



Applications of Derivatives: Related Rates and Optimization

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

In this booster

38.1 Related Rates

38.2 Optimization

38.3 Curve Analysis: Increasing, Decreasing, Concavity

38.4 Mean Value Theorem, Linear Approximation, and Motion

Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

38.1 Related Rates

Learn a reliable method for related rates problems: connect the variables with an equation, differentiate with respect to time, and substitute last, for ladders, cones, shadows, balloons, and angles.

Why This Matters

Real quantities rarely change alone. As a balloon inflates, its radius, surface area, and volume all change together. As a ladder slides, the height of its top and the distance of its base change together. **Related rates** problems use the chain rule to connect these changes: if you know how fast one quantity changes, you can find how fast a connected quantity changes at a particular moment. Engineers, scientists, and economists use this idea whenever quantities are tied together by a formula.

Students usually get stuck for four reasons. They do not know which formula connects the variables. They plug in a number too early, which turns a changing quantity into a constant and kills its derivative. They mix up the signs of rates (a decreasing quantity has a negative rate). And they forget units, which are part of the answer on every AP free-response question. The method below handles each of these.

Core Ideas

Every variable in a related rates problem is a function of time t . The radius of a circle is really $r(t)$, so the area $A = \pi r^2$ is a composite function, and the chain rule gives $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. The whole topic is that one idea: **differentiate the connecting equation with respect to t** .

The Related Rates Method

1. Draw a picture and label every quantity that changes with a variable name. Write the known rates and the rate you want, with units.
2. Write one equation connecting the variables, true at every moment (a geometry formula, the Pythagorean Theorem, similar triangles, or trigonometry).
3. Differentiate both sides with respect to t , using the chain rule on every variable.
4. Only now substitute the values that hold at the instant in question, and solve for the unknown rate. State the answer with units and a sentence.

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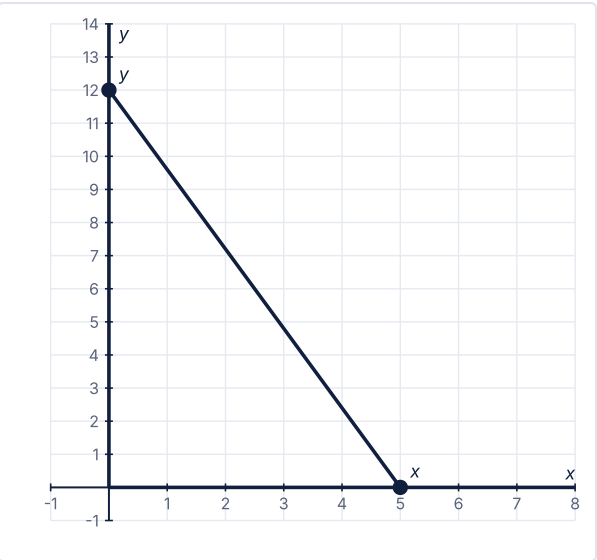
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Situation	Useful equation
Circle	$A = \pi r^2, C = 2\pi r$
Sphere	$V = \frac{4}{3}\pi r^3, S = 4\pi r^2$
Cone (full)	$V = \frac{1}{3}\pi r^2 h$
Right triangle	$x^2 + y^2 = z^2$
Two similar triangles	$\frac{\text{side}_1}{\text{side}_2} = \frac{\text{side}_3}{\text{side}_4}$
Angle of elevation	$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$

Sign and Units of a Rate

A positive rate means the quantity is increasing. A negative rate means it is decreasing. Units are always (units of the quantity) per (unit of time).

Example: if a ladder's top slides down at 3 feet per second, then $\frac{dy}{dt} = -3$ ft/s.



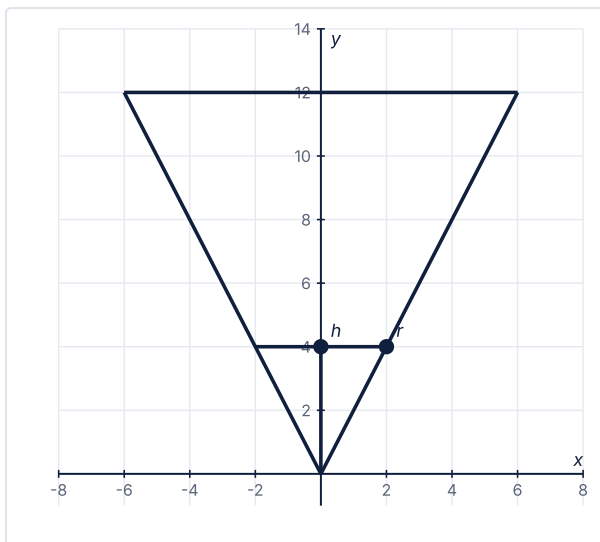
A 13-foot ladder against a wall. The base is x feet from the wall and the top is y feet high, with $x^2 + y^2 = 169$. At the instant shown, $x = 5$ and $y = 12$.

Similar triangles appear in shadow and cone problems. Because the ratio of corresponding sides stays the same at every moment, you can use it to eliminate a variable before differentiating. For a cone whose radius is always half its height, replace r by $\frac{h}{2}$, so the volume becomes a function of h alone.

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A cone tank 12 feet tall with top radius 6 feet. Water at depth h has surface radius $r = h/2$ by similar triangles.

Worked Examples

EXAMPLE 1

The radius of a circular oil slick grows at 2 cm per second. How fast is the area increasing when $r = 5$ cm?

SOLUTION

Known: $\frac{dr}{dt} = 2$ cm/s. Want: $\frac{dA}{dt}$ at $r = 5$.

Connecting equation: $A = \pi r^2$. Differentiate: $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$.

Substitute: $\frac{dA}{dt} = 2\pi(5)(2) = 20\pi$.

The area is increasing at $20\pi \approx 62.8$ square centimeters per second.

EXAMPLE 2

A 13-foot ladder leans against a vertical wall. The base slides away from the wall at 2 feet per second. How fast is the top sliding down when the base is 5 feet from the wall?

SOLUTION

Variables: x = distance of base from wall, y = height of top. Known: $\frac{dx}{dt} = 2$. Want: $\frac{dy}{dt}$ when $x = 5$.

Equation: $x^2 + y^2 = 169$. Differentiate: $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$, so $\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$.

When $x = 5$, $y = \sqrt{169 - 25} = 12$.

$\frac{dy}{dt} = -\frac{5}{12}(2) = -\frac{5}{6}$. The top is sliding down at $\frac{5}{6}$ foot per second.

EXAMPLE 3

Water is poured into the cone tank in the figure (height 12 feet, top radius 6 feet) at 3 cubic feet per minute. How fast is the water level rising when the depth is 4 feet?

SOLUTION

Known: $\frac{dV}{dt} = 3$. Want: $\frac{dh}{dt}$ at $h = 4$.

Similar triangles: $\frac{r}{h} = \frac{6}{12}$, so $r = \frac{h}{2}$.

Volume: $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3$.

Differentiate: $\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$. At $h = 4$: $3 = \frac{\pi}{4}(16) \frac{dh}{dt} = 4\pi \frac{dh}{dt}$.

$\frac{dh}{dt} = \frac{3}{4\pi} \approx 0.239$ feet per minute.

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EXAMPLE 4

A lamp is 6 meters above the ground. A person 1.8 meters tall walks away from the base of the lamp at 1.5 meters per second. How fast is the tip of the shadow moving, and how fast is the shadow lengthening?

SOLUTION

Let x = distance from lamp to person, s = length of shadow. The tip is $x + s$ from the lamp.

Similar triangles: $\frac{s}{1.8} = \frac{x+s}{6}$, so $6s = 1.8x + 1.8s$ and $4.2s = 1.8x$, giving $s = \frac{3}{7}x$.

Tip position: $x + s = \frac{10}{7}x$. Differentiate: tip speed = $\frac{10}{7} \cdot 1.5 = \frac{15}{7} \approx 2.14$ m/s.

Shadow length: $\frac{ds}{dt} = \frac{3}{7}(1.5) = \frac{9}{14} \approx 0.64$ m/s. These rates do not depend on how far the person has walked.

EXAMPLE 5

Car A leaves an intersection heading east at 60 mph. Car B leaves the same intersection heading north at 80 mph. How fast is the distance between them changing after 0.5 hour?

SOLUTION

After 0.5 hour, $x = 30$ miles and $y = 40$ miles, so the distance is $z = 50$ miles.

Equation: $z^2 = x^2 + y^2$. Differentiate: $2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$.

Substitute: $50 \frac{dz}{dt} = 30(60) + 40(80) = 1800 + 3200 = 5000$.

$\frac{dz}{dt} = 100$. The cars are separating at 100 miles per hour.

EXAMPLE 6

An observer stands 100 meters from a launch pad. A rocket rises straight up at 20 meters per second. How fast is the angle of elevation θ from the observer to the rocket changing when the rocket is 100 meters high?

SOLUTION

Known: $\frac{dh}{dt} = 20$. Equation: $\tan \theta = \frac{h}{100}$.

Differentiate: $\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{100} \frac{dh}{dt}$.

When $h = 100$, $\theta = \frac{\pi}{4}$ and $\sec^2 \theta = 2$.

$2 \frac{d\theta}{dt} = \frac{20}{100} = 0.2$, so $\frac{d\theta}{dt} = 0.1$ radian per second.

Coach's Playbook

- 1 Draw and label.** A clear figure with variable names is half the solution. Mark constants (such as the ladder length) differently from variables.
- 2 List knowns and the unknown.** Write them as derivatives: $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = ?$. This keeps you from mixing up which rate is which.
- 3 Eliminate extra variables first.** If the equation has more variables than you can handle, use similar triangles or a given relationship to reduce it before differentiating.
- 4 Differentiate before substituting.** Numbers that hold only at one instant, such as $r = 5$, are substituted after differentiation, never before.
- 5 Check signs and size.** A ladder sliding down should give a negative $\frac{dy}{dt}$. A growing circle should have a positive area rate. If the sign is wrong, recheck.

Common traps

1. Plugging in the instant's values before differentiating, which makes the derivative zero.
2. Using the wrong sign for a decreasing quantity.
3. Treating the radius of a cone as a constant when it changes with the height.
4. Leaving out units or not answering the question that was asked (for instance, giving $\frac{dh}{dt}$ when the question asks about the volume).

Mistake Clinic

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Spot the error

A student solves Example 1 this way: "When $r = 5$, $A = \pi(5)^2 = 25\pi$. Differentiating, $\frac{dA}{dt} = 0$ because 25π is a constant." The error is substituting the instant's value before differentiating. At that instant $r = 5$, but r is changing, so $A = \pi r^2$ must be differentiated first: $\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(5)(2) = 20\pi$.

Spot the error

In the ladder problem of Example 2, a student gets $\frac{dy}{dt} = +\frac{5}{6}$ and says the top rises at $\frac{5}{6}$ foot per second. The sign is wrong. As the base moves away from the wall, the top must come down, so y is decreasing. The equation gave $\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} = -\frac{5}{6}$, a negative number, so the top is falling.

Spot the error

In Example 3 a student writes $V = \frac{1}{3}\pi r^2 h$ and differentiates as if $r = 2$ were fixed: $3 = \frac{4\pi}{3} \frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{9}{4\pi}$. The error is treating r as constant. The radius of the water surface grows with the depth, $r = \frac{h}{2}$, so the volume must be written in terms of h alone: $V = \frac{\pi}{12} h^3$. The correct answer is $\frac{3}{4\pi}$, three times smaller.

Real-World Connection

A spherical balloon is inflated at 100 cubic centimeters per second. Using $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. When $r = 10$ cm, $100 = 4\pi(100) \frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{1}{4\pi} \approx 0.080$ cm per second. A bigger balloon grows more slowly in radius for the same airflow, which is why the first breaths into a balloon seem to do the most work.

Challenge Corner

Two sides of a triangle have fixed lengths 6 and 8 centimeters. The angle θ between them increases at 0.1 radian per second. Find the rate of change of the area and of the third side when $\theta = \frac{\pi}{3}$. Solution: $A = \frac{1}{2}(6)(8) \sin \theta = 24 \sin \theta$, so $\frac{dA}{dt} = 24 \cos \theta \frac{d\theta}{dt} = 24 \left(\frac{1}{2}\right) (0.1) = 1.2$ square centimeters per second. For the third side c , the Law of Cosines gives $c^2 = 100 - 96 \cos \theta$. Differentiate: $2c \frac{dc}{dt} = 96 \sin \theta \frac{d\theta}{dt}$. At $\theta = \frac{\pi}{3}$, $c^2 = 100 - 48 = 52$, so $c = 2\sqrt{13}$. Then $\frac{dc}{dt} = \frac{96 \cdot \frac{\sqrt{3}}{2} \cdot 0.1}{2 \cdot 2\sqrt{13}} = \frac{1.2\sqrt{3}}{\sqrt{13}} \approx 0.576$ centimeters per second.

Practice**PRACTICE Related Rates**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- The side of a square increases at 3 cm per second. How fast is the area increasing when the side is 5 cm?
(A) 30 cm²/s (B) 15 cm²/s (C) 75 cm²/s (D) 25 cm²/s
- A spherical ball's radius grows at 0.5 cm per second. How fast is its volume growing when $r = 4$ cm?
(A) $\frac{128\pi}{3}$ cm³/s (B) 32π cm³/s (C) 64π cm³/s (D) 16π cm³/s
- A 10-foot ladder has its base 6 feet from a wall and sliding away at 2 feet per second. How fast is the top moving down the wall?
(A) $\frac{3}{2}$ ft/s (B) $-\frac{8}{3}$ ft/s (C) $-\frac{3}{2}$ ft/s (D) $-\frac{3}{4}$ ft/s
- Two ships leave a port at the same time, one sailing north at 30 mph and the other east at 40 mph. How fast is the distance between them changing after 2 hours?
(A) 70 mph (B) 10 mph (C) 35 mph (D) 50 mph
- An oil spill is a circle whose area grows at 12π square meters per minute. How fast is the radius growing when $r = 3$ meters?
(A) 4 m/min (B) 1 m/min (C) 2 m/min (D) 12 m/min

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6. A child 1.5 m tall walks away from a lamp post 4.5 m tall at 1 m/s. How fast is the length of the child's shadow increasing?
(A) 1 m/s (B) 0.5 m/s (C) 1.5 m/s (D) 3 m/s
7. The length of a rectangle increases at 2 cm/s while the width decreases at 1 cm/s. How fast is the area changing when the length is 8 cm and the width is 5 cm?
(A) 18 cm²/s (B) -2 cm²/s (C) -6 cm²/s (D) 2 cm²/s
8. Water fills a cone-shaped cup (point down) of height 18 cm and top radius 6 cm, so $r = \frac{h}{3}$, at 2 cm³ per second. How fast is the depth rising when $h = 6$ cm?
(A) $\frac{1}{2\pi}$ cm/s (B) $\frac{3}{2\pi}$ cm/s (C) $\frac{2}{\pi}$ cm/s (D) $\frac{1}{6\pi}$ cm/s
9. An observer is 300 m from the base of a vertical cliff. A climbing drone rises at 30 m/s. How fast is the angle of elevation changing when the drone is 400 m high?
(A) 0.1 rad/s (B) 0.036 rad/s (C) 0.278 rad/s
(D) 0.06 rad/s
10. The edge of a cube grows at 2 cm per second. When the edge is 10 cm, find the rate of change of the volume and of the total surface area.
11. A 15-foot ladder leans on a wall. The top slides down the wall at 1.5 feet per second. How fast is the base moving away from the wall when the top is 9 feet high?
12. A spherical snowball melts so that its volume decreases at 20 cubic centimeters per minute. Find the rate of change of its radius and of its surface area when $r = 5$ cm.
13. A 6-foot person walks toward the base of a street lamp 18 feet high at 4 feet per second. At what rate is the length of the shadow changing, and how fast is the tip of the shadow moving?
14. An observer is 40 meters from a point directly below a balloon that rises at 3 meters per second. How fast is the angle of elevation changing when the balloon is 30 meters high?
15. Two sides of a triangle are 5 cm and 12 cm long, and the angle between them increases at 0.2 radian per second. How fast is the area changing when the angle is $\frac{\pi}{6}$?

38.2 Optimization

Learn to set up optimization problems with an objective and a constraint, find critical points, and justify absolute maxima and minima in the style of an AP free-response answer.

Why This Matters

Optimization asks for the best possible value of something: the largest area for a fixed amount of fence, the smallest cost for a package, the shortest distance from a point to a curve, the price that earns the most revenue. Calculus turns these questions into a clean routine. You write the quantity as a function of one variable, find where its derivative is zero, and then **prove** that you found a maximum or minimum. Businesses, engineers, and scientists solve problems like these every day.

Students usually get stuck at the setup, not the calculus. They are unsure which quantity to optimize, they do not use the constraint to reduce two variables to one, and they forget the **domain**. They also find a critical point and stop, without justifying that it is the maximum or minimum. On the AP exam the justification earns its own points. This section teaches the setup and the justification together.

Core Ideas

An optimization problem has two parts. The **objective** is the quantity you want to make as large or small as possible, such as area A , volume V , cost C , or distance d . The **constraint** is a given condition that links the variables, such as a fixed amount of fencing or a fixed volume. The constraint lets you eliminate one variable so the objective becomes a function of a single variable on a stated domain.

The Optimization Method

1. Draw a picture. Name the variables and write the objective as a formula.
2. Use the constraint to write the objective in terms of one variable. State the domain from the context (lengths are positive, and so on).
3. Find the critical points by solving $f'(x) = 0$ and checking where f' does not exist. Include the endpoints of the domain if it is closed.
4. Justify that the candidate gives the maximum or minimum (see the next box).
5. Answer the question that was asked, with units.

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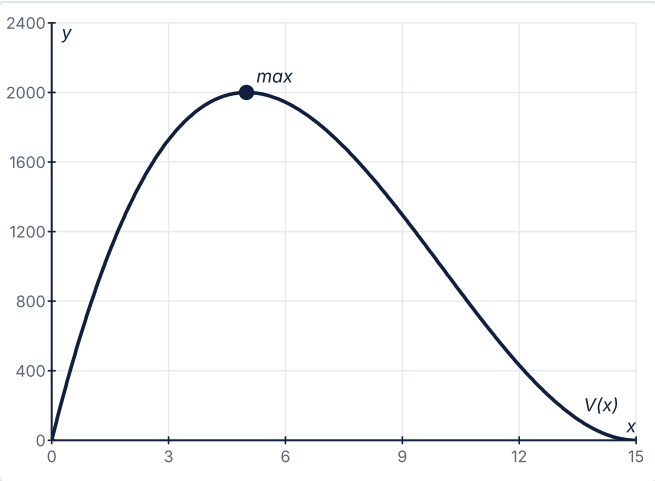
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Justifying an Absolute Maximum or Minimum

Candidates test (closed interval). If f is continuous on $[a, b]$, compare f at the critical numbers and the endpoints. The largest value is the absolute maximum and the smallest is the absolute minimum.

First derivative test. If f' changes from positive to negative at c , f has a relative maximum at c . If f' changes from negative to positive, a relative minimum.

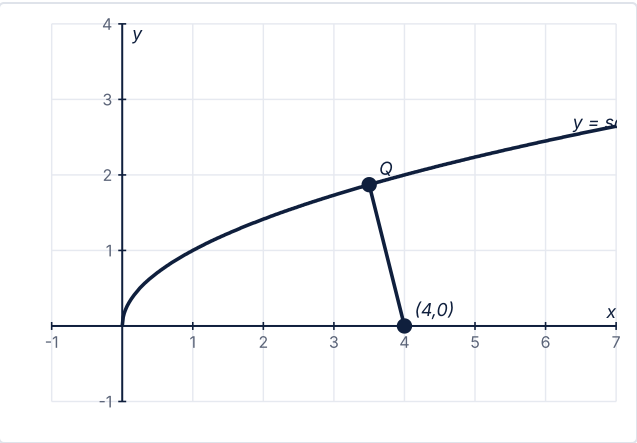
Only one critical point. If c is the only critical point on an interval and f has a relative maximum (or minimum) there, it is the absolute maximum (or minimum) on that interval. A second derivative check works too: $f''(c) < 0$ means a max and $f''(c) > 0$ a min.



Volume $V(x) = x(30 - 2x)^2$ of an open box made by cutting squares of side x from a 30 by 30 sheet. The maximum is 2000 at $x = 5$.

Why the domain matters. In the box problem, x is the side of a cut-out square, so $0 < x < 15$. The derivative is zero at $x = 5$ and at $x = 15$, but $x = 15$ is an endpoint that gives volume 0. Stating the domain rules out answers that do not make sense and tells you which endpoints to test.

Distance problems. To find the point on a curve closest to a given point, minimize the **square** of the distance. The square root is increasing, so d and d^2 are smallest at the same place, and the derivative of d^2 avoids a messy radical.



The point Q on $y = \sqrt{x}$ closest to $(4, 0)$. Minimizing the squared distance gives $x = 3.5$.

Problem type	Objective	Typical constraint
Fencing	Area	Fixed length of fence
Box or can	Volume or surface area	Fixed material or fixed volume
Closest point	Squared distance	Point lies on a curve
Business	Revenue, cost, or profit	Price-demand relationship

Worked Examples

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EXAMPLE 1

A farmer has 200 meters of fence for a rectangular pen against a straight river. No fence is needed along the river. What dimensions give the greatest area?

SOLUTION

Let x be the length of each of the two sides perpendicular to the river. The side parallel to the river is $200 - 2x$.

Objective: $A(x) = x(200 - 2x) = 200x - 2x^2$ on $0 < x < 100$.

$A'(x) = 200 - 4x = 0$ gives $x = 50$.

Justify: $A''(x) = -4 < 0$, so A is concave down and $x = 50$ is the absolute maximum. $A' > 0$ for $x < 50$ and $A' < 0$ for $x > 50$.

The pen is 50 m by 100 m (parallel to the river), with maximum area 5000 square meters.

EXAMPLE 2

A 30 cm by 30 cm square sheet has a square of side x cut from each corner, and the sides are folded up to make an open box. Find x for the largest volume.

SOLUTION

The base is $(30 - 2x)$ by $(30 - 2x)$ and the height is x , so $V(x) = x(30 - 2x)^2$ for $0 < x < 15$.

$V'(x) = (30 - 2x)^2 + x \cdot 2(30 - 2x)(-2) = (30 - 2x)(30 - 6x)$.

$V'(x) = 0$ at $x = 5$ (and $x = 15$, outside the open domain).

Justify: $V' > 0$ for $0 < x < 5$ and $V' < 0$ for $5 < x < 15$, so V has an absolute maximum at $x = 5$.

The maximum volume is $V(5) = 5 \cdot 20^2 = 2000$ cubic centimeters.

EXAMPLE 3

Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(4, 0)$.

SOLUTION

Squared distance from $(x, \sqrt{x})$ to $(4, 0)$: $D(x) = (x - 4)^2 + x$ for $x \geq 0$.

$D'(x) = 2(x - 4) + 1 = 2x - 7$, which is zero at $x = \frac{7}{2}$.

Justify: $D' < 0$ for $x < \frac{7}{2}$ and $D' > 0$ for $x > \frac{7}{2}$, so D has an absolute minimum at $x = \frac{7}{2}$. (Also $D(0) = 16$ is larger.)

The closest point is $(\frac{7}{2}, \sqrt{\frac{7}{2}}) \approx (3.5, 1.871)$, and the minimum distance is $\sqrt{D(3.5)} = \sqrt{3.75} \approx 1.936$.

EXAMPLE 4

[Calculator] A closed cylindrical can must hold 1000 cubic centimeters. Find the radius and height that minimize the surface area.

SOLUTION

Volume: $\pi r^2 h = 1000$, so $h = \frac{1000}{\pi r^2}$.

Surface area: $S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{2000}{r}$ for $r > 0$.

$S'(r) = 4\pi r - \frac{2000}{r^2} = 0$ gives $r^3 = \frac{500}{\pi}$, so $r \approx 5.419$ cm.

Justify: $S''(r) = 4\pi + \frac{4000}{r^3} > 0$ for all $r > 0$, so S is concave up and the critical point is an absolute minimum.

Then $h = \frac{1000}{\pi r^2} \approx 10.839$ cm, which is $2r$. The minimum surface area is about 553.6 square centimeters.

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EXAMPLE 5

A rectangular enclosure must have area 600 square meters. The front side costs 12 dollars per meter and the other three sides cost 8 dollars per meter. Find the dimensions of least cost.

SOLUTION

Let the front have length x and the depth be y , so $xy = 600$ and $y = \frac{600}{x}$.

Cost: $C = 12x + 8x + 8(2y) = 20x + 16y = 20x + \frac{9600}{x}$ for $x > 0$.

$C'(x) = 20 - \frac{9600}{x^2} = 0$ gives $x^2 = 480$, so $x = 4\sqrt{30} \approx 21.91$ m.

Justify: $C''(x) = \frac{19200}{x^3} > 0$, so $x = 4\sqrt{30}$ gives the absolute minimum.

Then $y = \frac{600}{x} \approx 27.39$ m and the minimum cost is $C = 160\sqrt{30} \approx 876.36$ dollars.

EXAMPLE 6

Find the absolute maximum and minimum of $f(x) = x^3 - 6x^2 + 9x + 1$ on $[0, 5]$.

SOLUTION

$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$, so the critical numbers are $x = 1$ and $x = 3$, both in the interval.

Candidates test: $f(0) = 1$, $f(1) = 5$, $f(3) = 1$, $f(5) = 21$.

Because f is continuous on the closed interval $[0, 5]$, the Extreme Value Theorem applies. The absolute maximum is 21 at $x = 5$. The absolute minimum is 1, attained at both $x = 0$ and $x = 3$.

Coach's Playbook

- 1 Name the objective in one line.** Write "Maximize A " or "Minimize C " at the top of your work. This protects you from optimizing the wrong thing.
- 2 Eliminate to one variable.** Solve the constraint for one variable and substitute. If you cannot, you are not ready to differentiate.
- 3 State the domain.** Use the context: lengths are positive, and a cut-out square cannot be larger than half the sheet.
- 4 Differentiate and solve $f' = 0$.** Factor when possible. Reject solutions outside the domain.
- 5 Justify in a sentence.** Use the first derivative test, the second derivative test, or the candidates test, and say why it gives an absolute extreme.
- 6 Answer the question.** Give all requested quantities (dimensions, maximum value) with units.

Common traps

1. Optimizing a function of two variables without using the constraint. 2. Skipping the justification and just writing "so this is the maximum." 3. Forgetting endpoints on a closed interval. 4. Giving the optimal variable value when the question asks for the optimal quantity (or the reverse).

Mistake Clinic

Spot the error

A student maximizes the area of a rectangle with perimeter 40 and writes: $A = xy$, $A' = y + x = 0$, so $x = -y$. The student treated $A = xy$ as a function of one variable. The constraint $2x + 2y = 40$ gives $y = 20 - x$, so $A(x) = x(20 - x)$ and $A'(x) = 20 - 2x = 0$ gives $x = 10$. The rectangle is a 10 by 10 square with area 100.

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Spot the error

A student finds $V'(x) = 0$ at $x = 5$ in Example 2 and writes: "So the answer is $x = 5$." That is not a justification. A complete answer must show why $x = 5$ gives the absolute maximum: $V' > 0$ on $(0, 5)$ and $V' < 0$ on $(5, 15)$, so V increases then decreases and the maximum volume is $V(5) = 2000$. Without this reasoning the response would lose justification credit on the exam.

Spot the error

A student maximizes $A(x) = x(20 - x)$ and correctly finds $x = 10$. Asked for the maximum area, the student writes "The maximum area is 10." The value $x = 10$ is where the maximum occurs, not the maximum itself. Substitute back: $A(10) = 10 \cdot 10 = 100$. The maximum area is 100.

Real-World Connection

A school fundraiser sells tickets. At a price of p dollars, the number of tickets sold is $700 - 20p$, so the revenue is $R(p) = p(700 - 20p)$ dollars for $0 < p < 35$. Then $R'(p) = 700 - 40p = 0$ gives $p = 17.5$, and $R'' = -40 < 0$ confirms a maximum. The best price is 17.50 dollars, and the revenue is $R(17.5) = 17.5 \cdot 350 = 6125$ dollars. Raising the price beyond that loses more in lost sales than it gains per ticket.

Challenge Corner

Find the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 5, with one side on the diameter. Solution: place the semicircle $y = \sqrt{25 - x^2}$ and let the rectangle have corners $(\pm x, 0)$ and $(\pm x, \sqrt{25 - x^2})$ for $0 < x < 5$. The area is $A(x) = 2x\sqrt{25 - x^2}$. Then $A'(x) = 2\sqrt{25 - x^2} - \frac{2x^2}{\sqrt{25 - x^2}} = \frac{2(25 - 2x^2)}{\sqrt{25 - x^2}}$. Setting $A' = 0$ gives $x = \frac{5}{\sqrt{2}}$. $A' > 0$ before and $A' < 0$ after this value, so it is the absolute maximum. The rectangle has width $5\sqrt{2}$ and height $\frac{5}{\sqrt{2}}$, so the maximum area is 25.

Practice**PRACTICE Optimization**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Two nonnegative numbers add to 20. What is the largest possible product?
(A) 10 (B) 100 (C) 20 (D) 400
- A farmer has 120 meters of fence and uses an existing wall as one side of a rectangular pen. What is the maximum area?
(A) 900 m² (B) 3600 m² (C) 1800 m² (D) 450 m²
- A 12 cm by 12 cm sheet has squares of side x cut from the corners to make an open box. Which x gives the maximum volume?
(A) 2 cm (B) 6 cm (C) 3 cm (D) 4 cm
- What is the minimum distance from the point $(0, 1)$ to the parabola $y = x^2$?
(A) 1 (B) $\frac{3}{4}$ (C) $\frac{1}{2}$ (D) $\frac{\sqrt{3}}{2}$
- The derivative of f satisfies $f'(2) = 0$. Which statement best justifies that f has a relative maximum at $x = 2$?
(A) $f'(2) = 0$. (B) f' changes from positive to negative at $x = 2$.
(C) $f(2)$ is greater than $f(0)$. (D) $f''(2) = 0$.
- A company's profit is $P(x) = -2x^2 + 120x - 500$ dollars for x items sold. How many items maximize profit?
(A) 60 (B) 15 (C) 30 (D) 1300
- Among all rectangles with perimeter 36, what is the smallest possible diagonal?
(A) $9\sqrt{2}$ (B) 18 (C) 9 (D) $18\sqrt{2}$

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8. What is the absolute maximum of $f(x) = x^3 - 12x$ on $[-3, 3]$?
 (A) 9 (B) 24 (C) -16 (D) 16
9. A rectangular field must have area 800 square meters. What is the least amount of fence needed for its perimeter?
 (A) 160 m (B) $80\sqrt{2}$ m (C) 80 m (D) $40\sqrt{2}$ m
10. Find two positive numbers whose product is 64 and whose sum is as small as possible. Justify your answer.
11. An open-top box with a square base must have volume 32 cubic meters. Find the base side and height that minimize the surface area, and state the minimum surface area.
12. Find the point on the line $y = 2x + 3$ that is closest to the origin, and the minimum distance.
13. A rectangular poster must have 150 square centimeters of printed area, with margins of 3 cm at the top and bottom and 2 cm on each side. Find the dimensions of the poster with least total area.
14. Find the absolute maximum and absolute minimum values of $f(x) = xe^{-x}$ on $[0, 3]$.
15. A window is a rectangle topped with a semicircle (the diameter of the semicircle is the top of the rectangle). Its total perimeter is 20 feet. Find the radius of the semicircle that maximizes the area of the window, and the maximum area.

38.3 Curve Analysis: Increasing, Decreasing, Concavity

Learn to read a function's behavior from its first and second derivatives: increasing and decreasing, relative extrema, concavity, inflection points, and the relationships among f , f' , and f'' .

Why This Matters

A graph tells a story, and calculus is the language that reads it. The first derivative reports whether a function is going up or down. The second derivative reports how the slope is changing, which tells you whether the graph bends like a cup or like a cap. Together they let you sketch a function, find its highs and lows, and explain why those are really highs and lows. This is the heart of AP free-response questions where you must **justify** a conclusion with a sign argument.

Students usually get stuck in these places. They believe that $f'(c) = 0$ always means a maximum or minimum. They think $f''(c) = 0$ always means an inflection point. They mix up the graph of f with the graph of f' and read the wrong one. And they write answers without the reasoning that the exam requires. Learn the sign charts below and these problems disappear.

Core Ideas

Increasing, Decreasing, and the First Derivative Test

If $f'(x) > 0$ on an interval, then f is **increasing** on that interval. If $f'(x) < 0$, then f is **decreasing**.

A **critical number** of f is a number c in the domain where $f'(c) = 0$ or $f'(c)$ does not exist. **First derivative test:** at a critical number c , if f' changes from positive to negative then f has a relative maximum at c . If f' changes from negative to positive, f has a relative minimum. If f' does not change sign, there is no extremum at c .

The sign chart is your main tool. Mark the critical numbers on a number line, pick a test value in each interval, and record the sign of f' . Then read off where f increases and decreases, and where it has relative extrema.

Concavity, Inflection Points, and the Second Derivative Test

If $f''(x) > 0$ on an interval, f is **concave up** (the graph bends like a cup, and f' is increasing). If $f''(x) < 0$, f is **concave down** (like a cap, and f' is decreasing).

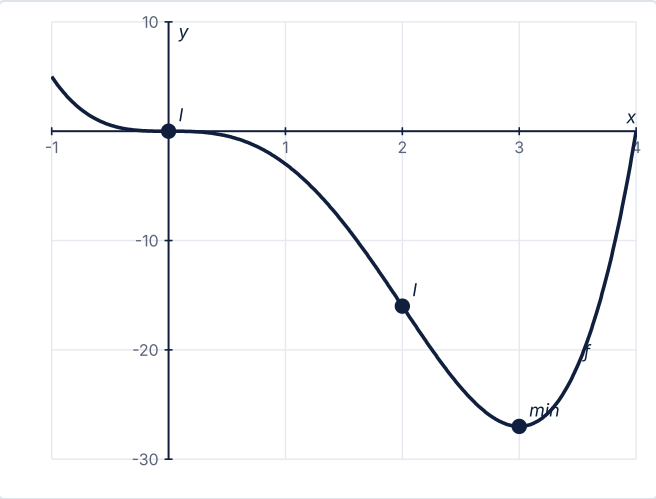
An **inflection point** is a point where the concavity changes, so f'' changes sign. Having $f''(c) = 0$ is not enough by itself.

Second derivative test: if $f'(c) = 0$ and $f''(c) > 0$, then f has a relative minimum at c . If $f'(c) = 0$ and $f''(c) < 0$, a relative maximum. If $f''(c) = 0$, the test is inconclusive; use the first derivative test.

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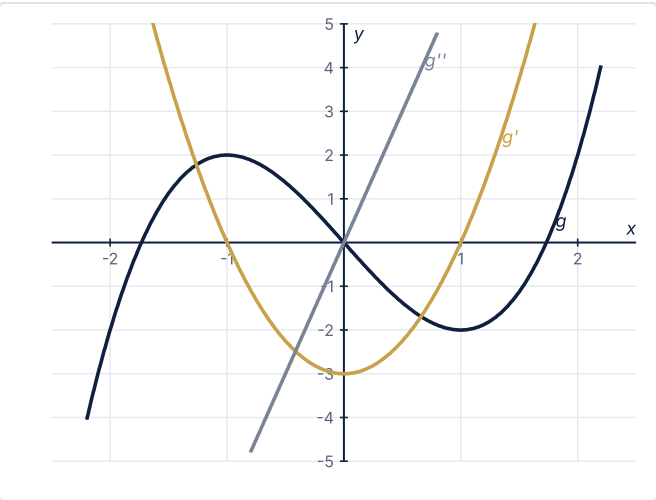
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The graph of $f(x) = x^4 - 4x^3$. It has a relative (and absolute) minimum at $x = 3$ and inflection points at $x = 0$ and $x = 2$.

Sign of f'	Sign of f''	What the graph of f does
+	+	Rising, bending up
+	-	Rising, bending down (leveling off)
-	+	Falling, bending up (leveling off)
-	-	Falling, bending down (steepening)

Reading three graphs at once. The functions f , f' , and f'' are linked. Where the graph of f' is above the x -axis, f rises. Where the graph of f' rises, f is concave up. Where f' crosses the axis going up, f has a relative minimum. Where f' has a peak or valley, f has an inflection point.



The function $g(x) = x^3 - 3x$ (navy), its derivative g' (gold), and g'' (gray, short). g' is zero where g has its peak and valley. g'' is zero where g changes concavity at the origin.

Absolute Extrema on a Closed Interval

If f is continuous on $[a, b]$, then f has an absolute maximum and an absolute minimum there. They occur at critical numbers in (a, b) or at the endpoints.

Compute f at each candidate and compare. This is the **candidates test**.

Worked Examples

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EXAMPLE 1

For $f(x) = x^3 - 6x^2 - 15x + 4$, find where f is increasing and decreasing, and locate the relative extrema.

SOLUTION

$f'(x) = 3x^2 - 12x - 15 = 3(x - 5)(x + 1)$, so the critical numbers are $x = -1$ and $x = 5$.

Sign chart for f' : for $x < -1$ it is positive; for $-1 < x < 5$ it is negative; for $x > 5$ it is positive.

So f is increasing on $(-\infty, -1)$ and $(5, \infty)$ and decreasing on $(-1, 5)$.

f' changes from $+$ to $-$ at $x = -1$, a relative maximum, with $f(-1) = 12$. It changes from $-$ to $+$ at $x = 5$, a relative minimum, with $f(5) = -96$.

EXAMPLE 2

Analyze $f(x) = x^4 - 4x^3$ for concavity and extrema.

SOLUTION

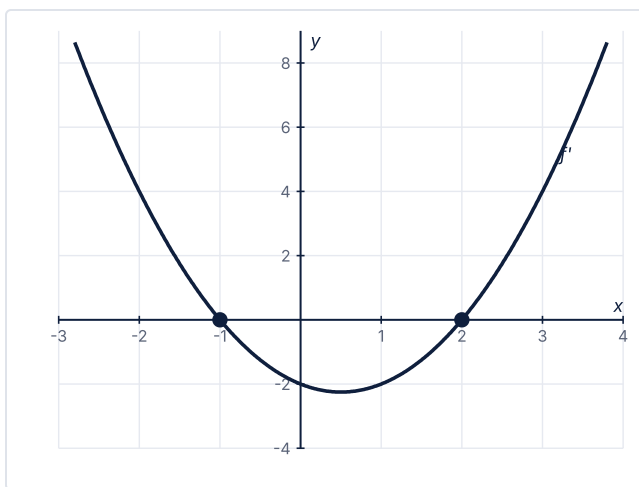
$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$. Critical numbers: $x = 0$ and $x = 3$.

$f''(x) = 12x^2 - 24x = 12x(x - 2)$, which is positive for $x < 0$ and $x > 2$, and negative for $0 < x < 2$.

Concave up on $(-\infty, 0)$ and $(2, \infty)$, concave down on $(0, 2)$. Concavity changes at $x = 0$ and $x = 2$, so both are inflection points: $(0, 0)$ and $(2, -16)$.

Second derivative test at $x = 3$: $f''(3) = 36 > 0$, so there is a relative minimum, $f(3) = -27$.

At $x = 0$, $f''(0) = 0$ is inconclusive. But $f'(x) = 4x^2(x - 3) < 0$ on both sides of 0, so f' does not change sign and there is no extremum at $x = 0$.



The graph of the derivative f' for Example 3. It crosses the x -axis at $x = -1$ and $x = 2$.

EXAMPLE 3

The graph above shows $f'(x) = x^2 - x - 2 = (x + 1)(x - 2)$. Describe where f is increasing, where it has relative extrema, and where it is concave up.

SOLUTION

$f' > 0$ for $x < -1$ and $x > 2$, so f is increasing there. $f' < 0$ on $(-1, 2)$, so f is decreasing there.

f' changes from $+$ to $-$ at $x = -1$: relative maximum. f' changes from $-$ to $+$ at $x = 2$: relative minimum.

Concavity comes from the slope of f' . The vertex of the parabola is at $x = \frac{1}{2}$, and $f''(x) = 2x - 1$. So f is concave up for $x > \frac{1}{2}$ and concave down for $x < \frac{1}{2}$, with an inflection point at $x = \frac{1}{2}$.

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EXAMPLE 4

Use the second derivative test on $f(x) = xe^{-x}$.

SOLUTION

$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$, which is zero only at $x = 1$ since $e^{-x} > 0$.

$f''(x) = -e^{-x} - (1-x)e^{-x} = (x-2)e^{-x}$.

$f''(1) = -e^{-1} < 0$, so f has a relative maximum at $x = 1$, with value $f(1) = \frac{1}{e}$.

Also f'' changes sign at $x = 2$, so $(2, 2e^{-2})$ is an inflection point.

EXAMPLE 5

A function f is differentiable with $f'(x) = (x-1)^2(x+2)$. Find all relative extrema of f and justify.

SOLUTION

Critical numbers: $x = 1$ and $x = -2$.

Since $(x-1)^2 \geq 0$, the sign of f' is the sign of $x+2$ (except at $x = 1$, where it is 0).

$f' < 0$ for $x < -2$, and $f' > 0$ for $-2 < x < 1$ and for $x > 1$.

Justification: f' changes from negative to positive at $x = -2$, so f has a relative minimum there. f' does not change sign at $x = 1$ (it is positive on both sides), so there is no relative extremum at $x = 1$.

f is increasing on $[-2, \infty)$.

EXAMPLE 6

Find constants a and b so that $f(x) = x^3 + ax^2 + bx$ has an inflection point at $x = 1$ and a relative maximum at $x = -1$.

SOLUTION

$f'(x) = 3x^2 + 2ax + b$ and $f''(x) = 6x + 2a$.

Inflection at $x = 1$ requires $f''(1) = 0$: $6 + 2a = 0$, so $a = -3$.

Relative max at $x = -1$ requires $f'(-1) = 0$: $3 - 2a + b = 0$, so $3 + 6 + b = 0$ and $b = -9$.

Check: $f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1)$ changes from $+$ to $-$ at $x = -1$, a relative max, and $f''(x) = 6x - 6$ changes sign at $x = 1$.

Coach's Playbook

- 1 Start with the domain.** Note any point where the function or its derivative is undefined. These split the number line, along with critical numbers.
- 2 Build a sign chart for f' .** Factor f' , mark zeros, test one point per interval, and write the signs. Use it to state increasing, decreasing, and relative extrema.
- 3 Build a sign chart for f'' .** Same process. Concavity and inflection points come from this chart.
- 4 Choose the right test.** Use the second derivative test when f'' is easy and nonzero at the critical number. Use the first derivative test when it fails or when f'' is messy.
- 5 Write the justification sentence.** " f' changes from negative to positive at $x = c$, so f has a relative minimum at $x = c$." Always cite the sign change.

Common traps

1. Assuming $f'(c) = 0$ guarantees a maximum or minimum (think of x^3 at 0).
2. Assuming $f''(c) = 0$ guarantees an inflection point (think of x^4 at 0).
3. Reading the graph of f' as if it were the graph of f .
4. Forgetting the endpoints when asked for an absolute maximum or minimum on a closed interval.

Mistake Clinic

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Spot the error

A student says: "For $f(x) = x^3$, $f'(0) = 0$, so f has a relative extremum at $x = 0$." The error is assuming that a zero derivative guarantees an extremum. Since $f'(x) = 3x^2 \geq 0$ on both sides of 0, f' does not change sign, and f is increasing through the origin. There is a horizontal tangent but no maximum or minimum.

Spot the error

A student says: "For $f(x) = x^4$, $f''(0) = 0$, so there is an inflection point at the origin." Compute $f''(x) = 12x^2 \geq 0$ for all x . Since f'' does not change sign, the graph is concave up on both sides of 0 and there is no inflection point. An inflection point requires a **sign change** in f'' .

Spot the error

The derivative of a function is $f'(x) = x - 4$ on $[0, 3]$. A student says: " f' is increasing, so f is increasing." The student confused the behavior of f' with the behavior of f . Because $f'(x) = x - 4 < 0$ on the whole interval, f is **decreasing**. The fact that f' is increasing ($f'' = 1 > 0$) tells us only that f is concave up.

Real-World Connection

A company's cumulative sales after t weeks of a campaign are $S(t) = -t^3 + 12t^2$ units. The sales rate is $S'(t) = -3t^2 + 24t$, which is positive for $0 < t < 8$, so sales keep growing. But $S''(t) = -6t + 24$ changes from positive to negative at $t = 4$. That is the **point of diminishing returns**: sales are still increasing, but the rate of increase is highest at $t = 4$ weeks, where $S'(4) = 48$ units per week, and then declines.

Challenge Corner

Find the inflection points of $f(x) = e^{-x^2}$ and describe the concavity. Solution: $f'(x) = -2xe^{-x^2}$ and $f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = (4x^2 - 2)e^{-x^2}$. Since $e^{-x^2} > 0$, the sign of f'' is the sign of $4x^2 - 2$. It is zero at $x = \pm \frac{1}{\sqrt{2}}$, positive for $|x| > \frac{1}{\sqrt{2}}$, and negative for $|x| < \frac{1}{\sqrt{2}}$. So the graph (a bell curve) is concave down on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and concave up outside. The inflection points are $\left(\pm \frac{1}{\sqrt{2}}, e^{-1/2}\right) \approx (\pm 0.707, 0.607)$, and f has an absolute maximum $f(0) = 1$.

Practice**PRACTICE** Curve Analysis: Increasing, Decreasing, Concavity

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

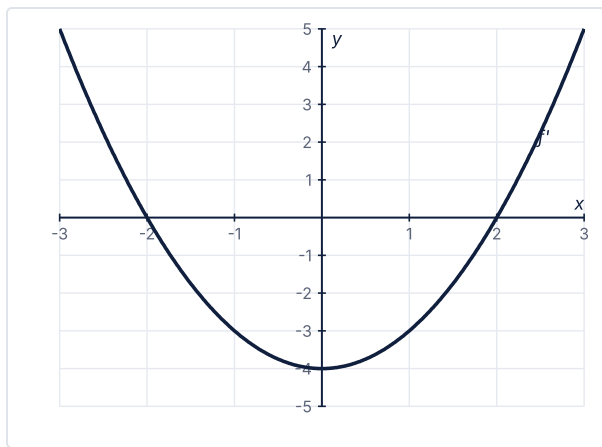
- The derivative of f is $f'(x) = (x - 2)(x + 3)$. On which interval is f decreasing?
 (A) $(-\infty, -3)$ (B) $(-3, 2)$ (C) $(2, \infty)$ (D) $(-2, 3)$
- For which values of x is $f(x) = x^3 - 3x^2$ concave up?
 (A) $x < 1$ (B) $0 < x < 2$ (C) $x > 1$ (D) $x > 2$
- Where does $f(x) = 2x^3 - 9x^2 + 12x$ have a relative maximum?
 (A) $x = 1$ (B) $x = 2$ (C) $x = 0$ (D) $x = \frac{3}{2}$
- For a twice differentiable function, $f'(3) = 0$ and $f''(3) = -4$. What does the second derivative test conclude?
 (A) f has a relative minimum at $x = 3$. (B) f has an inflection point at $x = 3$.
 (C) The test is inconclusive. (D) f has a relative maximum at $x = 3$.
- How many inflection points does the graph of $f(x) = x^4 - 6x^2$ have?
 (A) 0 (B) 2 (C) 1 (D) 3

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6. The graph of f' is shown, where $f'(x) = x^2 - 4$. At what value of x does f have a relative minimum?



Graph of f' .

- (A) $x = -2$ (B) $x = 0$ (C) $x = 2$ (D) $x = 4$
7. What is the absolute maximum value of $f(x) = xe^{-x}$ on $[0, 5]$?
 (A) $\frac{1}{e}$ (B) 1 (C) e (D) $5e - 5$
8. On an interval, $f'(x) > 0$ and $f''(x) < 0$. Which describes f on that interval?
 (A) Increasing and concave up (B) Decreasing and concave down
 (C) Decreasing and concave up (D) Increasing and concave down
9. What is the absolute minimum value of $f(x) = x^3 - 3x$ on $[-2, 2]$?
 (A) 2 (B) -2 (C) 0 (D) -4
10. For $f(x) = 2x^3 - 3x^2 - 12x + 5$, find the intervals where f is increasing and decreasing, and the relative extreme values.
11. Find the intervals of concavity and the inflection points of $f(x) = x^4 - 8x^2$.
12. Use the second derivative test to classify the critical numbers of $f(x) = x^3 - 3x^2 - 9x + 5$ and give the extreme values.
13. A differentiable function g has $g'(x) = x(x - 2)^2$. Find all relative extrema of g , and justify your answer with a sign argument.
14. The total sales of a product after t weeks are $S(t) = -t^3 + 9t^2$ units for $0 \leq t \leq 6$. At what time is the sales rate greatest, and what is that rate?
15. A continuous function f satisfies: $f'(x) > 0$ on $(-\infty, 1)$ and $(3, \infty)$; $f'(x) < 0$ on $(1, 3)$; $f''(x) < 0$ on $(-\infty, 2)$ and $f''(x) > 0$ on $(2, \infty)$. Describe the relative extrema, inflection point, and concavity of f .

38.4 Mean Value Theorem, Linear Approximation, and Motion

Use the Mean Value Theorem, tangent line approximations, and derivatives of position, velocity, and acceleration to analyze motion and justify conclusions in AP style.

Why This Matters

The derivative connects what happens at one instant to what happens over a whole interval. The **Mean Value Theorem** says that a smooth function must, at some moment, change at exactly its average rate. It is the reason a speed camera can prove a driver sped without ever clocking the car. **Linear approximation** uses the tangent line to estimate hard values, which is how calculators and engineers compute quickly. And **particle motion** applies the derivative to position, velocity, and acceleration, the most common application on the AP exam.

Students usually get stuck in these places. They apply the Mean Value Theorem without checking that its two hypotheses (continuity on a closed interval and differentiability on the open interval) are met. They write the linearization with the wrong base point or forget the term $f(a)$. They are unsure when speed is increasing, confusing it with positive acceleration. And they confuse displacement with total distance. The sections below fix each of these.

Core Ideas

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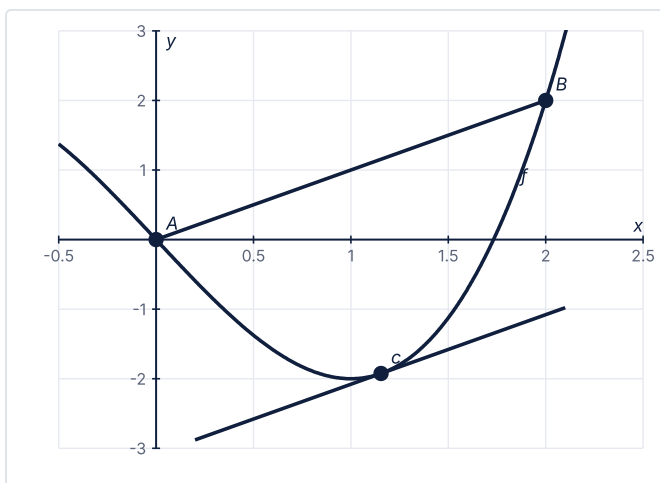
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The Mean Value Theorem (MVT)

If f is **continuous** on $[a, b]$ and **differentiable** on (a, b) , then there is at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Geometrically, some tangent line is parallel to the secant line through $(a, f(a))$ and $(b, f(b))$. **Rolle's Theorem** is the special case $f(a) = f(b)$, which gives $f'(c) = 0$.



For $f(x) = x^3 - 3x$ on $[0, 2]$, the secant AB has slope 1. The tangent at $x = 2/\sqrt{3}$, about 1.155, is parallel to it.

Always write the hypotheses when you use the theorem. A typical AP justification reads: "Since f is continuous on $[a, b]$ and differentiable on (a, b) , the Mean Value Theorem guarantees a c in (a, b) with $f'(c) = \dots$ " The theorem tells you that such a c exists, but not where it is or how many there are.

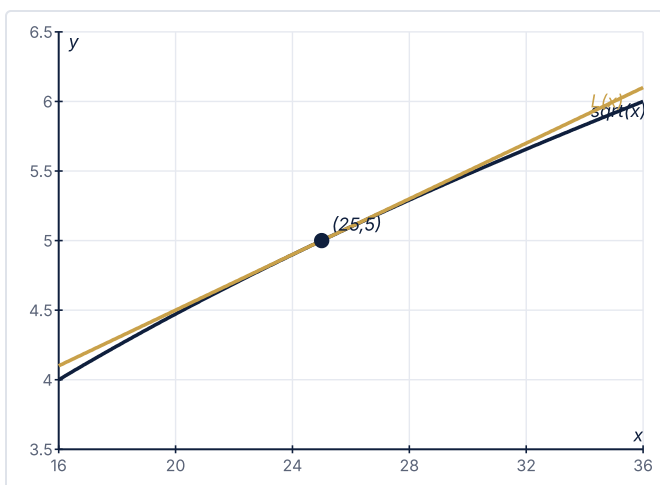
Linearization (Tangent Line Approximation)

If f is differentiable at a , the **linearization** of f at a is

$$L(x) = f(a) + f'(a)(x - a),$$

and $f(x) \approx L(x)$ for x near a .

If $f'' > 0$ near a (concave up), the tangent line lies **below** the curve, so L underestimates f . If $f'' < 0$ (concave down), the tangent line lies **above** the curve and L overestimates f .



The graph of $f(x) = \sqrt{x}$ and its tangent line L at $x = 25$. Because the graph is concave down, the tangent line lies above it.

The same idea written with differentials: $dy = f'(x) dx$. If a measured input x has a small error dx , the resulting error in $f(x)$ is approximately dy .

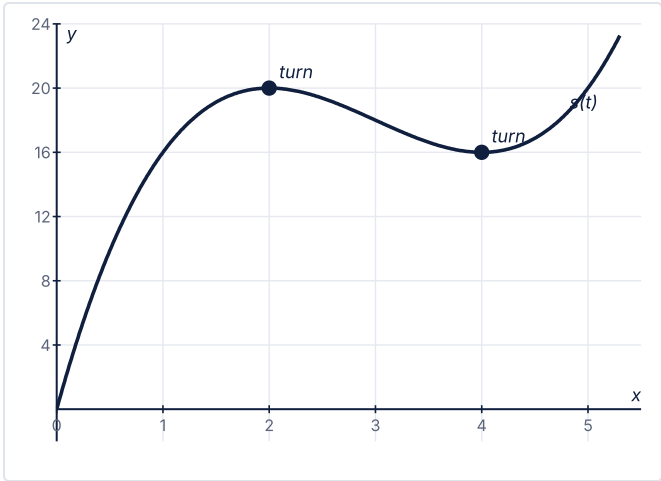
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Particle Motion

For a particle with position $s(t)$ on a line: **velocity** $v(t) = s'(t)$, **acceleration** $a(t) = v'(t) = s''(t)$, and **speed** $|v(t)|$.
 The particle moves right when $v > 0$ and left when $v < 0$. It is momentarily at rest when $v = 0$. **Speed is increasing when v and a have the same sign, and decreasing when they have opposite signs.** Displacement is $s(b) - s(a)$. Total distance is the sum of the lengths of each leg between turning points.



Position $s(t) = t^3 - 9t^2 + 24t$. The particle turns around at $t = 2$ and $t = 4$, where s has a relative maximum and minimum.

Interval	v	a	Motion	Speed
$0 < t < 2$	+	-	Right	Decreasing
$2 < t < 3$	-	-	Left	Increasing
$3 < t < 4$	-	+	Left	Decreasing
$4 < t < 5$	+	+	Right	Increasing

Worked Examples

EXAMPLE 1

Verify that the Mean Value Theorem applies to $f(x) = x^3 - 3x$ on $[0, 2]$ and find all values of c it guarantees.

SOLUTION
 f is a polynomial, so it is continuous on $[0, 2]$ and differentiable on $(0, 2)$.
 Average rate of change: $\frac{f(2)-f(0)}{2-0} = \frac{2-0}{2} = 1$.
 Solve $f'(c) = 3c^2 - 3 = 1$: $c^2 = \frac{4}{3}$, so $c = \pm \frac{2}{\sqrt{3}}$.
 Only $c = \frac{2}{\sqrt{3}} \approx 1.155$ lies in $(0, 2)$.

EXAMPLE 2

Explain why the Mean Value Theorem does not apply to $f(x) = |x - 1|$ on $[0, 3]$, and show that its conclusion fails.

SOLUTION
 f is continuous on $[0, 3]$, but it is not differentiable at $x = 1$, which is in $(0, 3)$. A hypothesis fails.
 Average rate: $\frac{f(3)-f(0)}{3-0} = \frac{2-1}{3} = \frac{1}{3}$.
 But $f'(x) = -1$ for $x < 1$ and $f'(x) = 1$ for $x > 1$. The value $\frac{1}{3}$ is never attained, so no c exists.
 The theorem gives a sufficient condition only. Here the missing differentiability allows the conclusion to fail.

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EXAMPLE 3

A car travels from mile marker 0 to mile marker 130 in 2 hours on a straight highway. Explain why the car's speed must have been exactly 65 mph at some moment.

SOLUTION

Let $D(t)$ be the distance traveled. The function is differentiable (hence continuous) on the interval, since the car has a velocity at every moment.

Average speed: $\frac{D(2)-D(0)}{2-0} = \frac{130}{2} = 65$ mph.

By the Mean Value Theorem, there is a c in $(0, 2)$ with $D'(c) = 65$, so the speedometer read exactly 65 mph at some moment.

EXAMPLE 4

Find the linearization of $f(x) = \sqrt{x}$ at $a = 25$, use it to estimate $\sqrt{26}$, and say whether the estimate is too high or too low.

SOLUTION

$f(25) = 5$ and $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(25) = \frac{1}{10}$.

$L(x) = 5 + \frac{1}{10}(x - 25)$.

$\sqrt{26} \approx L(26) = 5 + 0.1 = 5.1$.

$f''(x) = -\frac{1}{4}x^{-3/2} < 0$, so f is concave down and the tangent line lies above the curve. The estimate 5.1 is **too high**. (The true value is about 5.099.)

EXAMPLE 5

The edge of a cube is measured as 10 cm with a possible error of 0.1 cm. Use differentials to estimate the error in the computed volume.

SOLUTION

$V = s^3$, so $dV = 3s^2 ds$.

With $s = 10$ and $ds = 0.1$: $dV = 3(100)(0.1) = 30$.

The volume (nominally 1000 cm³) could be off by about 30 cubic centimeters, a 3% error.

EXAMPLE 6

A particle moves along a line with $s(t) = t^3 - 9t^2 + 24t$ for $0 \leq t \leq 5$. Find when it is at rest, when it moves left, and the total distance traveled.

SOLUTION

$v(t) = 3t^2 - 18t + 24 = 3(t - 2)(t - 4)$, so the particle is at rest at $t = 2$ and $t = 4$.

$v < 0$ for $2 < t < 4$, so it moves left on $(2, 4)$ and right on $(0, 2)$ and $(4, 5)$.

Positions: $s(0) = 0$, $s(2) = 20$, $s(4) = 16$, $s(5) = 20$.

Distance: $|20 - 0| + |16 - 20| + |20 - 16| = 20 + 4 + 4 = 28$. (Displacement is only $s(5) - s(0) = 20$.)

Coach's Playbook

- 1 Check hypotheses first.** For the MVT or Rolle's Theorem, write "continuous on $[a, b]$ " and "differentiable on (a, b) " before computing anything.
- 2 Write $L(x)$ with all three parts.** Base value $f(a)$, slope $f'(a)$, and the offset $(x - a)$. Then substitute the target x .
- 3 Use concavity to judge the estimate.** Concave up means the tangent line is below (underestimate). Concave down means above (overestimate).
- 4 Make a sign table for motion.** List intervals with the signs of v and a . Direction comes from v . Speeding up or slowing down comes from comparing both.
- 5 Separate displacement from distance.** Find turning points ($v = 0$ with a sign change), add up the absolute change on each leg for distance, and use only endpoints for displacement.

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Common traps

1. Using the MVT without verifying continuity and differentiability. 2. Writing $L(x) = f'(a)(x - a)$ and leaving out $f(a)$. 3. Saying the particle speeds up whenever $a > 0$ (you must compare with the sign of v). 4. Reporting displacement when the question asks for total distance.

Mistake Clinic**Spot the error**

A student says: "By the MVT, since $f(x) = |x|$ has average rate $\frac{1}{3}$ on $[-1, 2]$, there is a c with $f'(c) = \frac{1}{3}$." The student did not check the hypotheses. The function $|x|$ is not differentiable at 0, which is inside the interval, and its derivative is only ever -1 or 1 . No c exists. The MVT cannot be used here.

Spot the error

A student estimates $\sqrt{26}$ using the linearization at 25 and writes $L(26) = \frac{1}{10} \cdot 26 + 5 = 7.6$. The tangent line was applied incorrectly. The linearization is $L(x) = 5 + \frac{1}{10}(x - 25)$, and the offset is $x - a = 26 - 25 = 1$, not 26. The correct estimate is $5 + 0.1 = 5.1$.

Spot the error

For the particle with $s(t) = t^3 - 9t^2 + 24t$, a student says: "At $t = 3.5$ the acceleration is $a(3.5) = 6(3.5) - 18 = 3 > 0$, so the particle is speeding up." The student ignored the velocity. Here $v(3.5) = 3(1.5)(-0.5) = -2.25 < 0$. The velocity is negative and the acceleration is positive, which are opposite signs, so the particle is **slowing down**. Speed increases only when v and a have the same sign.

Real-World Connection

A highway camera system records a car passing a mile marker 10 at 2:00 pm and mile marker 45 at 2:30 pm. The average speed is $\frac{45-10}{0.5} = 70$ miles per hour. If the speed limit is 65 mph, the Mean Value Theorem guarantees that at some moment between the two cameras the car's speed was exactly 70 mph, so the car broke the limit even though no camera saw it at that moment.

Challenge Corner

Use the Mean Value Theorem to prove that $\sqrt{1+x} < 1 + \frac{x}{2}$ for all $x > 0$, then use it to bound $\sqrt{1.21}$. Solution: let $f(t) = \sqrt{1+t}$, continuous on $[0, x]$ and differentiable on $(0, x)$. By the MVT there is a c in $(0, x)$ with $f(x) - f(0) = f'(c)x = \frac{x}{2\sqrt{1+c}}$. Since $c > 0$, $\sqrt{1+c} > 1$, so $\frac{x}{2\sqrt{1+c}} < \frac{x}{2}$. Therefore $\sqrt{1+x} - 1 < \frac{x}{2}$. With $x = 0.21$: $\sqrt{1.21} < 1.105$. (The exact value is 1.1.)

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Practice

PRACTICE Mean Value Theorem, Linear Approximation, and Motion

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- The Mean Value Theorem is applied to $f(x) = x^2 - 4x$ on $[0, 6]$. What is the value of c ?
 (A) 2 (B) 6 (C) 3 (D) 4
- Which function satisfies the hypotheses of the Mean Value Theorem on $[-1, 1]$?
 (A) $f(x) = x^3 + 1$ (B) $f(x) = |x|$ (C) $f(x) = \frac{1}{x}$
 (D) $f(x) = x^{2/3}$
- Use the tangent line to $f(x) = e^x$ at $x = 0$ to approximate $e^{0.2}$.
 (A) 1.22 (B) 1.02 (C) 0.2 (D) 1.2
- A differentiable function satisfies $f(3) = 5$ and $f'(3) = -2$. What is the linear approximation of $f(3.1)$?
 (A) 5.2 (B) 4.8 (C) 4.98 (D) -0.2
- A particle has position $s(t) = t^3 - 6t^2 + 9t$. At what times is it at rest?
 (A) $t = 2$ only (B) $t = 0$ and $t = 3$ (C) $t = 1$ and $t = 3$
 (D) $t = 1$ only
- A particle has velocity $v(t) = t^2 - 4t$. At $t = 1$, what is true about its speed?
 (A) It is increasing, because velocity and acceleration are both negative.
 (B) It is decreasing, because the velocity is negative.
 (C) It is decreasing, because the acceleration is negative.
 (D) It is constant.
- A particle has position $s(t) = t^2 - 4t$ on $0 \leq t \leq 5$. What is the total distance traveled?
 (A) 5 (B) 9 (C) 17 (D) 13
- A differentiable function has $f(1) = 4$ and $f(5) = 16$. Which statement must be true?
 (A) $f'(x) = 3$ for every x in $(1, 5)$. (B) $f'(c) = 3$ for some c in $(1, 5)$.
 (C) $f'(x) > 3$ for every x in $(1, 5)$. (D) $f(3) = 10$.
- Use a tangent line approximation to estimate $\sqrt{16.4}$.
 (A) 4.05 (B) 4.1 (C) 4.4 (D) 4.025
- Find the value of c guaranteed by the Mean Value Theorem for $f(x) = \sqrt{x+1}$ on $[0, 8]$.
- Does the Mean Value Theorem apply to $f(x) = x^{2/3}$ on $[-1, 8]$? Explain, and check whether a number c with $f'(c) = \frac{f(8)-f(-1)}{9}$ exists in $(-1, 8)$.
- Find the linearization of $f(x) = \sqrt[3]{x}$ at $a = 27$, use it to approximate $\sqrt[3]{28}$, and state whether the approximation is an overestimate or an underestimate.
- The radius of a sphere is measured as 6 cm with a possible error of 0.05 cm. Use differentials to estimate the error in the volume $V = \frac{4}{3}\pi r^3$.
- A particle moves with position $s(t) = 2t^3 - 15t^2 + 24t$ for $0 \leq t \leq 5$. (a) When is it at rest? (b) When does it move left? (c) Is its speed increasing or decreasing at $t = 2$? (d) What is the total distance traveled?
- Suppose f is differentiable, $f(1) = 3$, and $f'(x) \leq 2$ for all x . Use the Mean Value Theorem to show that $f(5) \leq 11$.

Answers Answers and Explanations

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

38.1 Related Rates

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PRACTICE · RELATED RATES

1. A. $A = s^2$, so $dA/dt = 2s \, ds/dt = 2(5)(3) = 30$. The value 15 drops the factor 2, and 75 uses s^2 times the rate.
2. B. $dV/dt = 4 \pi r^2 \, dr/dt = 4 \pi (16)(0.5) = 32 \pi$. The choice $128 \pi / 3$ multiplies V itself by the rate instead of differentiating.
3. C. $y = 8$, and $dy/dt = -(x/y)(dx/dt) = -(6/8)(2) = -3/2$. The choice $-8/3$ inverts x/y , and $-3/4$ forgets to multiply by the rate 2.
4. D. After 2 hours the legs are 60 and 80 miles, so $z = 100$. Then $dz/dt = (80 \cdot 40 + 60 \cdot 30)/100 = 50$. Adding the speeds to get 70 ignores that the ships move at right angles.
5. C. $dA/dt = 2 \pi r \, dr/dt$, so $12 \pi = 6 \pi \, dr/dt$ and $dr/dt = 2$. The choice 4 forgets the factor 2 in $2 \pi r$, and 12 is just the area rate.
6. B. Similar triangles: $s/1.5 = (x+s)/4.5$, which gives $s = x/2$. So $ds/dt = (1/2)(1) = 0.5$. The tip of the shadow moves at 1.5 m/s, which is a different quantity.
7. D. $A = LW$, so $dA/dt = L'W + LW' = (2)(5) + (8)(-1) = 2$. The choice 18 treats the width as increasing; -2 reverses the final sign.
8. A. $V = (1/3) \pi (h/3)^2 h = \pi h^3/27$, so $dV/dt = \pi h^2/9 \, dh/dt$. At $h=6$: $2 = 4 \pi \, dh/dt$, so $dh/dt = 1/(2 \pi)$. The choice $3/(2 \pi)$ treats r as fixed.
9. B. $\tan(\theta) = h/300$, so $\sec^2(\theta) \, d(\theta)/dt = 30/300 = 0.1$. Here $\tan(\theta) = 4/3$, so $\sec^2 = 25/9$, and $d(\theta)/dt = 0.1(9/25) = 0.036$. The choice 0.1 forgets to divide by $\sec^2$; 0.278 multiplies by it.
10. Volume: 600 cm^3 per second. Surface area: 240 cm^2 per second. $V = s^3$ gives $V' = 3s^2 \, s' = 3(100)(2) = 600$, and $S = 6s^2$ gives $S' = 12 \, s \, s' = 12(10)(2) = 240$.
11. $9/8$ feet per second (1.125 ft/s). $x^2 + y^2 = 225$, and when $y = 9$, $x = 12$. Then $2x \, x' + 2y \, y' = 0$ gives $x' = -(y/x) \, y' = -(9/12)(-1.5) = 1.125$.
12. Radius: $-1/(5 \pi) \text{ cm}$ per minute (about -0.064). Surface area: -8 cm^2 per minute. $V = 4 \pi r^2 \, r'$ gives $-20 = 100 \pi \, r'$, so $r' = -1/(5 \pi)$. Then $S' = 8 \pi r \, r' = 8 \pi (5)(-1/(5 \pi)) = -8$.
13. The shadow is shrinking at 2 feet per second ($ds/dt = -2$), and the tip moves toward the lamp at 6 feet per second. Similar triangles: $s/6 = (x+s)/18$, so $s = x/2$. With $dx/dt = -4$: $ds/dt = -2$, and the tip position $x + s = 3x/2$ has rate $(3/2)(-4) = -6$.
14. 0.048 radian per second. $\tan(\theta) = h/40$, so $\sec^2(\theta) \, \theta' = 3/40$. With $h = 30$, $\tan(\theta) = 3/4$ and $\sec^2 = 25/16$. Then $\theta' = (3/40)(16/25) = 0.048$.
15. $3 \sqrt{3}$, about 5.196 cm^2 per second. $A = (1/2)(5)(12) \sin(\theta) = 30 \sin(\theta)$, so $A' = 30 \cos(\theta) \, \theta' = 30 (\sqrt{3}/2)(0.2) = 3 \sqrt{3}$.

38.2 Optimization

PRACTICE · OPTIMIZATION

1. B. With $y = 20 - x$ the product is $x(20-x)$, which is largest at $x = 10$ and equals 100. The choice 400 multiplies 20 by 20 and ignores the sum condition.
2. C. If each end is x meters, the side parallel to the wall is $120 - 2x$, so $A = x(120-2x)$ and $A' = 120 - 4x = 0$ at $x = 30$. $A(30) = 30(60) = 1800$. The value 900 is the area if all four sides needed fence.
3. A. $V = x(12-2x)^2$, so $V' = (12-2x)(12-6x) = 0$ at $x = 2$ or $x = 6$. At $x = 6$ the volume is zero, so $x = 2$ gives the maximum, with $V = 128$.
4. D. Minimize $D = x^2 + (x^2-1)^2 = x^4 - x^2 + 1$. $D' = 4x^3 - 2x = 0$ gives $x^2 = 1/2$ ($x = 0$ is a local maximum of D). Then $D = 3/4$ and the distance is $\sqrt{3}/2$. The choice $3/4$ forgets the square root.
5. B. A zero derivative only marks a critical point; it could be a minimum or neither. A sign change of f' from positive to negative is the first derivative test condition for a relative maximum.
6. C. $P'(x) = -4x + 120 = 0$ gives $x = 30$. The value 1300 is the maximum profit, not the number of items, and 60 is the sum of the two zeros of P , not the midpoint where the maximum occurs.
7. A. With sides x and $18-x$, the squared diagonal is $x^2 + (18-x)^2$, minimized at $x = 9$, giving 162. The diagonal is $\sqrt{162} = 9 \sqrt{2}$. The value 9 is a side, not the diagonal.
8. D. $f'(x) = 3x^2 - 12 = 0$ at $x = -2$ and $x = 2$. Candidates: $f(-3) = 9$, $f(-2) = 16$, $f(2) = -16$, $f(3) = -9$. The largest is 16. The choice 9 compares only the endpoints.
9. B. $P = 2x + 1600/x$, so $P' = 2 - 1600/x^2 = 0$ gives $x = \sqrt{800} = 20 \sqrt{2}$, a square. The perimeter is $4(20 \sqrt{2}) = 80 \sqrt{2}$. The choice $40 \sqrt{2}$ is only half the perimeter.
10. 8 and 8 (sum 16). With $y = 64/x$, $S = x + 64/x$, $S' = 1 - 64/x^2 = 0$ gives $x = 8$. $S'' = 128/x^3 > 0$, so the sum is a minimum.
11. Base 4 m by 4 m, height 2 m, minimum surface area 48 m^2 . $x^2 h = 32$, so $S = x^2 + 4xh = x^2 + 128/x$. $S' = 2x - 128/x^2 = 0$ gives $x = 4$, $h = 2$. $S'' > 0$ confirms a minimum, and $S(4) = 16 + 32 = 48$.
12. The point $(-6/5, 3/5)$, and the minimum distance is $3/\sqrt{5}$, about 1.342. $D = x^2 + (2x+3)^2$ has $D' = 10x + 12 = 0$, so $x = -6/5$ and $y = 3/5$. D' changes from negative to positive there, so it is a minimum.
13. The print area is 10 cm wide by 15 cm tall, so the poster is 14 cm by 21 cm, total area 294 cm^2 . With printed width x , $A = (x+4)(150/x + 6) = 174 + 6x + 600/x$. $A' = 6 - 600/x^2 = 0$ gives $x = 10$, and $A'' > 0$ confirms a minimum.
14. Absolute maximum $1/e$ (about 0.368) at $x = 1$; absolute minimum 0 at $x = 0$. $f'(x) = (1-x)e^{-x} = 0$ at $x = 1$. Candidates: $f(0) = 0$, $f(1) = 1/e$, $f(3) = 3e^{-3}$, about 0.149.
15. $r = 20/(4+\pi)$, about 2.80 ft, and the maximum area is $200/(4+\pi)$, about 28.0 square feet. With rectangle height h , the perimeter $2h + 2r + \pi r = 20$, so $A = 20r - (2 + \pi/2) r^2$. $A' = 20 - (4+\pi) r = 0$, and $A'' < 0$ confirms a maximum.

38.3 Curve Analysis: Increasing, Decreasing, Concavity

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PRACTICE · CURVE ANALYSIS: INCREASING, DECREASING, CONCAVITY

1. B. f' is negative between its zeros -3 and 2 . The other intervals $(-\infty, -3)$ and $(2, \infty)$ are where f' is positive. The choice $(-2, 3)$ swaps the signs of the zeros.
2. C. $f''(x) = 6x - 6$ is positive when $x > 1$. The choice $x > 2$ mistakes a zero of f' ($x = 0$ and $x = 2$) for the concavity change.
3. A. $f'(x) = 6(x-1)(x-2)$. f' changes from positive to negative at $x = 1$, a maximum, and from negative to positive at $x = 2$, a minimum. The choice $x = 2$ is the minimum.
4. D. A negative second derivative means the graph is concave down at a horizontal tangent, so it is the top of a hill: a relative maximum. A positive value would give a minimum.
5. B. $f''(x) = 12x^2 - 12 = 12(x-1)(x+1)$, which changes sign at both $x = -1$ and $x = 1$. Two sign changes mean two inflection points.
6. C. f' changes from negative to positive at $x = 2$, so f has a relative minimum there. At $x = -2$, f' changes from positive to negative, a relative maximum.
7. A. $f'(x) = (1-x)e^{-x} = 0$ at $x = 1$. Candidates: $f(0) = 0$, $f(1) = 1/e$, $f(5) = 5e^{-5}$, about 0.034 . The largest is $1/e$. The choice $5e^{-5}$ is the endpoint value only.
8. D. $f' > 0$ means f is increasing. $f'' < 0$ means f' is decreasing, so f is concave down, rising but leveling off.
9. B. $f'(x) = 3x^2 - 3 = 0$ at $x = -1$ and $x = 1$. Candidates: $f(-2) = -2$, $f(-1) = 2$, $f(1) = -2$, $f(2) = 2$. The smallest is -2 , attained at $x = -2$ and $x = 1$.
10. Increasing on $(-\infty, -1)$ and $(2, \infty)$; decreasing on $(-1, 2)$. Relative maximum $f(-1) = 12$ and relative minimum $f(2) = -15$. $f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$ changes from $+$ to $-$ at -1 and from $-$ to $+$ at 2 .
11. Concave up on $(-\infty, -2/\sqrt{3})$ and $(2/\sqrt{3}, \infty)$; concave down on $(-2/\sqrt{3}, 2/\sqrt{3})$. Inflection points at $x = -2/\sqrt{3}$ and $x = 2/\sqrt{3}$ ($f = -80/9$ at both). $f''(x) = 12x^2 - 16$ changes sign at $x = \pm 2/\sqrt{3}$.
12. Relative maximum at $x = -1$ with $f(-1) = 10$; relative minimum at $x = 3$ with $f(3) = -22$. $f'(x) = 3(x-3)(x+1)$ and $f''(x) = 6x - 6$. $f''(-1) = -12 < 0$ and $f''(3) = 12 > 0$.
13. g has a relative minimum at $x = 0$ and no relative extremum at $x = 2$. $g'(x)$ is negative for $x < 0$ and positive on $(0, 2)$ and $(2, \infty)$. g' changes from negative to positive at $x = 0$, but does not change sign at $x = 2$.
14. The rate is greatest at $t = 3$ weeks, and the rate is 27 units per week. $S'(t) = -3t^2 + 18t$ and $S''(t) = -6t + 18 = 0$ at $t = 3$. S'' changes from positive to negative there, so S' has its maximum, $S'(3) = 27$.
15. f has a relative maximum at $x = 1$ and a relative minimum at $x = 3$ (f' changes $+$ to $-$ at 1 and $-$ to $+$ at 3). f is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$, so the inflection point is at $x = 2$ (between the max and the min). f is increasing on $(-\infty, 1)$ and $(3, \infty)$ and decreasing on $(1, 3)$.

38.4 Mean Value Theorem, Linear Approximation, and Motion**PRACTICE · MEAN VALUE THEOREM, LINEAR APPROXIMATION, AND MOTION**

1. C. The average rate is $(12-0)/6 = 2$. Setting $f'(c) = 2c - 4 = 2$ gives $c = 3$. The choice 2 is the average rate itself, not the location c .
2. A. $x^3 + 1$ is a polynomial, so it is continuous and differentiable everywhere. $|x|$ and $x^{2/3}$ fail differentiability at 0 , and $1/x$ is not even defined at 0 .
3. D. $L(x) = f(0) + f'(0)x = 1 + x$, so $e^{0.2}$ is approximately 1.2 . The choice 0.2 forgets $f(0) = 1$, and 1.02 uses 0.02 .
4. B. $L(3.1) = 5 + (-2)(0.1) = 4.8$. The choice 5.2 has the wrong sign, and -0.2 omits $f(3) = 5$.
5. C. $v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3) = 0$ at $t = 1$ and $t = 3$. The value $t = 2$ is where the acceleration $a(t) = 6t - 12$ is zero.
6. A. $v(1) = -3$ and $a(t) = 2t - 4$ gives $a(1) = -2$. Same sign means the speed is increasing. A negative velocity alone or a negative acceleration alone does not decide the speed.
7. D. $v = 2t - 4$ changes sign at $t = 2$. $s(0) = 0$, $s(2) = -4$, $s(5) = 5$. Distance = $|-4 - 0| + |5 - (-4)| = 4 + 9 = 13$. The value 5 is the displacement.
8. B. The MVT gives a c with $f'(c) = (16-4)/(5-1) = 3$. It guarantees at least one such c , not every x , and says nothing about particular function values like $f(3)$.
9. A. Use $a = 16$: $L(x) = 4 + (x - 16)/8$. Then $L(16.4) = 4 + 0.4/8 = 4.05$. The choice 4.025 divides by 16 instead of 8 , which forgets the factor 2 in $f'(x) = 1/(2\sqrt{x})$.
10. $c = 3$. $f(8) = 3$ and $f(0) = 1$, so the average rate is $2/8 = 1/4$. Set $f'(c) = 1/(2\sqrt{c+1}) = 1/4$, so $\sqrt{c+1} = 2$ and $c = 3$, which is in $(0, 8)$. The function is continuous on $[0, 8]$ and differentiable on $(0, 8)$.
11. No. $f(x) = x^{2/3}$ is not differentiable at $x = 0$, which is in $(-1, 8)$. The average rate is $(4-1)/9 = 1/3$, and $f'(c) = 2/(3c^{1/3}) = 1/3$ requires $c = 8$, which is an endpoint and not in the open interval. So the conclusion fails.
12. $L(x) = 3 + (x-27)/27$, and cube root of 28 is approximately $3 + 1/27$, about 3.0370 . It is an overestimate because $f''(x) = -(2/9)x^{-5/3} < 0$, so f is concave down and the tangent line lies above the curve (true value about 3.0366).
13. $dV = 4\pi r^2 dr = 4\pi (36)(0.05) = 7.2\pi$, about 22.6 cubic centimeters.
14. (a) At rest at $t = 1$ and $t = 4$, since $v(t) = 6t^2 - 30t + 24 = 6(t-1)(t-4)$. (b) Moves left for $1 < t < 4$. (c) At $t = 2$, $v = -12$ and $a = 12(2) - 30 = -6$, the same sign, so the speed is increasing. (d) $s(0) = 0$, $s(1) = 11$, $s(4) = -16$, $s(5) = -5$, so distance = $11 + 27 + 11 = 49$.
15. Since f is differentiable, the MVT applies on $[1, 5]$: there is a c in $(1, 5)$ with $f(5) - f(1) = f'(c)(5-1)$. Because $f'(c) \leq 2$, $f(5) - 3 \leq 2(4) = 8$, so $f(5) \leq 11$.

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