\lim_{x \rightarrow 3^+} f(x) = 5$ and $f(3) = 2$, what is $\lim_{x \rightarrow 3} f(x)$?
 (A) 2 (B) 7 (C) 5
 (D) It does not exist.
7. Let $f(x) = 2x + 1$ for $x < 1$ and $f(x) = x^2 + 2$ for $x \geq 1$. What is $\lim_{x \rightarrow 1} f(x)$?
 (A) 3 (B) 4 (C) 5
 (D) It does not exist.
8. For which value of k does $\lim_{x \rightarrow 3} f(x)$ exist, where $f(x) = kx - 2$ for $x < 3$ and $f(x) = x^2 - 2$ for $x \geq 3$?
 (A) $k = 1$ (B) $k = 3$ (C) $k = \frac{5}{3}$ (D) $k = 7$
9. Numerically, what is $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$? (Check $x = \pm 0.1, \pm 0.01$.)
 (A) 1 (B) $\frac{1}{2}$ (C) It does not exist.
 (D) 0
10. Find $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5}$.
11. Find $\lim_{x \rightarrow -1} \frac{x^2 + 3x + 2}{x + 1}$.
12. Let $f(x) = x^2 - 3$ for $x < 2$ and $f(x) = 2x + 1$ for $x \geq 2$. Find both one-sided limits at $x = 2$ and decide whether $\lim_{x \rightarrow 2} f(x)$ exists.
13. Describe $\lim_{x \rightarrow 0^-} \frac{1}{x}$.
14. Find k so that $\lim_{x \rightarrow 1} f(x)$ exists for $f(x) = kx + 2$ when $x < 1$ and $f(x) = 3x^2 - k$ when $x \geq 1$.

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15. Use a table to estimate $\lim_{x \rightarrow 4} \frac{\sqrt{x}-2}{x-4}$, then give the exact value.

36.2 Evaluating Limits Algebraically

Learn to evaluate limits exactly with substitution, factoring, conjugates, special trigonometric limits and degree comparison for limits at infinity, and to show the work AP graders expect.

Why This Matters

Tables and graphs estimate limits; algebra **proves** them. On the AP exam you are expected to evaluate limits exactly, without a calculator, and to show enough work that a reader can follow your reasoning. Students get stuck because they see $\frac{0}{0}$ and freeze, they cancel terms instead of factors, and they apply rules for limits at infinity without comparing the degrees of numerator and denominator.

The same techniques reappear everywhere in calculus. The definition of the derivative gives $\frac{0}{0}$ at $h = 0$ and is solved by exactly the factoring, conjugate and simplification moves in this section. Limits at infinity describe long-run behavior: the cost per item of a large production run, the steady level of a drug in the blood, the speed a falling object tends toward.

Core Ideas

Limit laws and direct substitution

If $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then the limit of a sum, difference, product, constant multiple, and quotient (denominator limit not 0) is the corresponding combination of the limits.

If f is a polynomial, rational function (with nonzero denominator at a), root, sine, cosine or exponential function that is defined at a , then $\lim_{x \rightarrow a} f(x) = f(a)$. **Always try substitution first.**

When substitution gives $\frac{0}{0}$ the limit may still exist, and you must rewrite the expression so the troublesome factor disappears. The table summarizes what to try:

Situation	Technique	Example
$\frac{0}{0}$ with polynomials	Factor and cancel the common factor	$\frac{x^2-9}{x-3} = x+3$
$\frac{0}{0}$ with a square root	Multiply by the conjugate	$\frac{\sqrt{x+9}-3}{x} = \frac{1}{\sqrt{x+9}+3}$
$\frac{0}{0}$ with fractions in the numerator	Combine into one fraction, then simplify	$\frac{\frac{1}{x+2}-\frac{1}{x}}{x} = \frac{-1}{2(x+2)}$
Trig with small angles	Use the special limits	$\frac{\sin 5x}{3x} \rightarrow \frac{5}{3}$
$x \rightarrow \pm\infty$	Compare degrees / divide by the highest power	$\frac{4x^2-x+1}{2x^2+5} \rightarrow 2$

Why is canceling allowed? A limit only cares about values of x **near** a , never at a . For every $x \neq a$ the original expression and the simplified one are exactly equal, so they approach the same number. That is also why the simplified form is evaluated at a at the end: you are reading off where the simplified function is heading.

The Squeeze Theorem

If $g(x) \leq f(x) \leq h(x)$ for all x near a (except possibly at a) and $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$, then $\lim_{x \rightarrow a} f(x) = L$.

Typical use: $-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$ and both bounds tend to 0, so $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$ even though $\sin \frac{1}{x}$ itself oscillates.

Special trigonometric limits

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0.$$

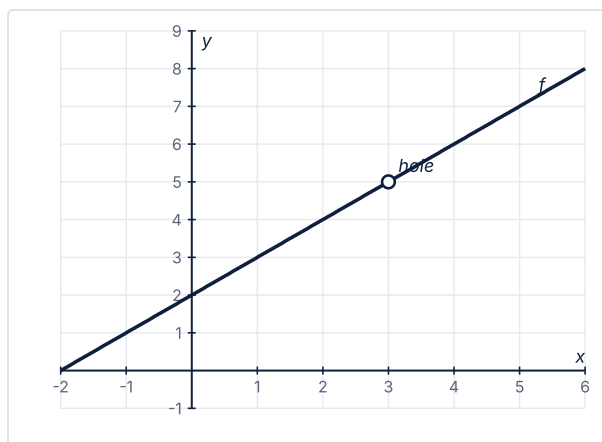
More generally $\lim_{x \rightarrow 0} \frac{\sin(kx)}{kx} = 1$, so $\lim_{x \rightarrow 0} \frac{\sin(kx)}{x} = k$. Also $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$.

A helpful picture is the graph of $f(x) = \frac{x^2-x-6}{x-3}$. Factoring, $f(x) = \frac{(x-3)(x+2)}{x-3} = x+2$ for $x \neq 3$. The graph is the line $y = x+2$ with a hole at $(3, 5)$, so $\lim_{x \rightarrow 3} f(x) = 5$.

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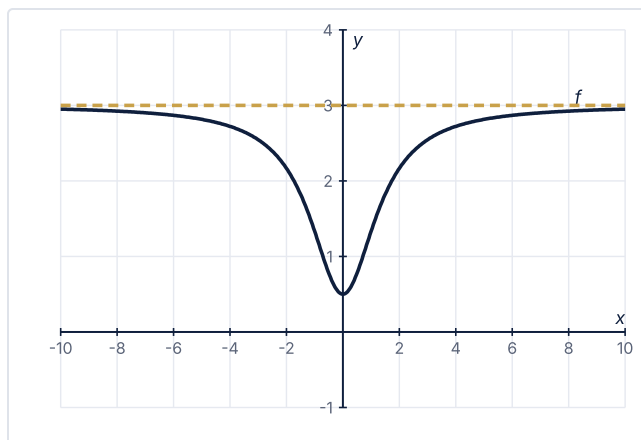
Factoring removes the 0/0: the limit equals the height of the hole.

Limits at infinity of rational functions

For $\frac{p(x)}{q(x)}$ with degrees m (numerator) and n (denominator), as $x \rightarrow \pm\infty$: if $m < n$ the limit is 0; if $m = n$ the limit is the ratio of the leading coefficients; if $m > n$ the values grow without bound (no finite limit).

The line $y = L$ is a **horizontal asymptote** when $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$. Basic fact: $\lim_{x \rightarrow \infty} \frac{1}{x^p} = 0$ for $p > 0$.

x	10	100	1000
$\frac{3x^2+1}{x^2+2}$	2.9510	2.9995	3.0000



The graph of $(3x^2 + 1)/(x^2 + 2)$ levels off at the horizontal asymptote $y = 3$.

Worked Examples

EXAMPLE 1: DIRECT SUBSTITUTION

Find $\lim_{x \rightarrow 2} \frac{x^3 - 3x + 1}{x + 1}$.

SOLUTION

The denominator is 3 at $x = 2$, not 0, so substitute.

$$\frac{8 - 6 + 1}{3} = 1.$$

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EXAMPLE 2: FACTOR AND CANCELFind $\lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x^2 - 9}$.**SOLUTION**Substitution gives $\frac{0}{0}$. Factor: $\frac{(x-2)(x-3)}{(x-3)(x+3)}$.Cancel $x - 3$ (valid since $x \neq 3$ in a limit): $\frac{x-2}{x+3}$.Substitute: $\frac{1}{6}$.**EXAMPLE 3: CONJUGATE**Find $\lim_{x \rightarrow 0} \frac{\sqrt{x+9}-3}{x}$.**SOLUTION**Multiply top and bottom by $\sqrt{x+9} + 3$: numerator becomes $(x+9) - 9 = x$.

$$\frac{x}{x(\sqrt{x+9}+3)} = \frac{1}{\sqrt{x+9}+3}$$

At $x = 0$: $\frac{1}{3+3} = \frac{1}{6}$.**EXAMPLE 4: COMPLEX FRACTION**Find $\lim_{x \rightarrow 0} \frac{\frac{1}{x+2} - \frac{1}{2}}{x}$.**SOLUTION**Combine the numerator: $\frac{1}{x+2} - \frac{1}{2} = \frac{2-(x+2)}{2(x+2)} = \frac{-x}{2(x+2)}$.Divide by x : $\frac{-1}{2(x+2)}$.Limit: $\frac{-1}{2 \cdot 2} = -\frac{1}{4}$.**EXAMPLE 5: SPECIAL TRIG LIMITS**Find (a) $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x}$ and (b) $\lim_{x \rightarrow 0} \frac{\tan x}{x}$.**SOLUTION**(a) Write $\frac{\sin 5x}{3x} = \frac{5}{3} \cdot \frac{\sin 5x}{5x} \rightarrow \frac{5}{3} \cdot 1 = \frac{5}{3}$.(b) $\frac{\tan x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x} \rightarrow 1 \cdot 1 = 1$.**EXAMPLE 6: LIMITS AT INFINITY**Find (a) $\lim_{x \rightarrow \infty} \frac{4x^2 - x + 1}{2x^2 + 5}$ and (b) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 4x} - x)$.**SOLUTION**(a) Divide every term by x^2 : $\frac{4 - \frac{1}{x} + \frac{1}{x^2}}{2 + \frac{5}{x^2}} \rightarrow \frac{4}{2} = 2$.(b) The form is $\infty - \infty$. Multiply by the conjugate: $\frac{(x^2 + 4x) - x^2}{\sqrt{x^2 + 4x} + x} = \frac{4x}{\sqrt{x^2 + 4x} + x}$.Divide by x (positive): $\frac{4}{\sqrt{1 + \frac{4}{x}} + 1} \rightarrow \frac{4}{1+1} = 2$.**Coach's Playbook**

- 1 Substitute first.** If you get a number, you are done. If you get number over zero, expect an infinite limit or DNE.
- 2 Spot $\frac{0}{0}$ and choose a tool.** Polynomials: factor. Roots: conjugate. Fractions: common denominator. Trig: special limits.
- 3 Simplify before substituting again.** Cancel factors, never terms.
- 4 For $x \rightarrow \infty$ compare degrees.** Divide by the highest power in the denominator, or just read leading terms.
- 5 Write the justification.** On the AP exam show the simplified form, then the limit: $' = \lim_{x \rightarrow 3} \frac{x-2}{x+3} = \frac{1}{6} '$.

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- ◆ A limit with $\frac{k}{0}$, $k \neq 0$, is never 0 and never a finite number: check the sides for $\pm\infty$ or DNE.
- ◆ For $x \rightarrow -\infty$ with a square root, $\sqrt{x^2} = |x| = -x$. Take care with the sign.
- ◆ Check a result with a table: it takes ten seconds and catches algebra slips.

Common traps

1) Canceling terms like the x^2 in $\frac{x^2-9}{x-3}$ instead of factoring. 2) Concluding every $\frac{0}{0}$ limit is finite. 3) Writing $\frac{\infty}{\infty} = 1$. 4) Treating $\sqrt{x^2}$ as x when $x \rightarrow -\infty$. 5) Using $\lim_{x \rightarrow 0} \frac{\sin kx}{x} = 1$ without the factor k .

Mistake Clinic

Spot the error 1

A student evaluates $\lim_{x \rightarrow 3} \frac{x^2-9}{x^2-6x+9}$. Factoring gives $\frac{(x-3)(x+3)}{(x-3)^2}$, and the student 'cancels' to get $\frac{x+3}{1} \rightarrow 6$. Only **one** factor of $x-3$ cancels, leaving $\frac{x+3}{x-3}$. At $x=3$ the numerator is 6 and the denominator is 0. From the left $x-3 < 0$ gives large negative values, from the right large positive values. The limit does not exist (it is unbounded).

Spot the error 2

A student writes $\lim_{x \rightarrow \infty} \frac{3x+1}{x^2+4} = \frac{\infty}{\infty} = 1$. The symbol $\frac{\infty}{\infty}$ is indeterminate, not 1. Divide by x^2 : $\frac{\frac{3}{x} + \frac{1}{x^2}}{1 + \frac{4}{x^2}} \rightarrow \frac{0}{1} = 0$. The denominator has the higher degree, so it wins and the limit is 0.

Spot the error 3

A student claims $\lim_{x \rightarrow 0} \frac{\sin 5x}{x} = 1$ because 'sine over its input goes to 1'. The input of the sine is $5x$, but the denominator is x . Write $\frac{\sin 5x}{x} = 5 \cdot \frac{\sin 5x}{5x} \rightarrow 5 \cdot 1 = 5$. The special limit needs the same expression inside the sine and in the denominator.

Real-World Connection

A factory has a setup cost of 500 dollars and each item costs 2 dollars to make. The average cost per item for n items is $A(n) = \frac{500+2n}{n} = \frac{500}{n} + 2$. As $n \rightarrow \infty$, $A(n) \rightarrow 2$. The long-run cost per item approaches 2 dollars, and the horizontal asymptote $y=2$ is the number a manager plans around.

Challenge Corner

Find $\lim_{x \rightarrow 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$. Solution: the form is $\frac{0}{0}$. Multiply top and bottom by the conjugate $\sqrt{1+x} + \sqrt{1-x}$. The numerator becomes $(1+x) - (1-x) = 2x$. So the expression equals $\frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})} = \frac{2}{\sqrt{1+x}+\sqrt{1-x}}$. As $x \rightarrow 0$ this tends to $\frac{2}{1+1} = 1$.

Practice

PRACTICE Evaluating Limits Algebraically

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is $\lim_{x \rightarrow 1} \frac{x^2+3x-2}{x+1}$?
 (A) 1 (B) 0
 (C) 2 (D) The limit does not exist.
- What is $\lim_{x \rightarrow -2} \frac{x^2-4}{x^2+5x+6}$?
 (A) 4 (B) 0 (C) -4 (D) $-\frac{4}{5}$
- What is $\lim_{x \rightarrow 4} \frac{\sqrt{x}-2}{x-4}$?
 (A) 0 (B) 4 (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

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4. What is $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$?
 (A) 1 (B) 3 (C) $\frac{1}{3}$ (D) 0
5. What is $\lim_{x \rightarrow \infty} \frac{5x^2+3}{2x^2-x}$?
 (A) 0 (B) 5 (C) $\frac{5}{2}$ (D) ∞
6. What is $\lim_{x \rightarrow \infty} \frac{4x+1}{x^2+3}$?
 (A) 0 (B) 4 (C) 1 (D) ∞
7. What is $\lim_{x \rightarrow 0} \frac{\frac{1}{x+3} - \frac{1}{3}}{x}$?
 (A) 0 (B) $\frac{1}{9}$ (C) $-\frac{1}{3}$ (D) $-\frac{1}{9}$
8. Which describes $\lim_{x \rightarrow \infty} \frac{x^2+1}{3x+2}$?
 (A) The limit is 0.
 (B) The limit is $\frac{1}{3}$.
 (C) The values grow without bound; there is no finite limit.
 (D) The limit is 1.
9. What is $\lim_{x \rightarrow 0} \frac{\tan 2x}{5x}$?
 (A) $\frac{5}{2}$ (B) $\frac{2}{5}$ (C) 2 (D) 0
10. Find $\lim_{x \rightarrow -3} \frac{x^2+x-6}{x+3}$.
11. Find $\lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3}$.
12. Find $\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 7x}$.
13. Find $\lim_{x \rightarrow \infty} (\sqrt{x^2+6x}-x)$.
14. Find $\lim_{x \rightarrow 1} \frac{\frac{1}{x}-1}{x-1}$.
15. Find $\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2+1}}{x+3}$.

36.3 Continuity and Discontinuities

Learn the three-condition test for continuity, classify removable, jump and infinite discontinuities, and find constants that make piecewise functions continuous.

Why This Matters

Informally, a function is **continuous** if you can draw its graph without lifting your pencil. That picture is useful, but calculus needs a precise test that works for formulas as well as pictures. Students get stuck because they treat 'defined at the point' and 'continuous at the point' as the same thing, because they cannot tell a hole from a jump from an asymptote, and because setting up equations for piecewise functions feels like a new kind of algebra.

Continuity is the hypothesis of the biggest theorems in calculus: the Intermediate Value Theorem, the Extreme Value Theorem and the Mean Value Theorem all begin with 'if f is continuous on...'. In applications it separates smooth physical processes, like temperature, from step-like ones such as a parking fee that jumps when an hour is crossed.

Core Ideas

Continuity at a point

A function f is **continuous at** $x = a$ if all three conditions hold:

- 1) $f(a)$ is defined. 2) $\lim_{x \rightarrow a} f(x)$ exists. 3) $\lim_{x \rightarrow a} f(x) = f(a)$.

If any condition fails, f is **discontinuous** at a . A function is continuous on an interval if it is continuous at every point of the interval (using one-sided limits at closed endpoints).

Polynomials are continuous everywhere. Rational functions are continuous everywhere their denominator is not zero. Roots, trigonometric, exponential and logarithmic functions are continuous on their domains. Sums, products, quotients (denominator nonzero) and compositions of continuous functions are continuous. So discontinuities of familiar formulas can only occur where something is undefined (a zero denominator, a negative under an even root) or where the formula changes (piecewise definitions).

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Types of discontinuity

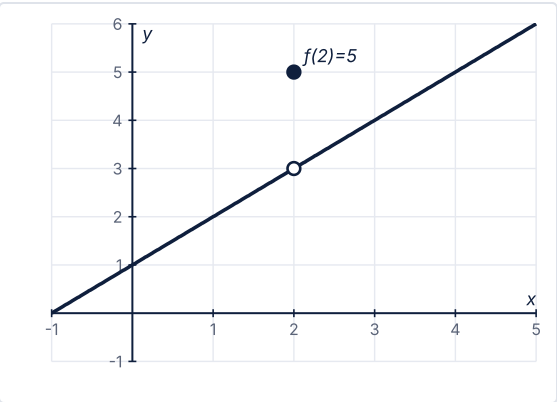
Removable: $\lim_{x \rightarrow a} f(x)$ exists (a finite number) but $f(a)$ is missing or different. The graph has a hole; redefining $f(a)$ to equal the limit repairs it.

Jump: the one-sided limits exist but are different. The graph jumps from one height to another.

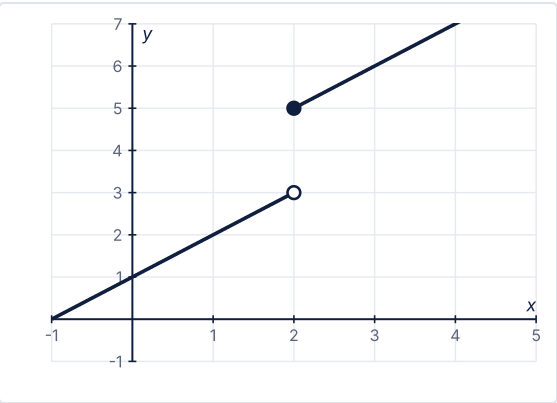
Infinite: at least one one-sided limit is $\pm\infty$ (a vertical asymptote).

A fourth case, **oscillating**, occurs for functions like $\sin \frac{1}{x}$ at 0.

Type	Limit at a	Picture	Fixable?
Removable	Exists, finite	Hole	Yes, redefine $f(a)$
Jump	One-sided limits differ	Step	No
Infinite	$\pm\infty$ from a side	Vertical asymptote	No
Oscillating	Does not exist	Wild wiggles	No



Removable: the limit at $x = 2$ is 3 but $f(2) = 5$.

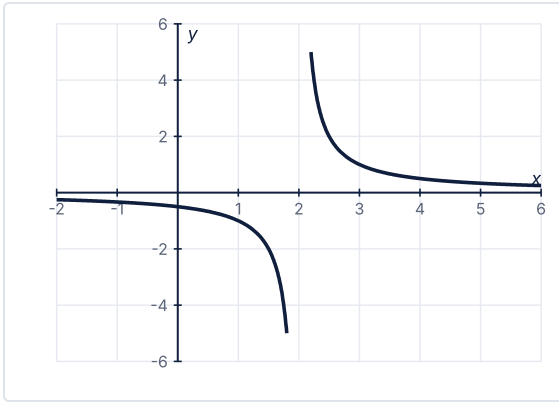


Jump: the left limit is 3 and the right limit is 5 at $x = 2$.

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Infinite: $f(x) = 1/(x - 2)$ has a vertical asymptote at $x = 2$.

A piecewise function can be continuous only if the pieces **meet** at each break. To test the break at $x = c$, compute the left limit with the left formula, the right limit with the right formula, and compare both with $f(c)$.

Two more ideas are worth knowing. First, **one-sided continuity**: f is continuous from the right at a if $\lim_{x \rightarrow a^+} f(x) = f(a)$, and from the left if $\lim_{x \rightarrow a^-} f(x) = f(a)$. A function is continuous at an interior point exactly when it is continuous from both sides, and on a closed interval $[a, b]$ it only needs right-continuity at a and left-continuity at b . Second, a continuous function can have a **sharp corner**, as $|x|$ does at 0. Continuity says there are no breaks; it does not say the graph is smooth, a distinction that matters when you study derivatives.

Making a function continuous

To choose constants so that a piecewise function is continuous at $x = c$, set the left-hand expression at c equal to the right-hand expression at c (and to the given value $f(c)$ if there is one). Solve the resulting equation or system for the constants.

Worked Examples

EXAMPLE 1: TEST THE THREE CONDITIONS

Is $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$ continuous at $x = 1$?

SOLUTION

Condition 1: $f(1) = 3$ is defined.

Condition 2: for $x \neq 1$, $\frac{x^2-1}{x-1} = x+1$, so $\lim_{x \rightarrow 1} f(x) = 2$.

Condition 3: $\lim f = 2 \neq 3 = f(1)$ fails. The function is discontinuous at 1 (removable); defining $f(1) = 2$ would repair it.

EXAMPLE 2: CLASSIFY DISCONTINUITIES OF A RATIONAL FUNCTION

Find and classify the discontinuities of $f(x) = \frac{x^2-x-6}{x^2-9}$.

SOLUTION

Factor: $f(x) = \frac{(x-3)(x+2)}{(x-3)(x+3)}$. The denominator is zero at $x = 3$ and $x = -3$.

At $x = 3$ the factor cancels: $\lim_{x \rightarrow 3} \frac{x+2}{x+3} = \frac{5}{6}$. **Removable**.

At $x = -3$: the factor $x + 3$ remains. From the left the quotient tends to $+\infty$ and from the right to $-\infty$. **Infinite** (vertical asymptote).

EXAMPLE 3: A JUMP IN A PIECEWISE FUNCTION

Let $f(x) = \begin{cases} 2x - 1, & x < 3 \\ x^2 - 2, & x \geq 3 \end{cases}$. Is f continuous at 3? Classify.

SOLUTION

Left limit: $2(3) - 1 = 5$. Right limit: $3^2 - 2 = 7$. Also $f(3) = 7$.

The one-sided limits differ, so $\lim_{x \rightarrow 3} f(x)$ does not exist. **Jump discontinuity** of size $7 - 5 = 2$.

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EXAMPLE 4: MAKE A FUNCTION CONTINUOUS

Find k so that $f(x) = \begin{cases} \frac{x^2-16}{x-4}, & x \neq 4 \\ k, & x = 4 \end{cases}$ is continuous at 4.

SOLUTION

For $x \neq 4$: $\frac{(x-4)(x+4)}{x-4} = x+4$, so $\lim_{x \rightarrow 4} f(x) = 8$.

Continuity requires $f(4) = \lim_{x \rightarrow 4} f(x)$, so $k = 8$.

EXAMPLE 5: TWO UNKNOWN CONSTANTS

Find a and b so that $f(x) = \begin{cases} x^2, & x < 1 \\ ax + b, & 1 \leq x \leq 3 \\ 12 - x, & x > 3 \end{cases}$ is continuous for all x .

SOLUTION

At $x = 1$: $1^2 = a(1) + b$, so $a + b = 1$.

At $x = 3$: $a(3) + b = 12 - 3$, so $3a + b = 9$.

Subtract: $2a = 8$, so $a = 4$, then $b = 1 - 4 = -3$.

EXAMPLE 6: CONTINUITY ON INTERVALS

State where $g(x) = \frac{\sqrt{x-2}}{x-5}$ is continuous.

SOLUTION

The square root needs $x - 2 \geq 0$, so $x \geq 2$. The denominator is zero at $x = 5$.

g is continuous on $[2, 5) \cup (5, \infty)$. At the endpoint $x = 2$ it is continuous from the right. At $x = 5$ there is an infinite discontinuity (the numerator is $\sqrt{3} \neq 0$).

Coach's Playbook

- 1 List the trouble spots.** Zeros of denominators, even-root restrictions, and break points of piecewise definitions.
- 2 Run the three-condition test at each spot.** Is it defined? Does the limit exist? Are they equal?
- 3 Factor to classify.** If a factor cancels, it is removable. If a factor stays in the denominator, it is infinite.
- 4 For parameter problems, write the equation first.** Left piece at c = right piece at c . Then solve.
- 5 Check by substitution.** Put your constants back and compute both sides.

- ◆ On the AP exam, justify continuity in a sentence: ' $f(a)$ exists, the limit exists, and they are equal.'
- ◆ If the question says 'continuous for all x ', every break point needs its own equation.
- ◆ Draw a quick sketch of the pieces near each break; seeing the gap often shows the answer.

Common traps

1) Saying 'defined at the point' means continuous there. 2) Calling every zero of a denominator a vertical asymptote (cancellation gives a hole). 3) Assuming a piecewise function made of continuous pieces is continuous. 4) Matching the pieces at the wrong x -value.

Mistake Clinic

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Spot the error 1

For $f(x) = \frac{x^2-1}{x-1}$ with $f(1) = 3$, a student says 'f(1) is defined, so f is continuous at 1'. Being defined is only the first condition. The limit is 2 but $f(1) = 3$, so the third condition fails. The function has a removable discontinuity.

Spot the error 2

A student says $\frac{x-3}{x^2-9}$ has vertical asymptotes at both $x = 3$ and $x = -3$. Factor: $\frac{x-3}{(x-3)(x+3)} = \frac{1}{x+3}$. At $x = 3$ the factor cancels, leaving a **hole** at $(3, \frac{1}{6})$. Only $x = -3$ is a vertical asymptote.

Spot the error 3

A student argues that $f(x) = 2x - 1$ for $x < 3$ and $f(x) = x^2 - 2$ for $x \geq 3$ is continuous everywhere 'because both pieces are polynomials'. Each piece is continuous, but they must also meet at $x = 3$. The left piece gives 5 and the right piece gives 7, so there is a jump at 3.

Real-World Connection

A parking garage charges 4 dollars for any time up to one hour and 7 dollars for time above one hour and up to two hours. The fee graph is a staircase: constant, then a jump of 3 dollars the instant one hour is passed. Fee functions are discontinuous at each break, while the temperature of a cooling drink is continuous. Knowing which model fits your situation tells you whether tools that need continuity can be used.

Challenge Corner

The function $f(x) = \begin{cases} \frac{x^2+ax+b}{x-2}, & x \neq 2 \\ 5, & x = 2 \end{cases}$ is continuous at 2. Find a and b . Solution: for the limit at 2 to exist, the numerator must vanish at 2 (otherwise the quotient is unbounded): $4 + 2a + b = 0$. Then $x^2 + ax + b = (x - 2)(x + m)$ and the limit is $2 + m$. Setting $2 + m = 5$ gives $m = 3$, so $(x - 2)(x + 3) = x^2 + x - 6$, which gives $a = 1$ and $b = -6$. Check: $4 + 2 - 6 = 0$ and $\lim_{x \rightarrow 2} \frac{x^2+x-6}{x-2} = \lim(x+3) = 5$.

Practice**PRACTICE** Continuity and Discontinuities

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

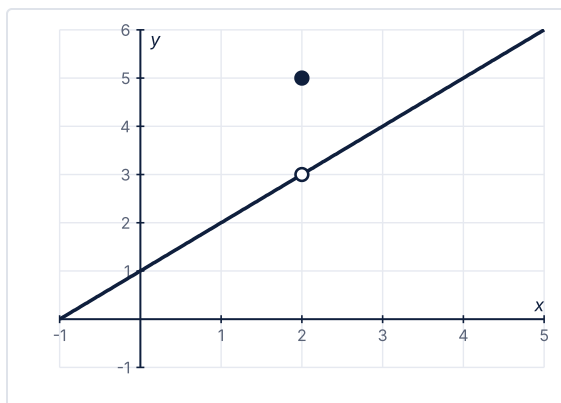
- Which list gives the three conditions for f to be continuous at $x = a$?
 - $f(a)$ is defined, $\lim_{x \rightarrow a} f(x)$ exists, and the two are equal.
 - $f(a)$ is defined and f is differentiable at a .
 - $f(a) = 0$ and the limit exists.
 - The limit exists and f is increasing at a .
- What type of discontinuity does $f(x) = \frac{x^2-4}{x-2}$ have at $x = 2$?
 - Jump
 - Removable
 - Infinite
 - No discontinuity
- What type of discontinuity does $f(x) = \frac{1}{x-3}$ have at $x = 3$?
 - Removable
 - Jump
 - Infinite
 - Oscillating
- Let $f(x) = x + 1$ for $x < 0$ and $f(x) = x^2 + 3$ for $x \geq 0$. What happens at $x = 0$?
 - A removable discontinuity
 - An infinite discontinuity
 - f is continuous.
 - A jump discontinuity

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5. For which value of k is $f(x) = \begin{cases} kx + 1, & x \leq 2 \\ x^2 - 1, & x > 2 \end{cases}$ continuous at $x = 2$?
- (A) $k = 1$ (B) $k = 3$ (C) $k = 2$ (D) $k = 0$
6. Where is $f(x) = \frac{x+1}{x^2-x-6}$ discontinuous?
- (A) $x = 3$ and $x = 2$ (B) $x = 3$ and $x = -2$ (C) $x = -3$ and $x = 2$
- (D) $x = -1$ only
7. The graph of f follows $y = x + 1$ except for an open circle at $(2, 3)$ and a filled dot at $(2, 5)$. What is true at $x = 2$?



- (A) f is continuous at 2 .
- (B) There is a jump discontinuity.
- (C) $\lim_{x \rightarrow 2} f(x) = 3$ but $f(2) = 5$, so there is a removable discontinuity.
- (D) The limit does not exist.
8. What value should be assigned to $f(0)$ so that $f(x) = \frac{\sin x}{x}$ for $x \neq 0$ becomes continuous at 0?
- (A) 0 (B) $\frac{1}{2}$ (C) ∞ (D) 1
9. Which function is continuous for all real x ?
- (A) $h(x) = \frac{x^2 - 1}{x - 1}$ (B) $h(x) = x^2 + 3|x|$ (C) $h(x) = \tan x$
- (D) $h(x) = \sqrt{x - 1}$
10. Find k so that $f(x) = \begin{cases} \frac{x^2 - 25}{x - 5}, & x \neq 5 \\ k, & x = 5 \end{cases}$ is continuous at $x = 5$.
11. Classify the discontinuity of $f(x) = \frac{x+2}{x^2+4x+4}$ at $x = -2$.
12. For $f(x) = \begin{cases} 2x - 1, & x < 3 \\ x^2 - 2, & x \geq 3 \end{cases}$ find the size of the jump at $x = 3$.
13. Find a so that $f(x) = \begin{cases} x^2 + a, & x < 2 \\ 3x - 1, & x \geq 2 \end{cases}$ is continuous at $x = 2$.
14. Find and classify all discontinuities of $g(x) = \frac{x^2 - 1}{x^2 - 4x + 3}$.
15. Find a and b so that $f(x) = \begin{cases} 2x + 3, & x < -1 \\ ax + b, & -1 \leq x \leq 2 \\ x^2, & x > 2 \end{cases}$ is continuous everywhere.

36.4 The Intermediate Value Theorem and Asymptotes

Use the Intermediate Value Theorem to prove solutions exist, write AP-style justifications, and find vertical, horizontal and slant asymptotes and end behavior of functions.

Why This Matters

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Two ideas close this booster: what a continuous function **must** do between two known values, and what a function does at the **edges** of its graph. The first is the Intermediate Value Theorem (IVT), which guarantees that equations have solutions even when you cannot find them by algebra. The second is the study of asymptotes and end behavior, which describes where a function blows up and where it settles down. Students get stuck by applying the IVT without checking continuity, by treating it as a way to find the solution rather than prove one exists, and by mixing up vertical asymptotes (inputs where the function blows up) with horizontal ones (outputs the function levels off to).

Root-finding software uses the IVT idea every day: if the sign of a continuous function changes on an interval, a root is trapped inside, and halving the interval again and again closes in on it. Asymptotes describe long-run behavior in physics, economics and biology: a falling object approaches terminal speed, an average cost approaches a floor, a population approaches a limit.

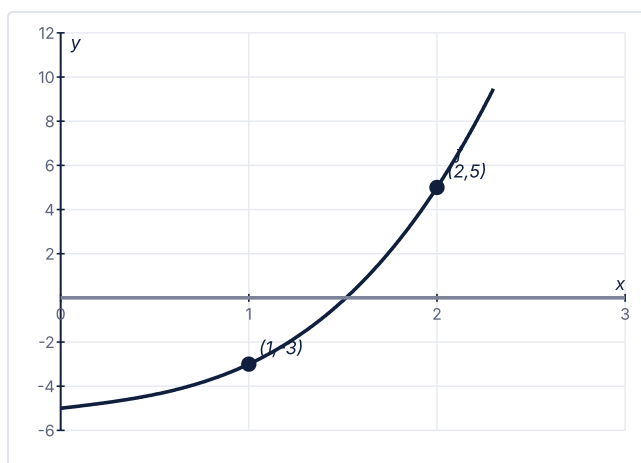
Core Ideas

The Intermediate Value Theorem

If f is **continuous on the closed interval** $[a, b]$ and N is any number strictly between $f(a)$ and $f(b)$, then there is at least one number c in (a, b) with $f(c) = N$.

Zero version: if f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then $f(c) = 0$ for some c in (a, b) .

The theorem is an existence statement. It does not say how many such c there are or where they are. Continuity is essential: $f(x) = \frac{1}{x}$ goes from -1 at $x = -1$ to 1 at $x = 1$ without ever being 0 , because it is not continuous at 0 .



$f(x) = x^3 + x - 5$ is continuous with $f(1) = -3$ and $f(2) = 5$, so the graph must cross the x -axis between $x = 1$ and $x = 2$.

AP-style IVT justification

Write four things: (1) f is continuous on $[a, b]$; (2) the values $f(a)$ and $f(b)$; (3) that the target value lies between them; (4) the conclusion 'by the Intermediate Value Theorem there is a c in (a, b) with $f(c) = N$ '.

Vertical and horizontal asymptotes

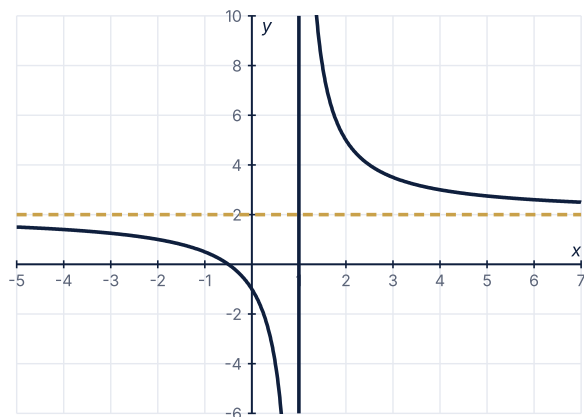
The line $x = a$ is a **vertical asymptote** of f if $\lim_{x \rightarrow a^-} f(x)$ or $\lim_{x \rightarrow a^+} f(x)$ is $+\infty$ or $-\infty$. For a rational function, check zeros of the denominator **after canceling common factors**.

The line $y = L$ is a **horizontal asymptote** if $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$. A function can have 0, 1 or 2 horizontal asymptotes (a rational function has at most one).

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$f(x) = (2x + 1)/(x - 1)$ has the vertical asymptote $x = 1$ and the horizontal asymptote $y = 2$.

Function type	End behavior as $x \rightarrow \pm\infty$
Rational, degree top < bottom	approaches $y = 0$
Rational, degrees equal	approaches ratio of leading coefficients
Rational, top degree one more	slant asymptote (divide polynomials)
Rational, top degree two or more greater	grows without bound, no line asymptote
$\sqrt{x^2 + \dots}$ in a quotient	may give different limits at $+\infty$ and $-\infty$

Growth rates

As $x \rightarrow \infty$: $\ln x$ grows slower than any positive power x^p , and x^p grows slower than e^x . So $\lim_{x \rightarrow \infty} \frac{\ln x}{x^p} = 0$ and $\lim_{x \rightarrow \infty} \frac{x^p}{e^x} = 0$ for every $p > 0$.

Worked Examples

EXAMPLE 1: SHOW A ROOT EXISTS

Show that $f(x) = x^3 + x - 5$ has a zero in $(1, 2)$.

SOLUTION

f is a polynomial, so it is continuous on $[1, 2]$.

$f(1) = 1 + 1 - 5 = -3 < 0$ and $f(2) = 8 + 2 - 5 = 5 > 0$.

Since 0 lies between -3 and 5 , by the Intermediate Value Theorem there is a c in $(1, 2)$ with $f(c) = 0$.

EXAMPLE 2: WHAT IS GUARANTEED?

A function f is continuous on $[0, 4]$ with $f(0) = 2$ and $f(4) = 10$. For which of $N = 7$ and $N = 12$ is a solution of $f(x) = N$ guaranteed?

SOLUTION

$N = 7$ lies between 2 and 10, so there is a c in $(0, 4)$ with $f(c) = 7$.

$N = 12$ is outside $[2, 10]$, so the theorem guarantees nothing (the function might still reach 12, but we cannot conclude it).

EXAMPLE 3: COUNT ZEROS FROM A TABLE

The continuous function g has values $g(0) = 3$, $g(2) = -1$, $g(4) = -4$, $g(6) = 2$, $g(8) = 5$. What is the least number of zeros of g on $[0, 8]$?

SOLUTION

Sign changes: $3 \rightarrow -1$ on $[0, 2]$ and $-4 \rightarrow 2$ on $[4, 6]$. No sign change on $[2, 4]$ or $[6, 8]$.

By the IVT there is at least one zero in $(0, 2)$ and one in $(4, 6)$: at least **2** zeros.

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EXAMPLE 4: VERTICAL ASYMPTOTE OR HOLE?

Find the vertical asymptotes of $f(x) = \frac{x^2-4}{x^2-x-6}$.

SOLUTION

Factor: $\frac{(x-2)(x+2)}{(x-3)(x+2)}$. Denominator zeros: $x = 3$ and $x = -2$.

At $x = -2$ the factor cancels: $\lim_{x \rightarrow -2} = \frac{-4}{-5} = \frac{4}{5}$, a hole.

At $x = 3$: from the right $\frac{x-2}{x-3} \rightarrow +\infty$ and from the left $\rightarrow -\infty$. The only vertical asymptote is $x = 3$.

EXAMPLE 5: HORIZONTAL ASYMPTOTES

Find the horizontal asymptotes of (a) $f(x) = \frac{6x^2-1}{2x^2+x}$ and (b) $g(x) = \frac{3x}{\sqrt{x^2+1}}$.

SOLUTION

(a) Equal degrees: the limit is $\frac{6}{2} = 3$. So $y = 3$.

(b) As $x \rightarrow \infty$: $\sqrt{x^2+1} \approx x$, so $g \rightarrow 3$. As $x \rightarrow -\infty$: $\sqrt{x^2+1} \approx |x| = -x$, so $g \rightarrow \frac{3x}{-x} = -3$.

Two horizontal asymptotes: $y = 3$ and $y = -3$.

EXAMPLE 6: SLANT ASYMPTOTE

Find the asymptotes of $f(x) = \frac{x^2+1}{x-1}$.

SOLUTION

Vertical: $x = 1$ (the numerator is $2 \neq 0$ there, so nothing cancels).

Divide: $x^2 + 1 = (x-1)(x+1) + 2$, so $f(x) = x+1 + \frac{2}{x-1}$.

As $x \rightarrow \pm\infty$ the fraction tends to 0, so f approaches the line $y = x+1$: a slant asymptote.

Coach's Playbook

- 1 Check continuity first.** For the IVT, name the function type or the reason it is continuous on the closed interval.
 - 2 Evaluate the endpoints.** Write both values with their signs.
 - 3 Compare the target.** Is N strictly between $f(a)$ and $f(b)$? If not, stop: nothing is guaranteed.
 - 4 Cancel before hunting for asymptotes.** Factor numerator and denominator, cancel common factors (holes), then read vertical asymptotes.
 - 5 Use degrees and signs for end behavior.** Compare degrees, and watch the sign when $x \rightarrow -\infty$ with roots.
- ◆ A bisection check: evaluate f at the midpoint and keep the half where the sign still changes.
 - ◆ A graph may cross a horizontal asymptote in the middle; asymptotes describe the **ends**.
 - ◆ Vertical asymptote: look at inputs. Horizontal asymptote: look at outputs as inputs get huge.

Common traps

1) Using the IVT on a function that is not continuous on the whole interval. 2) Believing 'same signs at the ends' proves there is no root. 3) Declaring every zero of the denominator a vertical asymptote. 4) Forgetting the second horizontal asymptote when roots give different limits at $\pm\infty$.

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Spot the error 1

A student writes: 'For $f(x) = \frac{1}{x}$ we have $f(-1) = -1$ and $f(1) = 1$. By the IVT there is a zero between -1 and 1 .' The theorem needs f to be continuous on $[-1, 1]$, but f is undefined at 0 and has an infinite discontinuity there. The conclusion is false: $\frac{1}{x}$ is never 0 .

Spot the error 2

A student says: 'Since $f(0) = -3$ and $f(4) = -3$ are both negative, f has no zero on $[0, 4]$.' The IVT only works in one direction. It says a sign change guarantees a zero; it does **not** say no sign change means no zero. The function could rise above 0 and come back down (for example $f(x) = -3 + x(4 - x)$ is 1 at $x = 2$, so it crosses zero twice).

Spot the error 3

A student reports that $g(x) = \frac{3x}{\sqrt{x^2+1}}$ has only the horizontal asymptote $y = 3$. For $x < 0$, $\sqrt{x^2+1} \approx |x| = -x$, so the quotient tends to $\frac{3x}{-x} = -3$. The graph has two horizontal asymptotes, $y = 3$ on the right and $y = -3$ on the left.

Real-World Connection

A car accelerates smoothly from 20 mph to 60 mph. Speed is a continuous function of time, so by the IVT the speedometer must read exactly 45 mph at some moment, even though no one wrote down when. Engineers use the same idea to trap a root: if a continuous measurement is below target at one time and above target at another, then the target was hit in between.

Challenge Corner

Show that $x = \cos x$ has a solution in $(0, 1)$ and narrow it to an interval of length 0.25 . Solution: let $f(x) = x - \cos x$, continuous everywhere. $f(0) = -1 < 0$ and $f(1) = 1 - \cos 1 \approx 0.460 > 0$, so a root lies in $(0, 1)$. Bisection: $f(0.5) \approx -0.378 < 0$, so the root is in $(0.5, 1)$. Next $f(0.75) \approx 0.018 > 0$, so the root lies in $(0.5, 0.75)$, an interval of length 0.25 . (The root is about 0.739 .)

Practice**PRACTICE The Intermediate Value Theorem and Asymptotes**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- A function f is continuous on $[1, 4]$ with $f(1) = -3$ and $f(4) = 6$. Which conclusion is guaranteed by the IVT?

(A) There is a c in $(1, 4)$ with $f(c) = 0$.	(B) $f(c) = 0$ for exactly one c in $(1, 4)$.
(C) f is increasing on $[1, 4]$.	(D) $f(c) = 10$ for some c in $(1, 4)$.
- On which interval does the IVT NOT apply to $f(x) = \frac{1}{x-1}$?

(A) $[2, 5]$	(B) $[0, 2]$	(C) $[-3, 0]$	(D) $[1.5, 4]$
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- For $g(x) = x^3 + 2x - 4$, which interval is guaranteed by the IVT to contain a zero?

(A) $[-1, 0]$	(B) $[0, 1]$	(C) $[1, 2]$	(D) $[2, 3]$
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- What are the vertical asymptotes of $f(x) = \frac{x-1}{x^2-1}$?

(A) $x = 1$ and $x = -1$	(B) $x = 1$ only	(C) There are none.
(D) $x = -1$ only		
- What is the horizontal asymptote of $f(x) = \frac{6x^2-1}{2x^2+x}$?

(A) $y = 6$	(B) $y = 3$	(C) $y = 0$	(D) $y = 2$
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- What is $\lim_{x \rightarrow -\infty} \frac{3x}{\sqrt{x^2+1}}$?

(A) 3	(B) 0	(C) -3	(D) $-\infty$
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7. What is $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$?
 (A) 0 (B) 1 (C) 2 (D) ∞
8. What is the slant asymptote of $f(x) = \frac{x^2+1}{x-1}$?
 (A) $y = x + 1$ (B) $y = x - 1$ (C) $y = x$ (D) $y = 1$
9. A function f is continuous on $[0, 4]$ with $f(0) = 2$ and $f(4) = 10$. For which value of N is a solution of $f(x) = N$ NOT guaranteed on $(0, 4)$?
 (A) 3 (B) 12 (C) 5 (D) 9
10. Find the integer interval $[n, n + 1]$ that contains the positive zero guaranteed by the IVT for $f(x) = x^3 - 4x - 2$. Justify with values.
11. Find all vertical asymptotes of $f(x) = \frac{x^2-9}{x^2-5x+6}$.
12. Find $\lim_{x \rightarrow \infty} \frac{4x-3}{\sqrt{x^2+9}}$ and $\lim_{x \rightarrow -\infty} \frac{4x-3}{\sqrt{x^2+9}}$.
13. Find $\lim_{x \rightarrow \infty} \frac{x^5}{e^x}$.
14. Find the slant asymptote of $f(x) = \frac{2x^2-3x+1}{x+1}$.
15. A continuous function g has $g(0) = -3$, $g(2) = 1$, $g(4) = -2$, $g(7) = 4$, $g(10) = -1$. What is the least number of zeros of g on $(0, 10)$?

Answers**Answers and Explanations**

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

36.1 The Idea of a Limit**PRACTICE · THE IDEA OF A LIMIT**

- B. The outputs approach 8 from both sides. Choice A is just one early value, and C is the input, not the output.
- D. Cancel: $(x-2)(x+2)/(x-2) = x + 2$, which approaches 4. Choice A stops at 0/0 and C believes 0/0 means DNE.
- A. The curve heads to height 3 from both sides. The filled point gives $f(2) = 1$, but the value does not affect the limit.
- C. For $x > 0$, $|x| = x$, so the quotient is 1. Choice A is the left limit, and D is the two-sided answer.
- B. $1/x^2$ is positive and huge near 0 from both sides. Choice A confuses the behavior with large x , and D has the wrong sign.
- C. Both one-sided limits equal 5, so the limit is 5. The value $f(3) = 2$ is irrelevant.
- A. Left: $2(1)+1 = 3$. Right: $1 + 2 = 3$. They agree. Choice B mixes up the pieces and D ignores that the sides match.
- B. Left $3k - 2$ must equal right 7, so $k = 3$. Choice D confuses k with the limit value 7, and C solves $3k = 5$ by forgetting to add back the 2.
- D. At $x = 0.1$ the value is 0.04996, at 0.01 it is 0.005, and at -0.1 it is -0.04996. The values approach 0 from both sides.
10. Factor: $\frac{(x-5)(x+5)}{x-5} = x + 5 \rightarrow 10$.
11. 1. Factor $x^2 + 3x + 2 = (x+1)(x+2)$, cancel, then $x + 2 \rightarrow 1$.
12. Left limit 1, right limit 5. They differ, so the limit does not exist.
13. The limit is $-\infty$: values like -100, -1000 grow without bound in the negative direction (no finite limit).
14. $k = \frac{1}{2}$. Left: $k + 2$. Right: $3 - k$. Setting $k + 2 = 3 - k$ gives $2k = 1$.
15. $\frac{1}{4}$. At $x = 4.01$ the value is about 0.24984 and at 3.99 about 0.25016. Exactly: multiply by the conjugate to get $\frac{1}{\sqrt{x}+2} \rightarrow \frac{1}{4}$.

36.2 Evaluating Limits Algebraically**PRACTICE · EVALUATING LIMITS ALGEBRAICALLY**

- A. The denominator is 2 at $x = 1$, so substitute: $(1 + 3 - 2)/2 = 1$. Choice D treats it as if the denominator vanished.
- C. Factor: $(x-2)(x+2)/((x+2)(x+3)) = (x-2)/(x+3)$, which is $-4/1 = -4$. Choice A loses the sign and B stops at 0/0.
- D. Multiply by the conjugate: $(x-4)/((x-4)(\sqrt{x}+2)) = 1/(\sqrt{x}+2)$, giving $1/4$. Choice C forgets to add the 2 in the denominator.
- B. $\sin(3x)/x = 3 \sin(3x)/(3x)$ tends to 3. Choice A ignores the 3, C inverts it.
- C. The degrees match, so the limit is the ratio of leading coefficients, $5/2$. Choice A is for a higher-degree denominator and D for a higher-degree numerator.
- A. The denominator has the higher degree, so the fraction tends to 0. Choice B uses the numerator's coefficient.
- D. Combine: $(3 - (x+3))/(3(x+3)) = -x/(3(x+3))$; divide by x to get $-1/(3(x+3))$ which tends to $-1/9$. Choice C forgets the factor $x+3$ in the denominator.
- C. The numerator has the higher degree, so the quotient grows like $x/3$. Choice A confuses which degree is higher and B uses only the leading coefficients.
- B. $\tan(2x)/(5x) = (2/5)(\tan 2x)/(2x)$, which tends to $(2/5)(1)$. Choice A inverts the ratio.
- 5. Factor $(x+3)(x-2)$, cancel, then $x - 2 \rightarrow -5$.
6. Multiply by $\sqrt{x} + 3$ to get $\sqrt{x} + 3 \rightarrow 6$ (or factor $x - 9 = (\sqrt{x} - 3)(\sqrt{x} + 3)$).
- $\frac{4}{7}$. Write $\frac{\sin 4x}{4x} \cdot \frac{7x}{\sin 7x} \cdot \frac{4}{7} \rightarrow \frac{4}{7}$.
3. Multiply by the conjugate: $\frac{6x}{\sqrt{x^2+6x}+x} = \frac{6}{\sqrt{1+6/x}+1} \rightarrow 3$.
- 1. $\frac{1}{x} - 1 = \frac{1-x}{x}$, so the quotient is $\frac{-1}{x} \rightarrow -1$.

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15. -2 . For $x < 0$, $\sqrt{4x^2 + 1} \approx 2|x| = -2x$, so the quotient is about $\frac{-2x}{x} = -2$.

36.3 Continuity and Discontinuities

PRACTICE · CONTINUITY AND DISCONTINUITIES

1. A. These are the three conditions of the definition. Choices B, C and D add requirements (differentiability, zero value, monotonicity) that are not part of continuity.
2. B. The expression simplifies to $x + 2$ with a hole at $x = 2$, so it is removable. Choice D forgets that $f(2)$ is undefined.
3. C. No factor cancels and the values grow without bound, so there is a vertical asymptote. A hole or a jump needs a finite one-sided limit.
4. D. The left limit is 1 and the right limit is 3, so the one-sided limits differ: a jump. Choice C fails because the limits do not agree.
5. A. Match the pieces: $2k + 1 = 4 - 1 = 3$, so $k = 1$. Choice B sets k equal to the right value 3.
6. B. The denominator factors as $(x-3)(x+2)$. Choice A and C flip signs, and D uses the numerator's zero.
7. C. The curve approaches 3 from both sides while the point is at 5. Choice B requires different one-sided limits, and D ignores the matching sides.
8. D. The limit of $\sin x / x$ at 0 is 1, so $f(0)$ must be 1. Choice A is tempting because $\sin 0 = 0$, but the limit is what matters.
9. B. Polynomials and absolute values are continuous everywhere. The others are undefined at some real numbers.
10. $k = 10$. $\frac{x^2 - 25}{x - 5} = x + 5 \rightarrow 10$.
11. Infinite. $f(x) = \frac{1}{x+2}$ after canceling one factor, so $x = -2$ is a vertical asymptote.
12. 2. The right limit is 7 and the left limit is 5.
13. $a = 1$. Left: $4 + a$. Right: 5. So $4 + a = 5$.
14. $x = 1$ is removable (limit -1); $x = 3$ is infinite (vertical asymptote). Factor: $\frac{(x-1)(x+1)}{(x-1)(x-3)}$.
15. $a = 1$, $b = 2$. At $x = -1$: $-a + b = 1$. At $x = 2$: $2a + b = 4$. Subtract: $3a = 3$.

36.4 The Intermediate Value Theorem and Asymptotes

PRACTICE · THE INTERMEDIATE VALUE THEOREM AND ASYMPTOTES

1. A. A sign change on a continuous function guarantees at least one zero. Choice B claims exactly one, C is unrelated and D uses a value outside -3 to 6.
2. B. The function is discontinuous at $x = 1$, which lies inside $[0, 2]$. The other intervals avoid $x = 1$, so f is continuous on them.
3. C. $g(1) = -1$ and $g(2) = 8$ have opposite signs. The other intervals have endpoint values with the same sign (-7 and -4 ; -4 and -1 ; 8 and 29).
4. D. Simplify: $(x-1)/((x-1)(x+1)) = 1/(x+1)$. The factor at $x = 1$ cancels to leave a hole, so only $x = -1$ is an asymptote.
5. B. Equal degrees give the ratio of leading coefficients $6/2 = 3$. Choice A uses only the numerator's coefficient and C treats the degrees as unequal.
6. C. For $x < 0$, $\sqrt{x^2 + 1}$ is about $|x| = -x$, so the quotient tends to -3 . Choice A forgets the sign change.
7. D. Exponential growth eventually beats any power, so the quotient is unbounded. Choice A would be right for x^2 / e^x .
8. A. Division gives $x^2 + 1 = (x - 1)(x + 1) + 2$, so $f(x) = x + 1 + 2/(x - 1)$. Choice B reuses the divisor's sign.
9. B. The theorem covers values strictly between 2 and 10. The number 12 is outside that range, so nothing is guaranteed.
10. $[2, 3]$. f is continuous, $f(2) = -2 < 0$ and $f(3) = 13 > 0$.
11. $x = 2$ only. Factor: $\frac{(x-3)(x+3)}{(x-2)(x-3)} = \frac{x+3}{x-2}$; at $x = 3$ there is a hole with limit 6.
12. 4 and -4 . Divide by $|x|$: for $x > 0$ the quotient is about 4; for $x < 0$ it is about -4 .
13. 0. The exponential grows faster than any power.
14. $y = 2x - 5$. Division: $2x^2 - 3x + 1 = (x + 1)(2x - 5) + 6$.
15. 4. The sign changes on $[0, 2]$, $[2, 4]$, $[4, 7]$ and $[7, 10]$ each guarantee a zero.

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