



Applications of Integration and Introduction to Series

Four focused lessons, worked examples, a mistake clinic, and 60 practice problems with full explanations.

In this booster

40.1 Area Between Curves

40.2 Volumes of Solids

40.3 Differential Equations and Slope Fields

40.4 Introduction to Sequences and Series for BC

Every section teaches the idea, walks through the most common problem types, flags the traps students fall into, and ends with fifteen practice problems. Full answers and explanations are at the back.

40.1 Area Between Curves

Set up and evaluate the area between two curves using top minus bottom or right minus left, find intersections, split at crossings, and choose between vertical and horizontal slices.

Why This Matters

In the last booster section you found the area between a curve and the x -axis. Many real regions are not bounded by an axis at all. The space between two roads, the cross-section of a bridge arch and its deck, the gap between two sales curves, or the difference in total output of two machines are all areas **between curves**. The method is the same idea of slicing and adding, with one new ingredient: you subtract the lower function from the upper one.

Students get stuck in four places. They cannot decide which curve is on top. They find the wrong intersection points, often by dividing away a solution. They set up one integral where the top and bottom switch roles, which cancels area. And they refuse to switch to horizontal slices when the region is awkward in terms of x . This skill feeds directly into volumes of solids and into probability, economics, and physics, where the area between two curves is a total difference.

Core Ideas

Picture a thin vertical slice of width dx at position x . Its height is the distance between the two curves, which is (top y) minus (bottom y). Its area is $[\text{top} - \text{bottom}] dx$, and adding all the slices from $x = a$ to $x = b$ gives an integral.

Area Between Curves (Vertical Slices)

If f and g are continuous and $f(x) \geq g(x)$ on $[a, b]$, the area of the region between them is

$$A = \int_a^b [f(x) - g(x)] dx.$$

The integrand is always **top minus bottom**. It is never negative where it counts, so the integral is positive.

Area Between Curves (Horizontal Slices)

If $x = R(y)$ is the right curve and $x = L(y)$ the left curve with $R(y) \geq L(y)$ on $[c, d]$, then

$$A = \int_c^d [R(y) - L(y)] dy.$$

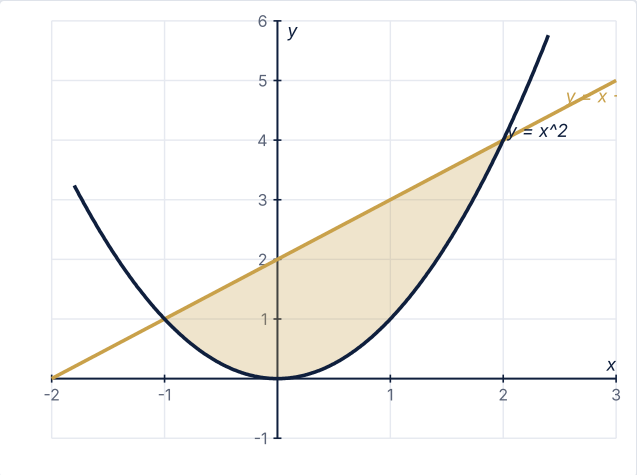
Use this when curves are given as functions of y , or when the region is bounded by one curve on the left and one on the right across its whole height.

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Here is the full routine. First sketch the curves. Second, find where they meet by setting the two expressions equal and solving; these points are your limits unless the problem gives vertical lines. Third, decide which curve is on top by testing a point between the limits. Fourth, integrate. If the curves cross inside the interval, the top and bottom swap, so split the integral at each crossing.



The line $y = x + 2$ is above the parabola between their intersections $x = -1$ and $x = 2$.

Step	What to do	Check
1. Sketch	Draw both curves and label them	Do they enclose a region?
2. Intersect	Set $f(x) = g(x)$ and solve	Did you keep every solution?
3. Order	Test a value between limits	Which is larger there?
4. Integrate	$\int(\text{top} - \text{bottom})$	Is the answer positive?
5. Interpret	Attach units	Does the size look reasonable?

Sometimes the region is cut by a boundary that is not a graph of one function, such as the x -axis becoming the bottom for part of the region. In that case either split the integral where the bottom changes or integrate with respect to y . Using y is usually easier when a sideways parabola such as $x = y^2$ is involved, because then each horizontal slice runs from one curve directly to another.

A last idea: the area between two **rate** curves is a quantity too. If $r_A(t)$ and $r_B(t)$ are rates, then $\int |r_A - r_B| dt$ is the total gap by which one process beats the other, while $\int (r_A - r_B) dt$ is the net difference. They agree only when one rate stays above the other.

Worked Examples

EXAMPLE 1

Find the area of the region bounded by $y = x + 2$ and $y = x^2$.

SOLUTION

Intersections: $x + 2 = x^2$ gives $x^2 - x - 2 = 0$, so $(x - 2)(x + 1) = 0$ and $x = -1, 2$. Testing $x = 0$: the line gives 2 and the parabola gives 0, so the line is on top.

$$A = \int_{-1}^2 (x + 2 - x^2) dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^2$$
$$= \left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) = \frac{10}{3} - \left(-\frac{7}{6} \right) = \frac{9}{2}.$$

EXAMPLE 2

Find the area enclosed by $y = 8 - x^2$ and $y = x^2$.

SOLUTION

Set $8 - x^2 = x^2$: $x^2 = 4$, so $x = \pm 2$. At $x = 0$, $8 - x^2 = 8$ is on top.

$$A = \int_{-2}^2 (8 - 2x^2) dx = 2 \int_0^2 (8 - 2x^2) dx \text{ by symmetry.}$$
$$= 2 \left[8x - \frac{2x^3}{3} \right]_0^2 = 2 \left(16 - \frac{16}{3} \right) = \frac{64}{3}.$$

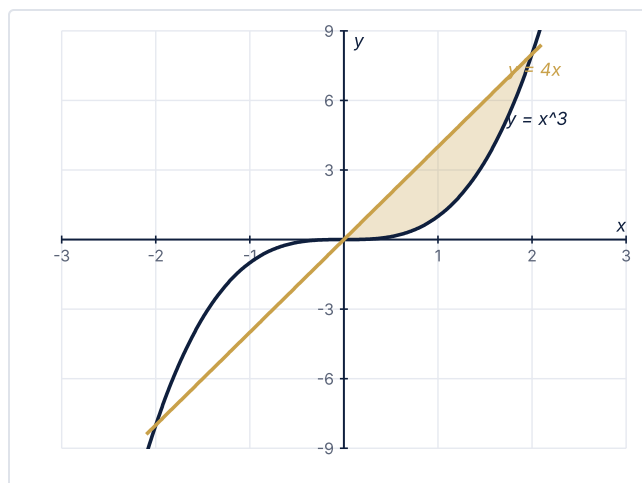
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EXAMPLE 3

Find the total area of the regions enclosed by $y = x^3$ and $y = 4x$.



The curves cross at $x = -2, 0, 2$. The line is on top on $[0, 2]$, and the cubic is on top on $[-2, 0]$.

SOLUTION

Intersections: $x^3 = 4x$ gives $x(x^2 - 4) = 0$, so $x = -2, 0, 2$. The curves cross at 0, so the top curve changes. On $[0, 2]$ the line is on top. On $[-2, 0]$ the cubic is on top (test $x = -1$: $x^3 = -1$, $4x = -4$).

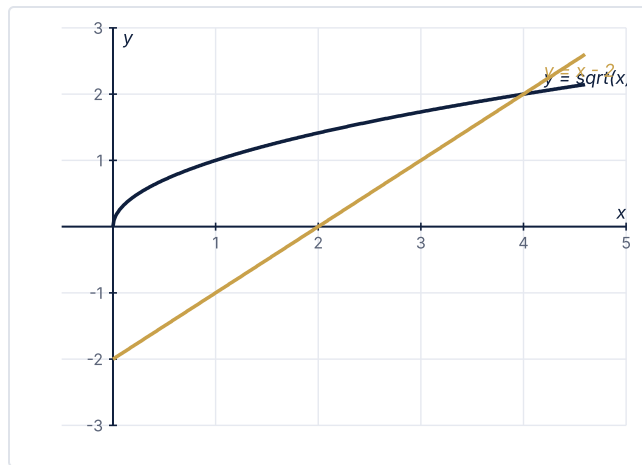
The picture is symmetric about the origin, so $A = 2 \int_0^2 (4x - x^3) dx$.

$$= 2 \left[2x^2 - \frac{x^4}{4} \right]_0^2 = 2(8 - 4) = 8.$$

A single integral $\int_{-2}^2 (4x - x^3) dx$ would give 0 because the two lobes cancel, so you must split.

EXAMPLE 4

Find the area of the region bounded by $y = \sqrt{x}$, $y = x - 2$, and the x -axis.



The region bounded by $y = \sqrt{x}$, $y = x - 2$ and the x -axis: it meets the axis at $x = 0$ and $x = 2$, and the curves meet at $(4, 2)$.

SOLUTION

Using x would need two integrals because the bottom boundary is the axis for $0 \leq x \leq 2$ and the line afterward. Slice horizontally instead. Solve for x : the left curve is $x = y^2$ and the right curve is $x = y + 2$.

The curves meet at $(4, 2)$ ($y^2 = y + 2$ gives $y = 2$ or -1 ; only $y = 2$ is on the region) and the region starts at $y = 0$.

$$A = \int_0^2 [(y + 2) - y^2] dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_0^2 = 2 + 4 - \frac{8}{3} = \frac{10}{3}.$$

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EXAMPLE 5

Find the area enclosed by $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq \frac{\pi}{2}$.

SOLUTION

They meet where $\sin x = \cos x$, at $x = \frac{\pi}{4}$. On $[0, \frac{\pi}{4}]$, cosine is on top; on $[\frac{\pi}{4}, \frac{\pi}{2}]$, sine is on top.

$$A = \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx.$$

First: $[\sin x + \cos x]_0^{\pi/4} = \sqrt{2} - 1$. Second: $[-\cos x - \sin x]_{\pi/4}^{\pi/2} = -1 + \sqrt{2}$.

$$A = 2\sqrt{2} - 2 \approx 0.828.$$

EXAMPLE 6

Find the area of the region bounded by $y = e^x$, $y = e^{-x}$, and $x = 1$.

SOLUTION

The curves meet at $x = 0$ (both equal 1), and $x = 1$ is the right edge. For $x > 0$, $e^x > e^{-x}$.

$$A = \int_0^1 (e^x - e^{-x}) dx = [e^x + e^{-x}]_0^1 = (e + e^{-1}) - 2.$$

$$A = e + \frac{1}{e} - 2 \approx 1.086.$$

Coach's Playbook

- 1 Sketch before you compute.** Even a rough sketch shows which curve is on top, how many regions there are, and whether y -slices would be easier.
 - 2 Solve for intersections carefully.** Move everything to one side and factor. Never divide by a variable factor, because you lose a solution.
 - 3 Test a point.** Pick an x between two limits, plug into both functions, and see which is larger. Do this on each piece.
 - 4 Write top minus bottom in parentheses.** Distribute the minus sign onto every term of the bottom function before integrating.
 - 5 Use symmetry.** If the region is symmetric about the y -axis or the origin, double a half-integral.
- ◆ **Choosing a variable:** if the same curve is on top across the entire interval, use x . If the left or right boundary changes but top and bottom are simple, try y .
 - ◆ **Bounded by lines:** vertical lines $x = a$ and $x = b$ give limits directly. Horizontal lines give limits for y -slices.
 - ◆ **Total versus net:** area is always positive. When curves cross, split. When you can't split, use absolute value on the calculator.
 - ◆ **Check:** the area should be less than the area of a rectangle that contains the region.

Common traps

(1) Subtracting in the wrong order and getting a negative area. (2) Using one integral across a crossing point, which cancels area. (3) Losing an intersection by dividing both sides by x . (4) Forgetting parentheses around the bottom function: for the bottom curve $x^2 - 2$, the difference $(6 - x^2) - (x^2 - 2)$ is $8 - 2x^2$, not $4 - 2x^2$.

Mistake Clinic**Spot the error**

Problem: Find the area bounded by $y = x + 2$ and $y = x^2$. **Student's work:** The limits are -1 and 2 . $\int_{-1}^2 (x^2 - (x + 2)) dx = -\frac{9}{2}$. So the area is $-\frac{9}{2}$. **What went wrong:** The student subtracted bottom minus top. Area cannot be negative. The parabola lies below the line between -1 and 2 , so the integrand must be (line) minus (parabola). **Correct solution:** $\int_{-1}^2 (x + 2 - x^2) dx = \frac{9}{2}$.

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Spot the error

Problem: Find the area between $y = x^3$ and $y = 4x$ from $x = -2$ to $x = 2$. **Student's work:** $\int_{-2}^2 (4x - x^3) dx = \left[2x^2 - \frac{x^4}{4} \right]_{-2}^2 = 0$. **What went wrong:** The curves cross at $x = 0$. On $[-2, 0]$ the cubic is on top, so $4x - x^3$ is negative there and cancels the positive part on $[0, 2]$. An area can never be 0 when the curves enclose regions. **Correct solution:** $2 \int_0^2 (4x - x^3) dx = 2(8 - 4) = 8$.

Spot the error

Problem: Find the area bounded by $y = \sqrt{x}$, $y = x - 2$, and the x -axis. **Student's work:** $\int_0^4 [\sqrt{x} - (x - 2)] dx = \frac{16}{3} - 0 = \frac{16}{3}$. **What went wrong:** On $[0, 2]$ the line $y = x - 2$ is below the x -axis, but the region's bottom boundary there is the axis, not the line. The student counted extra area under the axis. **Correct solution:** Integrate by y : $\int_0^2 [(y + 2) - y^2] dy = \frac{10}{3}$, which is smaller than $\frac{16}{3}$.

Real-World Connection

Two taps fill a tank for 4 minutes. Tap A delivers $6 + 2t$ gallons per minute and tap B delivers 10. The net difference in volume is $\int_0^4 (2t - 4) dt = 0$, so they deliver the same total. But the rates differ along the way: tap B leads for the first 2 minutes and tap A leads for the last 2. The total gap between the rate graphs is $\int_0^4 |2t - 4| dt = 4 + 4 = 8$ gallons of 'lead' that changed hands over time.

Challenge Corner

The region between $y = x^2$ and $y = 4$ is cut into two equal areas by a horizontal line $y = m$. Find m . **Solution:** The total area is $\int_{-2}^2 (4 - x^2) dx = \frac{32}{3}$. Use horizontal slices: the part of the region below $y = m$ has width $2\sqrt{y}$ at height y , so its area is $\int_0^m 2\sqrt{y} dy = \frac{4}{3}m^{3/2}$. Set this equal to half the total, $\frac{16}{3}$: $m^{3/2} = 4$, so $m = 4^{2/3} = \sqrt[3]{16} \approx 2.520$.

Practice**PRACTICE Area Between Curves**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is the area between $y = x$ and $y = x^2$ for $0 \leq x \leq 1$?
 (A) $\frac{1}{3}$ (B) $\frac{1}{6}$ (C) $\frac{1}{2}$ (D) $\frac{5}{6}$
- Which integral gives the area enclosed by $y = 6 - x^2$ and $y = x^2 - 2$?
 (A) $\int_{-2}^2 (2x^2 - 8) dx$ (B) $\int_{-2}^2 (4 - 2x^2) dx$ (C) $\int_0^2 (8 - 2x^2) dx$
 (D) $\int_{-2}^2 (8 - 2x^2) dx$
- What is the area of the region bounded by $y = x^2 - 4$ and the x -axis?
 (A) $-\frac{32}{3}$ (B) $\frac{16}{3}$ (C) $\frac{32}{3}$ (D) 8
- What is the area enclosed by $x = y^2$ and $x = 2y$?
 (A) $\frac{4}{3}$ (B) $\frac{8}{3}$ (C) $\frac{16}{3}$ (D) 4
- What is the total area enclosed by $y = x^3$ and $y = x$?
 (A) 0 (B) $\frac{1}{4}$ (C) 1 (D) $\frac{1}{2}$
- What is the area between $y = e^x$ and $y = 1$ for $0 \leq x \leq \ln 3$?
 (A) $2 + \ln 3$ (B) $\ln 3 - 2$ (C) $3 - \ln 3$ (D) $2 - \ln 3$

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7. What is the area enclosed by $y = \sqrt{x}$ and $y = \frac{x}{2}$?
- (A) $\frac{8}{3}$ (B) $\frac{16}{3}$ (C) $\frac{4}{3}$ (D) 4
8. What is the area between $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ for $1 \leq x \leq 2$?
- (A) $\ln 2 + \frac{1}{2}$ (B) $\ln 2 - \frac{1}{2}$ (C) $\frac{1}{2} - \ln 2$ (D) $\ln 2 - 1$
9. A region lies between a top curve f and bottom curve g . At $x = 0, 1, 2, 3$, f is 5, 6, 6, 4 and g is 1, 2, 3, 3. What is the trapezoid estimate of the area?
- (A) 11 (B) 8 (C) 9.5 (D) 12
10. Find the area enclosed by $y = x^2 + 1$ and $y = 3 - x^2$.
11. Find the area enclosed by $y = x^2$ and $y = x + 6$.
12. Find the area enclosed by $y = \sin x$ and $y = \sin 2x$ for $0 \leq x \leq \frac{\pi}{3}$.
13. Find the area enclosed by $x = 2y^2$ and $x = y + 1$.
14. Find the total area enclosed by $y = x^3 - 9x$ and the x -axis.
15. The vertical line $x = k$ divides the region under $y = x^2$ and above the x -axis on $[0, 2]$ into two equal areas. Find k .

40.2 Volumes of Solids

Compute volumes by slicing: discs, washers, rotation about lines other than the axes, and solids built on a base with square, semicircle, or triangle cross sections.

Why This Matters

Area is flat; volume is where calculus meets the real world. How much concrete fills a curved column? How much liquid fits in a bowl, a vase, a funnel, a fuel tank? Engineers answer these questions by slicing the solid into thin pieces whose volumes they can compute, then adding the pieces with an integral. Every method in this section is the same recipe: **volume equals the integral of cross-sectional area**.

Students struggle for predictable reasons. They forget to square the radius, they forget the factor π , they mix up the outer and inner radius in a washer, and they use the wrong radius when the axis of rotation is not the x -axis or y -axis. With cross sections they use the wrong area formula for squares, semicircles, or triangles. The good news is that every one of these is fixable with a drawing, a labeled radius, and a short checklist. Volume integrals are a favorite free-response topic, and they lead directly to physics (moments, tank problems) and to higher-dimensional integrals in college.

Core Ideas

A thin slice of the solid, perpendicular to the axis of rotation, has area A and thickness dx (or dy). Its volume is $A dx$. Adding slices gives $V = \int A dx$. For a solid of revolution, each slice is a disc (a circle) or a washer (a circle with a hole).

Disc and Washer Methods

Rotate the region under $y = R(x) \geq 0$ on $[a, b]$ about the x -axis (disc):

$$V = \pi \int_a^b [R(x)]^2 dx.$$

If the region lies between an outer curve $R(x)$ and an inner curve $r(x)$ (washer):

$$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) dx.$$

Subtract the **squares**, not the square of the difference. For rotation about the y -axis, swap the roles of x and y and integrate with respect to y .

When the axis of rotation is a horizontal line $y = k$ (or vertical line $x = k$), a radius is a **distance from the axis**. For a point on the curve $y = f(x)$ and the axis $y = k$, the radius is $|f(x) - k|$. The table shows the usual cases.

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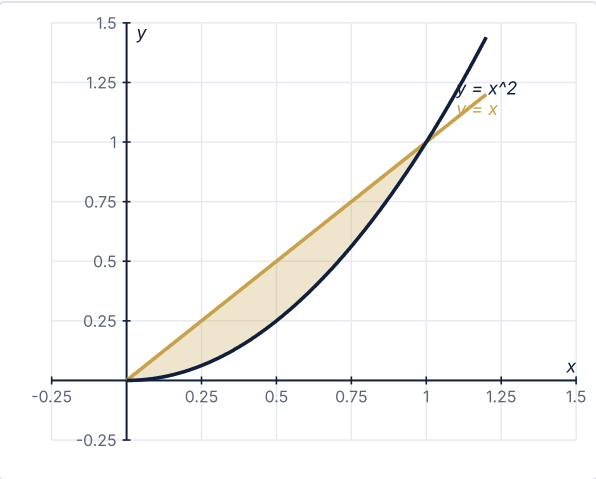
Axis of rotation	Region above or below?	Radius of a curve $y = f(x)$
$y = 0$ (the x -axis)	Above the axis	$f(x)$
$y = -2$	Above the axis	$f(x) + 2$
$y = 5$	Below the axis	$5 - f(x)$
$x = 4$ (vertical axis, integrate dy)	Left of the axis	$4 - x(y)$

Volume by Known Cross Sections

If the area of the cross section perpendicular to the x -axis at x is $A(x)$, then

$$V = \int_a^b A(x) \, dx.$$

Let s be the side or diameter lying in the base region (top curve minus bottom curve). Then: square $A = s^2$; semicircle on diameter s : $A = \frac{\pi}{8}s^2$; equilateral triangle: $A = \frac{\sqrt{3}}{4}s^2$; isosceles right triangle with leg s : $A = \frac{1}{2}s^2$.



The region between $y = x$ and $y = x^2$. Spun about the x -axis, it makes a washer with outer radius x and inner radius x^2 .

Method	When to use it	Slice area
Disc	Region touches the axis of rotation	πR^2
Washer	A gap between region and axis	$\pi(R^2 - r^2)$
Cross sections	A solid built on a base region	$A(x)$ from the shape

A reliable routine: (1) sketch the region and the axis. (2) Draw a representative slice perpendicular to the axis and label its radii. (3) Write the slice area. (4) Find the limits from the region. (5) Integrate. The slice is perpendicular to the axis: that tells you whether to integrate with respect to x or y .

Worked Examples

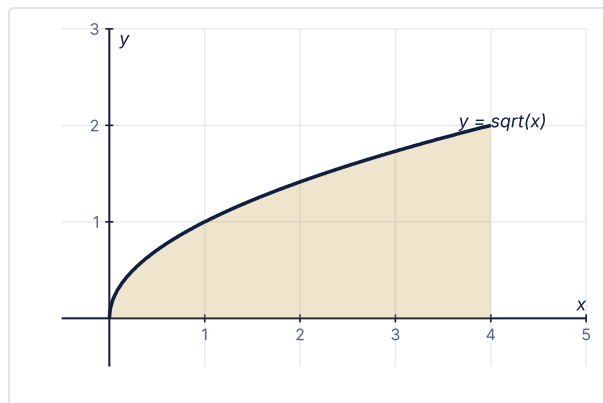
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EXAMPLE 1

The region under $y = \sqrt{x}$ on $[0, 4]$ is rotated about the x -axis. Find the volume.



The region under $y = \sqrt{x}$ on $[0, 4]$, rotated about the x -axis.

SOLUTION

Each slice is a disc of radius $\sqrt{x}$, so the area is $\pi(\sqrt{x})^2 = \pi x$.

$$V = \pi \int_0^4 x \, dx = \pi \cdot \frac{x^2}{2} \Big|_0^4 = 8\pi.$$

EXAMPLE 2

The region bounded by $y = x^2$, the y -axis, and $y = 4$ is rotated about the y -axis. Find the volume.

SOLUTION

The slices are perpendicular to the y -axis, so integrate in y . Solve $y = x^2$ for $x = \sqrt{y}$ (with $x \geq 0$): each disc has radius $\sqrt{y}$.

$$V = \pi \int_0^4 (\sqrt{y})^2 \, dy = \pi \int_0^4 y \, dy = \pi \cdot 8 = 8\pi.$$

EXAMPLE 3

The region between $y = x$ and $y = x^2$ is rotated about the x -axis. Find the volume.

SOLUTION

The curves meet at $x = 0$ and $x = 1$, and $y = x$ is above. The outer radius is x and the inner radius is x^2 .

$$V = \pi \int_0^1 (x^2 - x^4) \, dx = \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15}.$$

EXAMPLE 4

The region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$ is rotated about the line $y = -2$. Find the volume.

SOLUTION

The axis is below the region. The outer radius is the distance from $y = -2$ to the curve: $\sqrt{x} + 2$. The inner radius is the distance to the x -axis boundary: 2.

$$V = \pi \int_0^4 [(\sqrt{x} + 2)^2 - 2^2] \, dx = \pi \int_0^4 (x + 4\sqrt{x}) \, dx.$$

$$= \pi \left[\frac{x^2}{2} + \frac{8}{3}x^{3/2} \right]_0^4 = \pi \left(8 + \frac{64}{3} \right) = \frac{88\pi}{3}.$$

EXAMPLE 5

A solid has as its base the region between $y = x$ and $y = x^2$. Cross sections perpendicular to the x -axis are squares. Find the volume.

SOLUTION

The side of the square at x is top minus bottom: $s = x - x^2$ (the base is the region in the graph above).

$$A(x) = (x - x^2)^2 = x^2 - 2x^3 + x^4.$$

$$V = \int_0^1 (x^2 - 2x^3 + x^4) \, dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{1}{30}.$$

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EXAMPLE 6

The base of a solid is the region between $y = x^2$ and $y = 8 - x^2$. Cross sections perpendicular to the x -axis are semicircles with diameter in the base. Find the volume.

SOLUTION

The curves meet where $x^2 = 8 - x^2$, so $x = \pm 2$. The diameter is $s = (8 - x^2) - x^2 = 8 - 2x^2$.

The semicircle has radius $\frac{s}{2}$, so $A = \frac{1}{2}\pi \left(\frac{s}{2}\right)^2 = \frac{\pi}{8}s^2 = \frac{\pi}{8}(8 - 2x^2)^2$.

$$V = \frac{\pi}{8} \int_{-2}^2 (8 - 2x^2)^2 dx = \frac{\pi}{8} \cdot 4 \int_{-2}^2 (4 - x^2)^2 dx = \frac{\pi}{2} \cdot \frac{512}{15} = \frac{256\pi}{15}.$$

Coach's Playbook

- 1 Draw the region and the axis.** Mark the line of rotation and shade the region. Everything else depends on this picture.
 - 2 Draw one slice perpendicular to the axis.** If the slice is vertical, integrate dx . If horizontal, integrate dy .
 - 3 Label radii as distances.** Outer radius is the far curve's distance from the axis, inner radius the near curve's distance. Subtract the axis value, not the curve value.
 - 4 Square, then subtract.** Write $R^2 - r^2$ with each radius in parentheses before squaring.
 - 5 Check for reasonableness.** The volume must be less than that of the cylinder or box that encloses the solid.
- ◆ **Disc or washer:** if the region touches the axis along its whole length, you get a disc. A gap between the region and the axis means a washer.
 - ◆ **Horizontal axis, dx ; vertical axis, dy :** this works for disc and washer when the slices are perpendicular to the axis.
 - ◆ **Cross sections:** compute s as (top minus bottom) in the base and use the shape formula. A semicircle's formula $\frac{\pi}{8}s^2$ comes from radius $\frac{s}{2}$ and half of a circle.
 - ◆ **Units:** length cubed. If the lengths are in centimeters the volume is in cubic centimeters.

Common traps

(1) Squaring the difference of the radii, $\pi \int (R - r)^2$, instead of subtracting the squares. (2) Leaving out the π or forgetting to square the radius. (3) Using the curve's value as the radius when the axis is not the x -axis. (4) Using πs^2 for a semicircular cross section, or $\frac{\pi}{8}s$ without squaring.

Mistake Clinic**Spot the error**

Problem: The region between $y = x$ and $y = x^2$ for $0 \leq x \leq 1$ is rotated about the x -axis. Find the volume. **Student's work:** $V = \pi \int_0^1 (x - x^2)^2 dx = \frac{\pi}{30}$. **What went wrong:** This is the volume of a solid of radius $x - x^2$ (the gap width), not a washer. The washer area is the outer circle minus the inner circle, which is $\pi(x^2) - \pi(x^4)$, because each circle's radius is measured from the axis. **Correct solution:** $\pi \int_0^1 (x^2 - x^4) dx = \frac{2\pi}{15}$.

Spot the error

Problem: The region under $y = \sqrt{x}$ on $[0, 4]$ is rotated about the x -axis. **Student's work:** $V = \int_0^4 \sqrt{x} dx = \frac{16}{3}$. **What went wrong:** The student integrated the radius. A volume integral integrates an **area**, so the radius must be squared and multiplied by π . Dimensionally, $\int \sqrt{x} dx$ is an area, not a volume. **Correct solution:** $\pi \int_0^4 (\sqrt{x})^2 dx = \pi \int_0^4 x dx = 8\pi$.

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Spot the error

Problem: The region bounded by $y = \sqrt{x}$, $y = 0$, $x = 4$ is rotated about $y = -2$. **Student's work:** $V = \pi \int_0^4 (\sqrt{x})^2 dx = 8\pi$.

What went wrong: The student used the curve's height as the radius, as if the axis were the x -axis. The axis is $y = -2$, so there is a hole of radius 2 and the outer radius is $\sqrt{x} + 2$. **Correct solution:** $\pi \int_0^4 [(\sqrt{x} + 2)^2 - 4] dx = \frac{88\pi}{3}$.

Real-World Connection

A parabolic bowl is made by rotating $y = x^2$ about the y -axis, with lengths in decimeters. Filled to height 4, it holds $\pi \int_0^4 y dy = 8\pi \approx 25.1$ cubic decimeters, which is about 25.1 liters. Designers of fuel tanks, wine glasses and funnels compute volumes this way before cutting any material.

Challenge Corner

For the bowl above, find the depth h at which the bowl is half full by volume. **Solution:** The volume up to depth h is $\pi \int_0^h y dy = \frac{\pi h^2}{2}$. Half of the full volume 8π is 4π . So $\frac{\pi h^2}{2} = 4\pi$, $h^2 = 8$, and $h = 2\sqrt{2} \approx 2.83$. A bowl that is half full by volume is more than half deep (since half of 4 is 2), because the bowl is wider near the top.

Practice**PRACTICE Volumes of Solids**

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- The region under $y = x^2$ on $[0, 2]$ is rotated about the x -axis. What is the volume?
(A) $\frac{8\pi}{3}$ (B) $\frac{32}{5}$ (C) $\frac{32\pi}{5}$ (D) 16π
- The region under $y = \sqrt{x}$ on $[0, 9]$ is rotated about the x -axis. What is the volume?
(A) 18π (B) $\frac{81\pi}{2}$ (C) 9π (D) $\frac{81}{2}$
- The region between $y = x^2$ and $y = 2x$ is rotated about the x -axis. What is the volume?
(A) $\frac{16\pi}{15}$ (B) $\frac{64}{15}$ (C) $\frac{64\pi}{15}$ (D) $\frac{8\pi}{3}$
- The region bounded by $y = x^2$, $y = 0$, and $x = 2$ is rotated about the y -axis. What is the volume?
(A) $\frac{32\pi}{5}$ (B) 16π (C) 4π (D) 8π
- A solid has as its base the region under $y = 3 - x$ on $[0, 3]$. Cross sections perpendicular to the x -axis are squares. What is the volume?
(A) $\frac{9}{2}$ (B) 3 (C) 27 (D) 9
- A solid has as its base the region under $y = \sqrt{x}$ on $[0, 4]$. Cross sections perpendicular to the x -axis are semicircles with diameter in the base. What is the volume?
(A) 8π (B) π (C) 2π (D) $\frac{\pi}{2}$
- The region under $y = x^2$ on $[0, 1]$ is rotated about the line $y = -1$. What is the volume?
(A) $\frac{28\pi}{15}$ (B) $\frac{\pi}{5}$ (C) $\frac{8\pi}{15}$ (D) $\frac{13\pi}{15}$
- The region between $y = x^2$ and $y = 4$ is rotated about the line $y = 4$. What is the volume?
(A) $\frac{512\pi}{15}$ (B) $\frac{256\pi}{15}$ (C) $\frac{128\pi}{3}$ (D) $\frac{64\pi}{3}$
- The base of a solid is the region under $y = \sqrt{x}$ on $[0, 4]$. Cross sections perpendicular to the x -axis are equilateral triangles with a side in the base. What is the volume?
(A) $\sqrt{3}$ (B) $4\sqrt{3}$ (C) $2\sqrt{3}$ (D) 8
- The region under $y = 3x$ on $[0, 2]$ is rotated about the x -axis. Find the volume.
- The region under $y = x^3$ on $[0, 2]$ is rotated about the x -axis. Find the volume.

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12. The region between $y = \sqrt{x}$ and $y = x$ is rotated about the x -axis. Find the volume.
13. The region bounded by $y = x^2$, $y = 0$, and $x = 2$ is rotated about the vertical line $x = 4$. Find the volume.
14. A solid has base the region between $y = x^2$ and $y = 4$. Cross sections perpendicular to the x -axis are isosceles right triangles with a leg in the base. Find the volume.
15. A solid has base the region between $y = x^2$ and $y = 2x$. Cross sections perpendicular to the x -axis are semicircles with diameter in the base. Find the volume.

40.3 Differential Equations and Slope Fields

Solve separable differential equations with initial conditions, read and sketch slope fields, and model exponential growth, decay, and Newton's cooling from a rate that is proportional to the quantity.

Why This Matters

A **differential equation** is an equation that contains a derivative. Instead of asking for a number, it asks for a function. 'The population grows at a rate proportional to its size,' 'a cup of coffee cools at a rate proportional to how much hotter it is than the room,' and 'a drug leaves the bloodstream at a rate proportional to its concentration' are all differential equations. Nearly every law of physics, biology, chemistry, and economics is written this way, so this is where calculus becomes a modeling language.

Students get stuck because they try to treat $\frac{dy}{dx}$ as a single symbol that cannot be moved. They also separate variables incorrectly, forget the constant C until it is too late, and lose the sign of the initial condition when taking a square root. Slope fields cause a different struggle: you are asked to read a picture of a rule and not compute a number. This section builds three skills: solving separable equations, reading and sketching slope fields, and modeling exponential growth and decay. All three appear regularly on the AP exam.

Core Ideas

A **solution** of a differential equation is a function that makes the equation true when you substitute it in. The **general solution** contains an arbitrary constant, and a **particular solution** uses an initial condition such as $y(0) = 3$ to fix the constant. To check a proposed solution, differentiate it and see if the equation holds.

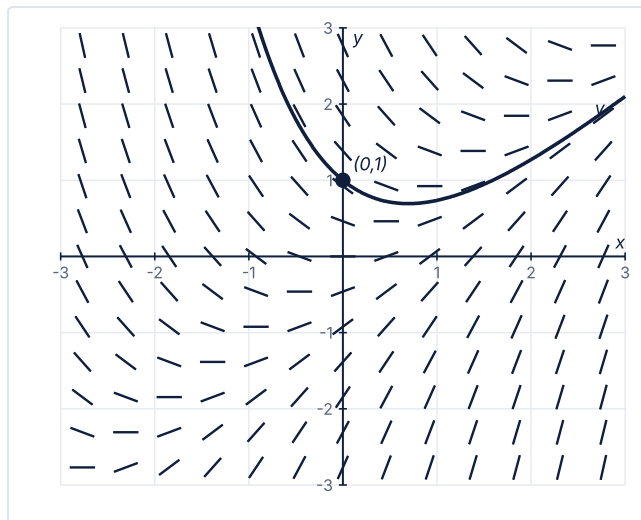
Separation of Variables

A differential equation is **separable** if it can be written

$$\frac{dy}{dx} = g(x)h(y).$$

Method: (1) Rewrite as $\frac{1}{h(y)} dy = g(x) dx$. (2) Integrate both sides: $\int \frac{dy}{h(y)} = \int g(x) dx$, with a single constant C . (3) Use the initial condition to find C . (4) Solve for y if possible.

A **slope field** (direction field) draws a short segment with slope $\frac{dy}{dx}$ at each point of a grid. You do not need to solve the equation to draw it: just plug in the coordinates. A solution curve follows the segments like a boat following a current. Slope fields tell you where solutions rise or fall, where they level off (slope 0), and what happens in the long run.



Slope field for $dy/dx = x - y$ with the solution through $(0, 1)$. Segments are flat along the line $y = x$.

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Point (x, y)	Slope $x - y$	Segment
$(0, 0)$	0	flat
$(1, 0)$	1	rises at 45 degrees
$(0, 1)$	-1	falls at 45 degrees
$(2, 2)$	0	flat
$(1, 3)$	-2	falls steeply

Exponential Growth and Decay

If a quantity changes at a rate proportional to itself,

$$\frac{dy}{dt} = ky,$$

then

$$y(t) = y_0 e^{kt},$$

where $y_0 = y(0)$. If $k > 0$ the quantity grows; if $k < 0$ it decays.

More generally, if $\frac{dy}{dt} = k(y - b)$, then $y = b + (y_0 - b)e^{kt}$: the quantity moves exponentially toward (or away from) the constant b . Newton's law of cooling has this form with b equal to the room temperature.

To get the formula from separation of variables: $\frac{dy}{y} = k dt$, so $\ln |y| = kt + C_1$, and exponentiating gives $y = Ce^{kt}$ with $C = \pm e^{C_1}$. Then $y(0) = C$. Doubling time and half-life come from solving $e^{kt} = 2$ or $\frac{1}{2}$: they equal $\frac{\ln 2}{k}$ and $\frac{\ln 2}{|k|}$.

Worked Examples

EXAMPLE 1

Show that $y = Ce^{2x} - 1$ is a solution of $\frac{dy}{dx} = 2y + 2$, and find the particular solution with $y(0) = 4$.

SOLUTION

$\frac{dy}{dx} = 2Ce^{2x}$. And $2y + 2 = 2(Ce^{2x} - 1) + 2 = 2Ce^{2x}$. They match for every C , so the family is a solution.

Use $y(0) = 4$: $C - 1 = 4$, so $C = 5$.

$$y = 5e^{2x} - 1.$$

EXAMPLE 2

Solve $\frac{dy}{dx} = 2xy$ with $y(0) = 3$.

SOLUTION

Separate: $\frac{dy}{y} = 2x dx$.

Integrate: $\ln |y| = x^2 + C$. Then $y = Ae^{x^2}$ with $A = \pm e^C$.

$y(0) = A = 3$, so

$$y = 3e^{x^2}.$$

EXAMPLE 3

Solve $\frac{dy}{dx} = \frac{x}{y}$ with $y(1) = 2$.

SOLUTION

Separate: $y dy = x dx$.

Integrate: $\frac{y^2}{2} = \frac{x^2}{2} + C$, so $y^2 = x^2 + K$ where $K = 2C$.

Use $(1, 2)$: $4 = 1 + K$, so $K = 3$. Since $y(1) = 2 > 0$, take the positive root:

$$y = \sqrt{x^2 + 3}.$$

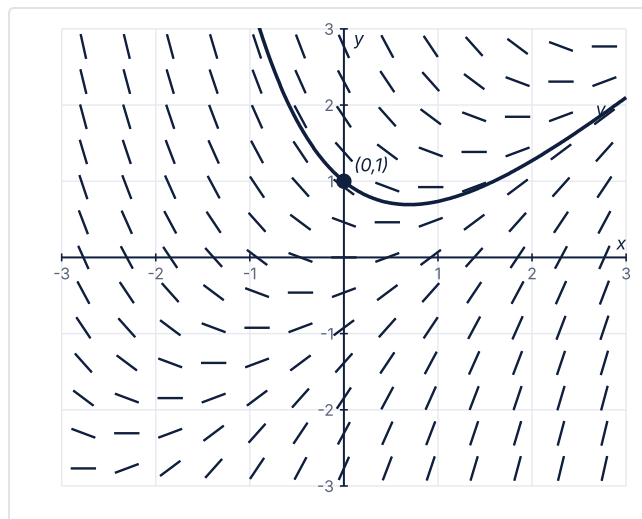
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EXAMPLE 4

For $\frac{dy}{dx} = x - y$, use the slope field to describe the solution through $(0, 1)$ and check with the formula $y = x - 1 + 2e^{-x}$.



Slope field for $dy/dx = x - y$ with the solution through $(0, 1)$. Segments are flat along the line $y = x$.

SOLUTION

At $(0, 1)$ the slope is $0 - 1 = -1$, so the solution starts by falling. It keeps falling until it meets the line $y = x$ (where the slope is 0); this happens where $x - 1 + 2e^{-x} = x$, that is $e^{-x} = \frac{1}{2}$, at $x = \ln 2 \approx 0.693$.

After that point the curve is below the line $y = x$, so $x - y > 0$ and the solution rises, approaching the line $y = x - 1$.

Check: $y' = 1 - 2e^{-x}$ and $x - y = 1 - 2e^{-x}$. They agree, and $y(0) = -1 + 2 = 1$.

EXAMPLE 5

A population satisfies $\frac{dP}{dt} = kP$. It is 500 at $t = 0$ and 800 at $t = 3$ years. Find $P(t)$, $P(6)$, and the doubling time.

SOLUTION

$P = 500e^{kt}$. From $800 = 500e^{3k}$ we get $e^{3k} = 1.6$, so $k = \frac{\ln 1.6}{3} \approx 0.157$ per year.

$P(6) = 500e^{6k} = 500(1.6)^2 = 1280$.

Doubling: $e^{kt} = 2$, so $t = \frac{\ln 2}{k} = \frac{3 \ln 2}{\ln 1.6} \approx 4.424$ years.

EXAMPLE 6

A cup of tea at 90 degrees is placed in a 20-degree room. Its temperature satisfies $\frac{dT}{dt} = -0.1(T - 20)$ with t in minutes. Find $T(t)$ and the time it takes to cool to 50 degrees.

SOLUTION

Separate with $u = T - 20$: $\frac{dT}{T-20} = -0.1 dt$. Then $\ln |T - 20| = -0.1t + C$, so $T - 20 = Ae^{-0.1t}$.

$T(0) = 90$ gives $A = 70$:

$$T = 20 + 70e^{-0.1t}.$$

Set $T = 50$: $30 = 70e^{-0.1t}$, so $e^{-0.1t} = \frac{3}{7}$ and $t = 10 \ln \frac{7}{3} \approx 8.473$ minutes.

Coach's Playbook

- 1 Identify the type.** Can you get all the y 's with dy on one side and all the x 's with dx on the other? If yes, separate. If the equation has the form $y' = k(y - b)$, use the shortcut.
- 2 Separate before integrating.** Never integrate a side that still has the other variable in it. Multiply or divide first.
- 3 Add one constant C .** Put $+C$ on the x side right after integrating. Do not add one on both sides.

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- 4 Plug in the initial condition early.** Find C before you solve for y when the algebra is easier.
- 5 Choose the branch.** When you take a square root, use the sign of the initial y -value to choose $+$ or $-$.

- ◆ **Slopes first:** to read a slope field, compute the slope at a few easy points such as the axes, the lines $y = x$ and $x = 0$, and compare with the picture.
- ◆ **Zero slope:** set $\frac{dy}{dx} = 0$ to find where segments are flat. For autonomous equations $y' = f(y)$, the solution $y = c$ with $f(c) = 0$ is a constant solution.
- ◆ **Sign of k :** growth if $k > 0$, decay if $k < 0$. If your formula doesn't match the story, recheck the sign.
- ◆ **Check with the equation:** differentiate your final answer and compare with the right-hand side. It takes a minute and catches nearly every mistake.

Common traps

(1) Writing $y = e^{2x} + C$ instead of $y = Ce^{2x}$ after exponentiating. (2) Splitting a product incorrectly, such as integrating $\frac{dy}{dx} = xy$ as $y = \frac{x^2}{2}y + C$. (3) Dropping the shift in Newton's cooling, so the temperature heads toward 0 instead of the room temperature. (4) Taking both signs after a square root, instead of using the initial condition to select the branch.

Mistake Clinic

Spot the error

Problem: Solve $\frac{dy}{dx} = 2y$ with $y(0) = 3$. **Student's work:** $\frac{dy}{y} = 2 dx$, so $\ln |y| = 2x + C$, so $y = e^{2x} + C$. With $y(0) = 3$: $1 + C = 3$, $C = 2$, so $y = e^{2x} + 2$. **What went wrong:** Exponentiating $\ln |y| = 2x + C$ gives $|y| = e^{2x} \cdot e^C$, a **product**, not a sum. Check: the student's answer has derivative $2e^{2x}$, but $2y = 2e^{2x} + 4$. **Correct solution:** $y = Ae^{2x}$, $A = 3$, so $y = 3e^{2x}$.

Spot the error

Problem: Solve $\frac{dy}{dx} = xy$. **Student's work:** Integrate both sides with respect to x : $y = \frac{x^2}{2}y + C$. **What went wrong:** The student integrated xy as if y were a constant. But y is a function of x , so it cannot be pulled out. The variables must be separated first. **Correct solution:** $\frac{dy}{y} = x dx$, so $\ln |y| = \frac{x^2}{2} + C$ and $y = Ae^{x^2/2}$.

Spot the error

Problem: A cup of tea at 90 degrees cools in a 20-degree room: $\frac{dT}{dt} = -0.1(T - 20)$. **Student's work:** $T = Ce^{-0.1t}$, and $T(0) = 90$ gives $T = 90e^{-0.1t}$. **What went wrong:** The solution of $T' = -0.1T$ is not a solution of $T' = -0.1(T - 20)$. The student's formula would cool the tea toward 0 degrees, below room temperature, which is impossible. **Correct solution:** Let $u = T - 20$, so $u' = -0.1u$ and $u = 70e^{-0.1t}$. Then $T = 20 + 70e^{-0.1t}$, which approaches 20.

Real-World Connection

Money in an account that compounds continuously at 4 percent satisfies $\frac{dA}{dt} = 0.04A$. Starting from 1000 dollars, $A(t) = 1000e^{0.04t}$, so after 10 years $A(10) = 1000e^{0.4} \approx 1491.82$ dollars. The same equation, with a negative k , describes the decay of medicine in the body.

Challenge Corner

A tank holds 100 liters of salt water containing 5 kilograms of salt. Pure water flows in at 2 liters per minute and the well-mixed solution flows out at 2 liters per minute, so the volume stays 100. Find when the amount of salt falls to 1 kilogram. **Solution:** The salt leaves at rate $2 \cdot \frac{S}{100} = \frac{S}{50}$ kilograms per minute, so $\frac{dS}{dt} = -\frac{S}{50}$ and $S = 5e^{-t/50}$. Setting $S = 1$: $e^{-t/50} = \frac{1}{5}$, so $t = 50 \ln 5 \approx 80.47$ minutes.

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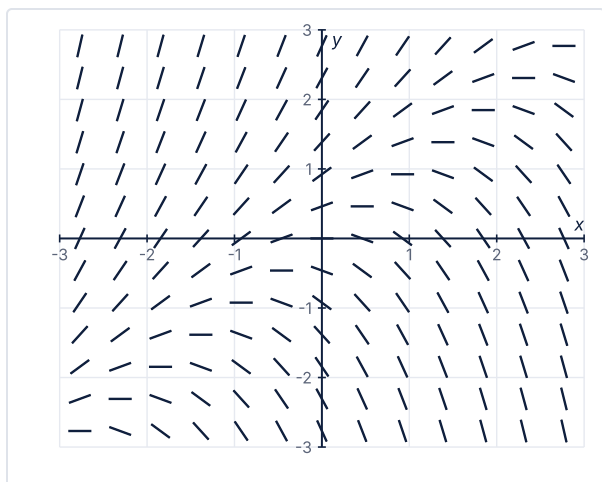
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Practice

PRACTICE Differential Equations and Slope Fields

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- Which function is a solution of $\frac{dy}{dx} = 3y$?
 (A) $y = 3e^{5x}$ (B) $y = 5e^{3x}$ (C) $y = e^{x^3}$ (D) $y = 3x + 5$
- Separating variables in $\frac{dy}{dx} = \frac{x}{y^2}$ gives which equation?
 (A) $y dy = x dx$ (B) $dy = xy^2 dx$ (C) $y^2 dy = x dx$
 (D) $\frac{dy}{y^2} = x dx$
- What is the general solution of $\frac{dy}{dx} = \frac{y}{x}$ for $x > 0$?
 (A) $y = x + C$ (B) $y = Ce^x$ (C) $y = C \ln x$ (D) $y = Cx$
- What is the solution of $\frac{dy}{dx} = 6x^2y$ with $y(0) = 2$?
 (A) $y = 2e^{2x^3}$ (B) $y = 2e^{6x^2}$ (C) $y = e^{2x^3} + 1$
 (D) $y = 2x^3 + 2$
- The slope field shown matches which differential equation?



Slope field. Flat along $y = x$, rising above it, falling below it.

- (A) $\frac{dy}{dx} = x - y$ (B) $\frac{dy}{dx} = y - x$ (C) $\frac{dy}{dx} = x + y$ (D) $\frac{dy}{dx} = xy$
- A culture satisfies $\frac{dP}{dt} = 0.2P$ with $P(0) = 50$. What is $P(5)$?
 (A) $50e$ (B) $50e^{0.2}$ (C) 250 (D) 51
- A sample of 80 grams of a substance loses half of its mass every 4 years. How many grams remain after 12 years?
 (A) 40 (B) 20 (C) 10 (D) 26.7
- If $\frac{dy}{dx} = \frac{e^x}{y}$ and $y(0) = 2$, what is $y(\ln 3)$?
 (A) $2\sqrt{2}$ (B) $\sqrt{8} + 2$ (C) $\sqrt{3}$ (D) 4
- Which pair lists all constant solutions of $\frac{dy}{dx} = (y - 1)(y - 3)$?
 (A) $y = -1$ and $y = -3$ (B) $y = 0$ only (C) $y = 1$ and $y = 3$
 (D) $y = 1$ only
- Solve $\frac{dy}{dx} = 3x^2y$ with $y(0) = 2$.
- Solve $\frac{dy}{dx} = \frac{x}{y}$ with $y(0) = -3$, and find $y(4)$.

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12. A population P satisfies $\frac{dP}{dt} = kP$, starts at 200, and doubles every 5 hours. Find $P(15)$.
13. [Calculator] Coffee at 95 degrees is in a 20-degree room. Its temperature obeys $T = 20 + 75e^{-kt}$, and it is 60 degrees after 10 minutes. Find the time (in minutes, to three decimals) when it reaches 30 degrees.
14. Let $y = f(x)$ be the solution of $\frac{dy}{dx} = xy$ with $f(1) = 2$. Use the tangent line at $x = 1$ to approximate $f(1.2)$.
15. Solve $\frac{dy}{dx} = 2y - 6$ with $y(0) = 5$, then find $y(\ln 2)$.

40.4 Introduction to Sequences and Series for BC

Meet infinite series: partial sums, geometric and p-series, a toolkit of convergence tests, and Taylor polynomials with error bounds, the entry point to BC power series.

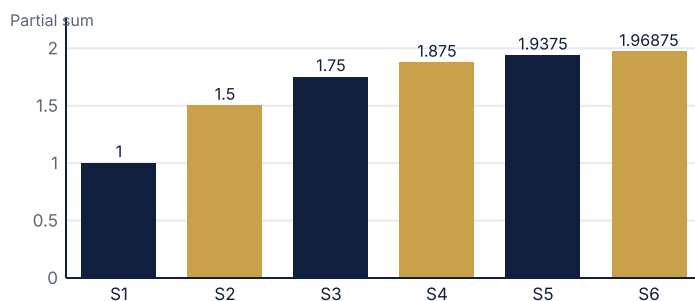
Why This Matters

Can you add infinitely many numbers and get a finite answer? Zeno's old puzzle says yes: walk half the distance to a wall, then half the remaining distance, and so on. The total $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ is exactly 1. Calculus makes this idea precise. An **infinite series** is a limit of partial sums, and it either converges to a number or it doesn't. Series are how calculators compute $\sin x$ and e^x , how engineers approximate complicated functions, and how scientists predict signals and waves.

Students struggle with series because it feels unlike everything before it. There are many tests, and choosing the right one is a skill. The most common errors are: believing that terms going to zero is enough to guarantee convergence, using the geometric formula with the wrong first term, using the ratio test when it gives no information, and mixing up the Taylor polynomial (an approximation) with the function itself. This introduction gives you the vocabulary, the two series you must know cold, a toolkit of tests, and the first steps of Taylor polynomials with a clear idea of how much error to expect.

Core Ideas

A **sequence** is an ordered list $a_1, a_2, a_3, \dots$. It **converges** if a_n approaches a single number as $n \rightarrow \infty$. A **series** adds the terms: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$. Its **partial sums** are $S_N = a_1 + \dots + a_N$, and the series converges to S if $S_N \rightarrow S$. If the partial sums do not approach a finite limit, the series **diverges**.



Partial sums of $1 + 1/2 + 1/4 + \dots$ climb toward 2 but never pass it.

Geometric Series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots = \frac{a}{1-r} \quad \text{if } |r| < 1,$$

and the series diverges if $|r| \geq 1$ (for $a \neq 0$).

Here a is the **first term** of the series as written and r is the common ratio. If the index starts at $n = 1$, the first term is ar , not a .

p-Series and the nth-Term Test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1 \text{ and diverges if } p \leq 1.$$

The case $p = 1$ is the **harmonic series**, which diverges.

nth-Term Test: if $\lim_{n \rightarrow \infty} a_n \neq 0$ (or does not exist), then $\sum a_n$ diverges. If the limit is 0, the test says nothing: the series may converge or diverge.

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The table collects the convergence tests you should know. Always start with the quick checks: is it geometric, is it a p -series, do the terms fail to go to zero?

Test	Use when	Conclusion
nth-Term	Always check first	$\lim a_n \neq 0 \Rightarrow$ diverges
Geometric	Terms are ar^n	Converges iff $ r < 1$
p -Series	Terms are $\frac{1}{n^p}$	Converges iff $p > 1$
Ratio	Factorials, exponentials	$L = \lim \left \frac{a_{n+1}}{a_n} \right $; $L < 1$ converges, $L > 1$ diverges, $L = 1$ no information
Comparison	Looks like a known series	Smaller than a convergent series converges; bigger than a divergent series diverges
Integral	$a_n = f(n)$ positive, decreasing	$\int_1^\infty f$ and $\sum a_n$ both converge or both diverge
Alternating Series	Signs alternate, $ a_n $ decreasing to 0	Converges; error is at most the first omitted term

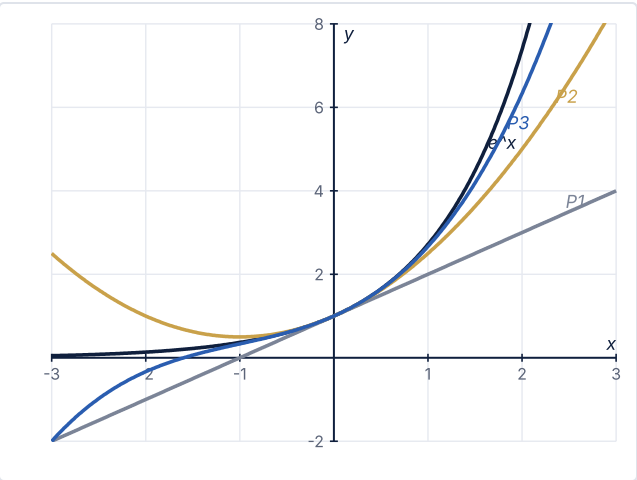
Next, **Taylor polynomials**. Near a point $x = a$, a smooth function looks like a polynomial built from its derivatives. The more terms you take, the better the match near a .

Taylor Polynomial

The Taylor polynomial of degree n for f centered at $x = a$ is
$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

When $a = 0$ it is also called a Maclaurin polynomial.

Lagrange error bound: if $|f^{(n+1)}(t)| \leq M$ for all t between a and x , then
$$|f(x) - P_n(x)| \leq \frac{M}{(n + 1)!} |x - a|^{n+1}.$$



Taylor polynomials of e^x at 0. Each added term hugs the curve over a wider interval.

Memorize the Maclaurin series that appear all the time: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$, and $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ for $|x| < 1$.

Worked Examples

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EXAMPLE 1

Find the sum of $\sum_{n=1}^{\infty} 4 \left(\frac{1}{3}\right)^n$.

SOLUTION

This is geometric with ratio $r = \frac{1}{3}$, which satisfies $|r| < 1$, so it converges.

The first term is the one at $n = 1$: $a = 4 \cdot \frac{1}{3} = \frac{4}{3}$.

$$S = \frac{4/3}{1 - 1/3} = \frac{4/3}{2/3} = 2.$$

EXAMPLE 2

Decide whether each series converges: (a) $\sum \frac{n}{3n+1}$, (b) $\sum \frac{1}{n^{3/2}}$, (c) $\sum \frac{1}{\sqrt{n}}$.

SOLUTION

(a) $\lim_{n \rightarrow \infty} \frac{n}{3n+1} = \frac{1}{3} \neq 0$, so by the nth-term test the series **diverges**.

(b) p -series with $p = \frac{3}{2} > 1$: **converges**.

(c) $\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ is a p -series with $p = \frac{1}{2} \leq 1$: **diverges**.

EXAMPLE 3

Use the ratio test on $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

SOLUTION

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2} \left(\frac{n+1}{n} \right)^2.$$

As $n \rightarrow \infty$, $\left(\frac{n+1}{n} \right)^2 \rightarrow 1$, so $L = \frac{1}{2} < 1$.

By the ratio test, the series **converges**.

EXAMPLE 4

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges. Approximate it with S_3 and bound the error.

SOLUTION

$$S_3 = 1 - \frac{1}{4} + \frac{1}{9} = \frac{31}{36} \approx 0.8611.$$

The terms $\frac{1}{n^2}$ decrease to 0 and the signs alternate, so the alternating series error bound applies: $|S - S_3| \leq a_4 = \frac{1}{16} = 0.0625$.

So the sum lies between $0.8611 - 0.0625$ and $0.8611 + 0.0625$.

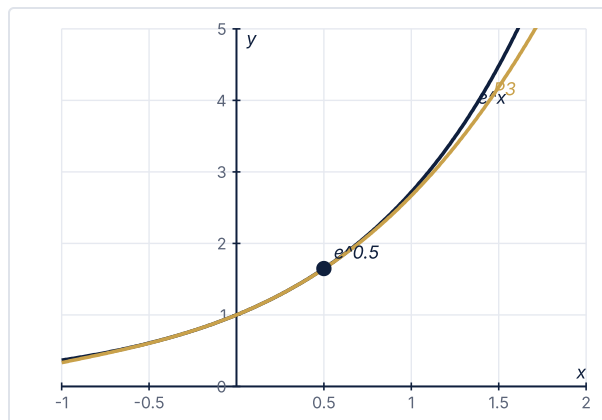
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EXAMPLE 5

Find the degree-3 Maclaurin polynomial of $f(x) = e^x$, use it to approximate $e^{0.5}$, and bound the error.



P_3 and e^x agree closely near the center $x = 0$.

SOLUTION

All derivatives of e^x at 0 equal 1: $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$.

$$P_3(0.5) = 1 + 0.5 + 0.125 + 0.02083 = 1.64583.$$

For $0 \leq t \leq 0.5$, $f^{(4)}(t) = e^t \leq e^{0.5} < 3$. So $|R_3| \leq \frac{3}{4!}(0.5)^4 = \frac{3}{24} \cdot \frac{1}{16} \approx 0.0078$.

The true value is 1.64872, so the actual error 0.0029 is inside the bound.

EXAMPLE 6

Find the degree-3 Taylor polynomial of $f(x) = \ln x$ centered at $x = 1$, and use it to approximate $\ln 1.2$ with an error bound.

SOLUTION

Derivatives at 1: $f(1) = 0$, $f'(x) = \frac{1}{x} \rightarrow 1$, $f''(x) = -\frac{1}{x^2} \rightarrow -1$, $f'''(x) = \frac{2}{x^3} \rightarrow 2$.

$$P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{2}{6}(x-1)^3 = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}.$$

$$P_3(1.2) = 0.2 - 0.02 + 0.00267 = 0.18267.$$

Error: $f^{(4)}(x) = -\frac{6}{x^4}$, and for $1 \leq t \leq 1.2$, $|f^{(4)}| \leq 6$. So $|R_3| \leq \frac{6}{24}(0.2)^4 = 0.0004$. The true value is 0.18232, error 0.00035.

Coach's Playbook

- 1 Name the series type.** Is it geometric? A p -series? Does it have factorials or exponentials (ratio test)? Does it alternate?
 - 2 Check the n th term first.** If a_n does not go to zero, you are done: diverges. If it does go to zero, you still need another test.
 - 3 Match to a known series.** Compare to a geometric or p -series by looking at the dominant terms for large n .
 - 4 Write the test and its conclusion in a sentence.** On the AP exam, say which test, what you computed, and what it implies.
 - 5 For Taylor polynomials, build a table of derivatives at the center.** List $f^{(k)}(a)$ and divide by $k!$ before writing the polynomial.
- ◆ **Geometric first term:** write out the first two terms by plugging in the starting index. The first one you see is a .
 - ◆ **Ratio test:** put the ratio over a common denominator and cancel factorials and powers. $\frac{(n+1)!}{n!} = n + 1$.
 - ◆ **Alternating error:** the error is at most the size of the first omitted term, and has the same sign as that term.
 - ◆ **Taylor sanity check:** the polynomial and the function must agree at the center. Check by plugging in $x = a$.
 - ◆ **Error bound:** find M by bounding the $(n + 1)$ th derivative on the interval between a and x .

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Common traps

(1) Concluding a series converges because $a_n \rightarrow 0$; the harmonic series $\sum \frac{1}{n}$ is the standard counterexample. (2) Using $\frac{a}{1-r}$ with the wrong a when the series starts at $n = 1$. (3) Saying the ratio test shows divergence when $L = 1$; it is inconclusive. (4) Forgetting the factorial in the Taylor coefficient, using $f^{(k)}(a)$ instead of $\frac{f^{(k)}(a)}{k!}$.

Mistake Clinic**Spot the error**

Problem: Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge? **Student's work:** The terms $\frac{1}{n}$ go to 0, so the series converges. **What went wrong:** Terms going to zero is necessary but not sufficient. This is the harmonic series, a p -series with $p = 1$, which diverges: its partial sums grow without bound, though very slowly. **Correct solution:** $p = 1 \leq 1$, so the series diverges. Compare: $\sum \frac{1}{n^2}$ has $p = 2 > 1$ and converges.

Spot the error

Problem: Find $\sum_{n=1}^{\infty} 4 \left(\frac{1}{3}\right)^n$. **Student's work:** $a = 4$, $r = \frac{1}{3}$, so the sum is $\frac{4}{1-1/3} = 6$. **What went wrong:** The formula $\frac{a}{1-r}$ requires a to be the first term. The series starts at $n = 1$, so the first term is $4 \cdot \frac{1}{3} = \frac{4}{3}$, not 4. The student's answer includes an extra term, 4, that is not in the series. **Correct solution:** $\frac{4/3}{2/3} = 2$. Check: $6 - 4 = 2$.

Spot the error

Problem: Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge? **Student's work:** Ratio test: $\left| \frac{a_{n+1}}{a_n} \right| = \frac{n^2}{(n+1)^2} \rightarrow 1$. Since $L = 1$, the series diverges. **What went wrong:** When $L = 1$, the ratio test gives **no information**: it is inconclusive. Divergence requires $L > 1$. **Correct solution:** Use a different test. This is a p -series with $p = 2 > 1$, so it converges.

Real-World Connection

A ball is dropped from 10 feet and bounces back to 60 percent of its previous height each time. The total distance traveled is the first drop plus each bounce up and down: $10 + 2 \sum_{k=1}^{\infty} 10(0.6)^k = 10 + 2 \cdot \frac{6}{1-0.6} = 10 + 30 = 40$ feet. An infinite number of bounces still adds up to a finite distance.

Challenge Corner

Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. **Solution:** Use partial fractions: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. The partial sum telescopes: $S_N = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \dots + (\frac{1}{N} - \frac{1}{N+1}) = 1 - \frac{1}{N+1}$. As $N \rightarrow \infty$, $S_N \rightarrow 1$. The series converges to 1.

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Practice

PRACTICE Introduction to Sequences and Series for BC

Show your work on paper. Use a calculator only when a problem says so, or if it is allowed at your level.

- What is $\sum_{n=0}^{\infty} 6 \left(\frac{1}{4}\right)^n$?
 (A) 8 (B) 24 (C) $\frac{24}{5}$ (D) $\frac{9}{2}$
- Which series converges?
 (A) $\sum_{n=1}^{\infty} \frac{1}{n}$ (B) $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ (C) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
 (D) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$
- Which statement is true?
 (A) If $a_n \rightarrow 0$, then $\sum a_n$ converges. (B) If $\sum a_n$ converges, then $a_n \rightarrow 0$.
 (C) If $a_n \rightarrow 0$, then $\sum a_n$ diverges. (D) If $\sum a_n$ diverges, then $a_n \rightarrow \infty$.
- What does the ratio test conclude about $\sum_{n=1}^{\infty} \frac{n!}{3^n}$?
 (A) Converges, because $L = \frac{1}{3}$ (B) Converges, because $L = 0$ (C) Diverges, because $L = \infty$
 (D) The test is inconclusive
- The alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ is approximated by S_4 . What is the best bound on the error?
 (A) $\frac{1}{125}$ (B) $\frac{1}{64}$ (C) $\frac{1}{216}$ (D) $\frac{1}{4}$
- What is the degree-2 Taylor polynomial of $f(x) = \sqrt{x}$ centered at $x = 4$?
 (A) $2 + \frac{x-4}{4} - \frac{(x-4)^2}{32}$ (B) $2 + \frac{x-4}{4} + \frac{(x-4)^2}{64}$ (C) $2 + \frac{x-4}{4} - \frac{(x-4)^2}{64}$
 (D) $2 + \frac{x-4}{2} - \frac{(x-4)^2}{64}$
- What is $\sum_{n=1}^{\infty} 3 \left(-\frac{1}{2}\right)^{n-1}$?
 (A) 6 (B) $\frac{3}{2}$ (C) 3 (D) 2
- What is the coefficient of x^3 in the Maclaurin series of e^{2x} ?
 (A) 8 (B) $\frac{4}{3}$ (C) $\frac{1}{3}$ (D) $\frac{8}{3}$
- Which test can be used to show that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges?
 (A) Comparison with the divergent series $\sum \frac{1}{n}$
 (B) The nth-term test
 (C) The ratio test, which gives $L = 1$ and so convergence
 (D) Comparison with the convergent series $\sum \frac{1}{n^2}$
- Find $\sum_{n=0}^{\infty} 7 \left(-\frac{2}{5}\right)^n$.
- Write the repeating decimal $0.272727\ldots$ as a fraction in lowest terms, using a geometric series.
- Determine whether $\sum_{n=1}^{\infty} \frac{2n+1}{5n-3}$ converges. State the test and the key limit.
- Use the degree-3 Taylor polynomial of $f(x) = \ln x$ centered at $x = 1$ to approximate $\ln(1.2)$. Give your answer to five decimal places.
- Use the degree-4 Maclaurin polynomial of $\cos x$ to approximate $\cos(1)$. Give an exact fraction.

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15. For the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$, what is the smallest number of terms N so that the alternating series bound guarantees $|S - S_N| < 0.005$?

Answers**Answers and Explanations**

Complete solutions for every practice problem, numbered to match. Read the explanation even when your answer is right.

40.1 Area Between Curves**PRACTICE · AREA BETWEEN CURVES**

- B. The line $y = x$ is above $y = x^2$ on $[0, 1]$: $\int_0^1 (x - x^2) dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. Choice A is $\int x^2$ alone.
- D. The curves meet at ± 2 (since $6 - x^2 = x^2 - 2$ gives $x^2 = 4$), and the first is on top: $(6 - x^2) - (x^2 - 2) = 8 - 2x^2$. Choice A has the order reversed.
- C. The parabola is below the axis between -2 and 2 , so the area is $\int_{-2}^2 (0 - (x^2 - 4)) dx = \frac{32}{3}$. Choice A is the signed integral, which is negative.
- A. The curves meet at $y = 0$ and $y = 2$, with $2y$ on the right: $\int_0^2 (2y - y^2) dy = 4 - \frac{8}{3} = \frac{4}{3}$.
- D. The curves cross at $-1, 0, 1$ and the area is $2 \int_0^1 (x - x^3) dx = 2 \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{1}{2}$. Choice A uses one integral across the crossing.
- D. $\int_0^{\ln 3} (e^x - 1) dx = (3 - 1) - \ln 3 = 2 - \ln 3$. Choice C forgets to subtract the lower value $e^0 = 1$.
- C. They meet at $x = 0$ and 4 , with $\sqrt{x}$ on top: $\int_0^4 (\sqrt{x} - \frac{x}{2}) dx = \frac{16}{3} - 4 = \frac{4}{3}$. Choice B is just $\int \sqrt{x}$.
- B. $\frac{1}{x} \geq \frac{1}{x^2}$ for $x \geq 1$: $\int_1^2 (\frac{1}{x} - \frac{1}{x^2}) dx = \ln 2 - \frac{1}{2}$. Choice A adds the integral of $\frac{1}{x^2}$ instead of subtracting it.
- C. The gap $f - g$ is $4, 4, 3, 1$. Trapezoids: $\frac{4+4}{2} + \frac{4+3}{2} + \frac{3+1}{2} = 4 + 3.5 + 2 = 9.5$. Choice A is the left sum and B is the right sum.
- $\frac{8}{3}$. They meet where $x^2 + 1 = 3 - x^2$, so $x = \pm 1$. Then $\int_{-1}^1 (2 - 2x^2) dx = 2 \left(2 - \frac{2}{3} \right) = \frac{8}{3}$.
- $\frac{125}{6}$. Set $x^2 = x + 6$, so $x = -2, 3$. Then $\int_{-2}^3 (x + 6 - x^2) dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_{-2}^3 = \frac{125}{6}$.
- $\frac{1}{4}$. On $[0, \frac{\pi}{3}]$, $\sin 2x \geq \sin x$ (they meet at 0 and $\frac{\pi}{3}$). $\int_0^{\pi/3} (\sin 2x - \sin x) dx = \left[-\frac{\cos 2x}{2} + \cos x \right]_0^{\pi/3} = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$.
- $\frac{9}{8}$. Set $2y^2 = y + 1$, so $y = -\frac{1}{2}$ and $y = 1$. The line $x = y + 1$ is on the right. $\int_{-1/2}^1 (y + 1 - 2y^2) dy = \frac{9}{8}$.
- $\frac{81}{2}$. The roots are $-3, 0, 3$. By symmetry the area is $2 \int_0^3 (9x - x^3) dx = 2 \left(\frac{81}{2} - \frac{81}{4} \right) = \frac{81}{2}$.
- $\sqrt[3]{4} \approx 1.587$. The total area is $\int_0^2 x^2 dx = \frac{8}{3}$. Need $\int_0^k x^2 dx = \frac{k^3}{3} = \frac{4}{3}$, so $k^3 = 4$.

40.2 Volumes of Solids**PRACTICE · VOLUMES OF SOLIDS**

- C. $\pi \int_0^2 x^4 dx = \pi \cdot \frac{32}{5}$. Choice A integrates x^2 (the radius not squared), and B drops the π .
- B. $\pi \int_0^9 x dx = \frac{81\pi}{2}$. Choice A is $\pi \int \sqrt{x}$, which never squares the radius.
- C. Outer radius $2x$, inner x^2 , meeting at 0 and 2 : $\pi \int_0^2 (4x^2 - x^4) dx = \pi \left(\frac{32}{3} - \frac{32}{5} \right) = \frac{64\pi}{15}$. Choice A squares the difference of radii.
- D. Use washers in y : outer radius 2 , inner radius $\sqrt{y}$, from 0 to 4 . $\pi \int_0^4 (4 - y) dy = \pi(16 - 8) = 8\pi$. Choice A is the volume about the x -axis.
- D. $s = 3 - x$, so $V = \int_0^3 (3 - x)^2 dx = 9$. Choice A forgets to square the side.
- B. Diameter $\sqrt{x}$ gives area $\frac{\pi}{8}(\sqrt{x})^2 = \frac{\pi x}{8}$. $V = \frac{\pi}{8} \int_0^4 x dx = \pi$. Choice C uses $\frac{\pi}{4}s^2$ (a full-radius error).
- D. Outer radius $x^2 + 1$, inner radius 1 : $\pi \int_0^1 [(x^2 + 1)^2 - 1] dx = \pi \left(\frac{1}{5} + \frac{2}{3} \right) = \frac{13\pi}{15}$. Choice A forgets the hole.
- A. Radius $4 - x^2$ from -2 to 2 : $\pi \int_{-2}^2 (4 - x^2)^2 dx = \frac{512\pi}{15}$. Choice B is half of this, from using only $[0, 2]$.
- C. Side $\sqrt{x}$: $\frac{\sqrt{3}}{4}x$. $V = \frac{\sqrt{3}}{4} \int_0^4 x dx = 2\sqrt{3}$. Choice B is twice the correct value, from using side $2\sqrt{x}$.
- 24π . $\pi \int_0^2 (3x)^2 dx = 9\pi \cdot \frac{8}{3} = 24\pi$ (a cone).
- $\frac{128\pi}{7}$. $\pi \int_0^2 x^6 dx = \pi \cdot \frac{128}{7}$.
- $\frac{\pi}{6}$. The curves meet at 0 and 1 , with $\sqrt{x}$ above x . $\pi \int_0^1 (x - x^2) dx = \pi \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{6}$.
- $\frac{40\pi}{3}$. Use y -slices: outer radius $4 - \sqrt{y}$, inner radius $4 - 2 = 2$. $\pi \int_0^4 [(4 - \sqrt{y})^2 - 4] dy = \pi \int_0^4 (12 - 8\sqrt{y} + y) dy = \pi \left(48 - \frac{128}{3} + 8 \right) = \frac{40\pi}{3}$.
- $\frac{256}{15}$. Leg $s = 4 - x^2$, area $\frac{1}{2}s^2$. $V = \frac{1}{2} \int_{-2}^2 (4 - x^2)^2 dx = \frac{1}{2} \cdot \frac{512}{15} = \frac{256}{15}$.
- $\frac{2\pi}{15}$. The curves meet at 0 and 2 , diameter $s = 2x - x^2$. $V = \frac{\pi}{8} \int_0^2 (2x - x^2)^2 dx = \frac{\pi}{8} \left(\frac{32}{3} - 16 + \frac{32}{5} \right) = \frac{\pi}{8} \cdot \frac{16}{15} = \frac{2\pi}{15}$.

40.3 Differential Equations and Slope Fields

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PRACTICE · DIFFERENTIAL EQUATIONS AND SLOPE FIELDS

1. B. For $y = 5e^{3x}$: $y' = 15e^{3x} = 3y$. Choice A has derivative $15e^{5x}$, not $3y$.
2. C. Multiply both sides by y^2 and by dx : $y^2 dy = x dx$. Choice D leaves a y factor on the wrong side, and A drops the square.
3. D. $\frac{dy}{y} = \frac{dx}{x}$ gives $\ln|y| = \ln x + C$, so $y = Cx$. Choice B would solve $y' = y$.
4. A. $\frac{dy}{y} = 6x^2 dx$ gives $\ln|y| = 2x^3 + C$, so $y = Ae^{2x^3}$, and $y(0) = 2$ gives $A = 2$. Choice B forgets to integrate the exponent.
5. B. At $(0, 1)$ the field shows a rising segment and at $(1, 0)$ a falling one, so the slope is $y - x$. Choice A has the opposite signs. Along the line $y = x$ the slope is 0 in the picture, which rules out $x + y$ and xy .
6. A. The solution is $P = 50e^{0.2t}$, so $P(5) = 50e^1 = 50e$. Choice B uses $t = 1$.
7. C. Twelve years is three half-lives: $80 \cdot (\frac{1}{2})^3 = 10$. Choice A applies only one half-life.
8. A. $y dy = e^x dx$ gives $\frac{y^2}{2} = e^x + C$; $y(0) = 2$ gives $C = 1$, so $y^2 = 2e^x + 2$. At $x = \ln 3$: $y^2 = 8$ and $y = 2\sqrt{2}$ (positive since $y(0) > 0$). Choice B adds the initial value.
9. C. Constant solutions make $\frac{dy}{dx} = 0$, so $(y - 1)(y - 3) = 0$ and $y = 1$ or $y = 3$. Choice D misses a root.
10. $y = 2e^{x^3}$. $\frac{dy}{y} = 3x^2 dx$ gives $\ln|y| = x^3 + C$, so $y = Ae^{x^3}$, and $A = 2$.
11. $y(4) = -5$. $y dy = x dx$ gives $y^2 = x^2 + K$; $y(0) = -3$ gives $K = 9$. Since $y(0) < 0$ we take the negative root: $y = -\sqrt{x^2 + 9}$, so $y(4) = -\sqrt{25} = -5$.
12. 1600. Doubling every 5 hours gives $P = 200 \cdot 2^{t/5}$, so $P(15) = 200 \cdot 2^3 = 1600$.
13. 32.053 minutes. From $60 = 20 + 75e^{-10k}$: $e^{-10k} = \frac{40}{75}$, so $k = \frac{\ln 1.875}{10} \approx 0.06286$. Then $10 = 75e^{-kt}$ gives $t = \frac{\ln 7.5}{k} \approx 32.053$.
14. 2.4. The slope at $(1, 2)$ is $1 \cdot 2 = 2$, so $L(x) = 2 + 2(x - 1)$ and $L(1.2) = 2 + 0.4 = 2.4$.
15. 11. Let $u = y - 3$: $u' = 2u$, $u = Ae^{2x}$, and $u(0) = 2$ gives $A = 2$, so $y = 3 + 2e^{2x}$. At $x = \ln 2$: $e^{2 \ln 2} = 4$, so $y = 3 + 8 = 11$.

40.4 Introduction to Sequences and Series for BC**PRACTICE · INTRODUCTION TO SEQUENCES AND SERIES FOR BC**

1. A. $a = 6$, $r = \frac{1}{4}$, so $S = \frac{6}{1 - 1/4} = \frac{6}{3/4} = 8$. Choice B divides by r instead of $1 - r$.
2. D. $\frac{1}{n^{3/2}}$ is a p -series with $p = \frac{3}{2} > 1$. Choices A and C have $p \leq 1$, and B has terms approaching $\frac{1}{2}$, not 0.
3. B. A convergent series must have terms that go to zero (nth-term test, contrapositive). The converse is false, as the harmonic series shows, so choice A is wrong.
4. C. $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} = \frac{n+1}{3} \rightarrow \infty > 1$, so the series diverges. Choice B reverses the idea.
5. A. The error is at most the first omitted term, $a_5 = \frac{1}{5^3} = \frac{1}{125}$. Choice B uses a_4 , the last term kept.
6. C. $f(4) = 2$, $f'(4) = \frac{1}{4}$, $f''(4) = -\frac{1}{32}$. Dividing by $2!$ gives $-\frac{1}{64}$. Choice A forgets the factorial.
7. D. The first term is 3 and the ratio is $-\frac{1}{2}$: $\frac{3}{1 + 1/2} = 2$. Choice C is just the first term.
8. B. $e^{2x} = \sum \frac{(2x)^n}{n!}$, so the x^3 term is $\frac{8}{3!}x^3 = \frac{4}{3}x^3$. Choice A forgets the factorial.
9. D. Since $\frac{1}{n^2+1} < \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges, the smaller series converges. Choice A compares to a divergent series, which proves nothing. Choice C is wrong because $L = 1$ is inconclusive.
10. 5. $r = -\frac{2}{5}$ and $a = 7$, so $S = \frac{7}{1 + 2/5} = \frac{7}{7/5} = 5$.
11. $\frac{3}{11}$. $0.2727 \dots = 0.27 + 0.0027 + \dots$ has $a = 0.27$ and $r = 0.01$, so $S = \frac{0.27}{0.99} = \frac{27}{99} = \frac{3}{11}$.
12. Diverges. $\lim_{n \rightarrow \infty} \frac{2n+1}{5n-3} = \frac{2}{5} \neq 0$, so by the nth-term test the series diverges.
13. 0.18267. $P_3(1.2) = 0.2 - \frac{0.04}{2} + \frac{0.008}{3} = 0.2 - 0.02 + 0.002667 = 0.18267$.
14. $\frac{13}{24}$. ≈ 0.5417 . $P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$, so $P_4(1) = 1 - \frac{1}{2} + \frac{1}{24} = \frac{13}{24}$.
15. 14. The error bound after N terms is $\frac{1}{(N+1)^2}$. We need $(N+1)^2 > 200$. Since $14^2 = 196 < 200$ but $15^2 = 225 > 200$, $N+1 \geq 15$, so $N = 14$.

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