



Integrals and the Fundamental Theorem of Calculus

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

In this booster

39.1 Antiderivatives and Indefinite Integrals

39.2 Riemann Sums and the Definite Integral

39.3 The Fundamental Theorem of Calculus

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Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

39.1 Antiderivatives and Indefinite Integrals

Learn to reverse differentiation: the power rule backwards, trig and exponential antiderivatives, the constant of integration, and how initial conditions single out one function from a whole family.

Why This Matters

Up to now, calculus has asked you one kind of question: given a function, find how fast it changes. This section turns the question around. Given how fast something changes, can you recover the original quantity? If you know a car's velocity at every moment, can you find where the car is? If you know the rate at which a tank fills, can you find how much water is inside? Finding the original function from its rate of change is called **antidifferentiation**, and it is the first step of integral calculus.

Students get stuck here for three predictable reasons. First, they forget the constant of integration, $+C$, and lose points on every problem. Second, they mix up the power rule for derivatives and the power rule for antiderivatives, multiplying by the exponent when they should divide. Third, they try to integrate products and quotients term by term as if integration worked like multiplication. Antiderivatives show up everywhere later: in area and volume, in physics (position from velocity), in economics (total cost from marginal cost), and in every differential equation you will meet in college STEM courses. A strong habit now, checking every answer by differentiating it, will protect you for the rest of the course.

Core Ideas

A function F is an **antiderivative** of f on an interval if $F'(x) = f(x)$ for every x in that interval. For example, $F(x) = x^2$ is an antiderivative of $f(x) = 2x$, but so is $x^2 + 5$ and $x^2 - 100$. Constants disappear when you differentiate, so an antiderivative is never unique. The next result tells you exactly how much freedom you have.

Family of Antiderivatives

If F is one antiderivative of f on an interval, then every antiderivative of f on that interval has the form $F(x) + C$ for some constant C .

The **indefinite integral** is the whole family:

$$\int f(x) dx = F(x) + C.$$

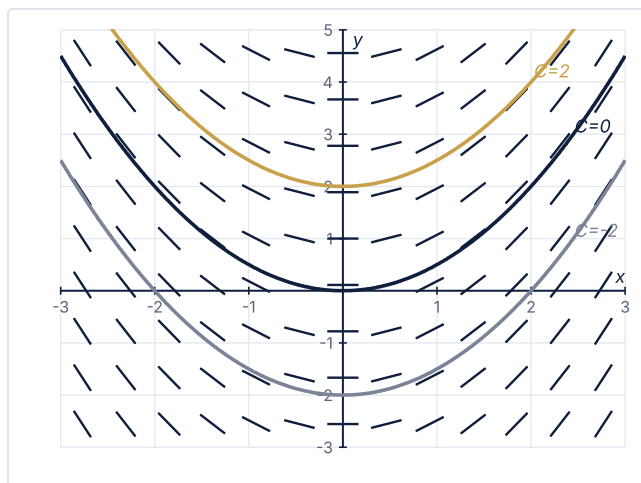
The symbol $\int$ means antidifferentiate, $f(x)$ is the **integrand**, and dx names the variable. The $+C$ is part of the answer, not an optional decoration.

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Geometrically, the graphs of $F(x) + C$ are vertical translates of each other. Every curve in the family has the same slope at the same x , because they all have derivative $f(x)$. The picture below shows the slope field for $\frac{dy}{dx} = x$ together with three members of the family $y = \frac{x^2}{2} + C$.



Every curve in the family has slope x at each x . The constant C only shifts the curve up or down.

Because differentiation rules reverse cleanly, each derivative formula you know gives an antiderivative formula. The table collects the ones you must know cold.

Integrand $f(x)$	Antiderivative family
x^n with $n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	$\ln x + C$
e^x	$e^x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$
k (a constant)	$kx + C$

Power Rule Backwards and Linearity

For $n \neq -1$:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

Add one to the exponent, then divide by the new exponent.

Integration is linear:

$$\int [af(x) + bg(x)] dx = a \int f(x) dx + b \int g(x) dx.$$

Constants pull out and sums split. Nothing similar holds for products or quotients, so expand or rewrite those first.

Notice the special case $n = -1$. The power rule would divide by zero, so the antiderivative of $x^{-1} = \frac{1}{x}$ is the natural logarithm, $\ln |x| + C$. The absolute value matters because $\frac{1}{x}$ is defined for negative x too. Roots and reciprocals must be rewritten as powers before you integrate: $\sqrt{x} = x^{1/2}$ and $\frac{1}{x^3} = x^{-3}$.

A single antiderivative can be pinned down by extra information. An **initial value problem** gives a derivative and one point on the curve. You integrate to get the family, then use the point to solve for C . For motion, with position $s(t)$, velocity $v(t) = s'(t)$ and acceleration $a(t) = v'(t)$, you integrate twice and need two conditions, one for v and one for s .

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Two notation habits will save you grief. First, always write dx at the end of an integral, because it tells you which variable you are integrating with respect to; in $\int (3t^2 + x) dt$ the letter x is a constant. Second, say what you are doing in words as you work: add one to the exponent, divide by the new exponent. Students who narrate their steps almost never multiply when they should divide. Finally, remember that an antiderivative answer is a claim you can test. If differentiating your answer does not return the integrand exactly, something is wrong, and you can find it before anyone else does.

Step in an initial value problem	What you do
1. Integrate	Write the general antiderivative, with $+C$
2. Substitute	Put the given input and output into your formula
3. Solve	Find the single value of C
4. Rewrite	State the final function with C replaced

Worked Examples

EXAMPLE 1

Find $\int (6x^2 - 4x + 5) dx$.

SOLUTION

Split the sum and apply the power rule to each term: $\int 6x^2 dx = 6 \cdot \frac{x^3}{3} = 2x^3$, $\int -4x dx = -4 \cdot \frac{x^2}{2} = -2x^2$, and $\int 5 dx = 5x$.

$$\int (6x^2 - 4x + 5) dx = 2x^3 - 2x^2 + 5x + C.$$

Check: the derivative of the answer is $6x^2 - 4x + 5$. Good.

EXAMPLE 2

Find $\int \left(4\sqrt{x} - \frac{6}{x^3} \right) dx$ for $x > 0$.

SOLUTION

Rewrite as powers: $4x^{1/2} - 6x^{-3}$.

$$\int 4x^{1/2} dx = 4 \cdot \frac{x^{3/2}}{3/2} = \frac{8}{3}x^{3/2}.$$

$$\int -6x^{-3} dx = -6 \cdot \frac{x^{-2}}{-2} = 3x^{-2}.$$

$$\frac{8}{3}x^{3/2} + 3x^{-2} + C = \frac{8}{3}x^{3/2} + \frac{3}{x^2} + C.$$

EXAMPLE 3

Find $\int (5 \cos x - 3 \sin x + 2e^x) dx$.

SOLUTION

Use linearity and the table: $\int \cos x dx = \sin x$, $\int \sin x dx = -\cos x$, $\int e^x dx = e^x$.

$$5 \sin x - 3(-\cos x) + 2e^x = 5 \sin x + 3 \cos x + 2e^x.$$

$$5 \sin x + 3 \cos x + 2e^x + C.$$

EXAMPLE 4

Find $\int \frac{t^3 - 4t + 7}{t} dt$ for $t > 0$.

SOLUTION

You cannot integrate a quotient directly, but a single-term denominator splits: $\frac{t^3}{t} - \frac{4t}{t} + \frac{7}{t} = t^2 - 4 + \frac{7}{t}$.

$$\int t^2 dt = \frac{t^3}{3}, \int -4 dt = -4t, \int \frac{7}{t} dt = 7 \ln |t|.$$

$$\frac{t^3}{3} - 4t + 7 \ln |t| + C.$$

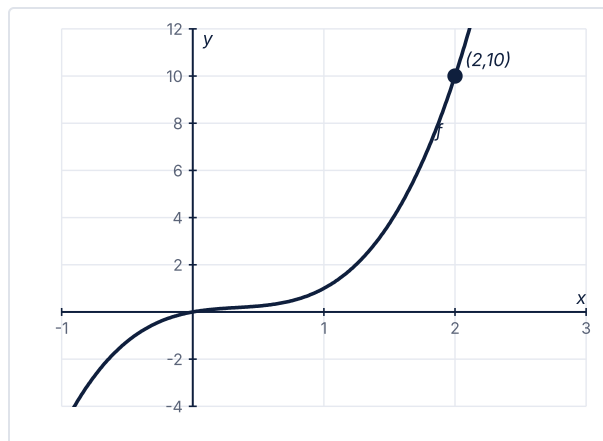
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EXAMPLE 5

Find f if $f'(x) = 6x^2 - 4x + 1$ and $f(2) = 10$.



The particular antiderivative through the point $(2,10)$.

SOLUTION

Integrate: $f(x) = 2x^3 - 2x^2 + x + C$.

Use $f(2) = 10$: $2(8) - 2(4) + 2 + C = 10$, so $10 + C = 10$ and $C = 0$.

$$f(x) = 2x^3 - 2x^2 + x.$$

Its graph is shown below and passes through $(2,10)$.

EXAMPLE 6

A particle moves on a line with acceleration $a(t) = 6t - 4$ in m/s^2 . At $t = 0$ its velocity is 3 m/s and its position is -2 m. Find its position at $t = 2$.

SOLUTION

Integrate acceleration: $v(t) = 3t^2 - 4t + C_1$. Since $v(0) = 3$, $C_1 = 3$, so $v(t) = 3t^2 - 4t + 3$.

Integrate velocity: $s(t) = t^3 - 2t^2 + 3t + C_2$. Since $s(0) = -2$, $C_2 = -2$, so $s(t) = t^3 - 2t^2 + 3t - 2$.

$s(2) = 8 - 8 + 6 - 2 = 4$. The particle is at 4 m.

Coach's Playbook

- 1 **Rewrite first.** Turn roots, reciprocals and quotients with one-term denominators into sums of powers of x . Expand products such as $x(x^2 - 4)$ before integrating.
 - 2 **Integrate term by term.** Add one to each exponent and divide by the new exponent. Treat $\frac{1}{x}$, e^x , and trig terms with the table.
 - 3 **Write +C immediately.** For an indefinite integral, put the constant in as you finish the last term so it never gets forgotten.
 - 4 **Check by differentiating.** This takes ten seconds and catches sign errors, wrong exponents, and missing coefficients.
 - 5 **For initial values, solve C last.** Only substitute the given point after you have the full general antiderivative.
- ◆ **Sign memory for trig:** the integral of $\sin x$ gets a minus sign, the integral of $\cos x$ does not. The derivative of $\cos x$ is $-\sin x$, so the minus travels back the other way.
 - ◆ **Units help:** if v is in meters per second and t in seconds, then s is in meters. Constants in motion problems often have a meaning, such as starting position.
 - ◆ **Coefficient sanity check:** after integrating x^n , the new coefficient should be the old one divided by $n + 1$. If you multiplied, you differentiated by accident.
 - ◆ **Two conditions, two constants:** integrating twice always produces two constants. Solve for the first one before integrating again.

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Common traps

(1) Forgetting $+C$. (2) Multiplying by the exponent instead of dividing: writing $\int x^3 dx = 3x^2$. (3) Writing $\int \frac{1}{x} dx = \frac{x^0}{0}$ instead of $\ln|x| + C$. (4) Integrating a product as a product of integrals, or a quotient as a quotient of integrals.

Mistake Clinic**Spot the error**

Problem: Find $\int \frac{1}{x^3} dx$. **Student's work:** $\int \frac{1}{x^3} dx = \ln|x^3| + C$. **What went wrong:** The student saw a fraction with x in the denominator and used the logarithm rule. The log rule applies only to $\frac{1}{x}$, which is x^{-1} . Here the exponent is -3 , so the power rule works. **Correct solution:** $\int x^{-3} dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$. Check: the derivative of $-\frac{1}{2}x^{-2}$ is x^{-3} .

Spot the error

Problem: Find $\int \sin x dx$. **Student's work:** $\int \sin x dx = \cos x + C$. **What went wrong:** The student remembered that sine and cosine swap, but forgot the sign. The derivative of $\cos x$ is $-\sin x$, not $\sin x$, so $\cos x + C$ differentiates to $-\sin x$. **Correct solution:** $\int \sin x dx = -\cos x + C$, because $\frac{d}{dx}(-\cos x) = \sin x$.

Spot the error

Problem: Find f if $f'(x) = 2x + 3$ and $f(1) = 7$. **Student's work:** $f(x) = x^2 + 3x$, then add the value: $f(x) = x^2 + 3x + 7$. **What went wrong:** The student treated the output 7 as the constant of integration. The constant is whatever makes the graph pass through $(1, 7)$, and it is found by substituting. Here $f(1) = 1 + 3 + 7 = 11$, not 7. **Correct solution:** $f(x) = x^2 + 3x + C$, and $f(1) = 1 + 3 + C = 7$ gives $C = 3$. So $f(x) = x^2 + 3x + 3$.

Real-World Connection

A cyclist's speed is $v(t) = 2t + 1$ meters per second, and at $t = 0$ she is 3 meters past a marker. Her distance from the marker is $s(t) = t^2 + t + 3$. After 5 seconds she is $25 + 5 + 3 = 33$ meters from the marker. Odometers, fuel gauges and bank balances all work by accumulating a rate, and antiderivatives are the tool that does it.

Challenge Corner

Find $f(x)$ if $f'(x) = (2x - 1)^2$ and $f(1) = 4$. **Solution:** You cannot yet integrate a power of a binomial by the power rule alone, so expand: $(2x - 1)^2 = 4x^2 - 4x + 1$. Integrate: $f(x) = \frac{4}{3}x^3 - 2x^2 + x + C$. Then $f(1) = \frac{4}{3} - 2 + 1 + C = 4$, so $C = 4 - \frac{4}{3} + 2 - 1 = \frac{11}{3}$. Therefore $f(x) = \frac{4}{3}x^3 - 2x^2 + x + \frac{11}{3}$.

Practice**PRACTICE Antiderivatives and Indefinite Integrals**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is $\int 8x^3 dx$?
 (A) $24x^2 + C$ (B) $2x^4 + C$ (C) $8x^4 + C$ (D) $4x^4 + C$
- What is $\int \frac{6}{x^4} dx$?
 (A) $-\frac{24}{x^5} + C$ (B) $\frac{2}{x^3} + C$ (C) $\frac{6}{x^3} + C$ (D) $-\frac{2}{x^3} + C$
- What is $\int (3 \sin x + 2 \cos x) dx$?
 (A) $-3 \cos x + 2 \sin x + C$ (B) $3 \cos x - 2 \sin x + C$ (C) $-3 \cos x - 2 \sin x + C$
 (D) $3 \cos x + 2 \sin x + C$

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4. For $x > 0$, what is $\int \left(5e^x - \frac{2}{x} \right) dx$?
- (A) $5e^x + \frac{2}{x^2} + C$ (B) $\frac{5e^{x+1}}{x+1} - 2\ln x + C$ (C) $5e^x - 2\ln x + C$
 (D) $5e^x + 2\ln x + C$
5. If $f'(x) = 4x - 3$ and $f(1) = 5$, what is $f(x)$?
- (A) $2x^2 - 3x + \frac{5}{2}$ (B) $2x^2 - 3x + \frac{5}{6}$ (C) $2x^2 - 3x + \frac{5}{8}$ (D) $4x^2 - 3x + \frac{5}{4}$
6. A particle has velocity $v(t) = 3t^2 - 2$ and starts at position 4 when $t = 0$. What is its position $s(t)$?
- (A) $t^3 - 2t + 4$ (B) $t^3 - 2t$ (C) $6t + 4$ (D) $t^3 - 2t - 4$
7. What is $\int x(x^2 - 4) dx$?
- (A) $\left(\frac{x^2}{2} \right) \left(\frac{x^3}{3} - 4x \right) + C$ (B) $\frac{x^4}{4} - 4x + C$ (C) $3x^2 - 4 + C$
 (D) $\frac{x^4}{4} - 2x^2 + C$
8. What is $\int (4\sec^2 x + 1) dx$?
- (A) $4\sec x \tan x + x + C$ (B) $4\tan x + x + C$ (C) $4\tan x + x + C$ (D) $4\tan x + 1 + C$
9. If $f''(x) = 6x$, $f'(0) = -1$ and $f(0) = 2$, what is $f(2)$?
- (A) 6 (B) 10 (C) 12 (D) 8
10. Find $\int (12x^3 - 6x^2 + 5) dx$.
11. Find $\int \left(3\sqrt{x} + \frac{4}{x^2} \right) dx$ for $x > 0$.
12. If $f'(x) = 3\cos x + 2$ and $f(0) = 1$, find $f\left(\frac{\pi}{2}\right)$.
13. A particle has acceleration $a(t) = 12t - 6$, velocity $v(1) = 2$, and position $s(0) = 5$. Find $s(3)$.
14. A curve has slope $\frac{dy}{dx} = 3x^2 - 2x + 4$ at each point and passes through $(1, 3)$. Find the y -value of the curve at $x = 2$.
15. Find $f(1)$ if $f''(x) = e^x + 6x$, $f'(0) = 3$ and $f(0) = 0$. Give an exact answer.

39.2 Riemann Sums and the Definite Integral

Approximate areas and accumulated change with left, right, midpoint and trapezoid sums, read tables and graphs, and see how the limit of these sums becomes the definite integral.

Why This Matters

How far did a car travel if you only know its speed at a few moments? How much water flowed into a tank if you only read the flow rate every few minutes? Real data rarely comes as a neat formula, and even when it does, the area under a curve may be hard to find exactly. A **Riemann sum** answers the question by slicing the region into thin pieces whose areas are easy to compute and adding them up. When the slices get infinitely thin, the sum becomes the **definite integral**.

Students struggle here for concrete reasons. They miscount the number of subintervals, use the wrong endpoints for left and right sums, apply the same width to intervals that are not equal, and forget that area below the x -axis counts as negative. They also mix up when an approximation is too big or too small. This section builds all of those skills, and the definite integral you meet here is the backbone of the Fundamental Theorem of Calculus, accumulation problems, and nearly every free-response question about a table of values.

Core Ideas

Suppose f is continuous on $[a, b]$. Split the interval into n subintervals. When they have equal width, that width is $\Delta x = \frac{b-a}{n}$ and the endpoints are $x_0 = a$, $x_1 = a + \Delta x$, $\dots$, $x_n = b$. Build a rectangle on each subinterval whose height is f evaluated at a **sample point** in that subinterval. The choice of sample point names the sum.

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Riemann Sums with n Equal Subintervals

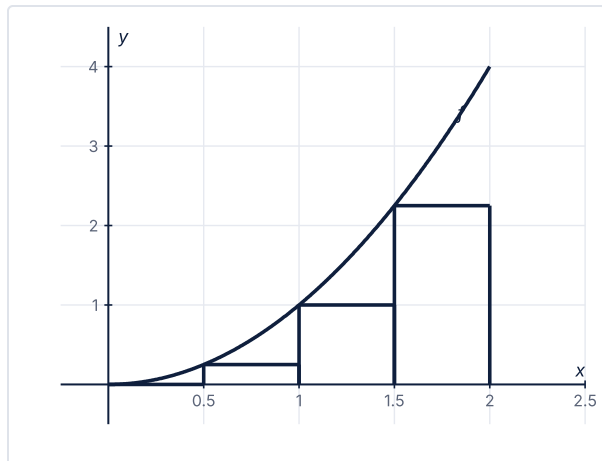
$$\text{Left: } L_n = \Delta x [f(x_0) + f(x_1) + \cdots + f(x_{n-1})]$$

$$\text{Right: } R_n = \Delta x [f(x_1) + f(x_2) + \cdots + f(x_n)]$$

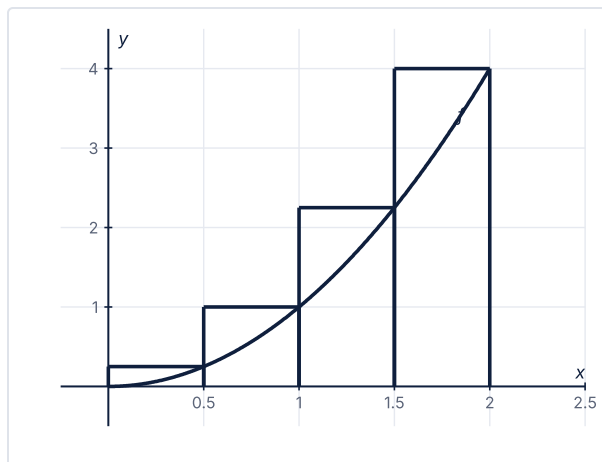
$$\text{Midpoint: } M_n = \Delta x [f(m_1) + \cdots + f(m_n)], \quad m_i = \text{midpoint of the } i\text{th subinterval}$$

Each sum has exactly n terms. The left sum skips the last endpoint; the right sum skips the first.

The next two figures show left and right sums for $f(x) = x^2$ on $[0, 2]$ with $n = 4$. In the first, each rectangle touches the curve at its left top corner and the curve rises above it, so the rectangles fit under the curve. In the second, the rectangles stick out above the curve.



Left sum, $n = 4$. The rectangles lie under the curve, so the left sum is an underestimate for an increasing function.



Right sum, $n = 4$. The rectangles stick out above the curve, so the right sum is an overestimate for an increasing function.

The Trapezoid Sum and the Definite Integral

Trapezoid sum on any partition $a = x_0 < x_1 < \cdots < x_n = b$ (widths may differ):

$$T = \sum_{i=1}^n \frac{f(x_{i-1}) + f(x_i)}{2} (x_i - x_{i-1}).$$

For equal widths, $T_n = \frac{L_n + R_n}{2}$.

The **definite integral** is the limit of Riemann sums:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x.$$

It measures net signed area: area above the x -axis counts positive, area below counts negative.

Which approximation is too big or too small? Look at the shape of the graph. The table summarizes the rules you can use on any free-response justification.

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Shape of f on $[a, b]$	Left sum	Right sum	Trapezoid	Midpoint
Increasing	Under	Over	depends on concavity	depends on concavity
Decreasing	Over	Under	depends on concavity	depends on concavity
Concave up			Over	Under
Concave down			Under	Over

Reading that table: a trapezoid connects two points on the curve with a chord, and a chord lies above a concave-up curve, so trapezoids overestimate there. The midpoint rectangle's top edge, by contrast, acts like a tangent line, which lies below a concave-up curve, so the midpoint sum underestimates.

Properties of Definite Integrals

$$\int_a^a f dx = 0, \quad \int_b^a f dx = -\int_a^b f dx, \quad \int_a^c f dx = \int_a^b f dx + \int_b^c f dx,$$

$$\int_a^b [kf(x) + g(x)] dx = k \int_a^b f dx + \int_a^b g dx, \quad \int_a^b k dx = k(b-a).$$

The **average value** of f on $[a, b]$ is $\frac{1}{b-a} \int_a^b f(x) dx$.

When a graph is made of lines and circles, you can often find an integral with geometry alone: triangle, rectangle, trapezoid, and circle areas. Area under the axis gets a minus sign.

Worked Examples

EXAMPLE 1

A cart's speed in feet per second is recorded every 2 seconds. The speeds at $t = 0, 2, 4, 6, 8$ are 3, 5, 6, 4, 2. Estimate the distance traveled from $t = 0$ to $t = 8$ with a left sum, a right sum, and a trapezoid sum using the four equal subintervals.

SOLUTION

$\Delta t = 2$. Left uses the first four values: $2(3 + 5 + 6 + 4) = 36$.

Right uses the last four values: $2(5 + 6 + 4 + 2) = 34$.

Trapezoid is the average: $\frac{36+34}{2} = 35$ feet. All three estimates are in feet, because (feet per second) times (seconds) is feet.

EXAMPLE 2

Use $n = 4$ to find the left, right, and midpoint sums for $f(x) = x^2$ on $[0, 2]$.

SOLUTION

$\Delta x = \frac{2-0}{4} = 0.5$, endpoints 0, 0.5, 1, 1.5, 2.

Left: $0.5[f(0) + f(0.5) + f(1) + f(1.5)] = 0.5(0 + 0.25 + 1 + 2.25) = 1.75$.

Right: $0.5[0.25 + 1 + 2.25 + 4] = 3.75$.

Midpoints are 0.25, 0.75, 1.25, 1.75: $M = 0.5(0.0625 + 0.5625 + 1.5625 + 3.0625) = 0.5(5.25) = 2.625$.

The exact area is $\frac{8}{3} \approx 2.667$, so $L < \frac{8}{3} < R$, and M is the closest.

EXAMPLE 3

For $f(x) = x^2$ on $[0, 2]$, decide whether the trapezoid sum with $n = 4$ is an overestimate or an underestimate of $\int_0^2 x^2 dx$, and find it.

SOLUTION

Trapezoid: $T_4 = \frac{L_4 + R_4}{2} = \frac{1.75 + 3.75}{2} = 2.75$.

Since $f''(x) = 2 > 0$, f is concave up, so chords lie above the curve and the trapezoid sum is an **overestimate**. Indeed $2.75 > 2.667$.

The midpoint sum 2.625 is an underestimate, which agrees with the table in Core Ideas.

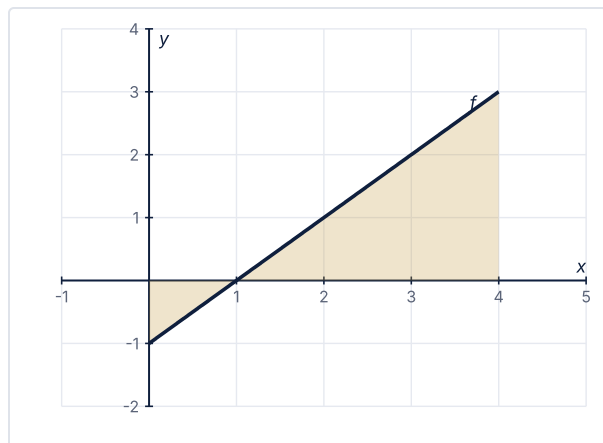
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EXAMPLE 4

Use geometry to evaluate $\int_0^4 (x-1) dx$.



The graph of $f(x) = x - 1$ on $[0, 4]$. The triangle below the axis counts negative.

SOLUTION

The graph is a line crossing the axis at $x = 1$ (see the figure).

From $x = 0$ to 1 the region lies below the axis: a triangle with base 1 and height 1, area $\frac{1}{2}$, counted as $-\frac{1}{2}$.

From $x = 1$ to 4 the region lies above: a triangle with base 3 and height 3, area $\frac{9}{2}$.

$$\int_0^4 (x-1) dx = \frac{9}{2} - \frac{1}{2} = 4.$$

EXAMPLE 5

Write $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(3 + \frac{2k}{n}\right)^2 \cdot \frac{2}{n}$ as a definite integral, then compute the right sum with $n = 4$.

SOLUTION

Compare with $\sum f(x_k) \Delta x$. Here $\Delta x = \frac{2}{n}$ and $x_k = 3 + \frac{2k}{n}$, so $a = 3$, $b = 3 + 2 = 5$, and $f(x) = x^2$.

$$\int_3^5 x^2 dx.$$

For $n = 4$, $\Delta x = 0.5$ and $x_k = 3.5, 4, 4.5, 5$: $0.5(12.25 + 16 + 20.25 + 25) = 0.5(73.5) = 36.75$.

The exact value of the integral is $\frac{98}{3} \approx 32.67$, so this right sum overestimates, as expected for an increasing function.

EXAMPLE 6

Suppose $\int_1^3 f(x) dx = 2$ and $\int_3^5 f(x) dx = 5$. Find (a) $\int_1^5 f(x) dx$, (b) $\int_5^3 f(x) dx$, (c) $\int_1^5 (2f(x) + 3) dx$, (d) the average value of f on $[1, 5]$.

SOLUTION

(a) Add adjacent pieces: $2 + 5 = 7$.

(b) Reversing limits flips the sign: -5 .

(c) $2(7) + 3(5 - 1) = 14 + 12 = 26$.

(d) $\frac{1}{5-1} \cdot 7 = \frac{7}{4}$.

Coach's Playbook

- 1 **Find the width first.** Compute $\Delta x = \frac{b-a}{n}$, or list the widths from a table. Everything else depends on it.
- 2 **List the endpoints.** Write all $n + 1$ endpoints in a row. Then circle which ones the sum uses: all but the last (left), all but the first (right).
- 3 **Count the terms.** A left or right sum with n subintervals has exactly n heights. If you have $n + 1$, you included an extra.
- 4 **Watch the units.** Rate times time gives an amount. Always attach units to a table-based answer.

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5 Justify with shape. For over or under, name the feature: increasing, decreasing, concave up or down. Say it in a sentence.

- ◆ **Unequal widths:** with a table, never use one common Δx . Multiply each rectangle's own width by its own height.
- ◆ **Trapezoid shortcut:** $T = \frac{L+B}{2}$ is a quick check when you already have both sums.
- ◆ **Net area:** break the region where the graph crosses the axis. Add areas above, subtract areas below.
- ◆ **Limits to integral:** match the pieces. The factor $\frac{b-a}{n}$ gives the width, the shift $a + k\Delta x$ gives a , and what multiplies Δx is f .

Common traps

(1) Using $n + 1$ heights in a left or right sum. (2) Using the same width for unequal intervals. (3) Counting area below the axis as positive. (4) Claiming that a right sum always overestimates; it underestimates when f is decreasing. (5) Forgetting the factor Δx , so the answer is a sum of heights instead of an area.

Mistake Clinic

Spot the error

Problem: Find the left sum for $f(x) = x^2$ on $[0, 2]$ with $n = 4$. **Student's work:** $\Delta x = 0.5$ and $0.5(0 + 0.25 + 1 + 2.25 + 4) = 3.75$. **What went wrong:** The student used all five endpoints. A left sum with four rectangles uses only the left endpoints 0, 0.5, 1, 1.5; the value at $x = 2$ belongs to the right sum. The total 3.75 happens to equal the right sum, because the extra term $f(2) = 4$ was added while the term $f(0) = 0$ contributed nothing. A correct left sum has exactly four heights, not five. **Correct solution:** $0.5(0 + 0.25 + 1 + 2.25) = 1.75$.

Spot the error

Problem: Evaluate $\int_0^4 (x - 1) dx$ using geometry. **Student's work:** Triangle on the left has area $\frac{1}{2}$, triangle on the right has area $\frac{9}{2}$, so the integral is $\frac{1}{2} + \frac{9}{2} = 5$. **What went wrong:** The integral measures signed area. The small triangle lies below the x -axis, where $f(x) < 0$, so it contributes $-\frac{1}{2}$. **Correct solution:** $\int_0^4 (x - 1) dx = -\frac{1}{2} + \frac{9}{2} = 4$. The value 5 is total area, which is a different question.

Spot the error

Problem: A function f is decreasing on $[0, 6]$. A student says, 'The right sum is always an overestimate, because it uses the larger x -values.' **What went wrong:** Bigger x does not mean bigger height. If f is decreasing, the right endpoint of each subinterval gives the smallest height on that subinterval, so each rectangle fits below the curve and the right sum is an **underestimate**. **Correct statement:** For a decreasing function, the left sum overestimates and the right sum underestimates. For an increasing function, it is the other way around.

Real-World Connection

A water tank is fed through a valve. The flow rate in gallons per minute is read at times 0, 1, 4, 6, 10 minutes as 8, 6, 5, 3, 2. The trapezoid sum with unequal widths gives $7 + 16.5 + 8 + 10 = 41.5$ gallons. Left and right sums give 48 and 35 gallons, which bracket the true amount because the flow rate is decreasing. Engineers use exactly this kind of estimate whenever they have discrete sensor readings.

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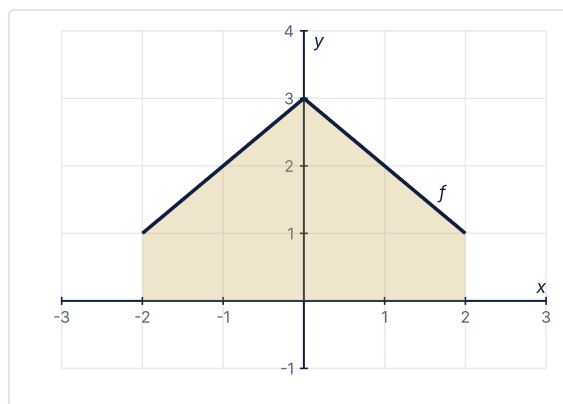
Challenge Corner

For $f(x) = \frac{1}{x}$ on $[1, 3]$, find the right sum with $n = 4$, decide whether it over- or underestimates $\int_1^3 \frac{1}{x} dx$, and compare it with $\ln 3 \approx 1.099$. **Solution:** $\Delta x = 0.5$ and the right endpoints are 1.5, 2, 2.5, 3. $R_4 = 0.5 \left(\frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3} \right) = 0.5(1.9) = 0.95$. Because $\frac{1}{x}$ is decreasing, the right sum is an **underestimate**. The true value is about 1.099, so the error is about 0.149.

Practice**PRACTICE Riemann Sums and the Definite Integral**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- A Riemann sum on $[2, 14]$ uses $n = 6$ equal subintervals. What is Δx ?
 (A) 12 (B) 2 (C) 6 (D) 3
- For $f(x) = x^2 + 1$ on $[0, 3]$ with $n = 3$, what is the left Riemann sum?
 (A) 17 (B) 12 (C) 8 (D) 18
- A rate $r(t)$ is recorded at $t = 0, 2, 5, 6$ with values 4, 6, 2, 8. What is the trapezoid sum for $\int_0^6 r(t) dt$?
 (A) 28 (B) 26 (C) 30 (D) 27
- Given $\int_1^4 f(x) dx = 5$ and $\int_4^7 f(x) dx = -3$, what is $\int_7^1 f(x) dx$?
 (A) -2 (B) 2 (C) 8 (D) -8
- Which definite integral equals $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(1 + \frac{3k}{n} \right)^3 \frac{3}{n}$?
 (A) $\int_0^3 x^3 dx$ (B) $\int_1^4 x^3 dx$ (C) $\int_1^3 x^3 dx$
 (D) $\int_1^4 3x^3 dx$
- The function f is increasing and concave down on $[a, b]$. Which statement is true?
 (A) The left sum is an overestimate. (B) The right sum is an underestimate.
 (C) The trapezoid sum is an underestimate. (D) The midpoint sum is an underestimate.
- What is $\int_{-2}^2 (3 - |x|) dx$? The graph is shown.



$y = 3 - |x|$ on $[-2, 2]$.

- (A) 8 (B) 6 (C) 10 (D) 12
- If $\int_0^4 f(x) dx = 10$, what is the average value of f on $[0, 4]$?
 (A) 10 (B) 40 (C) 0.4 (D) 2.5
- What is the midpoint sum for $f(x) = x^3$ on $[0, 4]$ with $n = 2$?
 (A) 16 (B) 144 (C) 56 (D) 64
- Find the right Riemann sum for $f(x) = 2x + 1$ on $[0, 5]$ with $n = 5$.

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11. A car's speed in meters per second is 2, 5, 7, 6, 3 at times $t = 0, 1, 3, 4, 8$ seconds. Use a trapezoid sum with the four subintervals to estimate the distance traveled.
12. Find the midpoint sum for $f(x) = \frac{1}{x}$ on $[1, 5]$ with $n = 2$.
13. Use the area of a circle to evaluate $\int_0^2 \sqrt{4 - x^2} dx$. Give an exact answer.
14. If $\int_0^3 f(x) dx = 4$ and $\int_0^3 g(x) dx = -2$, find $\int_0^3 [2f(x) - 3g(x) + 1] dx$.
15. For $f(x) = 3x$ on $[0, 6]$, the right sum minus the left sum with n equal subintervals equals $\frac{108}{n}$. What is the smallest whole number n that makes the difference less than 0.5?

39.3 The Fundamental Theorem of Calculus

Connect derivatives and integrals: evaluate definite integrals with antiderivatives, differentiate accumulation functions with the chain rule, and model rate in minus rate out for tanks and motion.

Why This Matters

The Fundamental Theorem of Calculus is the bridge between the two halves of the course. Derivatives measure rates of change; integrals measure accumulated change; the theorem says that these two operations undo each other. Before the theorem, finding an area meant building and taking limits of Riemann sums, which is slow. After it, a definite integral often takes one line: find an antiderivative and subtract two numbers.

Students get stuck in four places. They reverse the subtraction and compute $F(a) - F(b)$. They forget the chain rule when the upper limit of an accumulation function is something like x^2 . They treat net change and total distance as the same thing. And they apply the theorem to a function that is not continuous on the interval. The theorem also powers questions where you never find an antiderivative at all, such as reading an accumulation function from a graph or deciding when the water in a tank is greatest. Every physics and engineering course you take later uses this idea, from work and energy to probability.

Core Ideas

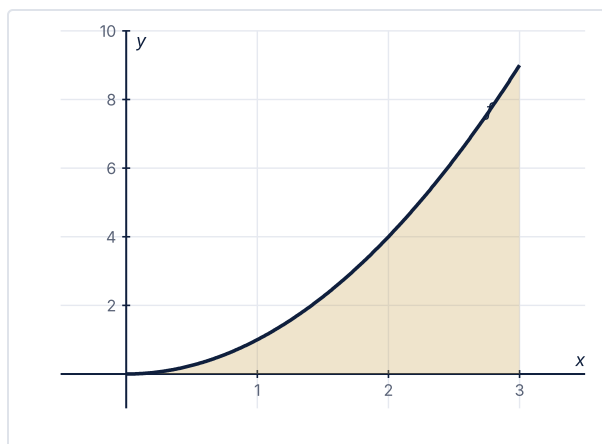
The theorem has two parts, and they answer two different questions. Part 2 is the evaluation tool: how do I compute $\int_a^b f$? Part 1 is the differentiation statement: what is the derivative of an integral whose upper limit moves?

Fundamental Theorem of Calculus, Part 2 (Evaluation)

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a) = [F(x)]_a^b.$$

Any antiderivative works, so you can ignore the $+C$: it cancels in the subtraction. Always take the value at the top limit minus the value at the bottom limit.



The shaded area under $y = x^2$ from 0 to 3 equals $F(3) - F(0) = 9 - 0$ with $F(x) = x^3/3$.

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Fundamental Theorem of Calculus, Part 1 (Accumulation)

If f is continuous on an interval containing a , and $g(x) = \int_a^x f(t) dt$, then

$$g'(x) = f(x).$$

With a moving upper limit $u(x)$, the chain rule gives

$$\frac{d}{dx} \int_a^{u(x)} f(t) dt = f(u(x)) u'(x).$$

If the variable is the lower limit, flip the integral and add a minus sign: $\frac{d}{dx} \int_x^b f(t) dt = -f(x)$.

An accumulation function $g(x) = \int_a^x f(t) dt$ tells you how much has built up from a to x . Since $g' = f$, the sign of f tells you whether g is increasing or decreasing, and f' tells you the concavity of g . The graph of f alone lets you answer questions about g : where it has a maximum, where it is concave down, and what its values are, using geometry for areas.

What you know about f	What it tells you about $g(x) = \int_a^x f$
$f(x) > 0$	g is increasing
$f(x) < 0$	g is decreasing
f changes from $+$ to $-$	g has a local maximum
f is increasing	g is concave up
f has a local extremum	g has an inflection point

Net Change and Rates

If F' is continuous, then

$$F(b) = F(a) + \int_a^b F'(x) dx.$$

The integral of a rate of change is the net change in the quantity.

For a tank: $\text{amount}(t) = \text{amount}(0) + \int_0^t [\text{rate in} - \text{rate out}] ds$. For motion: displacement is $\int v dt$ and total distance is $\int |v| dt$.

The quantity is at a maximum or minimum when the net rate changes sign. For a tank, that happens when rate in equals rate out and the net rate switches between positive and negative.

Worked Examples**EXAMPLE 1**

Evaluate $\int_1^3 (4x^3 - 2x) dx$.

SOLUTION

An antiderivative is $F(x) = x^4 - x^2$.

$F(3) = 81 - 9 = 72$ and $F(1) = 1 - 1 = 0$.

$$\int_1^3 (4x^3 - 2x) dx = 72 - 0 = 72.$$

EXAMPLE 2

Evaluate $\int_0^{\pi/2} (3 \cos x + \sin x) dx$.

SOLUTION

An antiderivative is $F(x) = 3 \sin x - \cos x$.

$F(\frac{\pi}{2}) = 3(1) - 0 = 3$ and $F(0) = 0 - 1 = -1$.

$3 - (-1) = 4$.

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EXAMPLE 3

Find (a) $\frac{d}{dx} \int_1^x \sqrt{t^3 + 8} dt$ at $x = 2$, (b) $\frac{d}{dx} \int_0^{x^2} \cos t dt$, and (c) $\frac{d}{dx} \int_x^5 \sqrt{t^3 + 8} dt$ at $x = 2$.

SOLUTION

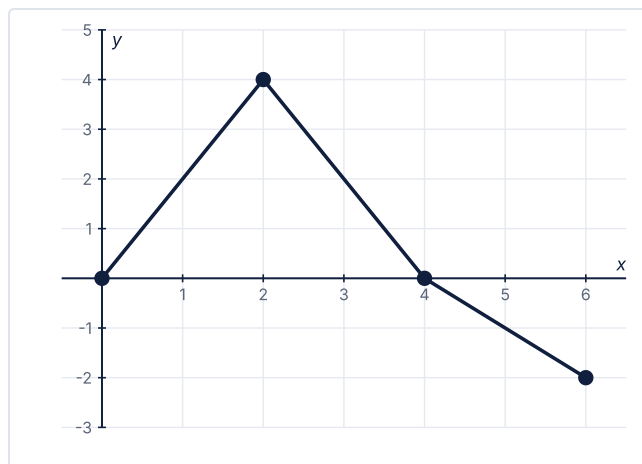
(a) Part 1 gives $\sqrt{x^3 + 8}$. At $x = 2$ this is $\sqrt{16} = 4$.

(b) The upper limit is x^2 , so use the chain rule: $\cos(x^2) \cdot 2x = 2x \cos(x^2)$.

(c) The variable is on the bottom, so there is a minus sign: $-\sqrt{x^3 + 8}$, which is -4 at $x = 2$.

EXAMPLE 4

The graph of f is made of line segments (see the figure). Let $g(x) = \int_0^x f(t) dt$. Find $g(2)$, $g(4)$, $g(6)$, where g has its maximum on $[0, 6]$, and where g is concave down.



Graph of f : rises from $(0,0)$ to $(2,4)$, falls to $(4,0)$, then continues to $(6,-2)$.

SOLUTION

Use triangle areas. $g(2) = \frac{1}{2}(2)(4) = 4$. $g(4) = 4 + \frac{1}{2}(2)(4) = 8$.

From 4 to 6 the graph is below the axis with height -2 at $x = 6$ and zero at $x = 4$, a triangle of area $\frac{1}{2}(2)(2) = 2$, counted negative: $g(6) = 8 - 2 = 6$.

Since $g' = f$ changes from positive to negative at $x = 4$, g has its maximum there: $g(4) = 8$ (this beats the endpoints $g(0) = 0$ and $g(6) = 6$).

$g'' = f'$ is negative where f is decreasing, which is on $(2, 6)$, so g is concave down on $(2, 6)$.

EXAMPLE 5

Water enters a tank at $3t^2$ gallons per minute and leaves at a constant 12 gallons per minute. The tank holds 20 gallons at $t = 0$. Find (a) the amount at $t = 5$ minutes and (b) the minimum amount during $0 \leq t \leq 5$.

SOLUTION

(a) amount(5) $= 20 + \int_0^5 (3t^2 - 12) dt = 20 + [t^3 - 12t]_0^5 = 20 + (125 - 60) = 85$ gallons.

(b) The net rate $3t^2 - 12$ is negative for $t < 2$ and positive for $t > 2$, so the amount decreases and then increases. The minimum is at $t = 2$: $20 + (8 - 24) = 4$ gallons.

Check the endpoints: the amount is 20 at $t = 0$ and 85 at $t = 5$, so the minimum is 4 gallons.

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EXAMPLE 6

A particle moves with velocity $v(t) = t^2 - 4t + 3$ meters per second for $0 \leq t \leq 4$. Find its displacement and the total distance traveled.

SOLUTION

An antiderivative is $V(t) = \frac{t^3}{3} - 2t^2 + 3t$. Displacement: $V(4) - V(0) = \frac{64}{3} - 32 + 12 = \frac{4}{3}$ meters.

Since $v(t) = (t-1)(t-3)$, the velocity is positive on $(0, 1)$, negative on $(1, 3)$, and positive on $(3, 4)$. We need $V(1) = \frac{4}{3}$, $V(3) = 0$, $V(4) = \frac{4}{3}$.

Distance = $|V(1) - V(0)| + |V(3) - V(1)| + |V(4) - V(3)| = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$ meters.

Coach's Playbook

- 1 Check continuity first.** The theorem requires f to be continuous on the whole interval. Look for division by zero, square roots of negatives, and breaks in a piecewise rule.
 - 2 Choose Part 1 or Part 2.** A number to compute from an integral with two numeric limits is Part 2. A derivative of an integral with a variable limit is Part 1.
 - 3 Evaluate top minus bottom.** Write $F(b)$ and $F(a)$ on separate lines, then subtract, parenthesizing the bottom value to protect negatives.
 - 4 For variable limits, plug the limit into f , then multiply by its derivative.** Write ' $f(u)$ times u prime' on your paper before computing anything.
 - 5 For distance, split at sign changes.** Find where $v = 0$, integrate on each piece, and add absolute values.
- ◆ **Reading graphs:** to analyze $g(x) = \int_a^x f$, talk about f : its sign for increasing or decreasing, its slope for concavity, and its areas for values.
 - ◆ **Units:** the integral of a rate has the units of rate times time, such as gallons per minute times minutes.
 - ◆ **Even and odd shortcut:** over $[-a, a]$, an odd function integrates to 0, but only after you confirm the function is continuous there.
 - ◆ **Check:** estimate $\int_a^b f$ by the sign and rough size of the graph. A negative answer for a positive function means you subtracted in the wrong order.

Common traps

(1) Computing $F(a) - F(b)$. (2) Dropping the $u'(x)$ factor in Part 1 when the limit is not just x . (3) Using displacement when the question asks for total distance. (4) Using the theorem across a discontinuity, such as $\int_{-1}^2 \frac{1}{x^2} dx$.

Mistake Clinic**Spot the error**

Problem: Evaluate $\int_1^3 2x \, dx$. **Student's work:** $F(x) = x^2$, so the answer is $F(1) - F(3) = 1 - 9 = -8$. **What went wrong:** The subtraction is reversed. The theorem says top minus bottom: $F(b) - F(a)$. A positive function over a left-to-right interval must give a positive integral, so -8 should have been a warning. **Correct solution:** $F(3) - F(1) = 9 - 1 = 8$.

Spot the error

Problem: Find $\frac{d}{dx} \int_0^{x^2} \sin t \, dt$. **Student's work:** By the Fundamental Theorem, the answer is $\sin(x^2)$. **What went wrong:** The student used Part 1 as if the upper limit were x . The upper limit is x^2 , so the chain rule contributes the factor $\frac{d}{dx}(x^2) = 2x$. **Correct solution:** $\sin(x^2) \cdot 2x = 2x \sin(x^2)$. You can verify: $\int_0^{x^2} \sin t \, dt = 1 - \cos(x^2)$, whose derivative is $2x \sin(x^2)$.

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Spot the error

Problem: Evaluate $\int_{-1}^2 \frac{1}{x^2} dx$. **Student's work:** $F(x) = -\frac{1}{x}$, so $F(2) - F(-1) = -\frac{1}{2} - 1 = -\frac{3}{2}$. **What went wrong:** The function $\frac{1}{x^2}$ is not defined at $x = 0$, which lies inside $[-1, 2]$, so f is not continuous on the interval and the theorem does not apply. The answer $-\frac{3}{2}$ is also impossible: $\frac{1}{x^2}$ is positive, so a correct integral could not be negative. **Correct statement:** The integral is improper and does not converge as written, so no finite value exists.

Real-World Connection

A file downloads at a speed of $6 - t$ megabytes per second for $0 \leq t \leq 4$ seconds. The total data received is $\int_0^4 (6 - t) dt = \left[6t - \frac{t^2}{2}\right]_0^4 = 24 - 8 = 16$ megabytes. Any time you see a rate and ask for a total, the Fundamental Theorem is working behind the scenes.

Challenge Corner

Let $A(x) = \int_1^x (3t^2 - 6t) dt$. Find the minimum value of A on $[0, 3]$ and justify it. **Solution:** By Part 1, $A'(x) = 3x^2 - 6x = 3x(x - 2)$, which is zero at $x = 0$ and $x = 2$. A' is negative on $(0, 2)$ and positive on $(2, 3)$, so A has a local minimum at $x = 2$. Evaluate: $A(x) = [t^3 - 3t^2]_1^x = x^3 - 3x^2 + 2$. Candidates: $A(0) = 2$, $A(2) = -2$, $A(3) = 2$. The minimum value is -2 , at $x = 2$.

Practice**PRACTICE The Fundamental Theorem of Calculus**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

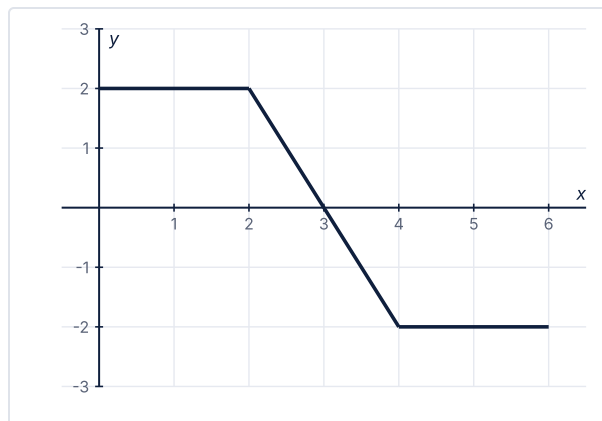
- What is $\int_0^2 (3x^2 + 1) dx$?
(A) 8 (B) 10 (C) 14 (D) 4
- What is $\frac{d}{dx} \int_1^x \sqrt{t^2 + 3} dt$?
(A) $\sqrt{x^2 + 3}$ (B) $\frac{x}{\sqrt{x^2 + 3}}$ (C) $\sqrt{x^2 + 3} - 2$
(D) $\frac{(x^2 + 3)^{3/2}}{3}$
- What is $\frac{d}{dx} \int_0^{x^2} e^t dt$?
(A) $e x^2$ (B) $e x^2 - 1$ (C) $2x(e^{x^2} - 1)$
(D) $2xe^{x^2}$
- What is $\int_{-1}^1 (x^3 + 2) dx$?
(A) 2 (B) 6 (C) 4 (D) 0
- Let $g(x) = \int_2^x (t^2 - 4t + 3) dt$. On which interval is g decreasing?
(A) $(-\infty, 1)$ (B) $(1, 3)$ (C) $(3, \infty)$ (D) $(2, \infty)$
- A tank holds 5 gallons at $t = 0$. Water flows in at $4t + 1$ gallons per minute. How many gallons are in the tank at $t = 3$?
(A) 26 (B) 21 (C) 13 (D) 31
- What is $\int_1^e \frac{3}{x} dx$?
(A) $3e$ (B) $3(e - 1)$ (C) $\frac{3}{e}$ (D) 3

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8. A particle has velocity $v(t) = 3 - t$ for $0 \leq t \leq 5$. What is the total distance it travels?
(A) 2.5 (B) 4.5 (C) 6.5 (D) 8
9. Let $G(x) = \int_1^{x^3} (2t + 1) dt$. What is $G'(2)$?
(A) 17 (B) 204 (C) 68 (D) 12
10. Evaluate $\int_1^4 (6\sqrt{x} - 2) dx$.
11. The graph of f is shown: it is the constant 2 on $[0, 2]$, then a line down to -2 at $x = 4$, then the constant -2 on $[4, 6]$. Let $g(x) = \int_0^x f(t) dt$. Find the maximum value of g on $[0, 6]$.

Graph of f .

12. Let $h(x) = \int_x^4 \sqrt{t^2 + 9} dt$. Find $h'(4)$.
13. Let $P(x) = \int_x^{x^2} t dt$. Find $P'(2)$.
14. A tank holds 30 liters at $t = 0$. Water enters at $2t$ liters per minute and leaves at 8 liters per minute for $0 \leq t \leq 10$. Find the minimum amount of water in the tank during this time.
15. A particle has velocity $v(t) = t^2 - 6t + 8$ meters per second for $0 \leq t \leq 5$. Find the total distance traveled.

39.4 u-Substitution

Master the reverse chain rule: recognize the inner function, convert dx to du , change bounds in definite integrals, and avoid the classic errors of dropped constants and unchanged limits.

Why This Matters

The antiderivative rules you know cover single pieces: x^n , $\sin x$, e^x . Real integrands are usually composites, like $(x^2 + 5)^4$ multiplied by $2x$, or e^{x^2} multiplied by x . **u-substitution** is the technique that undoes the chain rule. It turns a messy-looking integral into a simple one by renaming the inner function as a single variable u .

Students get stuck for three reasons. They cannot spot which part of the integrand is the inner function. They forget to convert dx into du , leaving x 's and u 's mixed in one integral. And in definite integrals they keep the old x -limits after substituting, which gives a wrong number with no warning. Substitution appears on nearly every AP free-response section, in physics (work, energy, decay), in probability, and later in every method of integration you will learn. Master it now and the next techniques will feel like extensions of the same idea.

Core Ideas

Recall the chain rule: $\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$. Read it backward and you get an antiderivative statement. Whenever an integrand looks like a function of $g(x)$ multiplied by the derivative $g'(x)$, you can swap in the single variable $u = g(x)$.

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u-Substitution (Indefinite Integrals)

If $u = g(x)$ is differentiable and f is continuous on the range of g , then

$$\int f(g(x)) g'(x) dx = \int f(u) du = F(u) + C = F(g(x)) + C.$$

Method: pick u , compute $du = g'(x) dx$, rewrite the **entire** integral in terms of u and du , integrate, then substitute back.

The key sentence is 'the entire integral.' Every x and the dx must disappear. If a leftover constant is missing, fix it by multiplying and dividing by that constant. If a leftover variable remains, the substitution is wrong, or it needs one more idea (see Example 6). The next table lists patterns to recognize on sight.

Why does this work? Suppose $F' = f$. By the chain rule, $\frac{d}{dx} F(g(x)) = f(g(x)) g'(x)$. So $F(g(x))$ is an antiderivative of $f(g(x)) g'(x)$. Substitution is nothing more than recognizing that product and writing $F(u)$ with $u = g(x)$. The differential notation $du = g'(x) dx$ is a bookkeeping device that makes the recognition mechanical, so treat it as a tool and always confirm your final answer by differentiating.

If you see...	Try $u =$	Because $du =$
$(\text{stuff})^n$ times the derivative of stuff	stuff	$(\text{stuff})' dx$
$\sqrt{\text{stuff}}$ times its derivative	the expression under the root	its derivative times dx
$e^{g(x)} \cdot g'(x)$	$g(x)$ (the exponent)	$g'(x) dx$
$\frac{g'(x)}{g(x)}$	$g(x)$ (the denominator)	$g'(x) dx$
$\sin(g(x)) g'(x)$ or $\cos(g(x)) g'(x)$	$g(x)$ (the inside)	$g'(x) dx$
$(\ln x)^n / x$	$\ln x$	$\frac{1}{x} dx$

Substitution with Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of g , then

$$\int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

Change the limits when you change the variable. The new limits are the values of u at the old x -limits. After that you never go back to x .

Two special cases come up constantly. For a linear inside function, $u = ax + b$ and $du = a dx$, so

$$\int f(ax + b) dx = \frac{1}{a} F(ax + b) + C.$$

For a quotient where the top is the derivative of the bottom, $\int \frac{g'(x)}{g(x)} dx = \ln |g(x)| + C$. A third trick, used in Example 6, is to solve $u = x + k$ for x and replace leftover x 's by $u - k$.

Old limits (x)	New limits (u)	When it happens
$x = a$	$u = g(a)$	Lower limit
$x = b$	$u = g(b)$	Upper limit
Order stays: top minus bottom	If g decreases the new top may be smaller than the new bottom	Do not swap them; evaluate top minus bottom as given

There are two correct ways to finish a definite integral, and you may use either one. Method 1, the one used in this section, changes the limits to u -values and never returns to x . Method 2 finds the indefinite antiderivative in terms of u , substitutes back to x , and then uses the original x -limits. For $\int_0^1 2x(x^2 + 1)^3 dx$, Method 2 gives $\left. \frac{(x^2 + 1)^4}{4} \right|_0^1 = \frac{16 - 1}{4} = \frac{15}{4}$, the same as Method 1. The mistake is to mix them: new variable with old limits, or old variable with new limits. When a free-response problem asks you to 'show your work,' Method 1 is cleaner because there is nothing to convert back.

Finally, remember what substitution cannot do. It cannot rescue an integral whose inside function's derivative is missing, as in $\int \sqrt{x^2 + 1} dx$ or $\int e^{x^2} dx$. Recognizing the integrals that need other tools is part of mastery, and the AP exam will not ask you to evaluate them with substitution alone.

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Worked Examples

EXAMPLE 1

Find $\int 2x(x^2 + 5)^4 dx$.

SOLUTION

The inner function is $x^2 + 5$ and its derivative $2x$ appears. Let $u = x^2 + 5$, so $du = 2x dx$.

$$\int u^4 du = \frac{u^5}{5} + C.$$

Substitute back:

$$\frac{(x^2 + 5)^5}{5} + C.$$

EXAMPLE 2

Find $\int x^2 \sqrt{x^3 + 1} dx$.

SOLUTION

Let $u = x^3 + 1$, so $du = 3x^2 dx$ and $x^2 dx = \frac{1}{3} du$.

$$\int \sqrt{u} \cdot \frac{1}{3} du = \frac{1}{3} \cdot \frac{u^{3/2}}{3/2} = \frac{2}{9} u^{3/2}.$$

$$\frac{2}{9} (x^3 + 1)^{3/2} + C.$$

EXAMPLE 3

Find $\int \frac{4x}{x^2 + 1} dx$.

SOLUTION

Let $u = x^2 + 1$, so $du = 2x dx$. The numerator $4x dx = 2 du$.

$$\int \frac{2}{u} du = 2 \ln |u| + C.$$

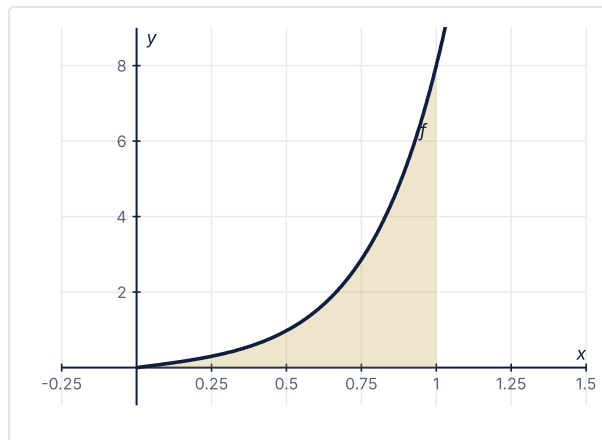
Since $x^2 + 1 > 0$:

$$2 \ln(x^2 + 1) + C.$$

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EXAMPLE 4Evaluate $\int_0^1 x(x^2 + 1)^3 dx$.The shaded area under $f(x) = x(x^2 + 1)^3$ from 0 to 1 is $15/8 = 1.875$.**SOLUTION**Let $u = x^2 + 1$, $du = 2x dx$, so $x dx = \frac{1}{2} du$. Limits: at $x = 0$, $u = 1$; at $x = 1$, $u = 2$.

$$\int_1^2 u^3 \cdot \frac{1}{2} du = \frac{1}{2} \cdot \frac{u^4}{4} \Big|_1^2 = \frac{1}{8}(16 - 1) = \frac{15}{8}.$$

The figure shows the region; its area is 1.875.

EXAMPLE 5Evaluate (a) $\int_0^{\ln 2} \frac{e^x}{1 + e^x} dx$ and (b) $\int_0^{\pi/2} \cos x \sin^2 x dx$.**SOLUTION**(a) Let $u = 1 + e^x$, $du = e^x dx$. Limits: $u = 2$ at $x = 0$ and $u = 3$ at $x = \ln 2$. $\int_2^3 \frac{1}{u} du = \ln 3 - \ln 2 = \ln \frac{3}{2}$.(b) Let $u = \sin x$, $du = \cos x dx$. Limits: $u = 0$ at $x = 0$ and $u = 1$ at $x = \frac{\pi}{2}$. $\int_0^1 u^2 du = \frac{1}{3}$.**EXAMPLE 6**Evaluate $\int_1^5 x\sqrt{x-1} dx$.**SOLUTION**Here the derivative of the inside is 1, so there is no extra factor, but a leftover x remains. Let $u = x - 1$. Then $x = u + 1$ and $dx = du$.Limits: $x = 1 \rightarrow u = 0$ and $x = 5 \rightarrow u = 4$.

$$\int_0^4 (u+1)u^{1/2} du = \int_0^4 (u^{3/2} + u^{1/2}) du = \left[\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} \right]_0^4$$

$$\frac{2}{5}(32) + \frac{2}{3}(8) = \frac{64}{5} + \frac{16}{3} = \frac{192+80}{15} = \frac{272}{15}.$$

Coach's Playbook

Here is the self-talk of a strong student facing $\int \frac{6x^2}{(x^3+7)^2} dx$. 'I see a power in the denominator. The inside is $x^3 + 7$. Its derivative is $3x^2$, and the numerator is $6x^2$, which is twice that. So $u = x^3 + 7$, $du = 3x^2 dx$, and $6x^2 dx = 2 du$. The integral is $\int 2u^{-2} du = -2u^{-1}$, so the answer is $-\frac{2}{x^3+7} + C$. Let me differentiate to check: $\frac{2 \cdot 3x^2}{(x^3+7)^2}$. Yes.' That whole process takes under a minute, and every move is something you can rehearse.

- 1 Scan for a composite with its derivative nearby.** Look for an inside function (under a root, in a power, in an exponent, in a denominator) and check whether its derivative, up to a constant, is also in the integrand.
- 2 Write u and du on paper.** Say it out loud: let $u = \dots$, so $du = \dots dx$. Solve for what you need, such as $x dx = \frac{1}{2} du$.
- 3 Rewrite everything.** After substituting, the integral should contain only u and du . Any x left over means you must change the plan.

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- 4 Integrate, then go back.** For an indefinite integral, replace u by the original expression. For a definite integral, do not go back; use the new limits.
- 5 Check by differentiating.** Differentiate the final answer. The chain rule must return the original integrand.
- ◆ **Decision flow:** (a) Is there an obvious inside function? (b) Is its derivative, up to a constant, also present? (c) If yes, substitute. If a lone x is left over and the inside is linear, solve for x . If not, try rewriting by algebra or expanding.
 - ◆ **Choosing u :** pick the inside of the messiest piece. If the first choice leaves leftover x 's, try the next candidate.
 - ◆ **Constants are free:** a missing constant factor is always fixable. A missing variable factor is not.
 - ◆ **Linear inside:** for $\int f(ax + b) dx$, just divide by a at the end.
 - ◆ **Decreasing inside:** if $u = g(x)$ decreases on $[a, b]$, then the new lower limit is larger than the new upper limit. That is fine; evaluate as written.
 - ◆ **Sanity check:** a positive integrand over a left-to-right interval must give a positive answer.

Common traps

(1) Forgetting the constant factor, such as dropping the $\frac{1}{2}$ in $x dx = \frac{1}{2} du$. (2) Leaving the x -limits in place after substituting into u . (3) Pulling a variable x outside the integral to match du ; only constants can come out. (4) Substituting u for the integral's inside but forgetting to substitute back in an indefinite integral.

Mistake Clinic

Spot the error

Problem: Find $\int x e^{x^2} dx$. **Student's work:** Let $u = x^2$, $du = 2x dx$. Then $\int e^u du = e^{x^2} + C$. **What went wrong:** The student replaced $x dx$ by du , but $du = 2x dx$, so $x dx = \frac{1}{2} du$. The constant $\frac{1}{2}$ was dropped. **Correct solution:** $\frac{1}{2} \int e^u du = \frac{1}{2} e^{x^2} + C$. Check: the derivative of $\frac{1}{2} e^{x^2}$ is $\frac{1}{2} e^{x^2} \cdot 2x = x e^{x^2}$.

Spot the error

Problem: Evaluate $\int_0^1 2x(x^2 + 1)^3 dx$. **Student's work:** Let $u = x^2 + 1$, so $\int_0^1 u^3 du = \frac{1}{4}$. **What went wrong:** The variable changed from x to u , but the limits stayed at the old x -values. When $x = 0$, $u = 1$, and when $x = 1$, $u = 2$. **Correct solution:** $\int_1^2 u^3 du = \frac{16-1}{4} = \frac{15}{4}$. The answer $\frac{1}{4}$ is far too small: the integrand $2x(x^2 + 1)^3$ climbs to 16 at $x = 1$, so the area must be much larger.

Spot the error

Problem: Find $\int \sqrt{x^2 + 1} dx$. **Student's work:** Let $u = x^2 + 1$, $du = 2x dx$, so $dx = \frac{du}{2x}$. Pull out $\frac{1}{2x}$: $\frac{1}{2x} \int \sqrt{u} du = \frac{1}{2x} \cdot \frac{2}{3} u^{3/2}$. **What went wrong:** A variable factor cannot be moved across an integral sign. Only constants can. The integrand has no x factor to supply du , so u -substitution does not apply. **Correct statement:** This integral needs a different technique (a trigonometric substitution) and not this one. Always check that the derivative of the inside function is present, up to a constant.

Real-World Connection

A factory's power use is $3t^2\sqrt{t^3 + 1}$ kilowatts at time t hours. The energy used in the first 2 hours is $\int_0^2 3t^2\sqrt{t^3 + 1} dt$. With $u = t^3 + 1$, $du = 3t^2 dt$, the limits are 1 to 9: $\int_1^9 \sqrt{u} du = \frac{2}{3}(27 - 1) = \frac{52}{3} \approx 17.3$ kilowatt-hours.

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Challenge Corner

Evaluate $\int_0^1 x^3 \sqrt{1-x^2} dx$. **Solution:** Let $u = 1 - x^2$. Then $du = -2x dx$, so $x dx = -\frac{1}{2} du$, and $x^2 = 1 - u$. Limits: $x = 0 \rightarrow u = 1$, $x = 1 \rightarrow u = 0$. Write $x^3 dx = x^2 \cdot x dx = (1 - u) \left(-\frac{1}{2}\right) du$. The integral becomes $-\frac{1}{2} \int_1^0 (1 - u) \sqrt{u} du = \frac{1}{2} \int_0^1 (u^{1/2} - u^{3/2}) du = \frac{1}{2} \left(\frac{2}{3} - \frac{2}{5}\right) = \frac{1}{2} \cdot \frac{4}{15} = \frac{2}{15}$.

Practice**PRACTICE u-Substitution**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is $\int 2x(x^2 + 3)^5 dx$?
 (A) $(x^2 + 3)^6 + C$ (B) $\frac{(x^2 + 3)^6}{6} + C$ (C) $12x(x^2 + 3)^4 + C$
 (D) $\frac{(x^2 + 3)^6}{12x} + C$
- What is $\int \cos(5x) dx$?
 (A) $5 \sin(5x) + C$ (B) $\sin(5x) + C$ (C) $-\frac{\sin(5x)}{5} + C$ (D) $\frac{\sin(5x)}{5} + C$
- What is $\int x^2 e^{x^3} dx$?
 (A) $\frac{e^{x^3}}{3} + C$ (B) $3e^{x^3} + C$ (C) $e^{x^3} + C$
 (D) $\frac{x^3 e^{x^3}}{3} + C$
- Which substitution is best for $\int \frac{x^2}{x^3 + 4} dx$?
 (A) $u = x^2$ (B) $u = x^3$ (C) $u = x^3 + 4$
 (D) $u = \frac{x^2}{x^3 + 4}$
- What is $\int \frac{4}{2x+1} dx$ for $x > 0$?
 (A) $2 \ln(2x + 1) + C$ (B) $4 \ln(2x + 1) + C$ (C) $\ln(2x + 1) + C$
 (D) $-\frac{4}{(2x + 1)^2} + C$
- What is $\int_1^e \frac{(\ln x)^2}{x} dx$?
 (A) $\frac{e^3 - 1}{3}$ (B) 1 (C) $\frac{1}{2}$ (D) $\frac{1}{3}$
- What is $\int_0^3 x \sqrt{x^2 + 16} dx$?
 (A) $\frac{61}{6}$ (B) $\frac{122}{3}$ (C) $\frac{61}{3}$ (D) $\frac{61}{2}$
- What is $\int \frac{\sin x}{\cos x} dx$?
 (A) $\ln |\cos x| + C$ (B) $-\ln |\cos x| + C$ (C) $\ln |\sin x| + C$ (D) $-\frac{1}{\cos x} + C$
- If $\int_2^6 f(u) du = 12$, what is $\int_1^3 f(2x) dx$?
 (A) 12 (B) 24 (C) 3 (D) 6
- Find $\int x^2(x^3 - 2)^4 dx$.
- Evaluate $\int_0^2 x e^{x^2} dx$. Give an exact answer.

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12. Evaluate $\int_0^{\pi/4} \tan x \sec^2 x \, dx$.
13. Evaluate $\int_e^{e^3} \frac{1}{x \ln x} \, dx$. Give an exact answer.
14. Evaluate $\int_0^5 x\sqrt{x+4} \, dx$.
15. Find $f(4)$ if $f'(x) = \frac{x}{\sqrt{x^2+9}}$ and $f(0) = 2$.

Answers**Answers and Explanations**

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

39.1 Antiderivatives and Indefinite Integrals**PRACTICE · ANTIDERIVATIVES AND INDEFINITE INTEGRALS**

- B. Add one to the exponent and divide: $8 \cdot \frac{x^4}{4} = 2x^4$. Choice A differentiates; C forgets to divide by 4.
- D. $6x^{-4}$ becomes $6 \cdot \frac{x^{-3}}{-3} = -2x^{-3}$. Choice B loses the negative sign, and C forgets to divide by -3 .
- A. $\int \sin x = -\cos x$, so $3 \sin x$ gives $-3 \cos x$; $\int \cos x = \sin x$, so $2 \cos x$ gives $2 \sin x$. Choices B, C and D each get a sign wrong.
- C. $\int e^x \, dx = e^x$ (no power rule on e^x) and $\int \frac{2}{x} \, dx = 2 \ln x$, with the minus sign kept. Choice B wrongly applies the power rule to e^x .
- B. $f(x) = 2x^2 - 3x + C$. From $f(1) = 2 - 3 + C = 5$ we get $C = 6$. Choice A uses the output 5 as the constant, and D forgot to divide by 2 when integrating $4x$.
- A. $s(t) = t^3 - 2t + C$ and $s(0) = 4$ gives $C = 4$. Choice B drops the constant, and C differentiates instead of integrating.
- D. Expand first: $x^3 - 4x$, then integrate to get $\frac{x^4}{4} - 2x^2$. Choice A integrates each factor separately, which is never valid.
- C. $\int \sec^2 x = \tan x$ and $\int 1 \, dx = x$. Choice D treats the integral of 1 as 1.
- D. $f'(x) = 3x^2 + C_1$ and $f'(0) = -1$ gives $f' = 3x^2 - 1$. Then $f = x^3 - x + C_2$ and $f(0) = 2$ gives $f = x^3 - x + 2$, so $f(2) = 8 - 2 + 2 = 8$. Choice B ignores the -1 ; A ignores the $+2$.
- $3x^4 - 2x^3 + 5x + C$. Divide each term by its new exponent: $12 \cdot \frac{x^4}{4}$, $-6 \cdot \frac{x^3}{3}$, and $\int 5 \, dx = 5x$.
- $2x^{3/2} - \frac{4}{x} + C$. Rewrite as $3x^{1/2} + 4x^{-2}$. Then $3 \cdot \frac{x^{3/2}}{3/2} = 2x^{3/2}$ and $4 \cdot \frac{x^{-1}}{-1} = -\frac{4}{x}$.
- $4 + \pi$. $f(x) = 3 \sin x + 2x + C$ and $f(0) = C = 1$. Then $f(\pi/2) = 3(1) + \pi + 1 = 4 + \pi$.
38. $v(t) = 6t^2 - 6t + C$; $v(1) = C = 2$. $s(t) = 2t^3 - 3t^2 + 2t + D$; $s(0) = D = 5$. So $s(3) = 54 - 27 + 6 + 5 = 38$.
11. $y = x^3 - x^2 + 4x + C$. At $(1, 3)$: $1 - 1 + 4 + C = 3$, so $C = -1$. At $x = 2$: $8 - 4 + 8 - 1 = 11$.
- $e + 2$. $f'(x) = e^x + 3x^2 + C_1$ and $f'(0) = 1 + C_1 = 3$ gives $C_1 = 2$. Then $f(x) = e^x + x^3 + 2x + C_2$ and $f(0) = 1 + C_2 = 0$ gives $C_2 = -1$. So $f(1) = e + 1 + 2 - 1 = e + 2$.

39.2 Riemann Sums and the Definite Integral**PRACTICE · RIEMANN SUMS AND THE DEFINITE INTEGRAL**

- B. $\Delta x = \frac{14-2}{6} = 2$. Choice A is the total length, and C is the number of subintervals.
- C. $\Delta x = 1$ and the left endpoints are 0, 1, 2: $f = 1, 2, 5$, sum 8. Choice A is the right sum (uses 1, 2, 3), and the others are arithmetic slips.
- D. Each trapezoid uses its own width: $2 \cdot \frac{4+6}{2} + 3 \cdot \frac{6+2}{2} + 1 \cdot \frac{2+8}{2} = 10 + 12 + 5 = 27$. Choice C wrongly uses one common width of 1.5.
- A. $\int_1^7 f = 5 + (-3) = 2$, and reversing the limits flips the sign: $\int_7^1 f = -2$. Choice B forgets the flip.
- B. The width is $\frac{3}{n}$, so $b - a = 3$, and $a = 1$ (the starting shift), so $b = 4$. The function is x^3 . Choice D wrongly moves the 3 into the integrand.
- C. A chord of a concave-down curve lies below the curve, so trapezoids underestimate. Choice D is true only for concave up, and for an increasing function the left sum underestimates (A is false) while the right sum overestimates (B is false).
- A. The region is a rectangle $4 \times 3 = 12$ with two corner triangles removed, each of area $\frac{1}{2} \cdot 2 \cdot 2 = 2$. So $12 - 4 = 8$. Choice D forgets to subtract the two corners.
- D. Average value is $\frac{1}{4-0} \cdot 10 = 2.5$. Choice A forgets to divide by the length of the interval.
- C. $\Delta x = 2$ and the midpoints are 1 and 3: $2(1 + 27) = 56$. Choice A is the left sum, B the right sum, and D is the exact integral, not a midpoint sum.
35. $\Delta x = 1$ and the right endpoints are 1, 2, 3, 4, 5, so the heights are 3, 5, 7, 9, 11. The sum is 35.
- 40 meters. The four trapezoids are $\frac{2+5}{2}(1) + \frac{5+7}{2}(2) + \frac{7+6}{2}(1) + \frac{6+3}{2}(4) = 3.5 + 12 + 6.5 + 18 = 40$.
- $\frac{3}{2}$. $\Delta x = 2$ and the midpoints are 2 and 4: $2(\frac{1}{2} + \frac{1}{4}) = \frac{3}{2}$.
- π . The graph of $y = \sqrt{4 - x^2}$ is the upper half of a circle of radius 2. From $x = 0$ to 2 we get a quarter circle: $\frac{1}{4}\pi(2)^2 = \pi$.
17. $2(4) - 3(-2) + \int_0^3 1 \, dx = 8 + 6 + 3 = 17$.
217. We need $\frac{108}{n} < 0.5$, so $n > 216$. At $n = 216$ the difference is exactly 0.5, which is not less, so the smallest whole number is 217.

39.3 The Fundamental Theorem of Calculus

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PRACTICE · THE FUNDAMENTAL THEOREM OF CALCULUS

1. B. $F(x) = x^3 + x$, so $F(2) - F(0) = 10$. Choice A integrates only $3x^2$ and drops the 1.
2. A. Part 1 says the derivative is the integrand evaluated at x . Choice B differentiates the integrand, and D integrates it.
3. D. The upper limit is x^2 , so the chain rule gives $e^{x^2} \cdot 2x$. Choice A omits $2x$.
4. C. $\int_{-1}^1 x^3 dx = 0$ (odd function) and $\int_{-1}^1 2 dx = 4$. Choice A treats the interval length as 1 instead of 2.
5. B. By Part 1, $g'(x) = x^2 - 4x + 3 = (x-1)(x-3)$, which is negative exactly on $(1, 3)$. The lower limit 2 does not change g' .
6. A. Amount $= 5 + \int_0^3 (4t + 1) dt = 5 + [2t^2 + t]_0^3 = 5 + 21 = 26$. Choice B forgets the starting 5 gallons.
7. D. $\int_1^e \frac{3}{x} dx = 3 \ln x \Big|_1^e = 3(1 - 0) = 3$. Choice B uses x in place of $\ln x$.
8. C. $v > 0$ on $(0, 3)$ and $v < 0$ on $(3, 5)$. Distance $= \int_0^3 (3-t) dt + \int_3^5 (t-3) dt = 4.5 + 2 = 6.5$. Choice A is the displacement $4.5 - 2 = 2.5$.
9. B. By the chain rule, $G'(x) = (2x^3 + 1) \cdot 3x^2$, so $G'(2) = 17 \cdot 12 = 204$. Choice A forgets the $3x^2$ factor.
10. 22. $F(x) = 4x^{3/2} - 2x$. $F(4) = 32 - 8 = 24$ and $F(1) = 4 - 2 = 2$, so the answer is $24 - 2 = 22$.
11. 5. $g' = f$ is positive for $x < 3$ (the line crosses zero at $x = 3$) and negative afterward, so the maximum is at $x = 3$. $g(3) = \int_0^2 2 dt + \frac{1}{2}(1)(2) = 4 + 1 = 5$.
12. -5. The variable is the lower limit, so $h'(x) = -\sqrt{x^2 + 9}$. At $x = 4$ this is $-\sqrt{25} = -5$.
13. 14. By Part 1 with two variable limits, $P'(x) = x^2 \cdot 2x - x \cdot 1 = 2x^3 - x$. At $x = 2$: $16 - 2 = 14$.
14. 14 liters. The net rate $2t - 8$ is negative until $t = 4$ and positive afterward, so the minimum is at $t = 4$: $30 + \int_0^4 (2t - 8) dt = 30 + (16 - 32) = 14$. The amount at $t = 10$ is 50, which is larger.
15. $\frac{28}{3}$ meters. $v = (t-2)(t-4)$ changes sign at $t = 2$ and 4 . With $V(t) = \frac{t^3}{3} - 3t^2 + 8t$: $V(0) = 0$, $V(2) = \frac{20}{3}$, $V(4) = \frac{16}{3}$, $V(5) = \frac{20}{3}$. Distance $= \frac{20}{3} + \frac{4}{3} + \frac{4}{3} = \frac{28}{3}$.

39.4 u-Substitution

PRACTICE · U-SUBSTITUTION

1. B. Let $u = x^2 + 3$, $du = 2x dx$: $\int u^5 du = \frac{u^6}{6}$. Choice A forgets to divide by 6, and C differentiates instead.
2. D. Let $u = 5x$, so $dx = \frac{1}{5} du$ and the result is $\frac{1}{5} \sin(5x)$. Choice A multiplies by 5 instead of dividing.
3. A. Let $u = x^3$, $du = 3x^2 dx$, so $\frac{1}{3} \int e^u du$. Choice C drops the $\frac{1}{3}$.
4. C. The derivative of $x^3 + 4$ is $3x^2$, which matches the numerator up to a constant. The other choices leave leftover variables.
5. A. Let $u = 2x + 1$, $du = 2 dx$: $4 \cdot \frac{1}{2} \ln u = 2 \ln(2x + 1)$. Choice B forgets the factor $\frac{1}{2}$ from du .
6. D. Let $u = \ln x$: the limits become 0 and 1, so $\int_0^1 u^2 du = \frac{1}{3}$. Choice A keeps the old limits 1 and e in $\frac{u^3}{3}$.
7. C. Let $u = x^2 + 16$, $x dx = \frac{1}{2} du$, limits 16 to 25: $\frac{1}{2} \cdot \frac{2}{3} (125 - 64) = \frac{61}{3}$. Choice A is exactly half the correct value, from dividing by 2 twice.
8. B. With $u = \cos x$, $du = -\sin x dx$, so the integral is $-\int \frac{du}{u} = -\ln |\cos x| + C$. Choice A misses the minus sign.
9. D. Let $u = 2x$, $dx = \frac{1}{2} du$, limits 2 to 6: $\frac{1}{2} \cdot 12 = 6$. Choice A forgets the factor $\frac{1}{2}$.
10. $\frac{(x^3 - 2)^5}{15} + C$. Let $u = x^3 - 2$, $x^2 dx = \frac{1}{3} du$: $\frac{1}{3} \cdot \frac{u^5}{5}$.
11. $\frac{e^4 - 1}{2}$. Let $u = x^2$, $x dx = \frac{1}{2} du$, limits 0 to 4: $\frac{1}{2} (e^4 - 1)$.
12. $\frac{1}{2}$. Let $u = \tan x$, $du = \sec^2 x dx$, limits 0 to 1: $\int_0^1 u du = \frac{1}{2}$.
13. $\ln 3$. Let $u = \ln x$, $du = \frac{1}{x} dx$, limits 1 to 3: $\int_1^3 \frac{1}{u} du = \ln 3 - \ln 1 = \ln 3$.
14. $\frac{506}{15}$. Let $u = x + 4$, so $x = u - 4$, limits 4 to 9: $\int_4^9 (u^{3/2} - 4u^{1/2}) du = \left[\frac{2}{5} u^{5/2} - \frac{8}{3} u^{3/2} \right]_4^9 = \left(\frac{486}{5} - 72 \right) - \left(\frac{64}{5} - \frac{64}{3} \right) = \frac{506}{15}$.
15. 4. Let $u = x^2 + 9$: $f(x) = \sqrt{x^2 + 9} + C$. From $f(0) = 3 + C = 2$, $C = -1$. Then $f(4) = 5 - 1 = 4$.

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