



Derivatives: Rules and Interpretation

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

In this booster

37.1 The Derivative as a Rate of Change

37.2 Power, Product, and Quotient Rules

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Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

37.1 The Derivative as a Rate of Change

Learn the limit definition of the derivative, how it measures slope and instantaneous rate of change, and how to estimate and interpret derivatives from graphs, tables, and stories.

Why This Matters

Calculus begins with one question: how fast is something changing right now? A speedometer answers it for a car, a thermometer trend answers it for the weather, and a growth chart answers it for a plant. The **derivative** is the exact tool for measuring change at a single instant, and every later idea in the course, from rules to optimization to motion, is built on it.

Students usually get stuck in four places. First, they confuse $f(a)$, the height of the graph, with $f'(a)$, the slope of the graph. Second, they forget that the difference quotient needs a **limit**, so they stop at an average rate instead of an instantaneous one. Third, they lose track of **units** when a derivative is interpreted in a story. Fourth, they assume that every smooth-looking function is differentiable everywhere. This section fixes all four.

Core Ideas

Start with average rate of change. If f is a function, the average rate of change from $x = a$ to $x = b$ is the change in output divided by the change in input. Graphically it is the slope of the **secant line** through the points $(a, f(a))$ and $(b, f(b))$. Now slide the second point closer and closer to the first. The secant lines settle toward a single line that just touches the curve at the first point. That line is the **tangent line**, and its slope is the derivative.

Definition of the Derivative

The derivative of f at $x = a$ is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists. An equivalent form is

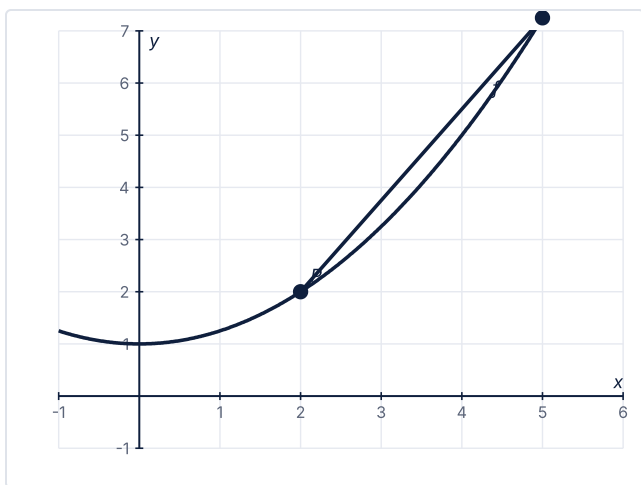
$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

If the limit exists, we say f is **differentiable** at a . If it does not exist, f is not differentiable at a .

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The secant line through P and Q. As Q slides toward P, the secant slope approaches the tangent slope at P.

The graph above shows the idea. The secant slope is the average rate over the interval from P to Q . The tangent slope at P is what the average rates approach as the interval shrinks to nothing. A table makes this concrete. For $f(x) = x^2$ at $a = 3$, the average rate from 3 to $3 + h$ is $\frac{(3+h)^2 - 9}{h} = 6 + h$, so the values get close to 6 as h gets close to 0.

h	1	0.1	0.01	0.001
Average rate $6 + h$	7	6.1	6.01	6.001

Slope, Tangent Line, and Interpretation

The number $f'(a)$ is the **slope of the tangent line** to $y = f(x)$ at $(a, f(a))$. The tangent line is

$$y - f(a) = f'(a)(x - a).$$

In context, $f'(a)$ is the **instantaneous rate of change** of f at a , measured in (units of f) per (unit of x). Its sign tells direction: positive means f is increasing, negative means decreasing.

Notation. The derivative function $f'(x)$ can also be written $\frac{dy}{dx}$, y' , or $\frac{d}{dx}f(x)$. The value at a point is written $f'(a)$ or $\left.\frac{dy}{dx}\right|_{x=a}$. All of these mean the same thing. Read $\frac{dy}{dx}$ as "the rate at which y changes with respect to x ."

Reading a derivative from a graph. Look at the steepness. Where the graph rises left to right, $f' > 0$. Where it falls, $f' < 0$. At a smooth peak or valley the tangent is horizontal, so $f' = 0$. A steeper graph means a larger $|f'|$.

Reading a derivative from a table. A table gives only some values, so you can only estimate. Use the two nearest known inputs on either side and compute the slope between them. Always divide by the change in x , and always write units.

Differentiability Requires Smoothness

If f is differentiable at a , then f is continuous at a . The converse is false.

A function fails to be differentiable at a point with a **corner** (like $|x|$ at 0), a **cusp**, a **vertical tangent** (like $\sqrt[3]{x}$ at 0), or a **discontinuity** (jump, hole, or asymptote).

Worked Examples

EXAMPLE 1

Use the limit definition to find $f'(2)$ for $f(x) = x^2 - 3x$.

SOLUTION

Write the difference quotient: $\frac{f(2+h) - f(2)}{h}$.

Compute $f(2) = 4 - 6 = -2$ and $f(2+h) = (2+h)^2 - 3(2+h) = 4 + 4h + h^2 - 6 - 3h = -2 + h + h^2$.

The numerator is $f(2+h) - f(2) = h + h^2$, so the quotient is $\frac{h+h^2}{h} = 1 + h$ for $h \neq 0$.

Take the limit: $f'(2) = \lim_{h \rightarrow 0} (1 + h) = 1$.

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EXAMPLE 2

Find the equation of the tangent line to $f(x) = \sqrt{x}$ at $x = 4$.

SOLUTION

The point is $(4, f(4)) = (4, 2)$.

Slope: $f'(4) = \lim_{x \rightarrow 4} \frac{\sqrt{x}-2}{x-4}$. Multiply top and bottom by $\sqrt{x} + 2$: $\frac{x-4}{(x-4)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2}$.

As $x \rightarrow 4$ this approaches $\frac{1}{4}$, so the slope is $\frac{1}{4}$.

Tangent line: $y - 2 = \frac{1}{4}(x - 4)$, or $y = \frac{1}{4}x + 1$.

t (min)	0	3	5	8	12
$T(t)$ (degrees F)	70	64	61	58	55

EXAMPLE 3

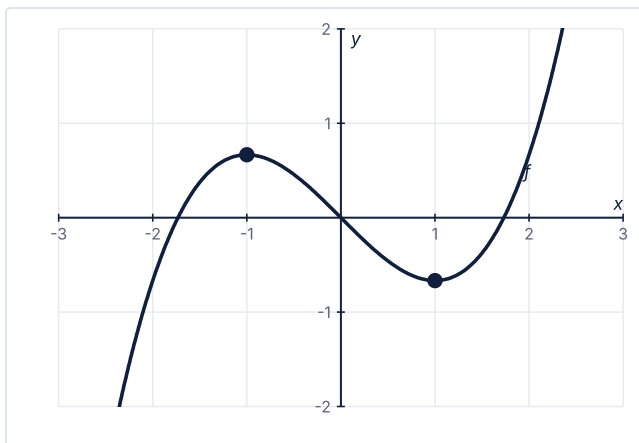
The temperature $T(t)$, in degrees Fahrenheit, of a drink in a refrigerator is recorded t minutes after it is put in. The table shows some values. Estimate $T'(6)$ and interpret it.

SOLUTION

Use the nearest inputs on either side of 6, which are $t = 5$ and $t = 8$.

$$T'(6) \approx \frac{T(8) - T(5)}{8 - 5} = \frac{58 - 61}{3} = -1.$$

Interpretation: at 6 minutes the temperature is falling at about 1 degree Fahrenheit per minute.



Graph of $f(x) = x^3/3 - x$. Horizontal tangents occur at $x = -1$ and $x = 1$.

EXAMPLE 4

The graph of $f(x) = \frac{x^3}{3} - x$ is shown below. For which values of x is $f'(x) = 0$, and on what interval is $f'(x) < 0$?

SOLUTION

$f'(x) = 0$ where the tangent line is horizontal, at the peak and valley of the graph.

The peak is at $x = -1$ and the valley is at $x = 1$, so $f'(-1) = 0$ and $f'(1) = 0$.

Between those points the graph falls from left to right, so $f'(x) < 0$ on $-1 < x < 1$. Check with $f'(x) = x^2 - 1$: it is negative exactly when $|x| < 1$.

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EXAMPLE 5

Let $f(x) = \begin{cases} x^2 + 1, & x \leq 1 \\ ax + b, & x > 1 \end{cases}$. Find a and b so that f is differentiable at $x = 1$.

SOLUTION

Differentiable implies continuous, so the two pieces must meet at $x = 1$: $1^2 + 1 = a(1) + b$, giving $a + b = 2$.

The slopes must also match. The left slope is the derivative of $x^2 + 1$ at 1, which is 2. The right slope is a . So $a = 2$.

Then $b = 2 - a = 0$. Check: the pieces are $x^2 + 1$ and $2x$, both equal to 2 at $x = 1$, both with slope 2.

EXAMPLE 6

The volume $V(t)$ of water in a tank, in liters, is changing with time t in minutes. It is known that $V(5) = 120$ and $V'(5) = -3$. Interpret $V'(5)$, and use the tangent line to estimate $V(5.5)$.

SOLUTION

$V'(5) = -3$ means that at $t = 5$ minutes the volume is decreasing at a rate of 3 liters per minute.

The tangent line at $t = 5$ is $V \approx 120 - 3(t - 5)$.

At $t = 5.5$: $V(5.5) \approx 120 - 3(0.5) = 118.5$ liters.

This is an estimate. It is reliable only for t close to 5.

Coach's Playbook

- 1 Name the object.** Before computing, say whether you need a function value $f(a)$, a slope $f'(a)$, or an equation of a line. These are three different things.
- 2 Choose your limit form.** For a polynomial, expand $f(a + h)$ and cancel h . For a square root, multiply by the conjugate. For a fraction, combine over a common denominator first.
- 3 Simplify before you take the limit.** The factor h must cancel from the numerator. If you still have $\frac{0}{0}$ after substituting, you have not finished simplifying.
- 4 Attach units.** A derivative in context always has units of output per unit of input. Write them, and use a full sentence.
- 5 Sanity check with a sketch.** If the graph rises at the point, your slope must be positive. If your answer disagrees with the picture, recheck.

Common traps

1. Writing $f'(3) = f(3)$ or reading the height of the graph as its slope. 2. Expanding $(x + h)^2$ as $x^2 + h^2$ and dropping the middle term $2xh$. 3. Dividing a table difference by the wrong number, such as an endpoint instead of the width of the interval. 4. Claiming a function is differentiable at a sharp corner because it is continuous there.

Mistake Clinic**Spot the error**

A student finds the slope of the tangent line to $f(x) = x^2$ at $x = 3$ and writes: "The slope is $f(3) = 9$." Where is the error? The student confused the output with the rate of change. The value $f(3) = 9$ is the height of the graph at $x = 3$. The slope requires the limit $\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \rightarrow 0} (6 + h) = 6$. The correct slope is 6, and the tangent line is $y - 9 = 6(x - 3)$.

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Spot the error

A student computes the difference quotient for $f(x) = x^2$ and writes: $\frac{(x+h)^2 - x^2}{h} = \frac{x^2 + h^2 - x^2}{h} = h$, so $f'(x) = 0$. The error is in the expansion. The square of a sum is $(x+h)^2 = x^2 + 2xh + h^2$, not $x^2 + h^2$. The correct work is $\frac{2xh + h^2}{h} = 2x + h$, and the limit is $f'(x) = 2x$.

Spot the error

From the table in Example 3, a student estimates $T'(6)$ as $\frac{58-61}{8} = -0.375$ degrees per minute. The error is the denominator. The slope between $t = 5$ and $t = 8$ must be divided by the width of the interval, $8 - 5 = 3$, not by the endpoint 8. The correct estimate is $\frac{58-61}{3} = -1$ degree per minute.

Real-World Connection

A car's trip computer records the distance driven $D(t)$ in miles. Over the interval from $t = 0.5$ to $t = 0.75$ hours it shows D rising from 28 to 39.5 miles. The average speed on that interval is $\frac{39.5-28}{0.25} = 46$ miles per hour. The speedometer, however, shows the **instantaneous** speed, which is $D'(t)$. As the time window shrinks, the average speed approaches the speedometer reading. This is exactly the limit in the definition of the derivative.

Challenge Corner

Find the equations of both lines through the point $(1, -3)$ that are tangent to $y = x^2$. Solution: let the point of tangency be (a, a^2) . The slope there is $2a$, so the tangent line is $y - a^2 = 2a(x - a)$, or $y = 2ax - a^2$. This line passes through $(1, -3)$, so $-3 = 2a - a^2$, which gives $a^2 - 2a - 3 = 0$ and $(a - 3)(a + 1) = 0$. So $a = 3$ or $a = -1$. The lines are $y = 6x - 9$ and $y = -2x - 1$. Check: both pass through $(1, -3)$, since $6 - 9 = -3$ and $-2 - 1 = -3$.

Practice**PRACTICE The Derivative as a Rate of Change**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Which limit gives $f'(3)$ for $f(x) = x^2$?
 (A) $\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{3^2 - 9}$ (B) $\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h}$ (C) $\lim_{h \rightarrow 0} \frac{(3+h)^2}{h}$
 (D) $\lim_{h \rightarrow 0} \frac{3^2 - 9}{h}$
- If $f(x) = x^2 - 5x$, what is $f'(4)$?
 (A) -4 (B) 8 (C) 3 (D) 11
- What is the slope of the tangent line to $y = \sqrt{x}$ at $x = 9$?
 (A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) 3 (D) $\frac{1}{18}$
- The table shows values of a differentiable function f . Which is the best estimate of $f'(5.5)$?

x	2	4	7	11
f(x)	10	16	25	31

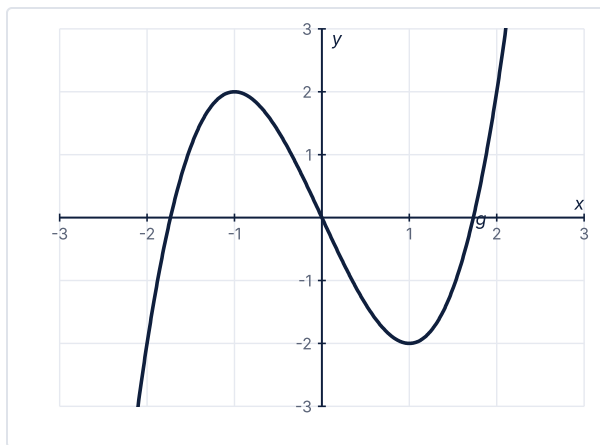
- (A) 9 (B) $\frac{1}{3}$ (C) $\frac{9}{11}$ (D) 3
- A population of bacteria is $P(t)$ thousand at t hours, and $P'(3) = 2.4$. What does this mean?
 (A) At 3 hours, the population is 2.4 thousand.
 (B) At 3 hours, the population is growing at about 2.4 thousand bacteria per hour.
 (C) The population grows by 2.4 thousand in the first 3 hours.
 (D) At 2.4 hours, the population is changing at 3 thousand per hour.

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6. The graph shows $g(x) = x^3 - 3x$. For which value of x is the slope of the tangent line negative?



Graph of g .

- (A) $x = -0.5$ (B) $x = -2$ (C) $x = 1$ (D) $x = 2$
7. Which function is NOT differentiable at $x = 0$?
- (A) $f(x) = x^3$ (B) $f(x) = \sin x$ (C) $f(x) = |x| + x^2$
 (D) $f(x) = x^4 + 1$
8. Let $f(x) = x^2$ for $x \leq 2$ and $f(x) = ax + b$ for $x > 2$. If f is differentiable at $x = 2$, what is $a + b$?
- (A) 4 (B) 0 (C) -4 (D) 8
9. What is the equation of the tangent line to $f(x) = x^3 - 2x$ at $x = 1$?
- (A) $y = x - 1$ (B) $y = -x$ (C) $y = x + 2$ (D) $y = x - 2$
10. Use the limit definition to find $f'(x)$ for $f(x) = 3x^2 - x$.
11. Use the limit definition to find $f'(2)$ for $f(x) = \frac{1}{x+1}$.
12. The distance $D(t)$, in kilometers, covered by a cyclist after t minutes is given in the table: $D(0) = 0$, $D(6) = 5$, $D(15) = 14$, $D(20) = 20$. Estimate $D'(10)$ using the nearest inputs and state the units.
13. Find the equation of the tangent line to $f(x) = \sqrt{x+3}$ at $x = 1$, then use it to approximate $f(1.2)$.
14. Let $f(x) = kx^2$ for $x \leq 1$ and $f(x) = mx + 1$ for $x > 1$. Find k and m so that f is differentiable at $x = 1$.
15. Using the limit definition, show that the derivative of $f(x) = x^3$ at $x = a$ is $3a^2$, then find all values of a where the tangent line has slope 12.

37.2 Power, Product, and Quotient Rules

Learn the power, product, and quotient rules and the derivatives of trigonometric, exponential, and logarithmic functions, and combine them to differentiate and find tangent lines.

Why This Matters

The limit definition is the truth behind the derivative, but nobody uses it for every problem. The **differentiation rules** turn the limit into a fast, reliable procedure. With them you can find the slope of almost any function you will meet in AP Calculus, and you will use them in every later topic: tangent lines, optimization, related rates, motion, and differential equations.

Students usually get stuck in these places. They try to differentiate a product by differentiating each factor separately. They mix up the order of the terms in the quotient rule, which changes a sign. They forget to rewrite roots and fractions as powers before using the power rule. And they apply the power rule to e^x when the variable is in the exponent. Learn the patterns below and these errors disappear.

Core Ideas

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Power Rule and Constant Rules

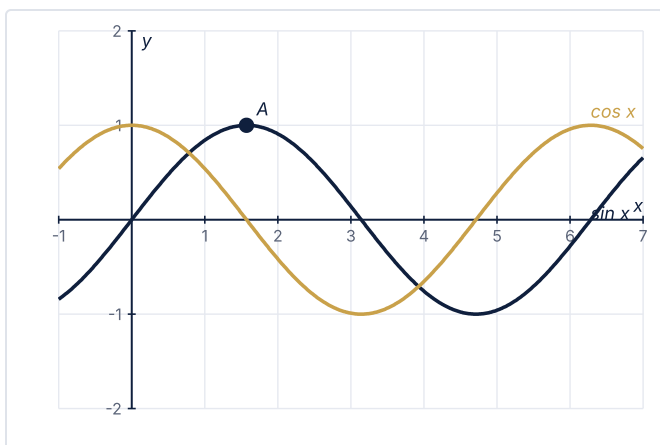
For any real number n ,

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Also $\frac{d}{dx}(c) = 0$, $\frac{d}{dx}(cf(x)) = cf'(x)$, and $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$.

The power rule works for negative and fractional exponents too, but only after you **rewrite** the expression: $\sqrt[3]{x} = x^{1/3}$, $\frac{1}{x^4} = x^{-4}$, and $\frac{5}{\sqrt{x}} = 5x^{-1/2}$. Then differentiate term by term.

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
e^x	e^x
$\ln x$	$\frac{1}{x}$ (for $x > 0$)



The cosine curve (gold) gives the slope of the sine curve (navy). At A, sine peaks and its tangent is horizontal, exactly where cosine crosses zero.

Product Rule and Quotient Rule

$$\frac{d}{dx}(fg) = f'g + fg' \qquad \frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2} \quad (g \neq 0).$$

Memory aid for the quotient: "low d-high minus high d-low, over low squared." The order in the numerator matters because subtraction is not commutative.

The product rule says: derivative of the first times the second, plus the first times the derivative of the second. A product of two different functions is **never** the product of their derivatives. You can use the quotient rule or rewrite a quotient as a product with a negative power. For a denominator that is only a constant, such as $\frac{x^3}{5}$, just pull out the constant.

Many AP problems give you values of f , f' , g , and g' in a table at a few points and ask for the derivative of a combination. You do not need the formulas for f and g . Just substitute the table values into the rule.

Choosing the right rule. Ask what the outermost operation is. If the expression is a sum or difference, differentiate term by term. If the outermost operation is multiplication of two non-constant functions, use the product rule. If it is division by a non-constant function, use the quotient rule. Many problems combine them. For $y = x^2 \sin x + \frac{e^x}{x}$, the sum splits into two pieces, the first needs the product rule, and the second needs the quotient rule.

Tangent lines with the rules. To write a tangent line at $x = a$, compute the point $(a, f(a))$, find $f'(x)$ with the correct rule, evaluate $f'(a)$, and then write $y - f(a) = f'(a)(x - a)$. To find where the tangent is horizontal, solve $f'(x) = 0$. To find where it is parallel to a given line, set $f'(x)$ equal to that line's slope.

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Situation	Rule to use
Sum or difference of terms	Differentiate term by term
Constant times a function	Pull the constant out
Two variable factors multiplied	Product rule
Variable function divided by a variable function	Quotient rule
Root or reciprocal of x	Rewrite as a power, then power rule

Worked Examples

EXAMPLE 1

Differentiate $y = 4x^5 - 3\sqrt{x} + \frac{2}{x^3} + 7$.

SOLUTION

Rewrite every term as a power: $y = 4x^5 - 3x^{1/2} + 2x^{-3} + 7$.

Differentiate term by term: $y' = 20x^4 - 3 \cdot \frac{1}{2}x^{-1/2} + 2(-3)x^{-4} + 0$.

Simplify: $y' = 20x^4 - \frac{3}{2\sqrt{x}} - \frac{6}{x^4}$.

EXAMPLE 2

Differentiate $y = x^3e^x$.

SOLUTION

This is a product of $f = x^3$ and $g = e^x$, so $f' = 3x^2$ and $g' = e^x$.

$y' = f'g + fg' = 3x^2e^x + x^3e^x$.

Factor: $y' = x^2e^x(3 + x)$.

EXAMPLE 3

Differentiate $y = \frac{2x+1}{x^2+3}$.

SOLUTION

Let $f = 2x + 1$ (so $f' = 2$) and $g = x^2 + 3$ (so $g' = 2x$).

$y' = \frac{2(x^2+3) - (2x+1)(2x)}{(x^2+3)^2}$.

Expand the numerator: $2x^2 + 6 - 4x^2 - 2x = -2x^2 - 2x + 6$.

$y' = \frac{-2x^2 - 2x + 6}{(x^2+3)^2}$.

EXAMPLE 4

Show that $\frac{d}{dx}(x \sin x + \cos x) = x \cos x$.

SOLUTION

Use the product rule on $x \sin x$: $\frac{d}{dx}(x \sin x) = 1 \cdot \sin x + x \cos x$.

The derivative of $\cos x$ is $-\sin x$.

Add: $\sin x + x \cos x - \sin x = x \cos x$.

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EXAMPLE 5

Find the x -coordinate where the graph of $y = \frac{\ln x}{x^2}$ has a horizontal tangent line.

SOLUTION

Quotient rule with $f = \ln x$, $f' = \frac{1}{x}$, $g = x^2$, $g' = 2x$: $y' = \frac{\frac{1}{x} \cdot x^2 - \ln x \cdot 2x}{x^4} = \frac{x - 2x \ln x}{x^4} = \frac{1 - 2 \ln x}{x^3}$.

A horizontal tangent needs $y' = 0$, so $1 - 2 \ln x = 0$ and $\ln x = \frac{1}{2}$.

Therefore $x = e^{1/2} = \sqrt{e}$, which is in the domain $x > 0$.

EXAMPLE 6

The table gives values of differentiable functions f and g at $x = 2$: $f(2) = 3$, $f'(2) = -1$, $g(2) = 4$, $g'(2) = 5$. Find $(fg)'(2)$ and $\left(\frac{f}{g}\right)'(2)$.

SOLUTION

Product rule: $(fg)'(2) = f'(2)g(2) + f(2)g'(2) = (-1)(4) + (3)(5) = 11$.

Quotient rule: $\left(\frac{f}{g}\right)'(2) = \frac{f'(2)g(2) - f(2)g'(2)}{g(2)^2} = \frac{-4 - 15}{16} = -\frac{19}{16}$.

Coach's Playbook

- 1 **Rewrite first.** Turn roots and reciprocals into powers, and split a fraction with a single-term denominator, for example $\frac{x^2+1}{x} = x + x^{-1}$. Often this removes the need for the quotient rule.
- 2 **Label the pieces.** Write f , f' , g , g' on the side before assembling the product or quotient rule. This prevents lost terms.
- 3 **Choose the lighter tool.** If a quotient has a single-term denominator, split it instead of using the quotient rule. For example $\frac{x^3+4x}{x} = x^2 + 4$ is much easier to differentiate than the original fraction.
- 4 **Factor your answer.** Factoring e^x or common powers makes it easy to solve $y' = 0$ afterward.
- 5 **Check with a value.** Pick $x = 1$ and compare your formula with a quick numerical slope estimate. A mismatch signals a sign or order error.

Common traps

1. Writing $(fg)' = f'g'$. 2. Reversing the numerator of the quotient rule and getting the opposite sign. 3. Using the power rule on e^x or on 2^x , where the variable is in the exponent. 4. Forgetting that $\frac{d}{dx} \cos x = -\sin x$ has a negative sign, and that $\ln x$ only exists for $x > 0$.

Mistake Clinic**Spot the error**

A student differentiates $y = x^2 \cdot x^3$ as $y' = 2x \cdot 3x^2 = 6x^3$. The error is treating the derivative of a product as the product of derivatives. The correct product rule gives $y' = 2x \cdot x^3 + x^2 \cdot 3x^2 = 2x^4 + 3x^4 = 5x^4$. This matches a quick check: $y = x^5$, so $y' = 5x^4$.

Spot the error

A student writes $\frac{d}{dx} \frac{x}{x+1} = \frac{x \cdot 1 - 1 \cdot (x+1)}{(x+1)^2} = \frac{-1}{(x+1)^2}$. The numerator is in the wrong order. The quotient rule is $f'g - fg'$, with the derivative of the top first. Here $f'g - fg' = 1 \cdot (x+1) - x \cdot 1 = 1$. The correct derivative is $\frac{1}{(x+1)^2}$, which is positive. That makes sense because $\frac{x}{x+1} = 1 - \frac{1}{x+1}$ is increasing for $x > -1$.

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Spot the error

A student writes $\frac{d}{dx}e^x = x e^{x-1}$ by using the power rule. The power rule applies only when the variable is in the base and the exponent is a constant. In e^x the variable is in the exponent, and the base e is a constant. The correct derivative is e^x itself. Compare x^3 , where the power rule gives $3x^2$.

Real-World Connection

A shop sells $q(p) = 500 - 20p$ items per week when the price is p dollars. The weekly revenue is $R(p) = p q(p) = 500p - 20p^2$. By the product rule, $R'(p) = q(p) + p q'(p) = 500 - 20p - 20p = 500 - 40p$. At $p = 10$, $R'(10) = 100$. This means raising the price from 10 dollars brings in about 100 more dollars of weekly revenue per dollar of increase. At $p = 12.5$ the derivative is zero, which is the price where revenue stops improving.

Challenge Corner

Find the equation of the line through the origin that is tangent to $y = e^x$. Solution: let the point of tangency be (a, e^a) . The slope there is e^a , so the tangent line is $y - e^a = e^a(x - a)$. It passes through $(0, 0)$, so $-e^a = -ae^a$, which gives $e^a(1 - a) = 0$. Since $e^a \neq 0$, we get $a = 1$. The slope is e and the line is $y = ex$, touching the curve at $(1, e)$.

Practice**PRACTICE Power, Product, and Quotient Rules**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is the derivative of $f(x) = 4x^3 - 6\sqrt{x}$?
 (A) $12x^2 - \frac{6}{3\sqrt{x}}$ (B) $12x^2 - \frac{6}{\sqrt{x}}$ (C) $12x^2 - \frac{3}{\sqrt{x}}$ (D) $12x^2 - \frac{3}{2\sqrt{x}}$
- What is $\frac{d}{dx}(x^2 \sin x)$?
 (A) $2x \sin x + x^2 \cos x$ (B) $2x \cos x$ (C) $2x \sin x - x^2 \cos x$
 (D) $x^2 \cos x$
- What is $\frac{d}{dx} \frac{x+2}{x-1}$?
 (A) $\frac{(x-1)^2 - 2}{(x-1)^2}$ (B) $-\frac{(x-1)^2 - 2}{(x-1)^2}$ (C) $\frac{(x-1)^2 - 2}{(x-1)^2}$ (D) $-\frac{3}{x-1}$
- If $f(x) = e^x \ln x$, what is $f'(1)$?
 (A) 0 (B) 1 (C) $2e$ (D) e
- What is the slope of the tangent line to $y = \tan x$ at $x = \frac{\pi}{4}$?
 (A) 2 (B) 1 (C) $\frac{1}{2}$ (D) $\sqrt{2}$
- Differentiable functions f and g satisfy $f(3) = 2$, $f'(3) = 4$, $g(3) = -1$, $g'(3) = 5$. What is $(fg)'(3)$?
 (A) 20 (B) -14 (C) 6 (D) -4
- At which values of x does $f(x) = x^3 - 3x^2 - 9x + 2$ have a horizontal tangent line?
 (A) $x = 1$ and $x = -3$ (B) $x = -1$ and $x = 3$ (C) $x = 0$ and $x = 3$
 (D) $x = -1$ and $x = 9$
- If $y = x^2 \ln x$, what is $\frac{dy}{dx}$ at $x = e$?
 (A) $3e$ (B) $2e$ (C) $2e + 1$ (D) e^2
- What is $\frac{d}{dx} \frac{\cos x}{x}$?
 (A) $\frac{-x \sin x + \cos x}{x^2}$ (B) $-\frac{\sin x}{x}$ (C) $\frac{x \sin x - \cos x}{x^2}$
 (D) $\frac{-x \sin x - \cos x}{x^2}$
- Find y' for $y = 3x^4 - \frac{5}{x^2} + 2\sqrt{x}$, and evaluate $y'(1)$.

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11. Let $f(x) = (x^2 + 1)e^x$. Find $f'(x)$, then write the equation of the tangent line at $x = 0$.
12. Let $g(x) = \frac{x^2}{\ln x}$ for $x > 1$. Find $g'(e)$.
13. Differentiable functions satisfy $f(1) = 2$, $f'(1) = 3$, $g(1) = 4$, $g'(1) = -2$. Find $(fg)'(1)$ and $\left(\frac{f}{g}\right)'(1)$.
14. Find all x in $[0, 2\pi]$ at which $y = \sin x + \cos x$ has a horizontal tangent line.
15. The parabola $y = ax^2 + bx$ passes through the point $(1, 3)$ and has slope 7 there. Find a and b .

37.3 The Chain Rule

Learn to differentiate composite functions by peeling layers from the outside in, including multi-layer problems, mixed rules, and problems given by tables.

Why This Matters

Almost every function you meet in real applications is a **composition**: a function inside another function. The temperature of a cooling drink might be $20 + 60e^{-0.2t}$, the area of a growing circle is πr^2 where r itself changes with time, and a sound wave can be modeled by $\sin(3x^2)$. The chain rule is the tool that differentiates these layered functions, and it is the most heavily used rule in AP Calculus. It reappears in implicit differentiation, related rates, integration by substitution, and differential equations.

Students usually get stuck in three places. They **forget the inner derivative**, so $\sin(3x)$ becomes $\cos(3x)$ instead of $3\cos(3x)$. They **differentiate the outer layer at the wrong place**, by plugging x instead of the inner function into the outer derivative. And they stall when there are **three or more layers**, or when the chain rule has to be mixed with the product or quotient rule. This section gives you a clear routine for all three.

Core Ideas

A **composite function** has the form $h(x) = f(g(x))$. The function g is the **inner** function, applied first. The function f is the **outer** function, applied to the result. For $h(x) = (3x^2 + 1)^5$, the inner function is $g(x) = 3x^2 + 1$ and the outer function is $f(u) = u^5$. Think of peeling an onion: you differentiate the outside layer first, leaving the inside alone, and then multiply by the derivative of the layer you just opened.

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then $h(x) = f(g(x))$ is differentiable at x and

$$h'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

The Leibniz form looks like fractions cancelling, which is a helpful memory aid: the rate of y with respect to x is the rate of y with respect to u , times the rate of u with respect to x . If a car uses 0.05 gallons per mile and drives 60 miles per hour, it uses $0.05 \cdot 60 = 3$ gallons per hour. Rates multiply along the chain.

Function	Outer / inner	Derivative
$(3x^2 + 1)^5$	$u^5 / 3x^2 + 1$	$5(3x^2 + 1)^4 \cdot 6x$
$\sin(4x)$	$\sin u / 4x$	$4 \cos(4x)$
e^{x^2}	e^u / x^2	$2x e^{x^2}$
$\ln(x^2 + 1)$	$\ln u / x^2 + 1$	$\frac{2x}{x^2 + 1}$
$\sqrt{5x - 2}$	$\sqrt{u} / 5x - 2$	$\frac{5}{2\sqrt{5x - 2}}$

Common Chain Rule Patterns

$$\begin{aligned} \frac{d}{dx} [g(x)]^n &= n[g(x)]^{n-1} g'(x), & \frac{d}{dx} e^{g(x)} &= e^{g(x)} g'(x), & \frac{d}{dx} \ln g(x) &= \frac{g'(x)}{g(x)}, \\ \frac{d}{dx} \sin g(x) &= \cos g(x) \cdot g'(x), & \frac{d}{dx} \cos g(x) &= -\sin g(x) \cdot g'(x). \end{aligned}$$

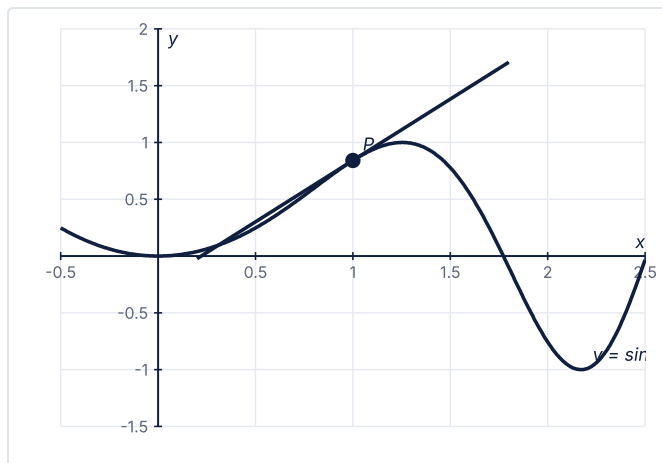
Why the rule works. Suppose $u = g(x)$ changes 3 units for every 1 unit of x , and $y = f(u)$ changes 5 units for every 1 unit of u . Then y changes $5 \cdot 3 = 15$ units for every 1 unit of x . The derivative is just this idea applied to instantaneous rates, which is why the two derivatives multiply rather than add.

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Powers of a function. An expression such as $\cos^4 x$ or $\sqrt{x^2 + 1}$ is a composition even though it does not look like one. The notation $\cos^4 x$ means $(\cos x)^4$, so the outer function is the fourth power and the inner function is $\cos x$. Writing the parentheses explicitly before you start is the cheapest way to avoid dropping a layer.



The graph of $y = \sin(x^2)$ and its tangent line at $P(1, \sin 1)$. The slope is $2 \cos 1$, about 1.081, from the chain rule.

Layers and tables. When f and g are given only by tables, use the rule directly: $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$. First find $g(a)$ from the table, then look up f' at that value, then multiply by $g'(a)$. **Three layers.** For $y = \sin(\cos(x^2))$, peel from the outside: derivative of $\sin$ at $\cos(x^2)$, times derivative of $\cos$ at x^2 , times derivative of x^2 .

Worked Examples

EXAMPLE 1

Differentiate $y = (3x^2 + 1)^5$.

SOLUTION

Outer function u^5 , inner function $u = 3x^2 + 1$.

$$y' = 5(3x^2 + 1)^4 \cdot \frac{d}{dx}(3x^2 + 1) = 5(3x^2 + 1)^4 \cdot 6x.$$

Simplify: $y' = 30x(3x^2 + 1)^4$.

EXAMPLE 2

Differentiate $y = \sin^3(2x)$.

SOLUTION

Rewrite as $y = [\sin(2x)]^3$. There are three layers: cube, sine, and $2x$.

Outer: $3[\sin(2x)]^2$. Middle: $\cos(2x)$. Inner: 2.

$$y' = 3 \sin^2(2x) \cdot \cos(2x) \cdot 2 = 6 \sin^2(2x) \cos(2x).$$

EXAMPLE 3

Differentiate $y = \ln(1 + e^{2x})$.

SOLUTION

Use $\frac{d}{dx} \ln g = \frac{g'}{g}$ with $g = 1 + e^{2x}$.

The derivative of g needs the chain rule too: $g' = e^{2x} \cdot 2 = 2e^{2x}$.

$$y' = \frac{2e^{2x}}{1 + e^{2x}}.$$

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EXAMPLE 4Differentiate $y = x^2\sqrt{1-x^2}$.**SOLUTION**This is a product with a chain rule inside. Let $f = x^2$, $g = (1-x^2)^{1/2}$.

$$f' = 2x \text{ and } g' = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1-x^2}}.$$

$$y' = 2x\sqrt{1-x^2} + x^2 \cdot \frac{-x}{\sqrt{1-x^2}}.$$

$$\text{Common denominator: } y' = \frac{2x(1-x^2) - x^3}{\sqrt{1-x^2}} = \frac{2x-3x^3}{\sqrt{1-x^2}}.$$

x	1	2	3
$f(x)$	2	3	1
$f'(x)$	4	-1	5
$g(x)$	3	1	2
$g'(x)$	2	5	-3

EXAMPLE 5Using the table above, let $h(x) = f(g(x))$ and $k(x) = g(f(x))$. Find $h'(2)$ and $k'(2)$.**SOLUTION** $h'(2) = f'(g(2)) \cdot g'(2)$. Since $g(2) = 1$, this is $f'(1) \cdot g'(2) = 4 \cdot 5 = 20$. $k'(2) = g'(f(2)) \cdot f'(2)$. Since $f(2) = 3$, this is $g'(3) \cdot f'(2) = (-3)(-1) = 3$.

Notice that the order of the composition changes the answer.

EXAMPLE 6Differentiate $y = \sin(\cos(x^2))$.**SOLUTION**Peel the layers: outer sin, middle cos, inner x^2 .

$$y' = \cos(\cos(x^2)) \cdot (-\sin(x^2)) \cdot 2x.$$

$$\text{Tidy up: } y' = -2x \sin(x^2) \cos(\cos(x^2)).$$

Coach's Playbook

- 1 Identify the outermost operation.** Ask: what is the very last thing done to x ? That is the outer layer. For $\sqrt{\sin(x^2)}$ the last step is the square root.
- 2 Differentiate the outer layer, copy the inside.** Write the outer derivative with the entire inside function unchanged, for example $\frac{1}{2\sqrt{\quad}}$ with the inside left in place.
- 3 Multiply by the derivative of the inside.** Repeat the same process on the inside if it is itself a composition. Keep going until you reach plain x .
- 4 Combine with other rules.** If a product or quotient is present, use that rule first, and apply the chain rule inside whichever pieces are composite.
- 5 Check by substituting.** Test your final formula at one easy value of x against a quick table estimate of the slope.

Common traps

1. Forgetting the inner derivative, such as writing $\frac{d}{dx}e^{3x} = e^{3x}$. 2. Changing the inside while differentiating the outside, such as writing $\frac{d}{dx}\sin(x^2) = \cos(2x)$. 3. Using $f'(a)$ instead of $f'(g(a))$ when reading a table. 4. Losing a layer in $\sin^2 x$, which is $(\sin x)^2$ and needs $2\sin x \cos x$.

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Mistake Clinic

Spot the error

A student writes $\frac{d}{dx} \sin(3x) = \cos(3x)$. The inner function $3x$ has derivative 3, which was left out. The correct derivative is $3 \cos(3x)$. A check: at $x = 0$ the slope of $\sin(3x)$ should be 3 (the curve is three times as steep as $\sin x$ near the origin), and $3 \cos 0 = 3$.

Spot the error

Using the table in Example 5, a student computes $h'(2)$ for $h(x) = f(g(x))$ as $f'(2) \cdot g'(2) = (-1)(5) = -5$. The outer derivative must be evaluated at the inner value $g(2) = 1$, not at 2. The correct work is $f'(g(2)) \cdot g'(2) = f'(1) \cdot 5 = 4 \cdot 5 = 20$.

Spot the error

A student writes $\frac{d}{dx} \sin^2 x = 2 \sin x$. The student differentiated the square but forgot the derivative of the inner function $\sin x$. The correct derivative is $2 \sin x \cos x$, which equals $\sin(2x)$. At $x = \frac{\pi}{2}$ the true slope is 0, since $\sin^2 x$ peaks there, but the wrong formula gives 2.

Real-World Connection

A circular ripple spreads on a pond. Its radius after t seconds is $r(t) = 2t + 1$ centimeters, so its area is $A = \pi r^2$. By the chain rule, $\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} = 2\pi(2t + 1) \cdot 2 = 4\pi(2t + 1)$. At $t = 2$ seconds this is 20π , about 62.8 square centimeters per second. The area is growing faster as time goes on because both the radius and the circumference are larger.

Challenge Corner

Let $f(x) = (2x + 1)^3(x - 2)^4$. Find all x where the graph has a horizontal tangent line. Solution: use the product rule with the chain rule in each factor. $f'(x) = 6(2x + 1)^2(x - 2)^4 + 4(2x + 1)^3(x - 2)^3$. Factor out the common part $2(2x + 1)^2(x - 2)^3$: $f'(x) = 2(2x + 1)^2(x - 2)^3 [3(x - 2) + 2(2x + 1)] = 2(2x + 1)^2(x - 2)^3(7x - 4)$. Setting $f'(x) = 0$ gives $x = -\frac{1}{2}$, $x = 2$, and $x = \frac{4}{7}$.

Practice

PRACTICE The Chain Rule

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is $\frac{d}{dx}(5x - 2)^4$?
 (A) $20(5x - 2)^3$ (B) $4(5x - 2)^3$ (C) $5(5x - 2)^3$
 (D) $20(5x - 2)^4$
- What is $\frac{d}{dx} \cos(4x^2)$?
 (A) $-\sin(4x^2)$ (B) $8x \sin(4x^2)$ (C) $-8x \sin(4x^2)$ (D) $-\sin(8x)$
- What is $\frac{d}{dx} e^{3x^2}$?
 (A) e^{3x^2} (B) $3x^2 e^{3x^2}$ (C) $6x e^{6x}$
 (D) $6x e^{3x^2}$

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4. What is $\frac{d}{dx} \ln(x^3 + 2x)$?
 (A) $\frac{x^3 + 1}{x^3 + 2x}$ (B) $\frac{3x^2 + 2}{x^3 + 2x}$ (C) $3x^2 + 2$
 (D) $\frac{x^3 + 2x}{3x^2 + 2}$
5. What is the slope of the tangent line to $y = \sqrt{4x + 1}$ at $x = 2$?
 (A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$
6. Differentiable functions satisfy $g(2) = 1$, $g'(2) = -3$, $f(1) = 5$, $f'(1) = 4$, $f'(2) = 7$. What is $(f \circ g)'(2)$?
 (A) -12 (B) -21 (C) 12 (D) 1
7. Let $h(x) = \sin^2(3x)$. What is $h'(\frac{\pi}{12})$?
 (A) 1 (B) 3 (C) 6 (D) $3\sqrt{2}$
8. What is the equation of the tangent line to $y = (x^2 - 3)^3$ at $x = 2$?
 (A) $y = 3x - 5$ (B) $y = 12x - 24$ (C) $y = 12x - 11$ (D) $y = 12x - 23$
9. What is $\frac{d}{dx}(x^2 e^{-x})$?
 (A) $x(2 - x)e^{-x}$ (B) $x(2 + x)e^{-x}$ (C) $-2xe^{-x}$
 (D) $2xe^{-x}$
10. Find the derivative of $y = (x^3 - 2x)^4$ and evaluate it at $x = 1$.
11. Find the derivative of $f(x) = \ln(x^2 + 1)$ and write the equation of the tangent line at $x = 1$.
12. Differentiate $y = \sin(\cos(x^2))$.
13. Differentiable functions satisfy $f(1) = 2$, $f(2) = 3$, $f(3) = 1$, $f'(1) = 4$, $f'(2) = -1$, $f'(3) = 5$ and $g(1) = 3$, $g(2) = 1$, $g(3) = 2$, $g'(1) = 2$, $g'(2) = 5$, $g'(3) = -3$. Find $\frac{d}{dx}f(g(x))$ at $x = 2$ and $\frac{d}{dx}g(f(x))$ at $x = 2$.
14. [Calculator] The temperature of a cooling cup of tea is $T(h) = 20 + 10e^{-0.3h}$ degrees Celsius, where h is hours. Find $T'(2)$ to three decimal places and state its units and meaning.
15. Find all x with $-2 < x < 2$ at which $g(x) = x\sqrt{4 - x^2}$ has a horizontal tangent line.

37.4 Implicit Differentiation and Inverse Functions

Learn to find slopes of curves defined implicitly, apply the inverse function derivative rule and inverse trigonometric derivatives, and compute first and second derivatives for AP problems.

Why This Matters

Not every curve is the graph of a function you can solve for y . A circle, an ellipse, and the curve $x^2y + y^3 = 10$ are all described by an equation that mixes x and y . **Implicit differentiation** lets you find slopes of such curves without solving for y . The same chain-rule thinking leads to the derivative of an **inverse function**, to the derivatives of inverse trigonometric functions, and to higher-order derivatives that describe how a slope itself changes.

Students usually get stuck in four places. They forget that y depends on x , so they differentiate y^2 as $2y$ instead of $2y \frac{dy}{dx}$. They drop the product rule on terms like xy . They mix up where to evaluate the inverse derivative formula, plugging in the wrong input. And they lose track of $\frac{dy}{dx}$ while computing a second derivative. The routines below remove these problems.

Core Ideas

In an equation such as $x^2 + y^2 = 25$, you treat y as a function of x that is not written out. When you differentiate a term with y in it, the chain rule requires a factor of $\frac{dy}{dx}$. For example, $\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$, because y^2 is an outer square applied to the inner function $y(x)$.

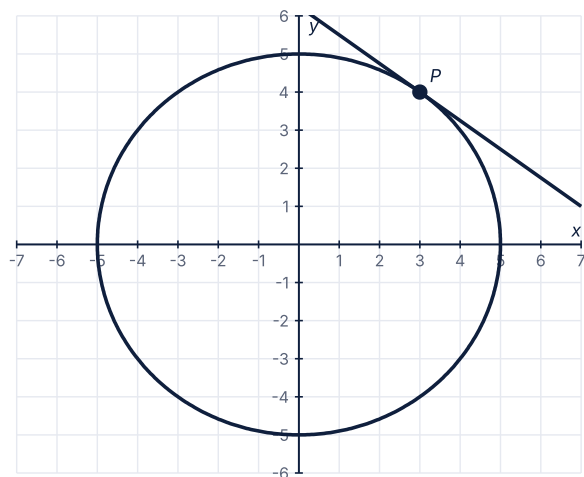
Implicit Differentiation Method

1. Differentiate both sides of the equation with respect to x . Every time you differentiate a term in y , multiply by $\frac{dy}{dx}$ (use the product rule for terms such as xy).
2. Collect all terms containing $\frac{dy}{dx}$ on one side and move everything else to the other side.
3. Factor out $\frac{dy}{dx}$ and divide. The answer usually contains both x and y , so you need a point (x, y) on the curve to get a number.

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The circle $x^2 + y^2 = 25$ and its tangent line at $P(3, 4)$. The slope is $-x/y = -3/4$.

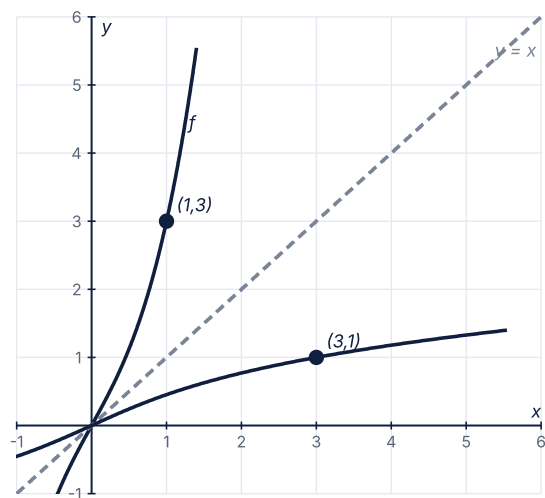
Derivative of an Inverse Function

Let f be differentiable and one-to-one near a with $f'(a) \neq 0$, and let $b = f(a)$. If $g = f^{-1}$, then

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{f'(g(b))}.$$

The slope of the inverse at the point (b, a) is the reciprocal of the slope of f at the matching point (a, b) .

Why does this make sense? The graph of $g = f^{-1}$ is the graph of f reflected across the line $y = x$. A reflection swaps run and rise, so the slope of the tangent line becomes the reciprocal. The most common error is evaluating f' at b instead of at $a = g(b)$. Always ask: which x -value of f matches the output you were given?



The graph of $f(x) = x^3 + 2x$ and its inverse, a reflection across the dashed line $y = x$. The slope at $(1, 3)$ is 5 and at $(3, 1)$ it is $1/5$.

Inverse Trigonometric Derivatives and Higher Orders

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx} \arctan x = \frac{1}{1+x^2}.$$

The **second derivative** $f''(x) = \frac{d}{dx}(f'(x))$ is also written $\frac{d^2y}{dx^2}$. Higher derivatives are written f''' , $f^{(4)}$, and so on. With the chain rule, $\frac{d}{dx} \arctan(u) = \frac{u'}{1+u^2}$.

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Second derivatives. For an explicit function, differentiate twice. For example, if $f(x) = x^4 - 2x^3$, then $f'(x) = 4x^3 - 6x^2$ and $f''(x) = 12x^2 - 12x$. The second derivative measures how fast the slope is changing, and it will tell you about concavity in a later section. For an implicit curve, differentiate $\frac{dy}{dx}$ again, then substitute the first derivative wherever $\frac{dy}{dx}$ appears.

Worked Examples

EXAMPLE 1

Find $\frac{dy}{dx}$ for $x^2 + y^2 = 25$, and write the equation of the tangent line at $(3, 4)$.

SOLUTION

Differentiate: $2x + 2y \frac{dy}{dx} = 0$.

Solve: $\frac{dy}{dx} = -\frac{x}{y}$.

At $(3, 4)$ the slope is $-\frac{3}{4}$.

Tangent line: $y - 4 = -\frac{3}{4}(x - 3)$.

EXAMPLE 2

Find the slope of the curve $x^2y + y^3 = 10$ at the point $(1, 2)$.

SOLUTION

Check the point: $1^2 \cdot 2 + 2^3 = 10$. Good.

Differentiate with the product rule on x^2y : $2xy + x^2 \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0$.

Collect: $\frac{dy}{dx}(x^2 + 3y^2) = -2xy$, so $\frac{dy}{dx} = \frac{-2xy}{x^2 + 3y^2}$.

At $(1, 2)$: $\frac{-4}{1 + 12} = -\frac{4}{13}$.

EXAMPLE 3

Find $\frac{dy}{dx}$ at $(1, 0)$ for the curve $e^y = x + y^2$.

SOLUTION

Check the point: $e^0 = 1$ and $1 + 0 = 1$. Good.

Differentiate: $e^y \frac{dy}{dx} = 1 + 2y \frac{dy}{dx}$.

Collect: $\frac{dy}{dx}(e^y - 2y) = 1$, so $\frac{dy}{dx} = \frac{1}{e^y - 2y}$.

At $(1, 0)$: $\frac{1}{1 - 0} = 1$.

EXAMPLE 4

Find the points on $x^2 + xy + y^2 = 12$ where the tangent line is horizontal.

SOLUTION

Differentiate: $2x + y + x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$.

A horizontal tangent needs the numerator to be zero (with the denominator nonzero): $2x + y = 0$, so $y = -2x$.

Substitute into the curve: $x^2 + x(-2x) + 4x^2 = 3x^2 = 12$, so $x = \pm 2$.

The points are $(2, -4)$ and $(-2, 4)$. At both, $x + 2y = -6$ or 6 , which is not zero, so the slope is 0.

EXAMPLE 5

Let $f(x) = x^3 + 2x$ and $g = f^{-1}$. Find $g'(3)$.

SOLUTION

First find a with $f(a) = 3$. Since $f(1) = 1 + 2 = 3$, $g(3) = 1$.

$f'(x) = 3x^2 + 2$, so $f'(1) = 5$. Since $f'(x) > 0$ for all x , f is increasing and therefore one-to-one.

$g'(3) = \frac{1}{f'(g(3))} = \frac{1}{f'(1)} = \frac{1}{5}$.

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EXAMPLE 6

For $x^2 + y^2 = 25$, find $\frac{d^2y}{dx^2}$ and evaluate it at $(3, 4)$.

SOLUTION

From Example 1, $y' = -\frac{x}{y}$.

Differentiate with the quotient rule: $y'' = -\frac{1 \cdot y - x y'}{y^2}$. Substitute $y' = -\frac{x}{y}$: $y'' = -\frac{y + \frac{x^2}{y}}{y^2} = -\frac{y^2 + x^2}{y^3}$.

Use the curve $x^2 + y^2 = 25$: $y'' = -\frac{25}{y^3}$.

At $(3, 4)$: $y'' = -\frac{25}{64}$. The negative sign tells you the upper half of the circle bends downward.

Coach's Playbook

- 1 Verify the point.** Substitute the given point into the original equation. If it does not satisfy the equation, the problem or your reading of it is off.
- 2 Mark every y .** Before differentiating, circle each y term. Each one will get a $\frac{dy}{dx}$ after it is differentiated.
- 3 Watch for products.** Terms such as xy or x^2y require the product rule. Write both pieces: one with y alone, one with $\frac{dy}{dx}$.
- 4 Solve for $\frac{dy}{dx}$ last.** Move all $\frac{dy}{dx}$ terms to one side, factor, and divide. Only then substitute the point.
- 5 For inverses, ask which input.** Write the equation $f(a) = b$ first, then use $g'(b) = \frac{1}{f'(a)}$. The input to f' is a , not b .

Common traps

1. Writing $\frac{d}{dx}y^2 = 2y$ without the factor $\frac{dy}{dx}$. 2. Writing $\frac{d}{dx}(xy) = y$ and omitting $x\frac{dy}{dx}$. 3. Using $\frac{1}{f'(b)}$ for the inverse derivative instead of $\frac{1}{f'(a)}$ where $f(a) = b$. 4. In a second derivative, leaving $\frac{dy}{dx}$ in the answer instead of replacing it with the first-derivative formula.

Mistake Clinic**Spot the error**

A student differentiates $x^2 + y^3 = 7$ and writes $2x + 3y^2 = 0$, which has no $\frac{dy}{dx}$ in it to solve for. The student forgot the chain-rule factor on the y term. The correct step is $2x + 3y^2\frac{dy}{dx} = 0$, which gives $\frac{dy}{dx} = -\frac{2x}{3y^2}$.

Spot the error

To find the slope of $xy = 12$, a student writes $\frac{d}{dx}(xy) = y$ and then concludes $y = 0$. The product rule was skipped. The correct derivative is $y + x\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{y}{x}$. At $(3, 4)$ the slope is $-\frac{4}{3}$.

Spot the error

For $f(x) = x^3 + 2x$, a student finds $(f^{-1})'(3)$ as $\frac{1}{f'(3)} = \frac{1}{29}$. The input to f' is wrong. The number 3 is an output of f , so you first need the x with $f(x) = 3$, which is $x = 1$. Then $(f^{-1})'(3) = \frac{1}{f'(1)} = \frac{1}{5}$.

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Real-World Connection

A balloon's height in meters after t minutes is $H(t)$, with $H(4) = 60$ and $H'(4) = 5$ meters per minute. The inverse function H^{-1} tells you the time needed to reach a given height. By the inverse rule, $(H^{-1})'(60) = \frac{1}{H'(4)} = \frac{1}{5}$. That means near a height of 60 meters, each extra meter takes about 0.2 minutes. A fast climb corresponds to a small time per meter, which is why the two rates are reciprocals.

Challenge Corner

The equation $y^3 + y = x$ defines y as a function of x near the point $(2, 1)$. Find $\frac{d^2y}{dx^2}$ at that point. Solution: differentiate once: $3y^2y' + y' = 1$, so $y' = \frac{1}{3y^2 + 1}$. At $(2, 1)$, $y' = \frac{1}{4}$. Differentiate again using the chain rule: $y'' = -\frac{6yy'}{(3y^2 + 1)^2} = -\frac{6y}{(3y^2 + 1)^3}$. At $y = 1$: $y'' = -\frac{6}{64} = -\frac{3}{32}$.

Practice

PRACTICE Implicit Differentiation and Inverse Functions

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- For the curve $x^2 + y^3 = 7$, what is $\frac{dy}{dx}$?
 (A) $-\frac{2x}{3y^2}$ (B) $\frac{2x}{3y^2}$ (C) $-\frac{3y^2}{2x}$ (D) $-2x$
- What is the slope of the tangent line to $x^2 + y^2 = 100$ at $(6, -8)$?
 (A) $-\frac{3}{4}$ (B) $\frac{4}{3}$ (C) $\frac{3}{4}$ (D) $-\frac{4}{3}$
- What is $\frac{dy}{dx}$ at $(3, 4)$ for $xy = 12$?
 (A) $\frac{4}{3}$ (B) $-\frac{3}{4}$ (C) 0 (D) $-\frac{4}{3}$
- For $x^2 + 3xy = 10$, what is $\frac{dy}{dx}$ at the point $(1, 3)$?
 (A) $-\frac{2}{3}$ (B) $-\frac{11}{3}$ (C) $\frac{11}{3}$ (D) $-\frac{3}{11}$
- Let $f(x) = x^3 + x$ and $g = f^{-1}$. What is $g'(2)$?
 (A) 4 (B) $\frac{1}{13}$ (C) $\frac{1}{4}$ (D) 2
- What is $\frac{d}{dx} \arctan(3x)$?
 (A) $\frac{3}{1+9x^2}$ (B) $\frac{1}{1+9x^2}$ (C) $\frac{3}{1+3x^2}$ (D) $\frac{1}{1+3x}$
- If $f(x) = x^4 - 2x^3$, what is $f''(2)$?
 (A) 8 (B) 24 (C) 36 (D) 12
- Differentiable function f satisfies $f(2) = 7$, $f'(2) = 4$, $f(7) = 10$, and $f'(7) = 3$. If $g = f^{-1}$, what is $g'(7)$?
 (A) $\frac{1}{3}$ (B) 4 (C) $\frac{1}{7}$ (D) $\frac{1}{4}$
- For $x^2 + y^2 = 25$, what is $\frac{d^2y}{dx^2}$ at $(0, 5)$?
 (A) $-\frac{1}{5}$ (B) $\frac{1}{5}$ (C) 0 (D) $-\frac{1}{25}$
- Find the equation of the tangent line to $x^2 + xy + y^2 = 7$ at $(1, 2)$.
- Find $\frac{dy}{dx}$ at the point $(2, 1)$ on the curve $xy + \ln y = 2$.
- Let $f(x) = x^5 + x^3 + x$ and $g = f^{-1}$. Find $g'(3)$.
- Show that $\frac{d}{dx} [x \arctan x - \frac{1}{2} \ln(1 + x^2)] = \arctan x$.
- Find $f''(x)$ and $f''(0)$ for $f(x) = x^2 e^x$.
- Find $\frac{d^2y}{dx^2}$ at $(2, 2)$ for the curve $xy = 4$.

Answers

Answers and Explanations

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

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37.1 The Derivative as a Rate of Change

PRACTICE · THE DERIVATIVE AS A RATE OF CHANGE

1. B. The difference quotient is $[f(3+h) - f(3)] / h$ with $f(3)=9$. The choice dividing by 3 is not a quotient by the change in input h .
2. C. $f'(x)=2x-5$, so $f'(4)=3$. The value -4 is $f(4)=16-20$, the height rather than the slope.
3. A. The slope is $1/(2 \sqrt{9}) = 1/6$. Choice $1/3$ forgets the factor 2 in the denominator; 3 is the height.
4. D. Use the nearest points $x=4$ and $x=7$: $(25-16)/(7-4)=3$. Choice 9 forgets to divide by 3; $1/3$ inverts the quotient; $9/11$ adds the inputs instead of subtracting.
5. B. P' is a rate, so it has units of thousands per hour at the stated time $t=3$. The other choices treat it as a population size, a total change, or swap the time and rate.
6. A. $g'(x)=3x^2-3$. At $x=-0.5$ it is -2.25 , negative. At $x=1$ it is 0 (the valley), and at $x=-2$ and $x=2$ it is 9, positive.
7. C. $|x|+x^2$ has a corner at 0 because $|x|$ does. The other three are polynomials or sine, which are smooth everywhere.
8. B. Matching slopes: $a=2 \cdot 2=4$. Matching values: $4=2a+b$ gives $b=-4$. So $a+b=0$. The value 4 is only a , and -4 is only b .
9. D. $f(1)=-1$ and $f'(1)=3-2=1$, so $y+1=1(x-1)$, giving $y=x-2$. Choice $y=x-1$ uses the slope but forgets the height $f(1)=-1$.
10. $6x - 1$. The quotient is $[3(x+h)^2 - (x+h) - 3x^2 + x]/h = (6xh + 3h^2 - h)/h = 6x + 3h - 1$, and the limit as h goes to 0 is $6x - 1$.
11. $-1/9$. The quotient is $[1/(3+h) - 1/3]/h = [(3 - (3+h))/(3(3+h))]/h = -1/(3(3+h))$, and the limit as h goes to 0 is $-1/9$.
12. 1 kilometer per minute. $D'(10)$ is approximately $(14-5)/(15-6) = 9/9 = 1$. The cyclist is moving at about 1 km per minute at $t=10$.
13. $y = 2 + (x-1)/4$, and $f(1.2)$ is approximately 2.05. $f(1)=2$ and $f'(x)=1/(2 \sqrt{x+3})$ gives $f'(1)=1/4$. Then $f(1.2)$ is about $2 + 0.2/4 = 2.05$.
14. $k = -1$ and $m = -2$. Continuity: $k = m+1$. Equal slopes: $2k = m$. Substituting gives $2k = k-1$, so $k=-1$ and $m=-2$. Check: both pieces give -1 at $x=1$ with slope -2 .
15. $a = 2$ or $a = -2$. The quotient is $[(a+h)^3 - a^3]/h = (3a^2 h + 3a h^2 + h^3)/h = 3a^2 + 3ah + h^2$, which tends to $3a^2$. Setting $3a^2=12$ gives $a^2=4$, so $a=2$ or $a=-2$.

37.2 Power, Product, and Quotient Rules

PRACTICE · POWER, PRODUCT, AND QUOTIENT RULES

1. C. Rewrite $6 \sqrt{x}$ as $6x^{1/2}$; its derivative is $3x^{-1/2}$. Choice with $6/\sqrt{x}$ forgets the factor $1/2$, and the $3/(2 \sqrt{x})$ choice applies it twice.
2. A. Product rule: $(2x)(\sin x) + (x^2)(\cos x)$. Choice $2x \cos x$ multiplies the derivatives, and the minus sign choice confuses $\cos$ with its negative.
3. B. Quotient rule: $[(1)(x-1) - (x+2)(1)]/(x-1)^2 = -3/(x-1)^2$. The positive version reverses the numerator; the other drops the square.
4. D. Product rule: $f'(x)=e^x \ln x + e^x/x$. At $x=1$, $\ln 1 = 0$, so $f'(1) = 0 + e = e$. The answer 0 forgets the second term.
5. A. The derivative is $\sec^2 x$. At $\pi/4$, $\sec = \sqrt{2}$, so $\sec^2 = 2$. The value 1 is $\tan(\pi/4)$, the height, not the slope.
6. C. Product rule: $f'g + fg' = (4)(-1) + (2)(5) = 6$. The value 20 multiplies f' and g' ; -14 subtracts instead of adding; -4 is only the first term.
7. B. $f'(x)=3x^2-6x-9=3(x-3)(x+1)$, which is zero at $x=-1$ and $x=3$. The choice 1 and -3 has the signs reversed.
8. A. Product rule: $y'=2x \ln x + x^2(1/x) = 2x \ln x + x$. At $x=e$ this is $2e+e=3e$. The choice $2e$ leaves out the second term.
9. D. Quotient rule: $[(-\sin x)(x) - (\cos x)(1)]/x^2$. The first wrong choice has a sign slip, the second differentiates top and bottom separately, and the last reverses the order, giving the opposite sign.
10. $y' = 12x^3 + 10x^{-3} + 1/\sqrt{x}$, so $y'(1) = 12 + 10 + 1 = 23$. Rewrite $-5/x^2$ as $-5x^{-2}$, whose derivative is $+10x^{-3}$.
11. $f'(x) = 2x e^x + (x^2+1)e^x = e^x(x+1)^2$. At $x=0$, $f(0)=1$ and $f'(0)=1$, so the tangent line is $y = x + 1$.
12. $g'(e) = e$. Quotient rule: $g'(x) = [2x \ln x - x^2(1/x)]/(\ln x)^2 = (2x \ln x - x)/(\ln x)^2$. At $x=e$, $\ln e = 1$, so $g'(e) = (2e - e)/1 = e$.
13. $(fg)'(1) = 8$ and $(f/g)'(1) = 1$. Product: $(3)(4) + (2)(-2) = 8$. Quotient: $[(3)(4) - (2)(-2)]/4^2 = 16/16 = 1$.
14. $x = \pi/4$ and $x = 5\pi/4$. $y' = \cos x - \sin x = 0$ gives $\tan x = 1$ (and $\cos x \neq 0$), so $x = \pi/4$ or $5\pi/4$ in the interval.
15. $a = 4$ and $b = -1$. The point gives $a+b=3$. The slope $y'=2ax+b$ gives $2a+b=7$ at $x=1$. Subtracting, $a=4$, then $b=-1$.

37.3 The Chain Rule

PRACTICE · THE CHAIN RULE

1. A. Outer derivative $4(5x-2)^3$ times the inner derivative 5 gives $20(5x-2)^3$. The choice $4(5x-2)^3$ forgets the inner derivative.
2. C. The derivative of $\cos u$ is $-\sin u$, times $u' = 8x$. The positive version loses the minus sign, $-\sin(4x^2)$ forgets $8x$, and $-\sin(8x)$ changes the inside function.
3. D. The derivative of e^u is e^u times u' , and $u' = 6x$. The choice $e^{(3x^2)}$ forgets u' . The choice with $e^{(6x)}$ changes the exponent.
4. B. For $\ln g$, the derivative is $g'/g = (3x^2+2)/(x^3+2x)$. The choice $1/(x^3+2x)$ forgets g' .
5. C. $y'=4/(2 \sqrt{4x+1}) = 2/\sqrt{4x+1}$; at $x=2$ this is $2/3$. The choice $1/6$ forgets the inner derivative 4 and the choice $1/3$ forgets only part of it.
6. A. $(f \circ g)'(2) = f'(g(2))$, $g'(2) = f'(1)(-3) = -12$. The value -21 uses $f'(2)$ instead of $f'(g(2))$.
7. B. $h'(x)=2 \sin(3x) \cos(3x)$ times 3 = $6 \sin(3x) \cos(3x)$. At $\pi/12$ both $\sin$ and $\cos$ of $\pi/4$ equal $\sqrt{2}/2$, so $h' = 6(1/2) = 3$. The value 1 forgets the inner factor 3.
8. D. At $x=2$, $y=1$. The slope is $3(x^2-3)^2(2x)=3(1)(4)=12$, so $y-1=12(x-2)$, or $y=12x-23$. The choice $y=3x-5$ forgets the inner derivative $2x$.
9. A. Product rule: $2x e^{(-x)} + x^2 (-e^{(-x)}) = x(2-x)e^{(-x)}$. The choice $x(2+x)e^{(-x)}$ forgets the minus sign from the chain rule on $e^{(-x)}$.
10. -4 . $y' = 4(x^3-2x)^3 (3x^2-2)$. At $x=1$, $x^3-2x = -1$ and $3x^2-2 = 1$, so $y'(1) = 4(-1)^3(1) = -4$.
11. $f'(x) = 2x/(x^2+1)$, and the tangent line is $y = \ln 2 + (x-1)$. At $x=1$, $f'(1) = 2/2 = 1$ and $f(1) = \ln 2$.

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12. $y' = -2x \sin(x^2) \cos(\cos(x^2))$. Outer: $\cos(\cos(x^2))$; middle: $-\sin(x^2)$; inner: $2x$. Multiply all three.
13. 20 and 3. For $f(g(x))$ at 2: $f'(g(2)) g'(2) = f'(1)(5) = 20$. For $g(f(x))$ at 2: $g'(f(2)) f'(2) = g'(3)(-1) = (-3)(-1) = 3$.
14. $T'(2)$ is approximately -1.646 degrees Celsius per hour. $T'(h) = 10(-0.3)e^{-0.3h} = -3e^{-0.3h}$, and at $h=2$ it is $-3e^{-0.6}$. The tea is cooling at about 1.646 degrees Celsius per hour at the 2 hour mark.
15. $x = \sqrt{2}$ and $x = -\sqrt{2}$. $g'(x) = \sqrt{4-x^2} - x^2/\sqrt{4-x^2} = (4-2x^2)/\sqrt{4-x^2}$. Setting the numerator to zero gives $x^2 = 2$.

37.4 Implicit Differentiation and Inverse Functions

PRACTICE · IMPLICIT DIFFERENTIATION AND INVERSE FUNCTIONS

- A. Differentiate: $2x + 3y^2 y' = 0$, so $y' = -2x/(3y^2)$. The choice $-2x$ drops the y -term entirely, and the others flip the sign or invert the fraction.
- C. $y' = -x/y = -6/(-8) = 3/4$. The choice $-3/4$ forgets that y is negative; the others invert the slope.
- D. Product rule: $y + x y' = 0$, so $y' = -y/x = -4/3$. The value 0 comes from skipping the $x y'$ term, and $4/3$ loses the sign.
- B. Differentiate: $2x + 3y + 3x y' = 0$. At $(1,3)$: $2 + 9 + 3y' = 0$, so $y' = -11/3$. The choice $-2/3$ forgets the $3y$ term from the product rule.
- C. $f(1)=2$, so $g(2)=1$. Then $g'(2)=1/f'(1)=1/4$. The choice $1/13$ uses $f'(2)$ instead of $f'(1)$.
- A. Use $u'/(1+u^2)$ with $u=3x$: $3/(1+9x^2)$. The choice $1/(1+9x^2)$ forgets the inner derivative.
- B. $f'(x)=4x^3-6x^2$ and $f''(x)=12x^2-12x$. At $x=2$: $48-24=24$. The value 8 is $f'(2)$, and 36 is $f'''(2)$.
- D. $g'(7)=1/f'(g(7))=1/f'(2)=1/4$ because $f(2)=7$. The choice $1/3$ evaluates f' at 7 instead of at $g(7)=2$.
- A. $y'' = -(x^2+y^2)/y^3 = -25/y^3$. At $y=5$ this is $-25/125 = -1/5$. The value 0 confuses y'' with y' , which is 0 at that point.
- $y - 2 = -(4/5)(x - 1)$, or $y = -(4/5)x + 14/5$. Differentiate: $2x + y + x y' + 2y y' = 0$, so $y' = -(2x+y)/(x+2y) = -4/5$ at $(1,2)$.
- $-1/3$. Differentiate: $y + x y' + y'/y = 0$, so $y' = -y/(x + 1/y)$. At $(2,1)$: $-1/(2+1) = -1/3$.
- $1/9$. $f(1)=3$, so $g(3)=1$. $f'(x)=5x^4+3x^2+1$, so $f'(1)=9$ and $g'(3)=1/9$.
- By the product rule, $d/dx[x \arctan x] = \arctan x + x/(1+x^2)$. Also $d/dx[(1/2)\ln(1+x^2)] = x/(1+x^2)$. Subtracting leaves $\arctan x$.
- $f''(x) = (x^2 + 4x + 2)e^x$ and $f''(0) = 2$. $f'(x) = (x^2+2x)e^x$, and differentiating again by the product rule gives $(x^2+4x+2)e^x$.
1. $y + x y' = 0$ gives $y' = -y/x$. Then $y'' = -(y' x - y)/x^2 = 2y/x^2$. At $(2,2)$: $2(2)/4 = 1$.

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